Integration-by-parts identities and multi-loop QCD amplitudes

Herschel A. Chawdhry

Cavendish Laboratory
University of Cambridge

IOP HEPP–APP Annual Conference
8th April 2019

Based on hep-ph/1805.09182 with M. A. Lim and A. Mitov

Introduction

- Precise calculations of scattering amplitudes in gauge theories (such as QCD) require the evaluation of multi-loop diagrams
- Multi-loop integrals are often evaluated using integration-by-parts (IBP) identities
- We introduce a new strategy for solving IBP identities
- As an example, we use it to reduce all of the planar integrals in any massless 2-loop 5-point QCD scattering amplitude

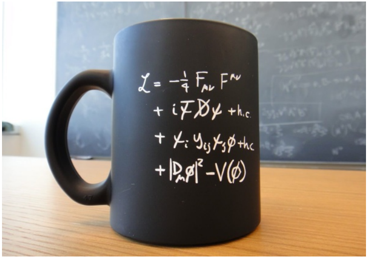
Outline

UNIVERSITY OF CAMBRIDGE

Outline

Why scattering amplitudes?

- Theory:

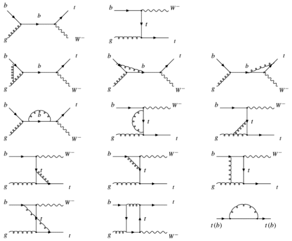


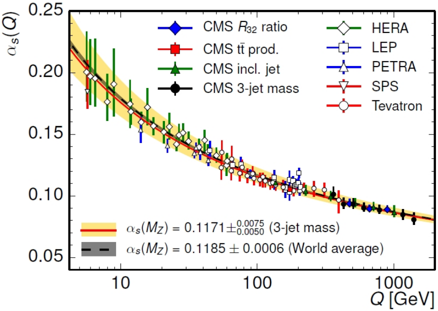
- Experiment:
  - masses
  - decay rates
  - cross-sections

Images: `www.quantumdiaries.org` & Andre-Pierre Olivier

UNIVERSITY OF CAMBRIDGE erc

Scattering amplitudes at particle colliders





$$\sigma_{\text{partonic}} = \sigma_{\text{LO}}\alpha_s^n + \sigma_{\text{NLO}}\alpha_s^{n+1} + \sigma_{\text{NNLO}}\alpha_s^{n+2} + \ldots$$

Images: Shouhua Zhu `arXiv:hep-ph/0109269` & CMS `arXiv:1412.1633`

Outline

Why 5-point QCD amplitudes?

- NNLO phenomenology at the LHC has cleared $2 \to 2$ processes
- However, $2 \to 3$ processes remain an open problem.
  - e.g. 3-jet production; $H + 2$ jets; $\gamma\gamma + 1$ jet
- 2-loop amplitudes are the biggest missing ingredient
- Many developments in recent years:
  - Badger, Frellesvig, Zhang (2013)
  - Ita (2015)
  - Badger, Mogull, Ochirov, O'Connell (2015)
  - Gehrmann, Henn, Presti (2015)
  - Dunbar, Perkins (2016)
  - Dunbar, Godwin, Jehu, Perkins (2017)
  - Badger, Brønnum-Hansen, Hartanto, Peraro (2017)
  - Abreu, Cordero, Ita, Page, Zeng (2017)
  - Böhm, Georgoudis, Larsen, Schönemann, Zhang (2018)
  - Kosower (2018)
  - HAC, Lim, Mitov (2018)
  - Badger, Brønnum-Hansen, Gehrmann, Hartanto, Henn, Lo Presti, Peraro (2018)
  - Abreu, Cordero, Ita, Page, Zeng (2018)
  - Gehrmann, Henn, Lo Presti (2018)
  - Abreu, Page, Zeng (2018)
  - Chicherin, Gehrmann, Henn, Lo Presti, Mitev, Wasser (2018)
  - Abreu, Cordero, Ita, Page, Sotnikov (2018)
  - Badger, Brönnum-Hansen, Hartanto, Peraro (2018)
  - Abreu, Dormans, Cordero, Ita, Page (2019)
  - Abreu, Dormans, Cordero, Ita, Page, Sotnikov (2019)
  - ...

Outline

Why IBPs?

- The IBPs are linear relations between Feynman integrals
- They contain an incredible amount of information about the problem
- They have been widely successful in the computation of multi-loop QCD amplitudes
- Some problems have so far remained beyond the reach of current IBP-solving methods
  - e.g. $2 \to 3$ at 2 loops; $2 \to 2$ at 3 loops; massive $2 \to 2$ at 2 loops
- Highly desirable to improve methods of solving IBPs
- Note: KIRA v1.1 makes steps in a similar direction

Outline

Integration-by-parts (IBP) identities

- Generic bare amplitude is a sum of many Feynman integrals

$$M = \sum_{i=1}^{N} f_i I_i$$

  - I_i are (wlog) scalar Feynman integrals

$$I_i = \int d^d k_1 \dots d^d k_L \left[\frac{1}{\Pi_1^{a_i} \Pi_2^{b_i} \dots} \right], \text{ where } \Pi \sim \left(q^2(p,k) - m^2 \right)$$

  - N is large: in our 2-loop 5-point case, we have $N \sim \mathcal{O}(10^4 - 10^5)$
  - f_i are rational functions, whereas I_i are dilogs/polylogs/etc.
- Integration-by-parts (IBP) identities:

$$\int d^d k_1 \dots d^d k_L \left[\frac{\partial}{\partial k_j^\mu} \left(\frac{v^\mu}{\Pi_1^a \Pi_2^b \dots} \right) \right] = 0$$

Hence, many linear relations between integrals.

Integration-by-parts (IBP) identities (continued)

- *The challenge:* Solve this system of equations to express all integrals in terms of a small basis of master integrals.

$$I_i = \sum_{m=1}^{\hat{N}} c_{i,m} \hat{I_m}$$

  - in our 2-loop 5-point case, we have $\hat{N} \sim \mathcal{O}(100)$
- Can hence write original amplitude in terms of master integrals

$$M = \sum_{m=1}^{\hat{N}} \hat{c_m} \hat{I_m}, \text{ with } \hat{c_m} = \sum_{i=1}^{N} c_{i,m} f_i$$

- In our work we focus on solving the IBPs. Evaluating the masters is a separate problem (but the IBP solutions help here too).
  - In our 5-point case, all planar masters are already known, as well as some non-planars.

Outline

Our strategy

- Recall:
  - We want to write each integral in terms of master integrals
    $$I_i = \sum_{m=1}^{\hat{N}} c_{i,m} \hat{I}_m$$
  - We'll use the IBP equations. They are linear and homogeneous.

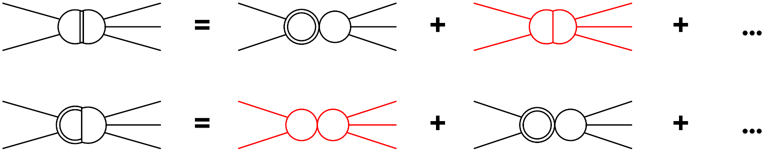
Our strategy:

- Generate the IBP equations
- Identify the master integrals (straight-forward e.g. using Reduze)
- Pick one master integral. Set all other masters to zero.
- Solve the (greatly simplified!) IBP equations
- This will give the projection of each integral onto the chosen master integral.
- Now repeat for each of the other masters; this way, we build up the full solution (i.e. we obtain each $c_{i,m}$).

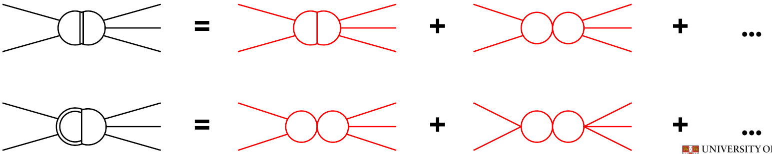
Note: we use Laporta algorithm but this is not essential.

Cartoon: Integration-by-parts identities (IBPs)

- IBP equations (coefficients not shown)



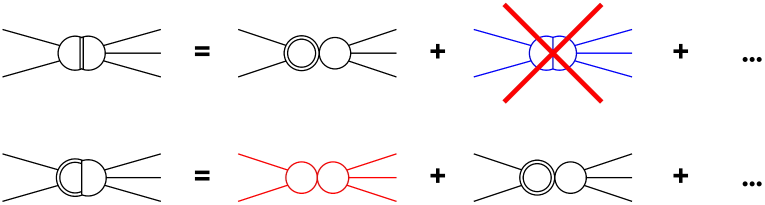
- Goal: Solve this system of equations to express all integrals in terms of a *small* basis of master integrals.



UNIVERSITY OF CAMBRIDGE

Cartoon: Our strategy

- Derive projections of the IBP equations onto a single master integral by setting all other master integrals to be zero

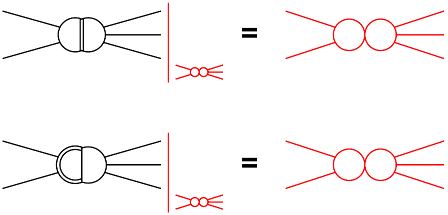


- This simplifies the IBP equations because many non-master integrals only project onto a subset of the master integrals

UNIVERSITY OF CAMBRIDGE

Cartoon: Our strategy (continued)

- By solving these simplified equations, one obtains the projections of all integrals onto a single master integral (coefficients still not shown)



- The full solution to the original IBP equations is obtained by repeating for each of the other master integrals and summing the solutions

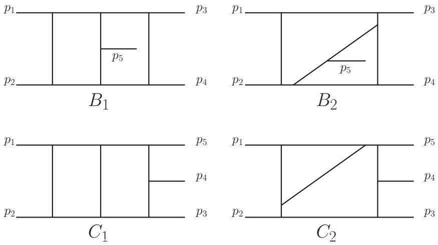
Benefits of this strategy

- Simplifies the problem
  - Many integrals only have projections onto a subset of all the possible masters. (A given integral, I, only projects onto a master, M, if the propagators of M are a subset of the propagators of I)
- Parallelisation
  - The IBP equations are "solved" many times – once for each master integral. These runs are independent of one another, so they can run in parallel.
  - Run times for different masters vary by several orders of magnitude. The overall running time is limited by a handful of 'difficult' masters.
- Reduced memory requirements
  - RAM usage is reduced, since far fewer coefficients need to be kept in memory at one time.

Outline

Application to 2 $\to$ 3: Setup

- We have implemented our strategy in a private C++ code and applied it to the 2-loop 5-point massless QCD amplitudes
- The most complicated 2-loop 5-point topologies have 8 propagators:



Outline

Application to 2 $\to$ 3: Results

- Full reduction for all planar (C_1 and C_2) integrals that contribute to 2-loop 5-point massless QCD processes.
- Some results for non-planar (B_1 and B_2) topologies:
  - Coefficients of the highest-weight masters, for all integrals with up to 6 numerator powers and up to 1 squared denominator
- Results available to download from: `www.precision.hep.phy.cam.ac.uk/results/amplitudes/`
  - Rational expressions are fully expanded
  - Files compressed (22GB)

Outline

Application to $2 \to 3$: Checks

- Computed $q\bar{q} \to Q\bar{Q}$ and cross-checked against `Reduze`
- Checked our results in the B_2 topology against those from `hep-th/1805.01873` (Böhm et al.)
- C_1 integrals with 5 numerator powers can be related to integrals with fewer numerator powers using `hep-th/1009.0472` (Gluza et al.) and `hep-th/1804.00131` (Kosower). We carried this out as a check and find full consistency with our solutions.

Outline

Conclusions and future work

- Proposed a new strategy for solving the IBP identities. Particularly useful for multi-scale problems.
- Derived analytic expressions for all integral coefficients needed to construct any planar 2-loop 5-point massless QCD amplitude with quarks and/or gluons. Results are now publicly available.
- Further reading – see our paper:

  `hep-ph/1805.09182`

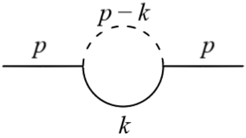
Outlook:

- We are working on the non-planar topologies
- Eventually intend to make our IBP-solving program public
- Ultimately, one would like a closed-form solution to the IBPs. This remains an open problem, although we hope our strategy could be of some help in this direction.

Section 6

Backup slides

Example: 1-loop bubble

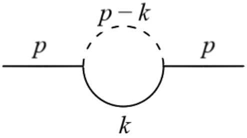


$$\mathcal{I}_{a,b}(p^2, m^2) = \int d^d k \frac{1}{\left[k^2 - m^2\right]^a \left[(p-k)^2\right]^b}$$

- Numerators absorbed: take $\int d^d k \left(\frac{N}{\left[k^2-m^2\right]^a\left[(p-k)^2\right]^b}\right)$, where N is:
  - m^2, p^2
  - $k^2 = (k^2 - m^2) + m^2$
  - $k \cdot p = -\frac{1}{2}(p-k)^2 + \frac{1}{2}p^2 + \frac{1}{2}k^2$
  - k^μ:
    - let $\int d^d k \left(\frac{k^\mu}{\left[k^2-m^2\right]^a\left[(p-k)^2\right]^b}\right) = Ap^\mu$
    - then $A = \frac{1}{p^2} \int d^d k \left(\frac{k \cdot p}{\left[k^2-m^2\right]^a\left[(p-k)^2\right]^b}\right)$
- More propagators upstairs → more complicated integral
- In general, might need extra propagators

Example: 1-loop bubble (2)

further details: see V. A. Smirnov, “Evaluating Feynman Integrals” (2004)

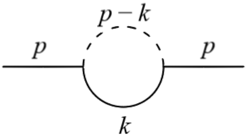


$$\mathcal{I}_{a,b}(p^2, m^2) = \int d^d k \frac{1}{\left[k^2 - m^2\right]^a \left[(p-k)^2\right]^b}$$

- Two IBP identities:
  1. $$\int d^d k \frac{\partial}{\partial k^\mu} \left(\frac{p^\mu}{[k^2 - m^2]^a \, [(p-k)^2]^b} \right) = 0$$
  2. $$\int d^d k \frac{\partial}{\partial k^\mu} \left(\frac{k^\mu}{[k^2 - m^2]^a \, [(p-k)^2]^b} \right) = 0$$

Example: 1-loop bubble (3)

further details: see V. A. Smirnov, "Evaluating Feynman Integrals" (2004)

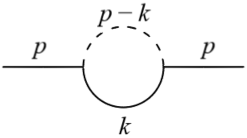


$$\mathcal{I}_{a,b}(p^2, m^2) = \int d^d k \frac{1}{\left[k^2 - m^2\right]^a \left[(p-k)^2\right]^b}$$

- Two IBP identities:
  1. $$(\mathrm{b}-\mathrm{a})\mathcal{I}_{\mathrm{a,b}} - (\mathrm{m}^2\mathrm{a} + \mathrm{p}^2\mathrm{a})\mathcal{I}_{\mathrm{a+1,b}} + \mathrm{a}\mathcal{I}_{\mathrm{a+1,b-1}} + (\mathrm{m}^2\mathrm{b} - \mathrm{p}^2\mathrm{b})\mathcal{I}_{\mathrm{a,b+1}} - \mathrm{b}\mathcal{I}_{\mathrm{a-1,b+1}} = 0$$
  2. $$(\mathrm{d}-2\mathrm{a}-\mathrm{b})\mathcal{I}_{\mathrm{a,b}} - 2\mathrm{m}^2\mathrm{a}\mathcal{I}_{\mathrm{a+1,b}} + (\mathrm{m}^2\mathrm{b} - \mathrm{p}^2\mathrm{b})\mathcal{I}_{\mathrm{a,b+1}} - \mathrm{b}\mathcal{I}_{\mathrm{a-1,b+1}} = 0$$
- Note: the identities are linear and homogeneous.

Example: 1-loop bubble (4)

further details: see V. A. Smirnov, "Evaluating Feynman Integrals" (2004)



$$\mathcal{I}_{a,b}(p^2, m^2) = \int d^d k \frac{1}{\left[k^2 - m^2\right]^a \left[(p-k)^2\right]^b}$$

General solution:

- $\mathcal{I}_{a,b} = f_{a,b}(p^2, m^2)\mathcal{I}_{1,0} + g_{a,b}(p^2, m^2)\mathcal{I}_{1,1}$

Special case: $b = 0$

- IBPs lead to recurrence relation: $\mathcal{I}_{a,0} = \frac{d-2a+2}{2(a-1)m^2}\mathcal{I}_{a-1,0}$
- Hence, closed-form solution: $\mathcal{I}_{a,0} = \frac{(-1)^a(1-d/2)}{\Gamma(a)(m^2)^{a-1}}\mathcal{I}_{1,0}$