Pseudo Entropy in dS/CFT and Time-like Entanglement Entropy

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Introduction

- **Holographic Principle** is an important approach to formulate the Quantum Gravity.
- **Quantum Entanglement** plays an important role to understand the non-perturbative features of holography.
- However, for de Sitter holography there has not been a consensus about holographic relation for quantum entanglement.
- In this talk, we propose a formulation of holographic relation for quantum entanglement in **dS/CFT**.

Contents

Holographic Entanglement Entropy

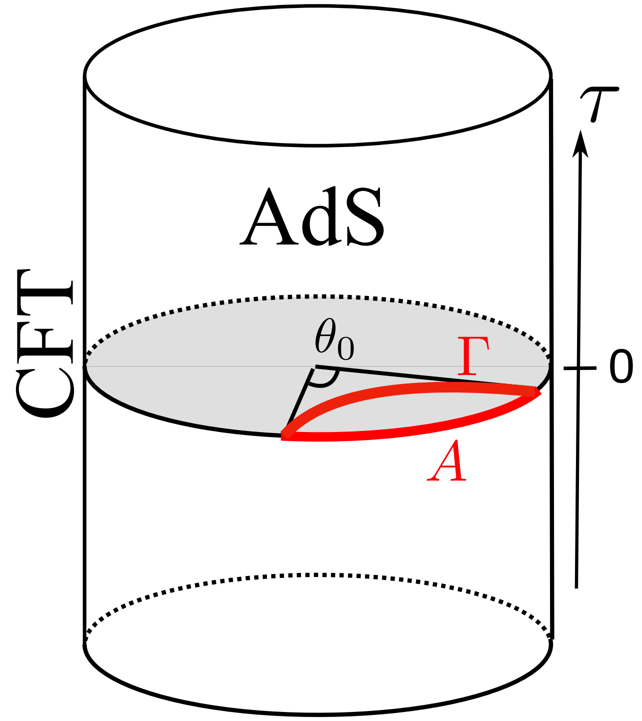
Ryu-Takayanagi formula

[Ryu, Takayanagi 2006]

$$S_A = \min_{\Gamma} \mathrm{ext} \left\{ \frac{\mathrm{Area}(\Gamma)}{4G_N} \,\middle|\, \Gamma : \text{bulk surfaces anchoring on } \partial A \right\}$$

- S_A: **Entanglement Entropy (EE)** for a subsystem A

$$S_A = -\mathrm{Tr}[\rho_A \log \rho_A] \, , \qquad \rho_A := \mathrm{Tr}_{\bar{A}}[|\psi\rangle\langle\psi|]$$



e.g.) EE of the vacuum in CFT on $\mathbb{R} \times \mathbb{S}^1$.

$$\mathrm{d}s^2 = R_{\mathrm{AdS}}^2(-\cosh^2\rho\, \mathrm{d}\tau^2 + \mathrm{d}\rho^2 + \sinh^2\rho\, \mathrm{d}\theta^2)$$

- Ryu-Takayanagi formula leads to

$$S_A = \frac{R_{\mathrm{AdS}}}{4G_N} \cosh^{-1}\left[\cosh^2\rho_\infty - \sinh^2\rho_\infty \cos^2\theta_0\right]$$

$$D(X,Y) = \cosh^{-1}(X^\mu Y_\mu)$$

$$\simeq \frac{R_{\mathrm{AdS}}}{2G_N} \log\left(\frac{2}{\epsilon}\sin\frac{\theta_0}{2}\right) = \frac{c}{3}\log\left(\frac{2}{\epsilon}\sin\frac{\theta_0}{2}\right) \qquad \left(\epsilon = 2e^{-\rho_\infty}, c = \frac{3R_{\mathrm{AdS}}}{2G_N}\right)$$

Pseudo Entropy

[Nakata, Takayanagi, Tamaoka, YT, Wei 2020]

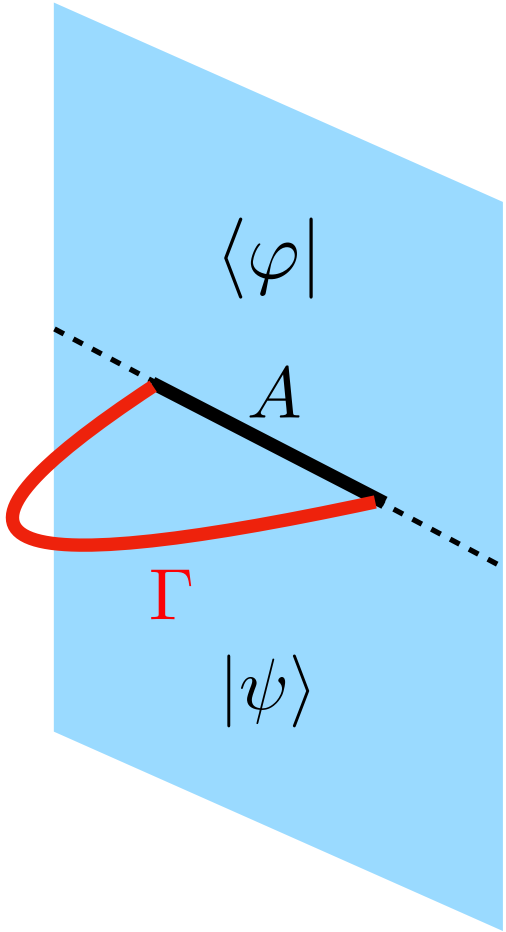
- **Pseudo Entropy (PE)** is a generalization of EE as follows:

Transition matrix: $\tau^{\psi|\varphi} := \frac{|\psi\rangle\langle\varphi|}{\langle\varphi|\psi\rangle}$ $\quad$ (for $\langle\varphi|\psi\rangle \neq 0$)

Pseudo entropy: $S_A = -\mathrm{Tr}\left[\tau_A^{\psi|\varphi} \log \tau_A^{\psi|\varphi}\right]$, $\quad \tau_A^{\psi|\varphi} := \mathrm{Tr}_{\bar{A}}\left[\tau^{\psi|\varphi}\right]$

- In general, PE takes complex-valued because transition matrix is non-Hermitian.

- The original motivation to introduce PE is to formulate the holographic relation for Euclidean geometry without time reversal symmetry.



Contents

dS/CFT

[Strominger 2001, Maldacena 2002]

Quantum Gravity
on dS_{d+1}

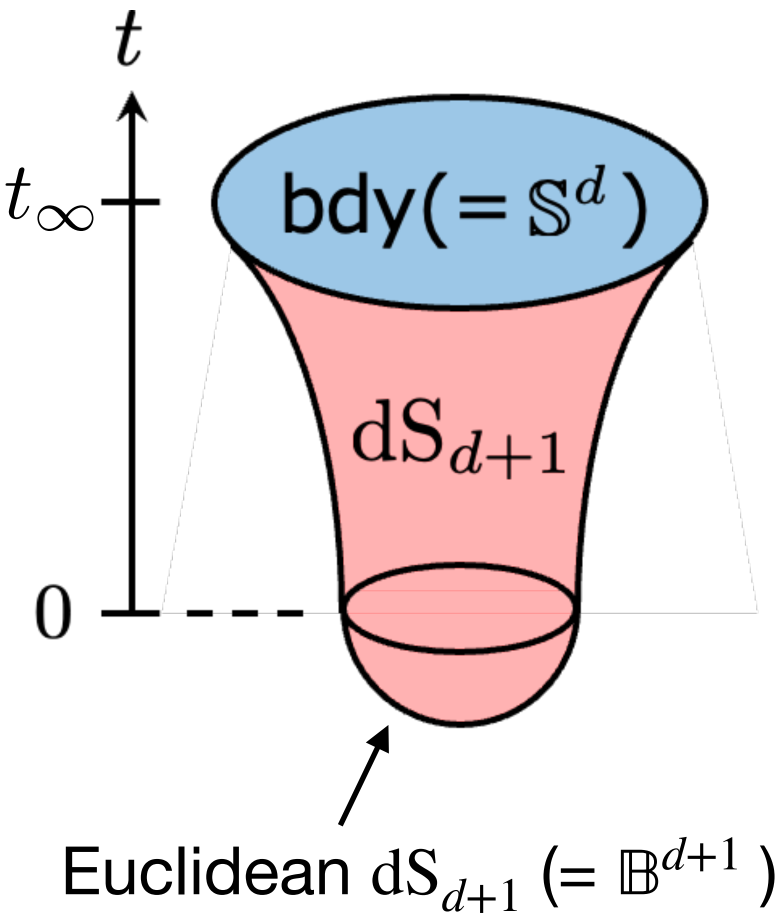
$\Longleftrightarrow$

Conformal Field Theory
on d-dim boundary

- de Sitter spacetime has the **space-like** boundary at the **temporal infinity.**

- GKPW relation holds on **the (Hartle-Hawking) wave functional of universe**:

$$\Psi_{\mathrm{dS}}\left[\phi|_{\mathrm{bdy}}=\phi^{(0)}\right]=\left\langle \exp\int \mathrm{d}^d x\ \phi^{(0)}(x)\mathcal{O}(x)\right\rangle_{\mathrm{CFT}}$$



Properties of dual CFT (d=2)

Euclidean AdS$_3$

$$ds^2 = R_{\mathrm{AdS}}^2 \frac{dz^2 + dt_{\mathrm{E}}^2 + dx^2}{z^2}$$

$$R_{\mathrm{AdS}} = -iR_{\mathrm{dS}}$$

$$z = -i\eta$$

$$\Longleftrightarrow$$

dS$_3$

$$ds^2 = R_{\mathrm{dS}}^2 \frac{-d\eta^2 + dt_{\mathrm{E}}^2 + dx^2}{\eta^2}$$

- The **central charge** c of CFT dual to dS$_3$: [Maldacena 2002]

$$c = \frac{3R_{\mathrm{AdS}}}{2G_{\mathrm{N}}} = -i\frac{3R_{\mathrm{dS}}}{2G_{\mathrm{N}}} \equiv -ic_{\mathrm{dS}} \quad \Longrightarrow \quad \text{non-unitary!}$$

[Brown, Henneaux 1986]

- Therefore the dual CFT is not “physical” theory. Nevertheless dS/CFT is useful in applications to
  - computation of cosmological correlators. [Maldacena 2002, …]
  - determination of complex saddles in wave function of universe. [Chen, Hikida, YT, Uetoko 2023]

Question

Can we formulate holographic relation for quantum entanglement in dS/CFT?

Holographic Entanglement Entropy in dS/CFT

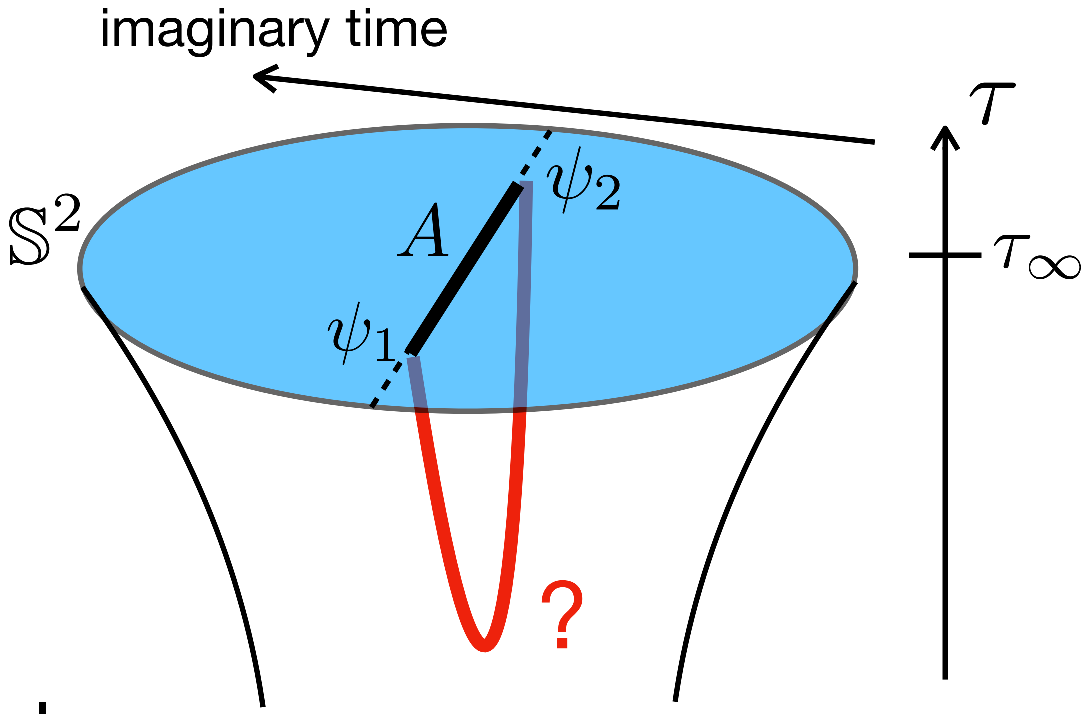
- In analogy with RT formula in AdS/CFT, EE in CFT dual to dS is also expected to be identical to the area of the extremal surface.

(c.f. [Narayan 2016] [Sato 2016])

global dS_3

$$ds^2 = R_{\mathrm{dS}}^2(-d\tau^2 + \cosh^2\tau d\Omega_{\mathbb{S}^2}^2)$$

$$d\Omega_{\mathbb{S}^2}^2 = d\psi^2 + \sin^2\psi\, d\phi^2$$



- However, this picture seems to have problems.

Geodesics emanating from an edge of A should be **time-like.**

(1) A geodesic emanating from one side of ∂A cannot go back to the other side of ∂A?

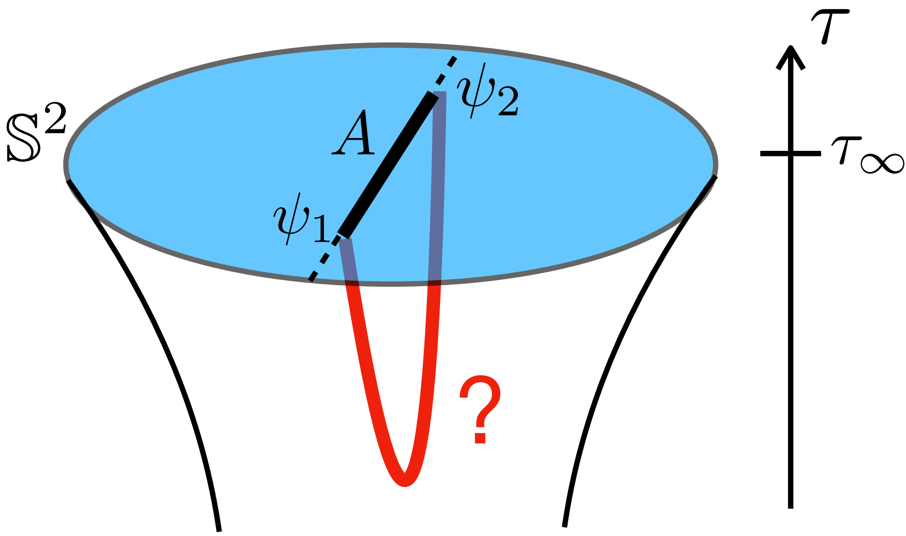
(2) The area $\int_\Gamma \sqrt{g}$ takes **imaginary-valued.**

Holographic Entanglement Entropy in dS/CFT

- Naively applying the formula to geodesic between (τ_∞, ψ_1) and (τ_∞, ψ_2)

$$\begin{aligned} S_A &= \frac{R_{\mathrm{dS}}}{4G_N} \cos^{-1}\left(1 - 2\sin^2\frac{\Delta\psi}{2}\cosh^2\tau_\infty\right) \qquad (\Delta\psi = \psi_1 - \psi_2) \\ &\simeq \frac{c_{\mathrm{dS}}}{6}\left[\pi + \cos^{-1}\left(\frac{2}{\epsilon^2}\sin^2\frac{\Delta\psi}{2}\right)\right] \qquad (\epsilon := 2e^{-\tau_\infty}, \epsilon \ll 1) \\ &\simeq \underbrace{\frac{\pi c_{\mathrm{dS}}}{6}}_{\text{real part}} + \underbrace{\frac{-ic_{\mathrm{dS}}}{3}\log\left(\frac{2}{\epsilon}\sin\frac{\Delta\psi}{2}\right)}_{\text{imaginary part}} \end{aligned}$$

→ there is no geodesic.



- However, this equals to analytic continuation of EE in CFT.

$$S_A = \frac{c_{\mathrm{AdS}}}{3}\log\left(\frac{L}{\epsilon_{\mathrm{AdS}}}\right) \xrightarrow[\substack{R_{\mathrm{AdS}} = -iR_{\mathrm{dS}} \\ \epsilon_{\mathrm{AdS}} = -i\epsilon_{\mathrm{dS}}}]{} S_A = \frac{\pi c_{\mathrm{dS}}}{6} + \frac{-ic_{\mathrm{dS}}}{3}\log\left(\frac{L}{\epsilon_{\mathrm{dS}}}\right)$$

Bulk complex geodesic

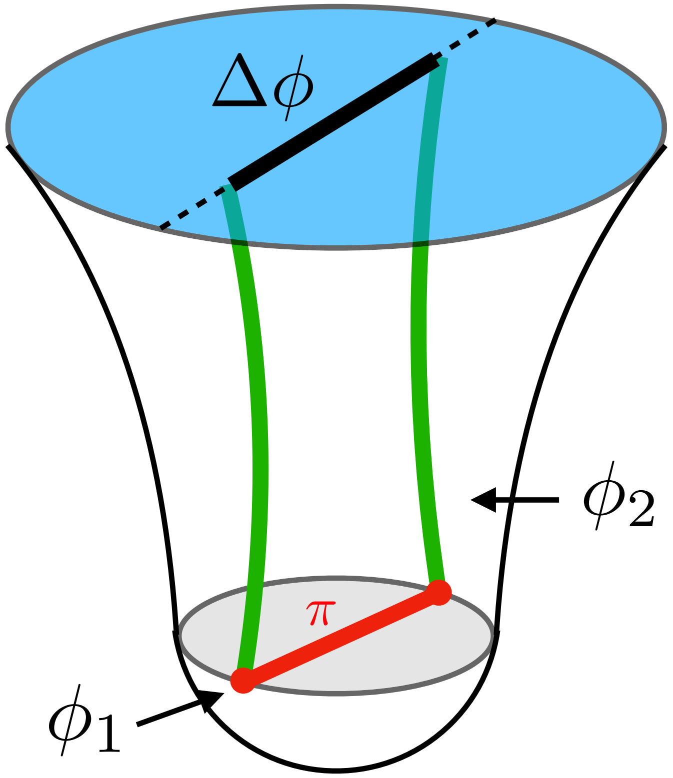
- How do we construct the geodesic configuration?

→ ansatz: **The complex geodesic goes through the Euclidean part.**

- Consider constant slice $\psi = 0$ of

$$ds^2 = R_{\rm dS}^2(-d\tau^2 + \cosh^2\tau(d\psi^2 + \cos^2\psi d\phi))$$

$$ds^2 = R_{\rm dS}^2(d\tau_E^2 + \cos^2\tau_E(d\psi^2 + \cos^2\psi d\phi))$$



- We have to impose “extremalization condition” for this ansatz.

- First we fix the joint points as ϕ_1 and ϕ_2.

Bulk complex geodesic

1. Solve the variation problem for each part.

Euclidean → Varying $D = R_{\mathrm{dS}} \int d\tau_E \sqrt{1 + \cos^2 \tau_E \, \phi'(\tau_E)^2}$

→ $D = \pi R_{\mathrm{dS}}, \quad \phi_2 - \phi_1 = \pi$

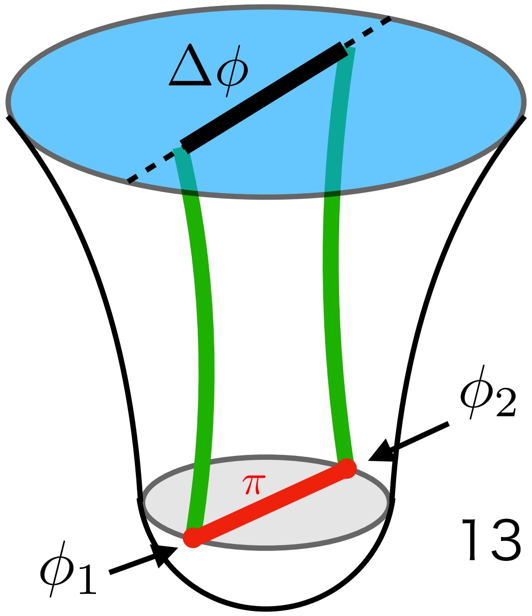
Lorentzian → Varying $D = R_{\mathrm{dS}} \int d\tau \sqrt{-1 + \cosh^2 \tau \, \phi'(\tau)^2}$

→ $D \simeq i R_{\mathrm{dS}} \log \left[-\frac{1}{2} e^{2\tau_\infty} \left(\cos \Delta\phi + \cos 2\phi_1 \right) \right]$

2. Solve $\frac{dD}{d\phi_1} = 0$, then we have

$$S_A = \underline{\frac{\pi c_{\mathrm{dS}}}{6}} + \frac{-i c_{\mathrm{dS}}}{3} \log \left(\frac{2}{\epsilon} \sin \frac{\Delta\phi}{2} \right)$$

A half of dS entropy



Higher dimensions

- We can straightforwardly extend the analysis to dS_{d+1}

$$ds^2 = R_{\mathrm{dS}}^2(-d\tau^2 + \cosh^2\tau d\Omega_{\mathbb{S}^d}^2)$$

with spherical subsystem A.

- The result is

$$= \frac{1}{2} \cdot \frac{R_{\mathrm{dS}}^{d-1}}{4G_N^{(d+1)}} \mathrm{Vol}(\mathbb{S}^{d-1})$$

A half of dS entropy

$$\underline{S_A = \frac{R_{\mathrm{dS}}^{d-1}}{4G_N^{(d+1)}} \mathrm{Vol}(\mathbb{S}^{d-2}) \frac{\sqrt{\pi}\Gamma\left(\frac{d-1}{2}\right)}{2\Gamma\left(\frac{d}{2}\right)}}$$

$$+i \frac{R_{\mathrm{dS}}^{d-1}}{4G_N^{(d+1)}} \mathrm{Vol}(\mathbb{S}^{d-2}) \begin{cases} \displaystyle\sum_{k=0}^{\frac{d-3}{2}} \binom{\frac{d-3}{2}}{k} \frac{1}{d-2k-2} \left(\frac{T_0}{2\epsilon}\right)^{d-2k-2} & (d : \mathrm{odd}) \\ \displaystyle\sum_{k=0}^{\frac{d}{2}-2} \binom{\frac{d-3}{2}}{k} \frac{1}{d-2k-2} \left(\frac{T_0}{2\epsilon}\right)^{d-2k-2} + \frac{\Gamma\left(\frac{d-1}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{d}{2}\right)} \log \frac{T_0}{2\epsilon} & (d : \mathrm{even}) \end{cases}$$

Holographic pseudo entropy in dS/CFT

- By definition, entanglement entropy should take real-valued.

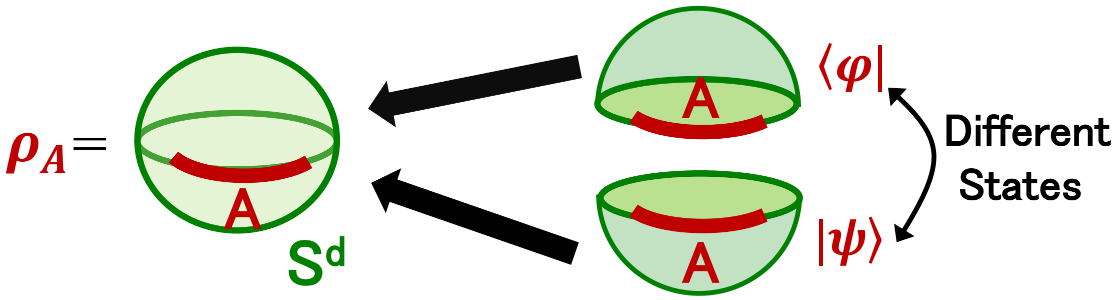
➡ Is the holographic formula in dS/CFT wrong?

- We claim that the quantity should be called “**pseudo entropy**” and it corresponds to a union of complex geodesics.

- This is because the dual CFT is a non-unitary theory associated with a non-Hermitian Hamiltonian $H^\dagger \neq H$.

➡ The bra and ket vectors prepared by path integral are different states:

$$|\Psi\rangle^\dagger \neq \langle\Psi|$$



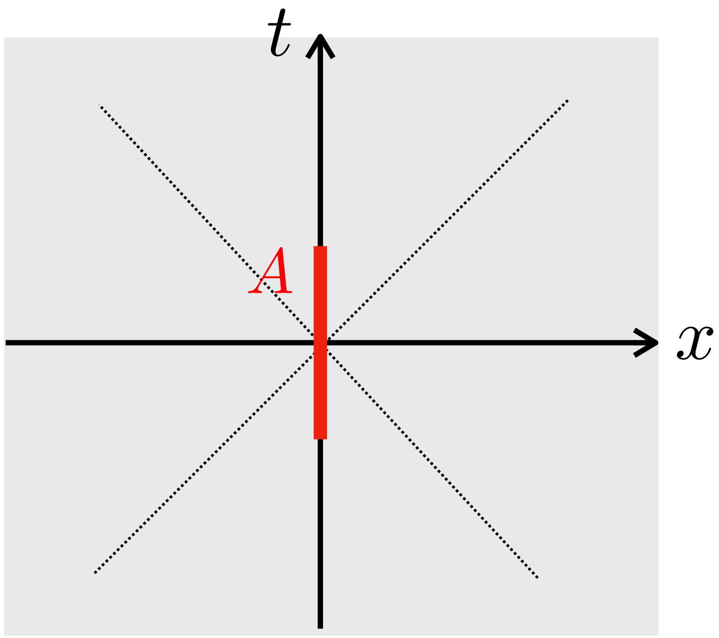
➡ $\rho = |\Psi\rangle\langle\Psi|$ is non-Hermitian, so it is rather **transition matrix.**

$S_A = -\mathrm{Tr}[\rho_A \log \rho_A]$ should be called **pseudo entropy.**

Contents

Timelike entanglement entropy

- We call the entanglement entropy for a timelike subsystem as **timelike entanglement entropy**



- We define it by using an analytic continuation.

- In 2d CFT, EE is given by

$$S_A = \frac{c}{3}\log\frac{L}{\epsilon}$$

Performing the Wick rotation $L \to iT_0$,

$$S_A = \frac{c}{3}\log\frac{iT_0}{\epsilon} = \frac{c}{3}\log\frac{T_0}{\epsilon} + \frac{i\pi c}{6}$$

- Similarly, we can also consider the finite temperature system.

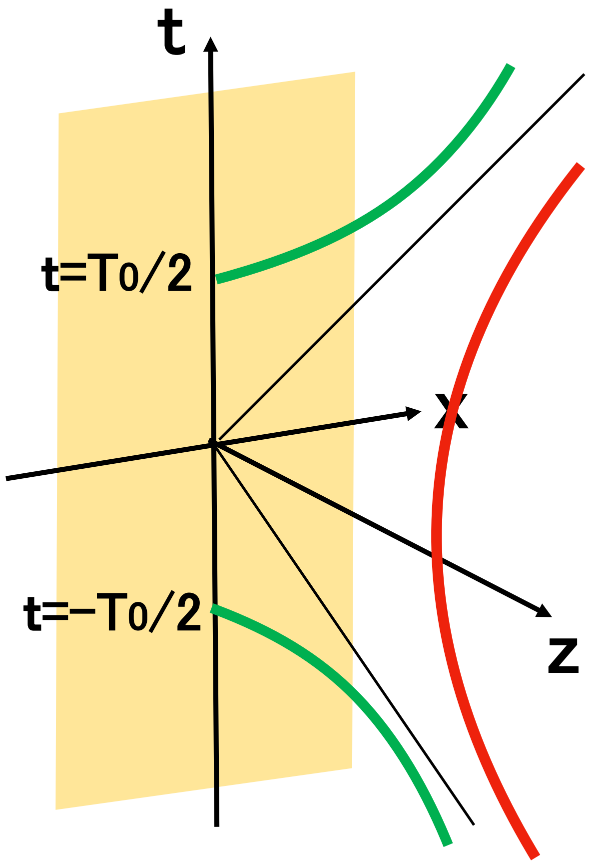
$$S_A = \frac{c}{3}\log\left[\frac{\beta}{\pi\epsilon}\sinh\frac{\pi T_0}{\beta}\right] + \frac{i\pi c}{6}$$

Time-like entanglement entropy in AdS$_3$/CFT$_2$

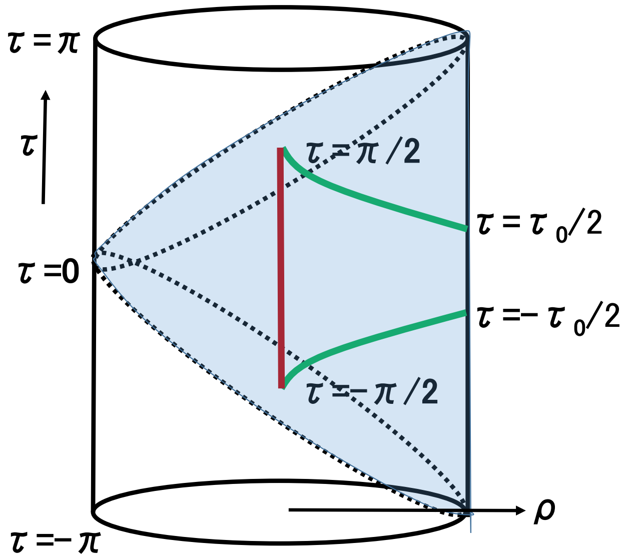
- We would like to give the bulk interpretation to

$$S_A = \underbrace{\frac{c}{3}\log\frac{T_0}{\epsilon}}_{\text{real part}} + \underbrace{\frac{i\pi c}{6}}_{\text{imaginary part}}$$

- We cannot connect two time-like separated points by single geodesic.



embedded in global AdS



space-like geodesic: $t^2 - z^2 = \left(\frac{T_0}{2}\right)^2$

$$\Rightarrow \quad \frac{\text{Area}}{4G_N} = \frac{2R_{\text{AdS}}}{4G_N}\int_{\epsilon}^{\infty} \mathrm{d}z \frac{\sqrt{1+t'(z)^2}}{z} = \frac{c_{\text{AdS}}}{3}\log\frac{T_0}{\epsilon}$$

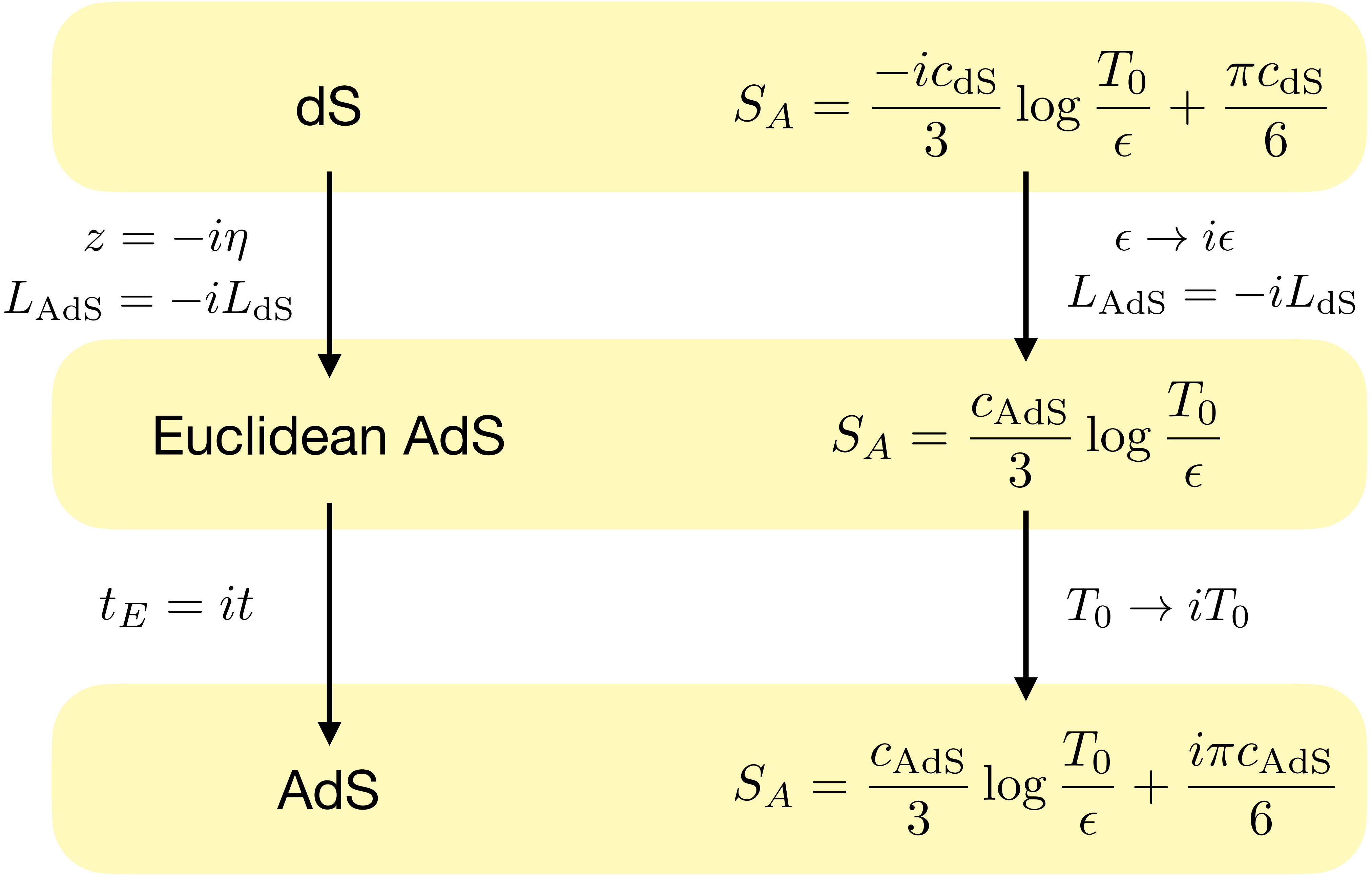
$$\left(c_{\text{AdS}} = \frac{3R_{\text{AdS}}}{2G_N}\right)$$

time-like geodesic: $z^2 - t^2 = \left(\frac{T_0}{2}\right)^2$

$$\Rightarrow \quad \frac{\text{Area}}{4G_N} = \frac{2R_{\text{AdS}}}{4G_N}\int_{T_0/2}^{\infty} \mathrm{d}z \frac{\sqrt{1+t'(z)^2}}{z} = \frac{i\pi c_{\text{AdS}}}{6}$$

Relation to pseudo entropy in dS/CFT

- We can relate the time-like entanglement entropy to pseudo entropy in dS/CFT by double analytic continuations.



dS

$$S_A = \frac{-ic_{\mathrm{dS}}}{3}\log\frac{T_0}{\epsilon} + \frac{\pi c_{\mathrm{dS}}}{6}$$

$z = -i\eta$

$L_{\mathrm{AdS}} = -iL_{\mathrm{dS}}$

$\epsilon \to i\epsilon$

$L_{\mathrm{AdS}} = -iL_{\mathrm{dS}}$

Euclidean AdS

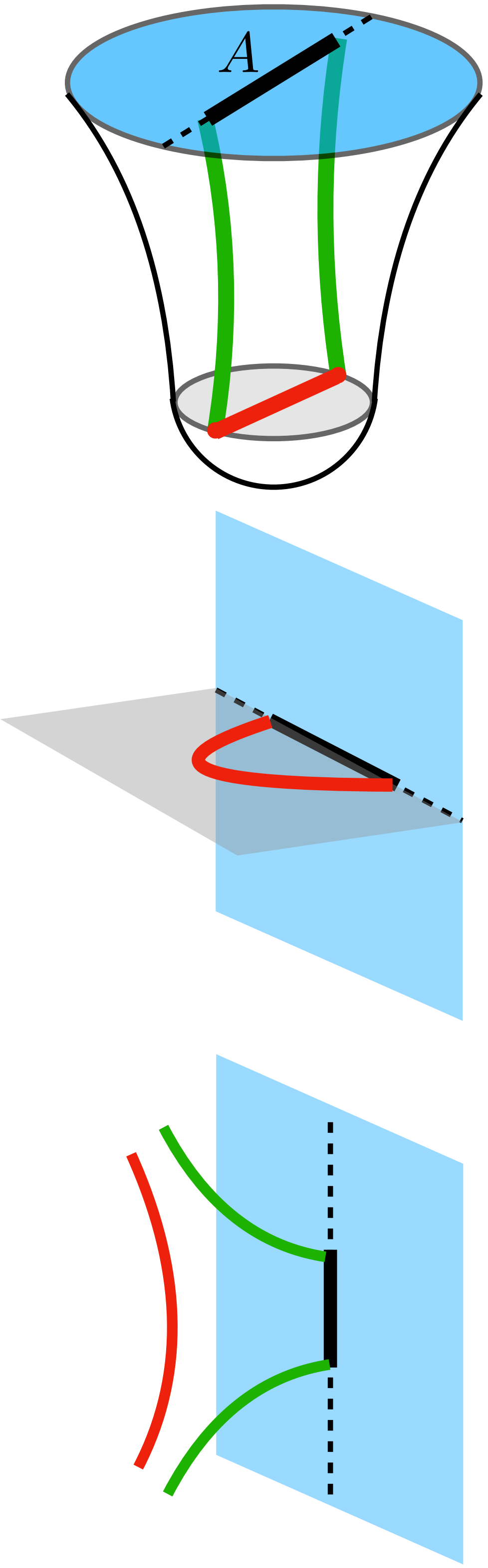
$$S_A = \frac{c_{\mathrm{AdS}}}{3}\log\frac{T_0}{\epsilon}$$

$t_E = it$

$T_0 \to iT_0$

AdS

$$S_A = \frac{c_{\mathrm{AdS}}}{3}\log\frac{T_0}{\epsilon} + \frac{i\pi c_{\mathrm{AdS}}}{6}$$



4. Summary and future problems

Summary

- We attempted to extend holographic EE formula to dS/CFT.
- We claim that the **counterpart of Ryu-Takayanagi formula in dS/CFT is given for pseudo entropy**, which takes complex-valued.
- The corresponding quantity of dS pseudo entropy in AdS/CFT is **time-like entanglement entropy**.

What we did for timelike EE

- In BTZ black hole
- Numerical study
- Higher dimensions