Non-perturbative Cosmological Bootstrap: Hilbert space and Källén–Lehmann representation

Kamran Salehi Vaziri
University of Amsterdam

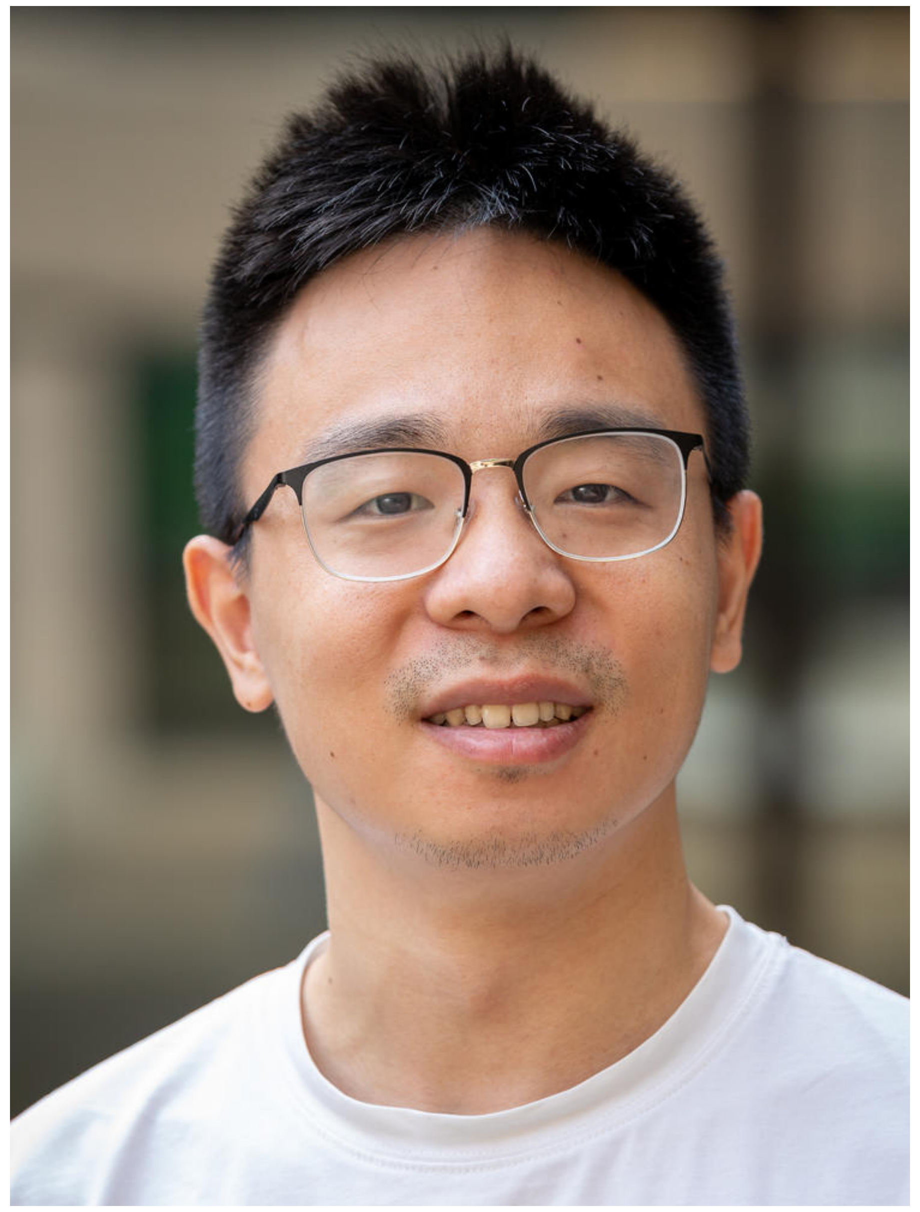
20 April 2023

Non-perturbative Cosmological Bootstrap: Hilbert space and Källén–Lehmann representation

(Not Cumrun)
Kamran Salehi Vaziri
University of Amsterdam

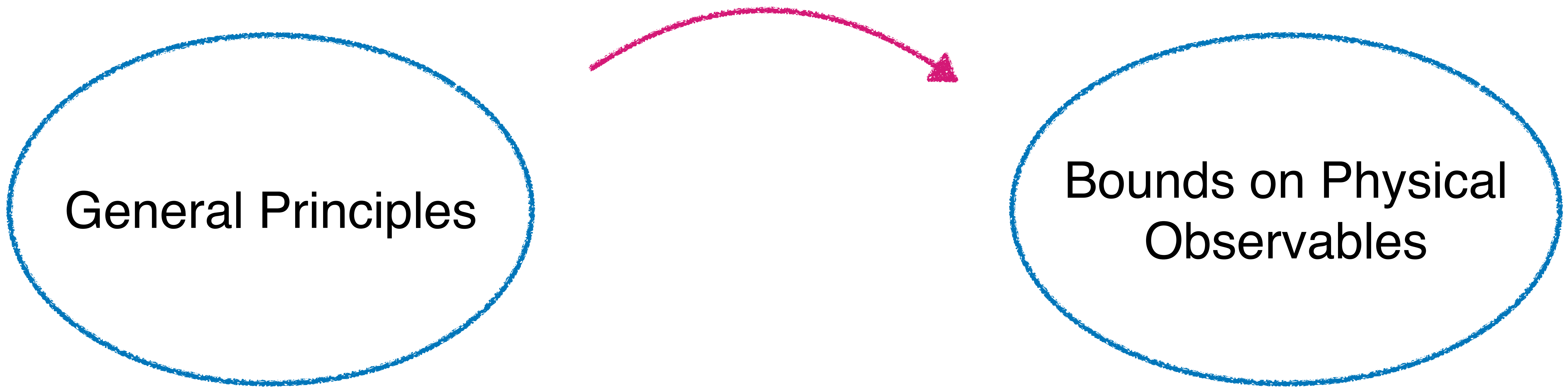
20 April 2023

arXiv:2107.13871 Matthijs Hogervorst, João Penedones, KSV
arXiv:2301.04146 João Penedones, KSV, Zimo Sun
Out soon: Manuel Loparco, João Penedones, KSV, Zimo Sun

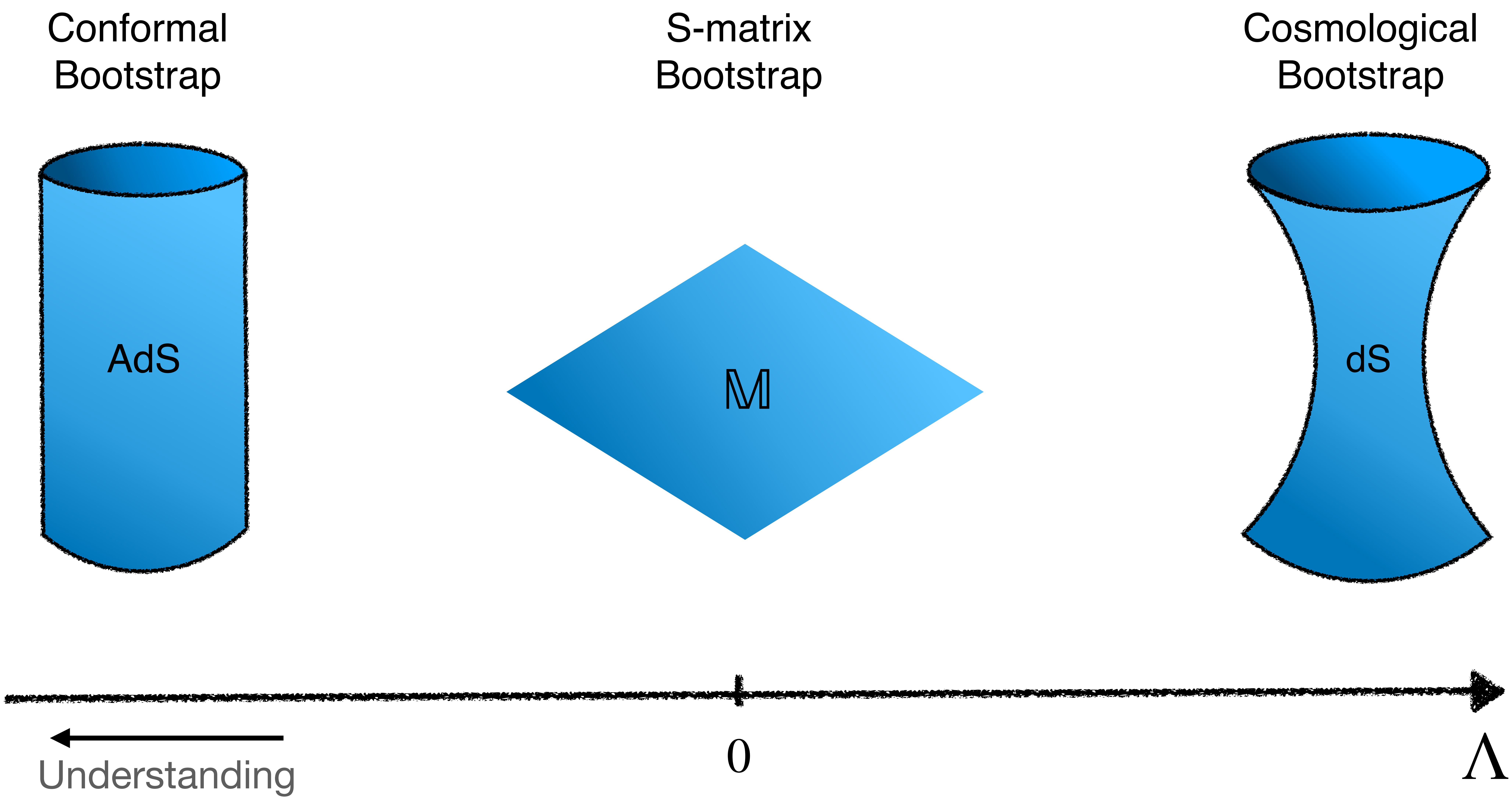



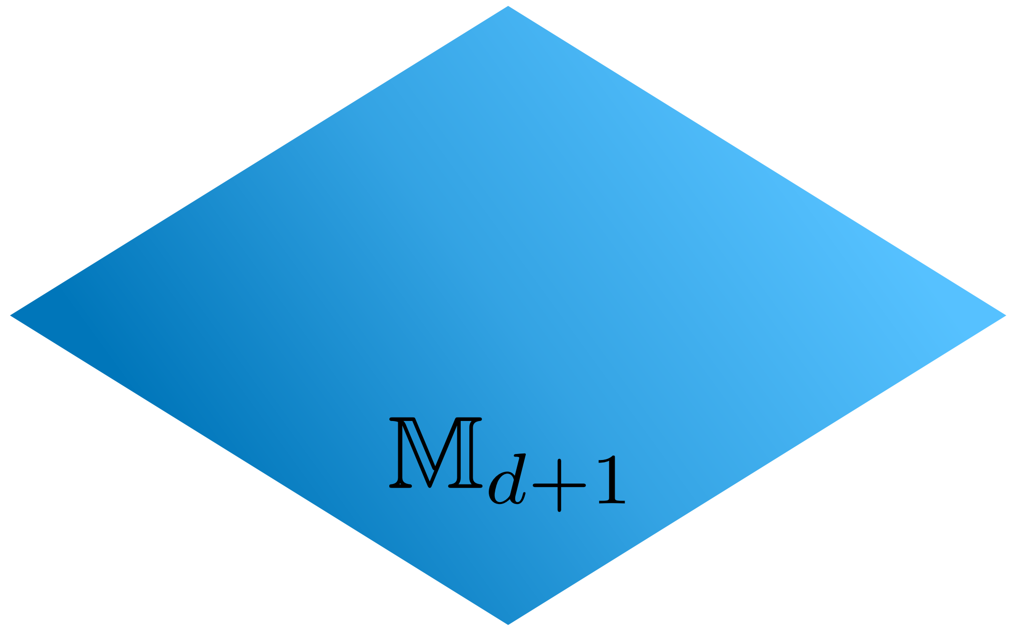
Motivation/Future direction:

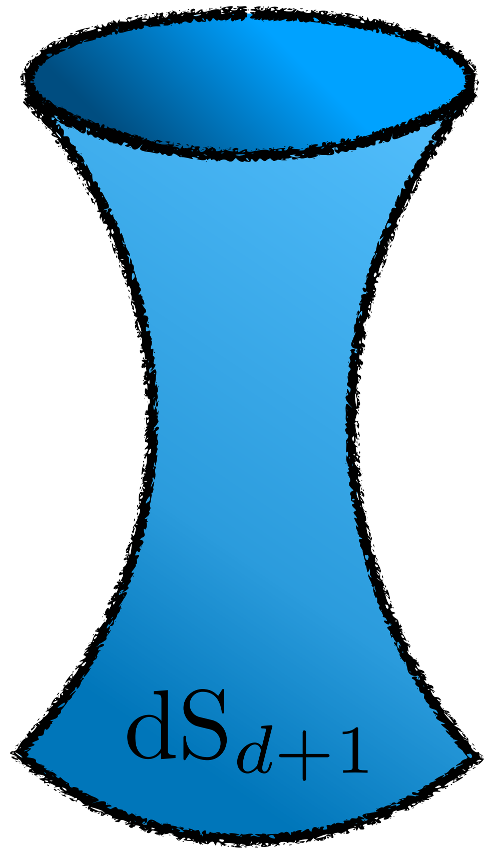
Non-perturbative study of QFT in dS_{d+1}



Cosmological bootstrap!

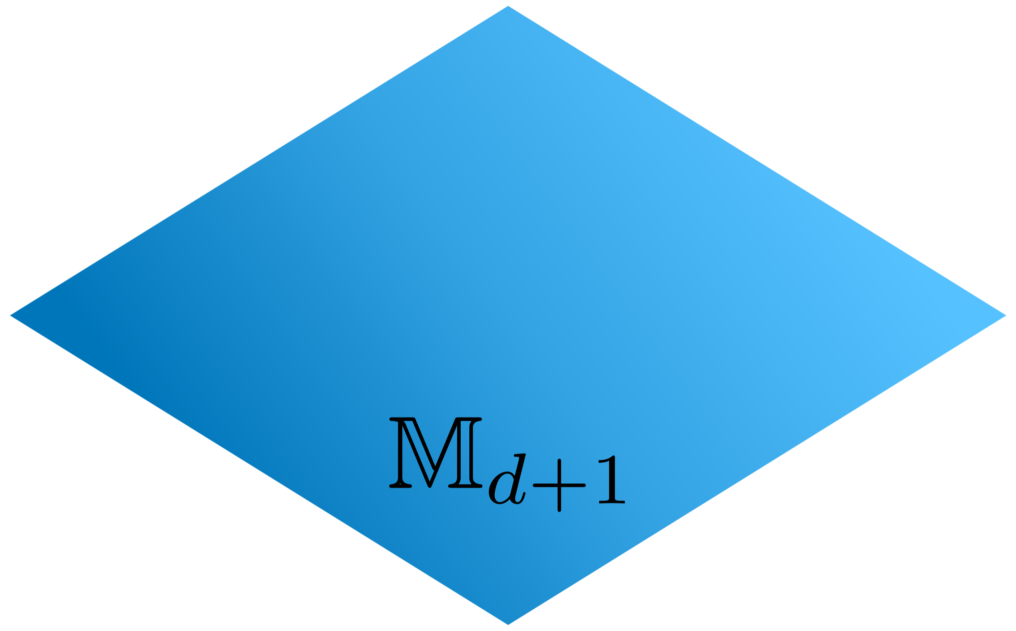

Conformal
Bootstrap
S-matrix
Bootstrap
Cosmological
Bootstrap
AdS
M
dS
Understanding
0
Λ

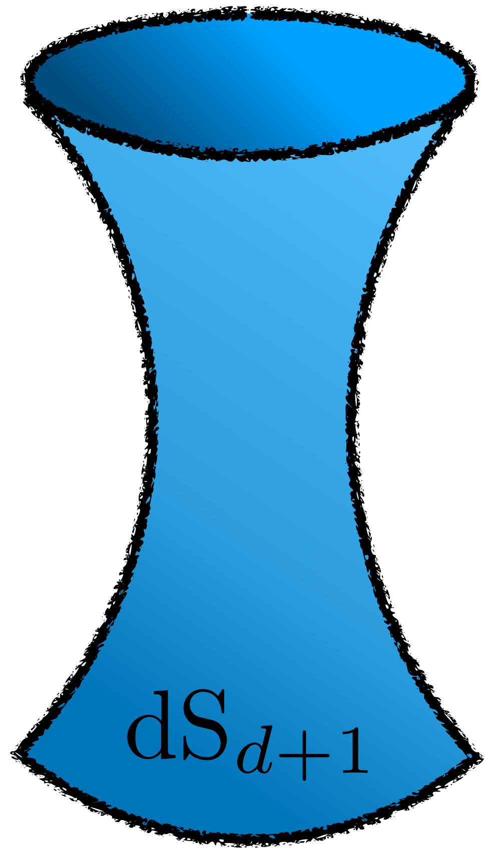




$$-Y_0^2 + Y_1^2 + \cdots + Y_d^2 = R^2$$

Group symmetry	Poincare group	Conformal group $SO(1, d+1)$
Isometries	$H \ , \ P_i \ , \ B_i \ , \ M_{ij}$	$D \ , \ P_i \ , \ K_i \ , \ M_{ij}$
Coordinate system	$ds^2 = -dt^2 + d\vec{x}^2$	$ds^2 = \dfrac{-d\eta^2 + d\vec{y}^2}{\eta^2}$ $\eta < 0 \ , \ \text{boundary} : \eta \to 0$
Hilbert space	$\lvert m, \ell, \vec{k} \rangle$	$\lvert \Delta, \ell, \vec{k} \rangle$





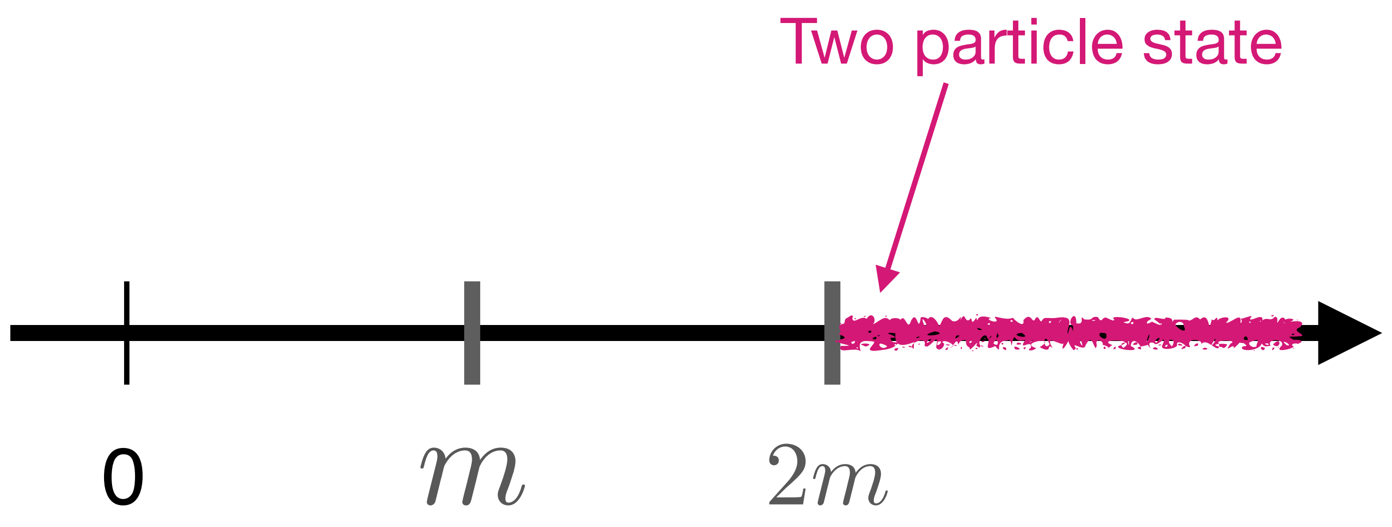
Hilbert space

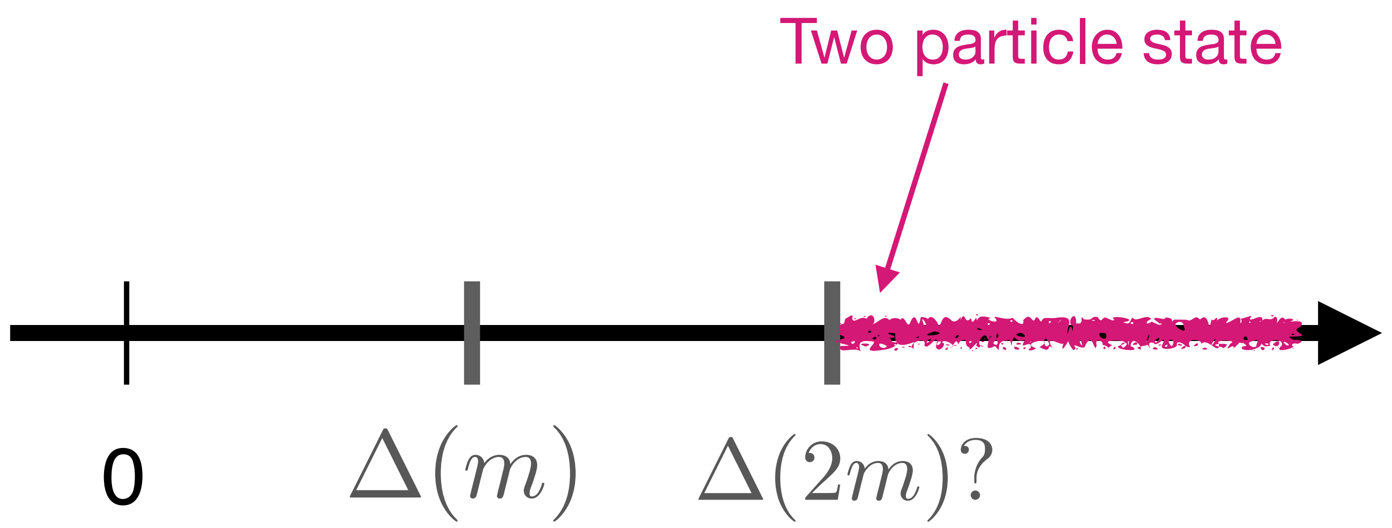
$$|m, \ell, \vec{k}\rangle$$

$$|\Delta, \ell, \vec{k}\rangle$$

$$\Delta(d-\Delta) = m^2 R^2 \text{ : free massive scalar}$$

Free theory warm up!



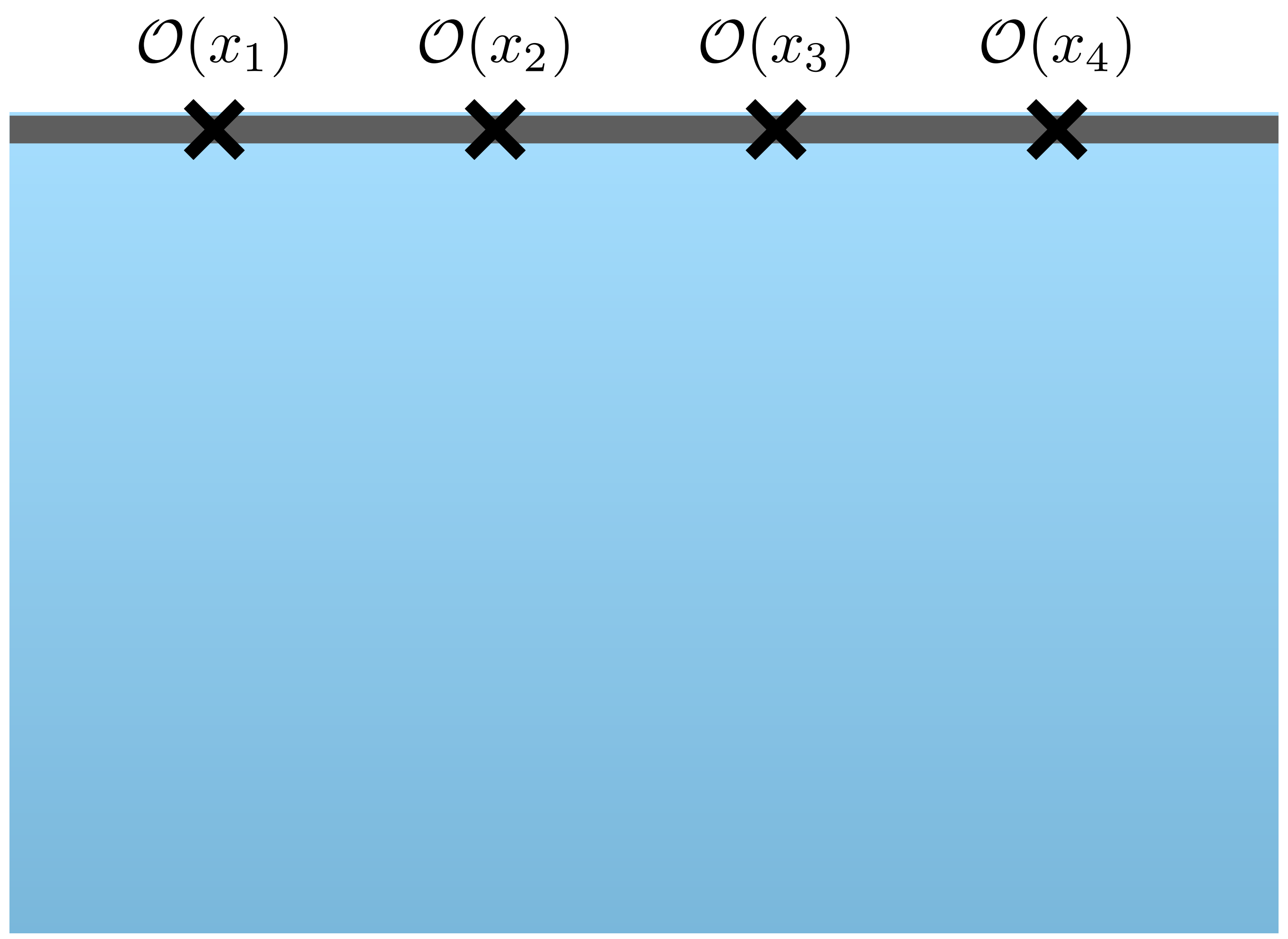


$$\Delta(m) = \frac{d}{2} \pm \sqrt{\frac{d^2}{4} - m^2 R^2}$$

Conformal Bootstrap vs Cosmological Bootstrap

- Conformal invariance
- Unitarity — Positivity
- Crossing symmetry

$$\langle \mathcal{O}(x_1) \ldots \mathcal{O}(x_4) \rangle_{\rm CFT} = \sum_{\Delta,\ell} \lambda^2_{\Delta,\ell} \, g_{\Delta,\ell}(z,\bar{z})$$

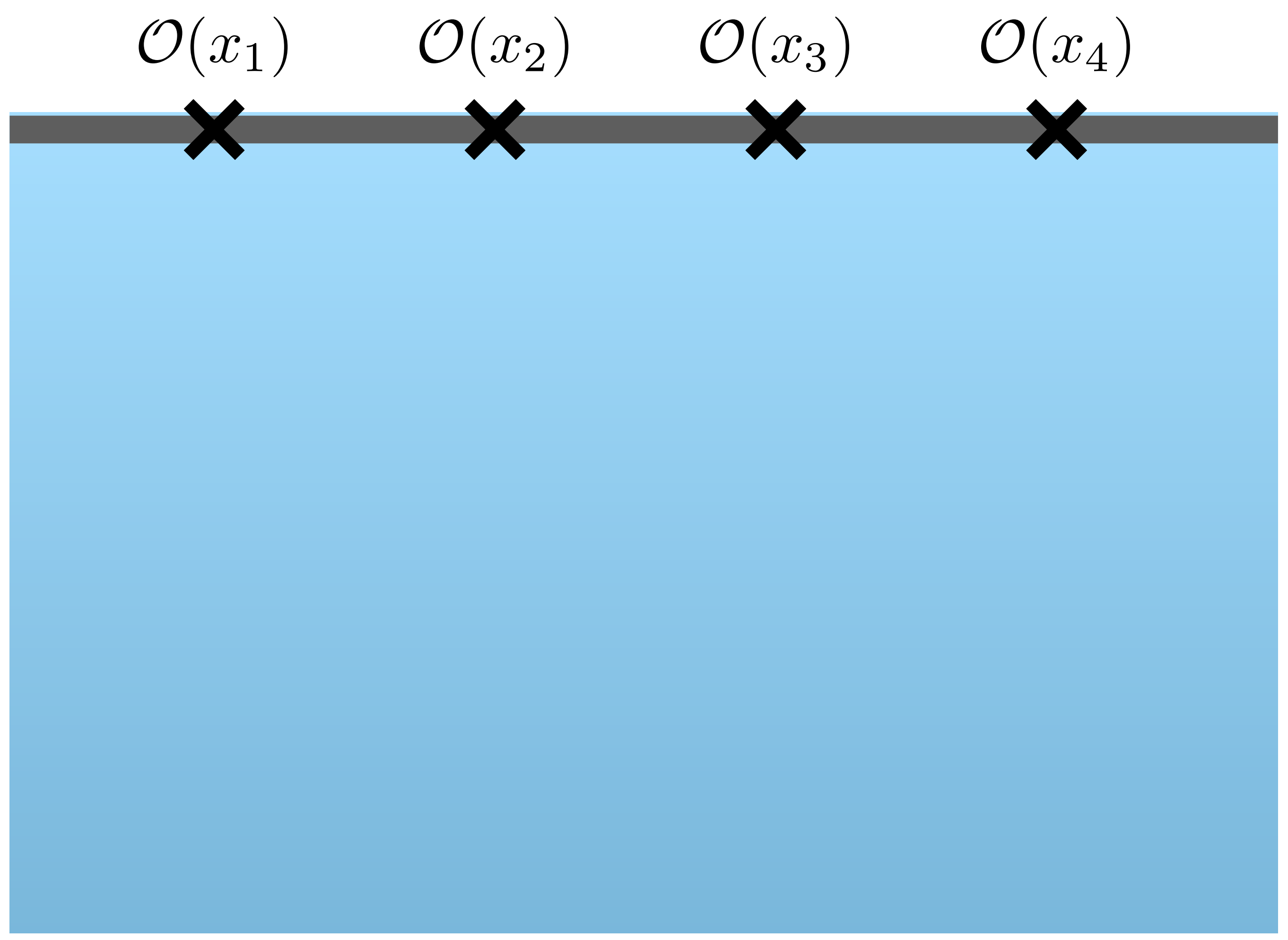


Conformal Bootstrap vs Cosmological Bootstrap

- Conformal invariance
- Unitarity — Positivity
- Crossing symmetry

$$\langle \mathcal{O}(x_1)\ldots\mathcal{O}(x_4)\rangle_{\rm CFT} = \sum_{\Delta,\ell} \lambda^2_{\Delta,\ell}\, g_{\Delta,\ell}(z,\bar{z})$$

$$\langle \mathcal{O}(x_1)\ldots\mathcal{O}(x_4)\rangle_{\rm dS} = \sum_{\ell}\int_{\Delta} I_{\Delta,\ell}\, \Psi_{\Delta,\ell}(z,\bar{z})$$



Conformal Bootstrap vs Cosmological Bootstrap

- Conformal invariance
- Unitarity — Positivity
- Crossing symmetry

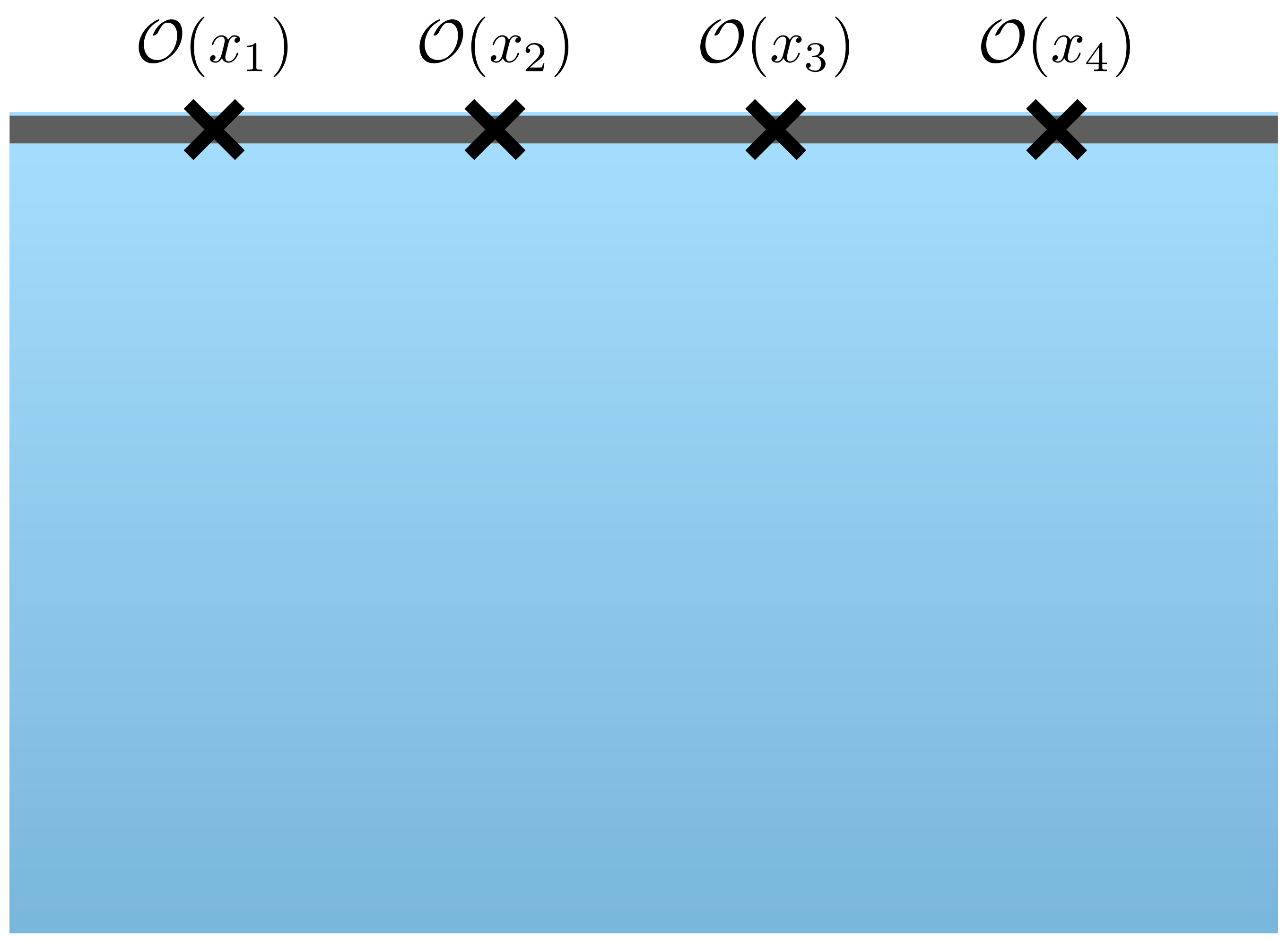
OPE

$$\langle \mathcal{O}(x_1)\dots\mathcal{O}(x_4)\rangle_{\rm CFT} = \sum_{\Delta,\ell} \lambda^2_{\Delta,\ell}\, g_{\Delta,\ell}(z,\bar z) \qquad \lambda^2_{\Delta,\ell} \geq 0$$

$$\langle \mathcal{O}(x_1)\dots\mathcal{O}(x_4)\rangle_{\rm dS} = \sum_{\ell}\int_{\Delta} I_{\Delta,\ell}\, \Psi_{\Delta,\ell}(z,\bar z) \qquad I_{\Delta,\ell} \geq 0$$

Boundary operators

Irreps



positivity + crossing → bounds on $I_{\Delta,\ell}$: $0 < I_{\Delta,\ell} < \#$

2d de Sitter [2107.1387]

Outline:

- Källén–Lehmann(KL): Non-perturbative

  Bulk two-point functions:
  1. Which irreps?
  2. Boundary operators
  3. Unitarity: Positivity (bounds)

- Representation theory:

  How does the Hilbert space of a QFT in dS decompose into unitary irreducible representations of SO(1,d+1)?

$$\langle \mathcal{O}(x_1)\ldots\mathcal{O}(x_4)\rangle = \sum_\ell \int_\Delta I_{\Delta,\ell}\ \Psi_{\Delta,\ell}(z,\bar{z})\ +\ \cdots$$

Which representations?

Källén–Lehmann spectral decomposition

KL decomposition in Minkowski:

A spectral decomposition of the two-point function into a sum/integral over free propagators (kinematical functions)

$$\langle \phi(x_1)\phi(x_2)\rangle = \int d\mu^2 \rho(\mu^2)\, G_{\text{free}}(x_{12}, \mu^2)$$

KL decomposition in Minkowski:

A spectral decomposition of the two-point function into a sum/integral over free propagators (kinematical functions)

$$\langle \phi(x_1)\phi(x_2)\rangle = \int d\mu^2 \rho(\mu^2)\, G_{\text{free}}(x_{12}, \mu^2)$$

$\rho(\mu^2) \geq 0$

- Non-perturbative!
- Symmetry fixes the x-dependence
- Unitarity ⟶ Positive density

Is it useful? Yes! Some examples:

1. No higher derivative terms in the UV complete Lagrangian
2. Bounds on EFT coefficients

Even more useful in dS!

KL decomposition in Minkowski:

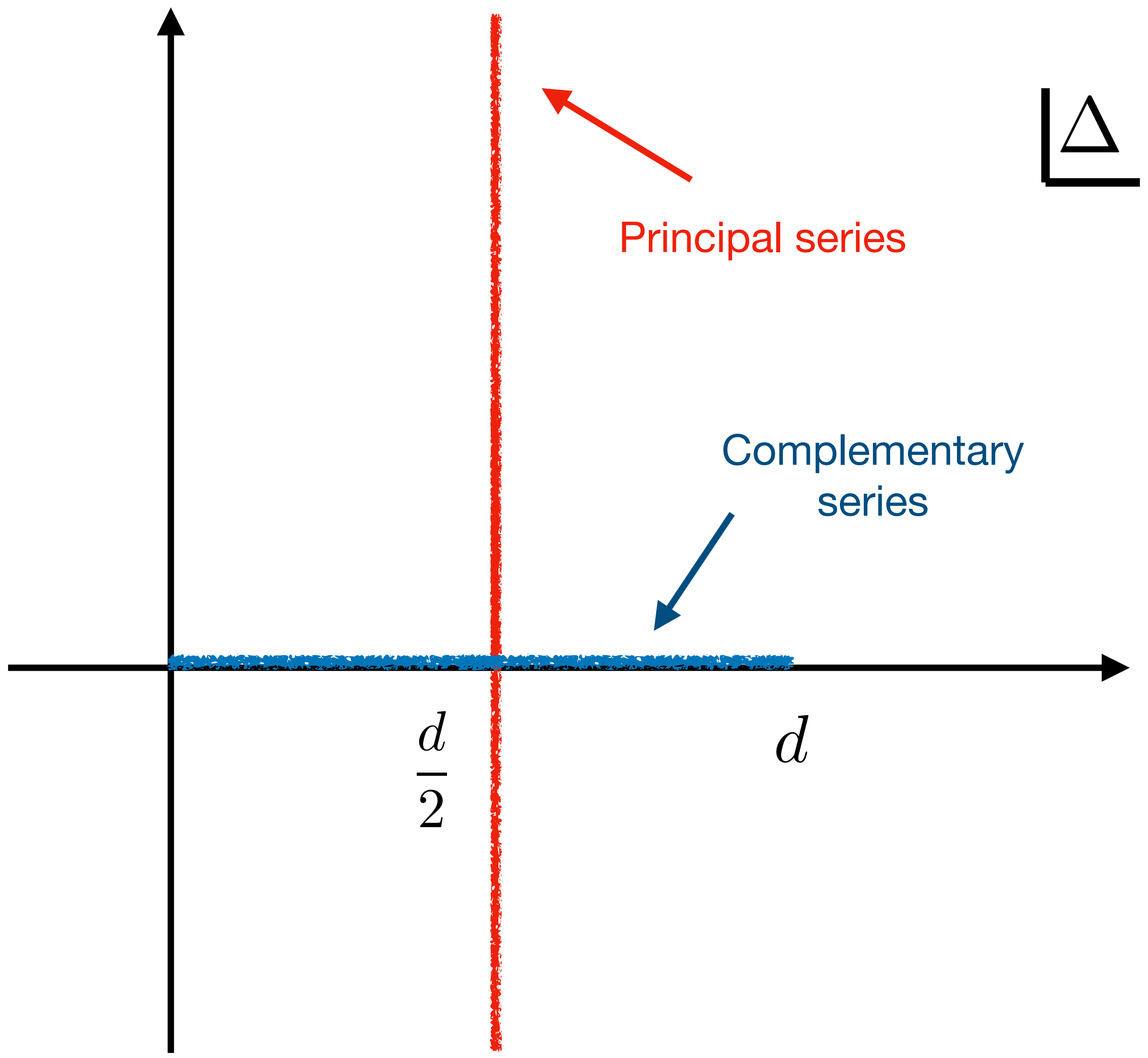
A spectral decomposition of the two-point function into a sum/integral over free propagators (kinematical functions).

$$\langle \phi(x_1)\phi(x_2)\rangle = \int d\mu^2 \rho(\mu^2)\, G_{\rm free}(x_{12},\mu^2)$$

KL decomposition in dS

$$\langle \phi(Y_1)\phi(Y_2)\rangle = \int_{\rm reps} \rho(\Delta)\, G_{\rm free}(Y_{12},\Delta)$$

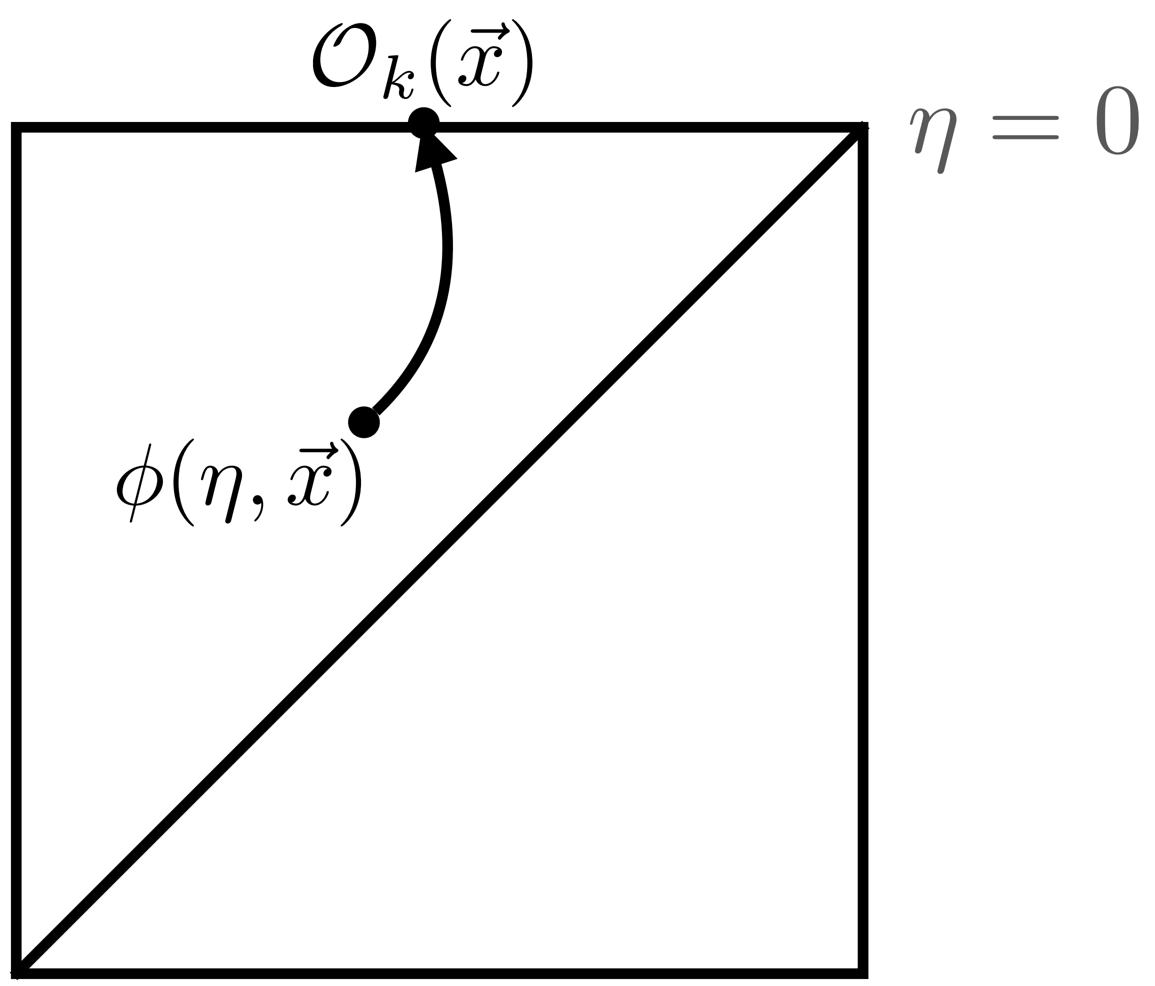
$$\langle T^{(J)}(Y_1)T^{(J)}(Y_2)\rangle = \sum_{\ell=0}^{J}\int_{\rm reps} \rho_\ell^{(J)}(\Delta)\, \nabla_1^{J-\ell}\nabla_2^{J-\ell}\, G_\ell(Y_{12},\Delta)$$



Spectral density and boundary operators:

$$\langle \phi(\eta, \vec{y}_1)\phi(\eta, \vec{y}_2)\rangle = \int_{\frac{d}{2}-i\infty}^{\frac{d}{2}+i\infty} d\Delta\, \rho(\Delta)\, G_{\rm free}(\eta, \vec{y}_{12})$$

$\eta \to 0$



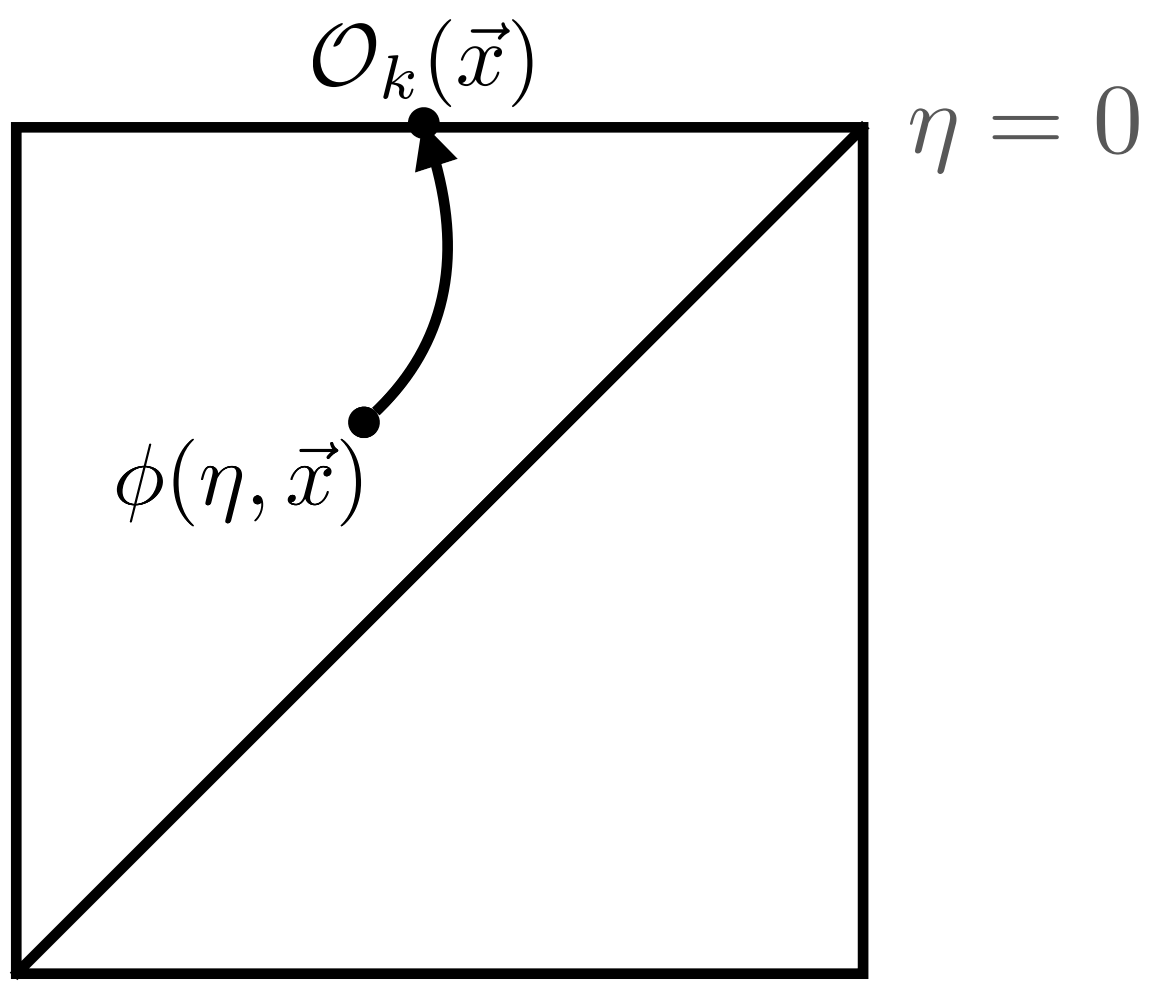
$$\phi(\eta, \vec{y}) = \sum_i (-\eta)^{\Delta_i}\, b_i \left[\mathcal{O}_i(\vec{y}) \; + \; \text{des}\right]$$

Spectral density and boundary operators:

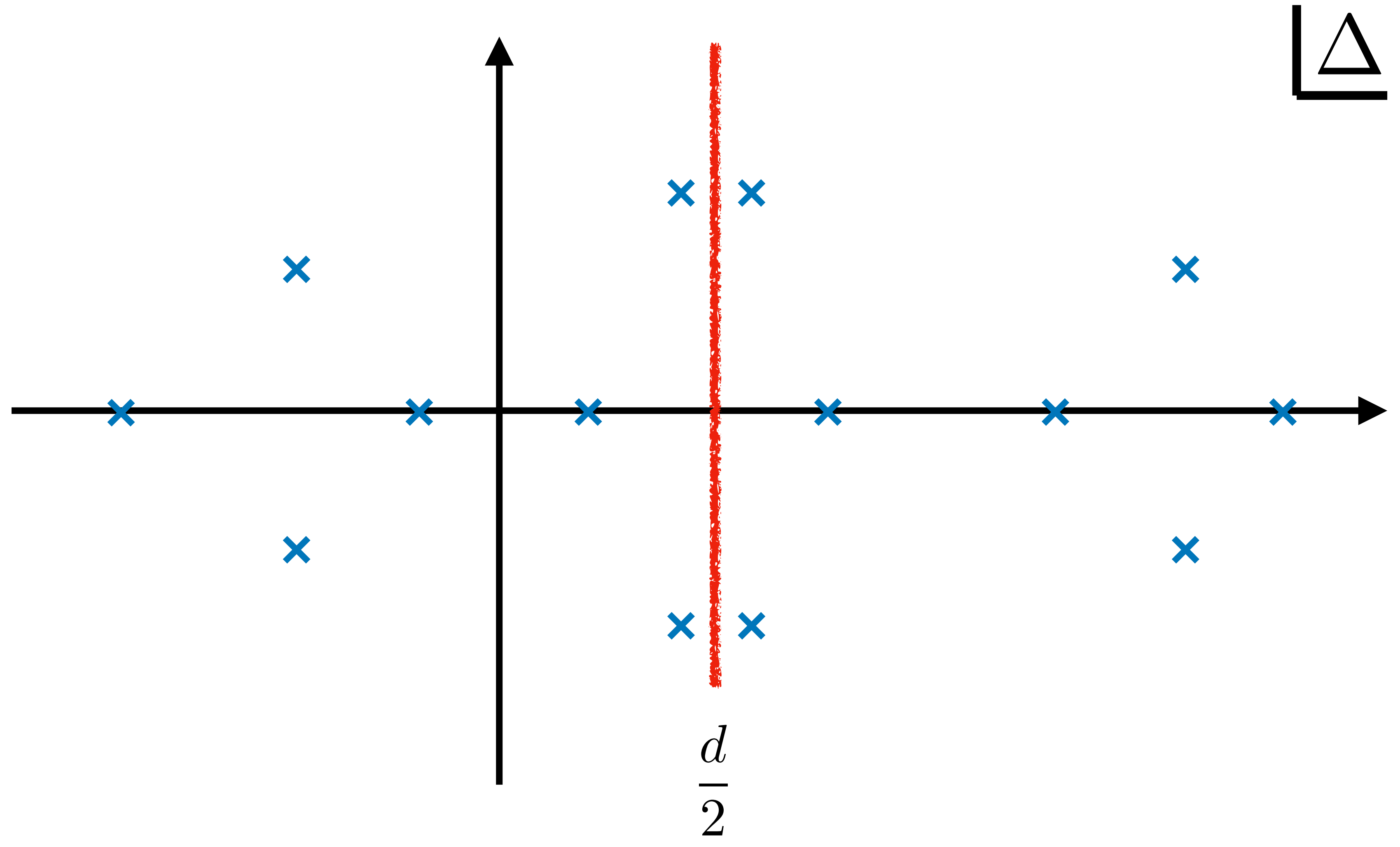
$$\langle \phi(\eta, \vec{y}_1) \phi(\eta, \vec{y}_2) \rangle = \int_{\frac{d}{2}-i\infty}^{\frac{d}{2}+i\infty} d\Delta \, \rho(\Delta) \, G_{\text{free}}(\eta, \vec{y}_{12})$$

$\eta \to 0$ contour deformation

$$\sum_i b_i^2 \langle \mathcal{O}_i(\vec{y}_1) \mathcal{O}_i(\vec{y}_2) \rangle \sim \sum_{j=\text{poles}} \text{Res}[\rho(\Delta_j)] \, \langle \mathcal{O}_j(\vec{y}_1) \mathcal{O}_j(\vec{y}_2) \rangle$$



$$\phi(\eta, \vec{y}) = \sum_i (-\eta)^{\Delta_i} b_i \left[\mathcal{O}_i(\vec{y}) \; + \; \text{des}\right]$$



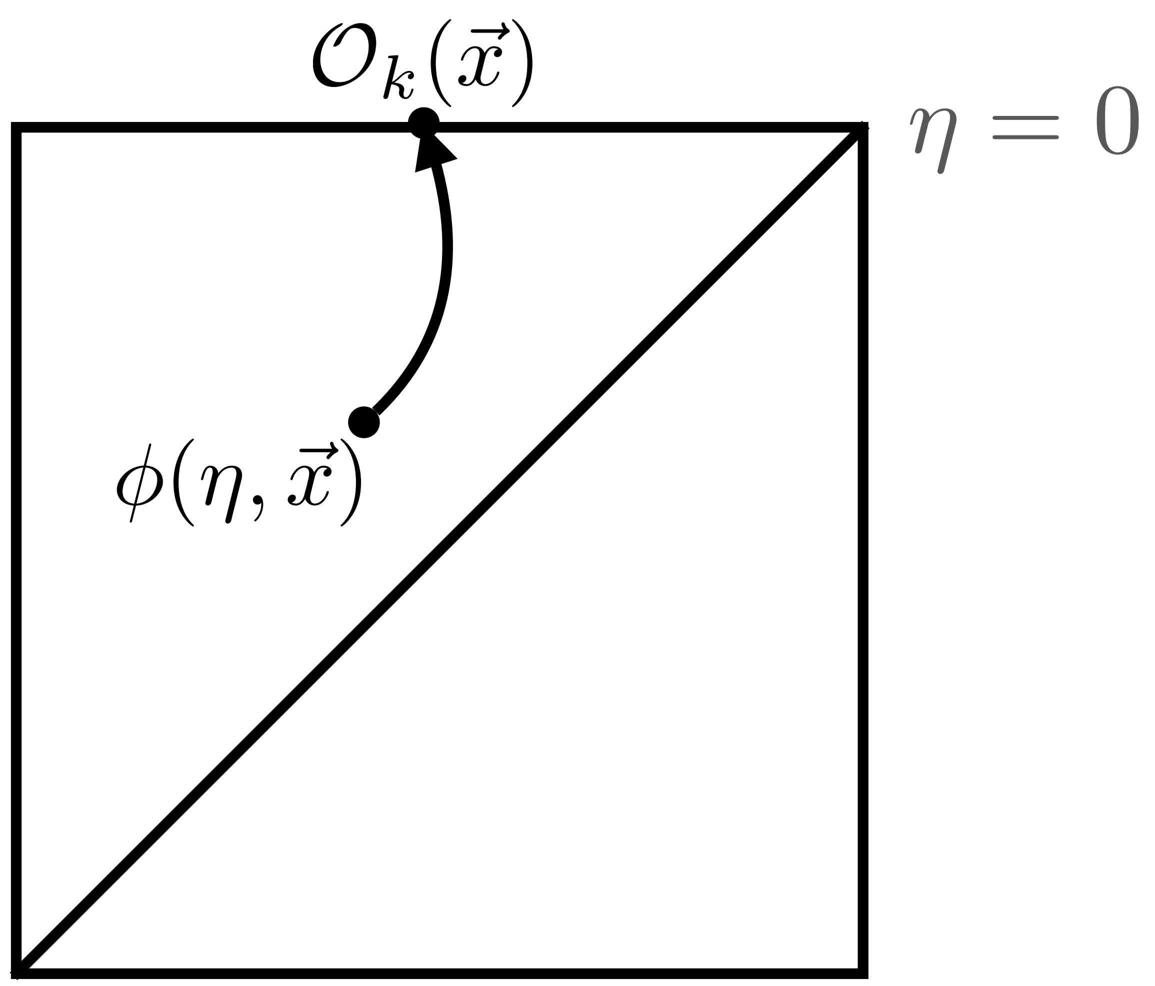
Spectral density and boundary operators:

$$\langle \phi(\eta, \vec{y}_1)\phi(\eta, \vec{y}_2)\rangle = \int_{\frac{d}{2}-i\infty}^{\frac{d}{2}+i\infty} d\Delta\, \rho(\Delta)\, G_{\text{free}}(\eta, \vec{y}_{12})$$

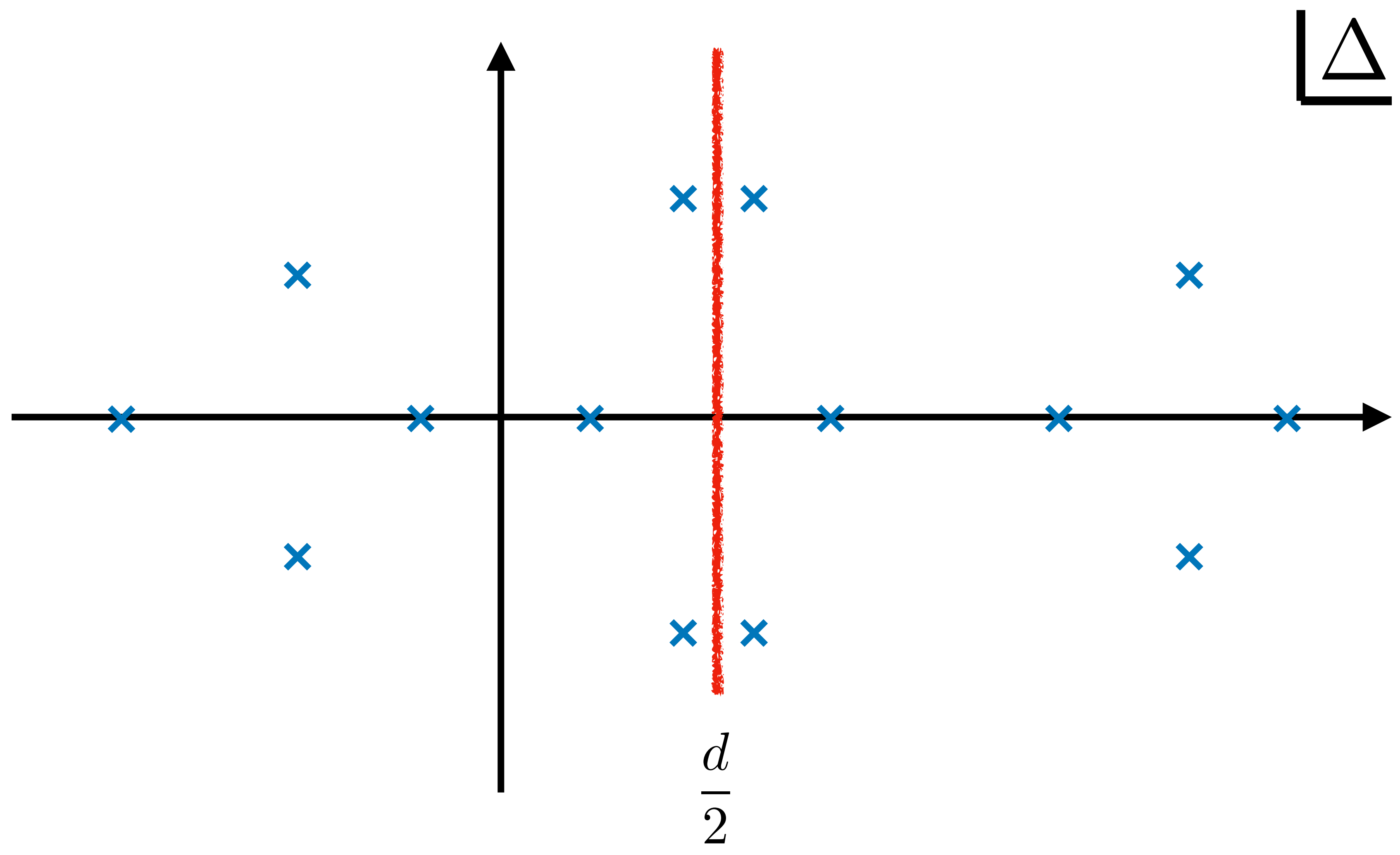
$\eta \to 0$ contour deformation

$$\sum_i b_i^2 \langle \mathcal{O}_i(\vec{y}_1)\mathcal{O}_i(\vec{y}_2)\rangle \sim \sum_{j=\text{poles}} \text{Res}[\rho(\Delta_j)]\, \langle \mathcal{O}_j(\vec{y}_1)\mathcal{O}_j(\vec{y}_2)\rangle$$

$$\Delta_i \sim \text{pole of } \rho(\Delta) \qquad \sum_i b_i^2 \sim \text{Res}[\rho(\Delta_i)]$$



$$\phi(\eta, \vec{y}) = \sum_i (-\eta)^{\Delta_i}\, b_i\, [\mathcal{O}_i(\vec{y}) \ + \ \text{des}]$$



How to find spectral density? An inversion formula

$$\rho(\Delta) \overset{?}{=} \int_{\text{space}} \mathcal{F}(\Delta)\ \langle \phi\phi \rangle$$

How to find spectral density? An inversion formula

Analytic continuation (Wick Rotation) to EAdS

$$ds^2 = \frac{-d\eta^2 + d\vec{y}^2}{\eta^2} \qquad\qquad ds^2 = \frac{dz^2 + d\vec{x}^2}{z^2}$$

$$Y = (\eta, \vec{y}) \quad \longrightarrow \quad X = (\pm iz, \vec{x})$$

The propagators in dS translate to Harmonic functions in EAdS

$$G_{\text{free}}(Y_{12}, \Delta) \quad \longrightarrow \quad \Omega_\Delta(X_1, X_2)$$

Harmonic functions: Orthogonal

$$\int_X \Omega_\Delta(X_1, X)\Omega_{\Delta'}(X, X_2) = \delta(\Delta - \Delta')\,\Omega_\Delta(X_1, X_2)$$

Power of analytic continuation to EAdS

$$\langle\phi(Y_1)\phi(Y_2)\rangle \sim \int_{\text{reps}} \rho(\Delta)\; G_{\text{free}}(Y_{12},\Delta) \quad \longrightarrow \quad \int_{\frac{d}{2}-i\infty}^{\frac{d}{2}+i\infty} d\Delta\; \rho_\ell(\Delta)\; \Omega_\Delta(X_1,X_2)$$

Completeness of principal series for square-integrable two-point functions

Should Decay fast enough at large distances

$$\int_{\text{EAdS}} |G|^2 < \infty$$

- Orthogonality of harmonic functions helps us to invert the KL decomposition of any spin to a one variable integral over (space-like) chordal distance!

- For example for spin 0:

$$\rho(\Delta) \sim \int_{-\infty}^{-1} d\sigma\,(\sigma^2-1)\;{}_2F_1(\Delta, d-\Delta, \frac{d+1}{2}, \frac{1+\sigma}{2})\langle\phi\phi\rangle$$

Spinning KL:

$$\langle T^{(J)}(Y_1)T^{(J)}(Y_2)\rangle \sim \sum_{\ell=0}^{J}\int_{\frac{d}{2}-i\infty}^{\frac{d}{2}=i\infty} \rho_\ell(\Delta)\,\nabla_1^{J-\ell}\nabla_2^{J-\ell}\,G_\ell(Y_{12},\Delta)$$

$$\rho_\ell^{(J)}(\Delta) = \int_{X_1} \Omega_{\Delta,\ell}(X_1,X_2)\nabla_1^{(J-\ell)}\nabla_2^{(J-\ell)}\langle T^{(J)}(X_1)T^{(J)}(X_2)\rangle$$

Examples:

Explicit expressions for spectral densities: the expected boundary operator content, manifestly positive and match with the flat-space limit

- Free theory composite operators two-point functions:

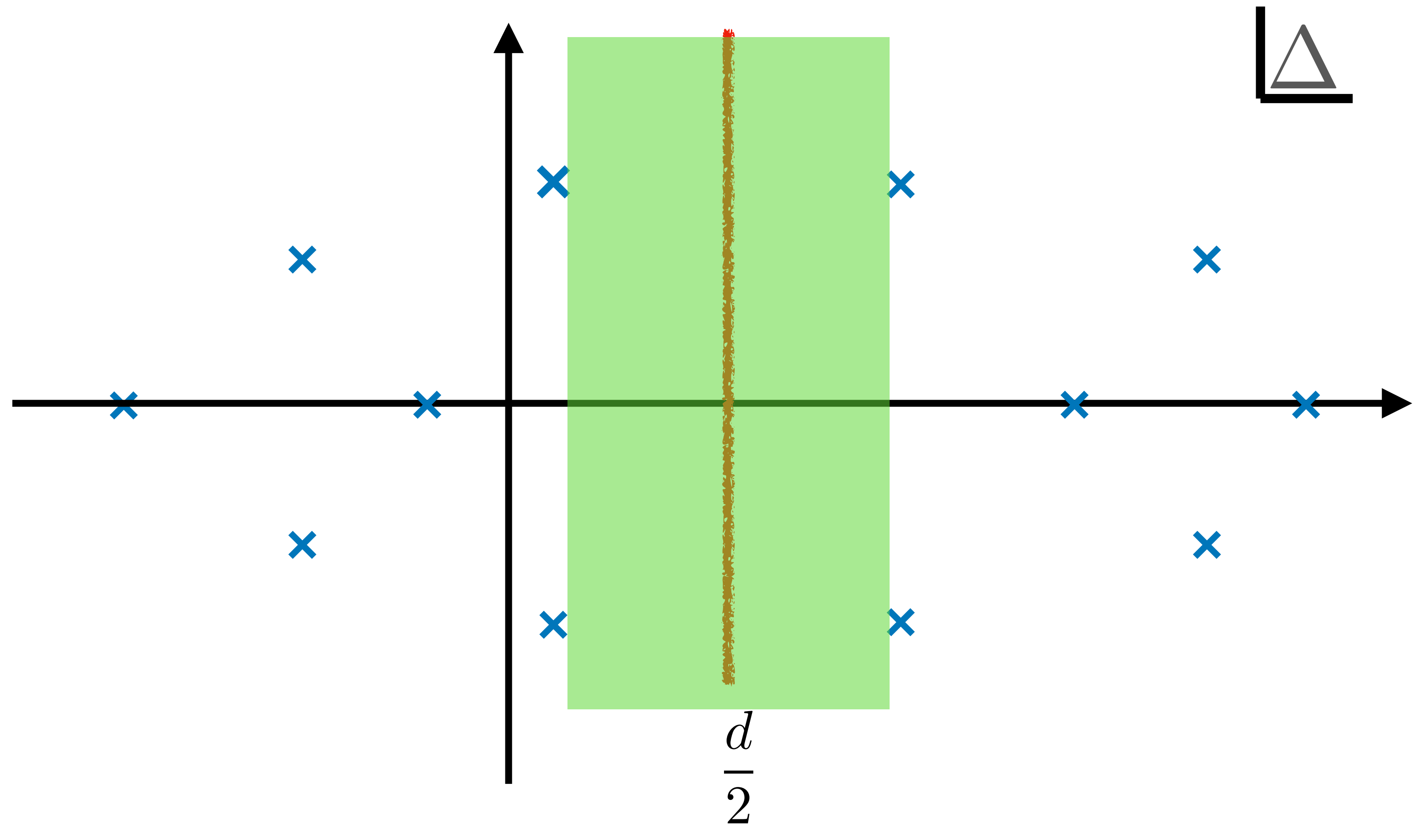
  1. $\langle \phi_1\phi_2(Y_1)\phi_1\phi_2(Y_2)\rangle = \langle\phi_1\phi_1\rangle\langle\phi_2\phi_2\rangle$
  2. $\langle V_\mu\phi(Y_1)V_\nu\phi(Y_2)\rangle = \langle V_\mu V_\nu\rangle\langle\phi\phi\rangle$
  3. $\langle \phi\nabla_\mu\phi(Y_1)\phi\nabla_\nu\phi(Y_2)\rangle = \langle\nabla_\mu\phi\nabla_\nu\phi\rangle\langle\phi\phi\rangle$

- Bulk CFT spin 0, 1, 2 $\quad \rho^{(2)}_{\mathrm{CFT},\ell=2}(\Delta) \sim \dfrac{|\Gamma(\Delta_T-\Delta)|^2}{|\Gamma(\Delta-\frac{d}{2})|^2}$

Fun facts:

- Large distance behaviour ⟶ Convergence of the inversion formula ⟶ a strip of analyticity in Δ plane.

- Complementary series:
  pole crossing over principal series. A discrete sum?!



- Anomalous dimensions

- Another way: analytic continuation from/to sphere. It is an integral of discontinuity of two-point function over time-like separated points — equivalent to the EAdS one!

Representation theory

Unitary irreducible representations SO(1,d+1) $\{\Delta, s\}$

Casimir: $\Delta(d-\Delta) - s(d+s-2)$

- **Principal series** $\mathcal{P}_{\Delta,s}$: $\Delta \in \frac{d}{2} + i\mathbb{R}$ and $s \geq 0$.

  Heavy massive scalars fields

- **Complementary series** $\mathcal{C}_{\Delta,s}$: $0 < \Delta < d$ when $s = 0$ and $1 < \Delta < d-1$ when $s \geq 1$.

  Light massive scalars fields

- **Type I exceptional series** $\mathcal{V}_{p,0}$: $\Delta = d + p - 1$ and $s = 0$ for $p \geq 1$.

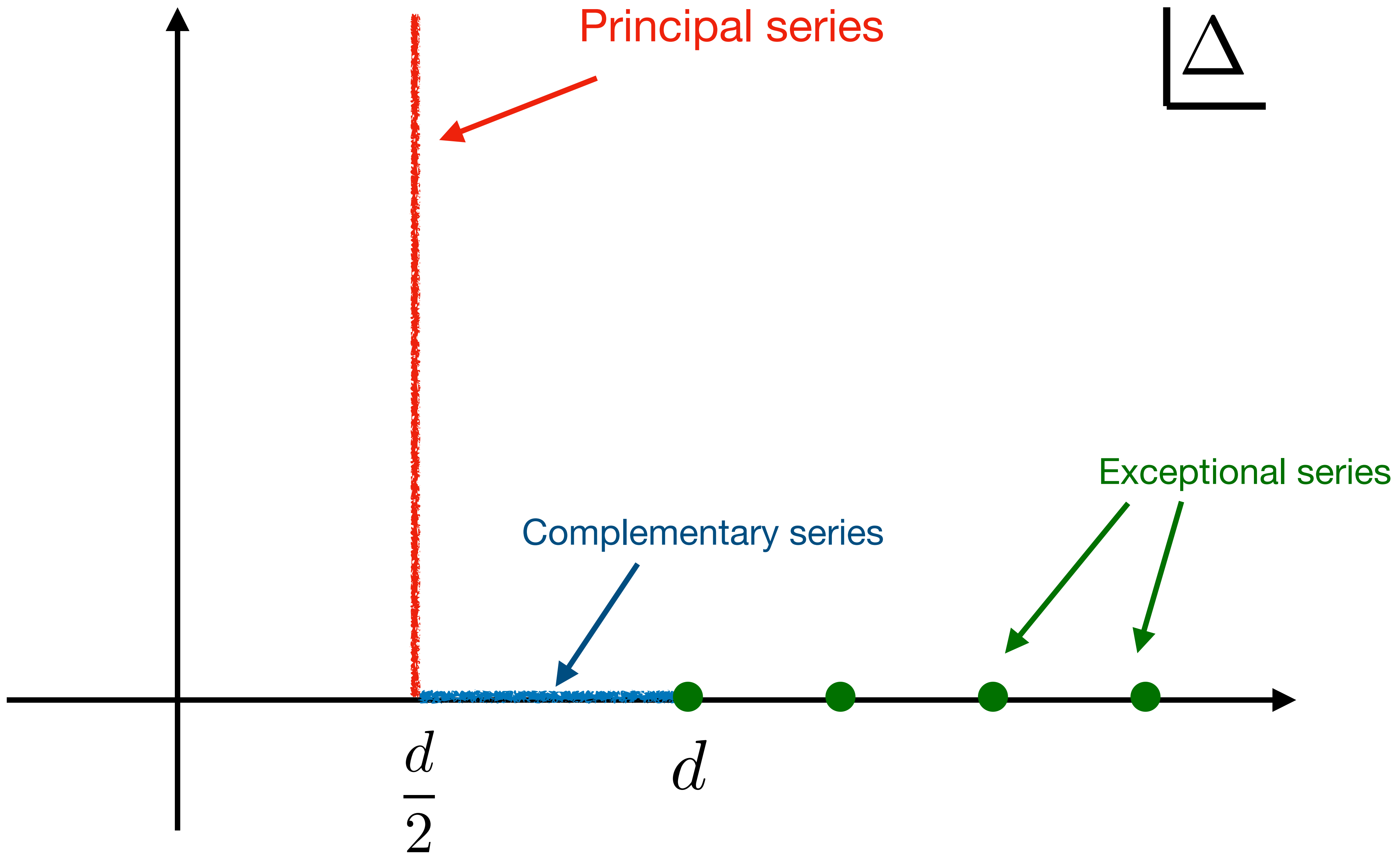
  Shift symmetric scalars in dS_{d+1}

- **Type II exceptional series** $\mathcal{U}_{s,t}$: $\Delta = d + t - 1$ and $s \geq 1$ with $t = 0, 1, 2 \cdots, s-1$.

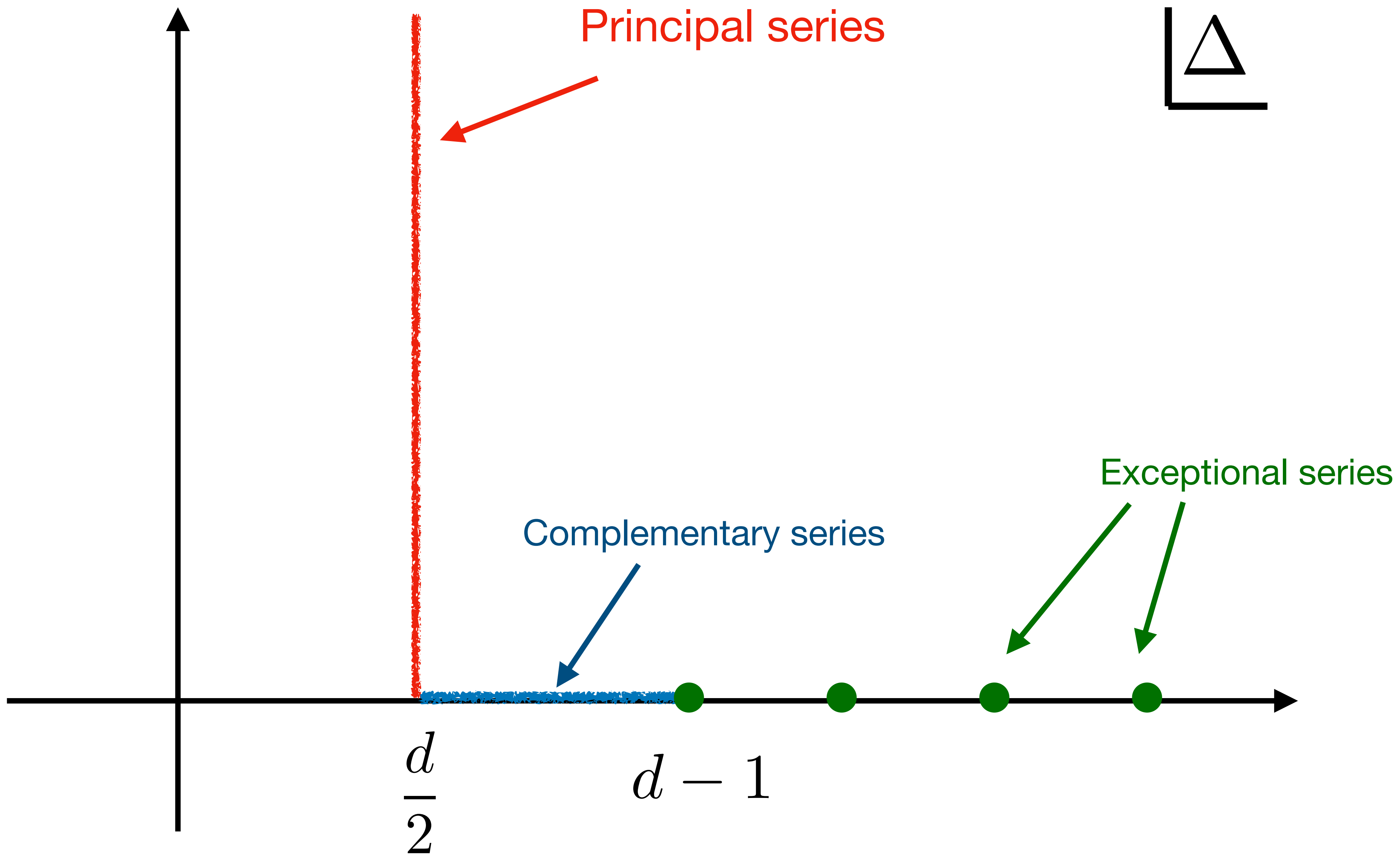
  Partially massless field of spin s and depth t in dS_{d+1}

Unitary irreducible representations SO(1,d+1) $\{\Delta, s\}$

Shadow symmetry: $\Delta \longleftrightarrow d - \Delta$



Scalar



Spinning

Hilbert space of QFT in dS$_{d+1}$

Two simple starting points:

- Free theory: Fock space

  Decomposition of tensor products (two particle states)

- CFT in dS:

  SO(2,d+1) ⟶ SO(1,d+1)

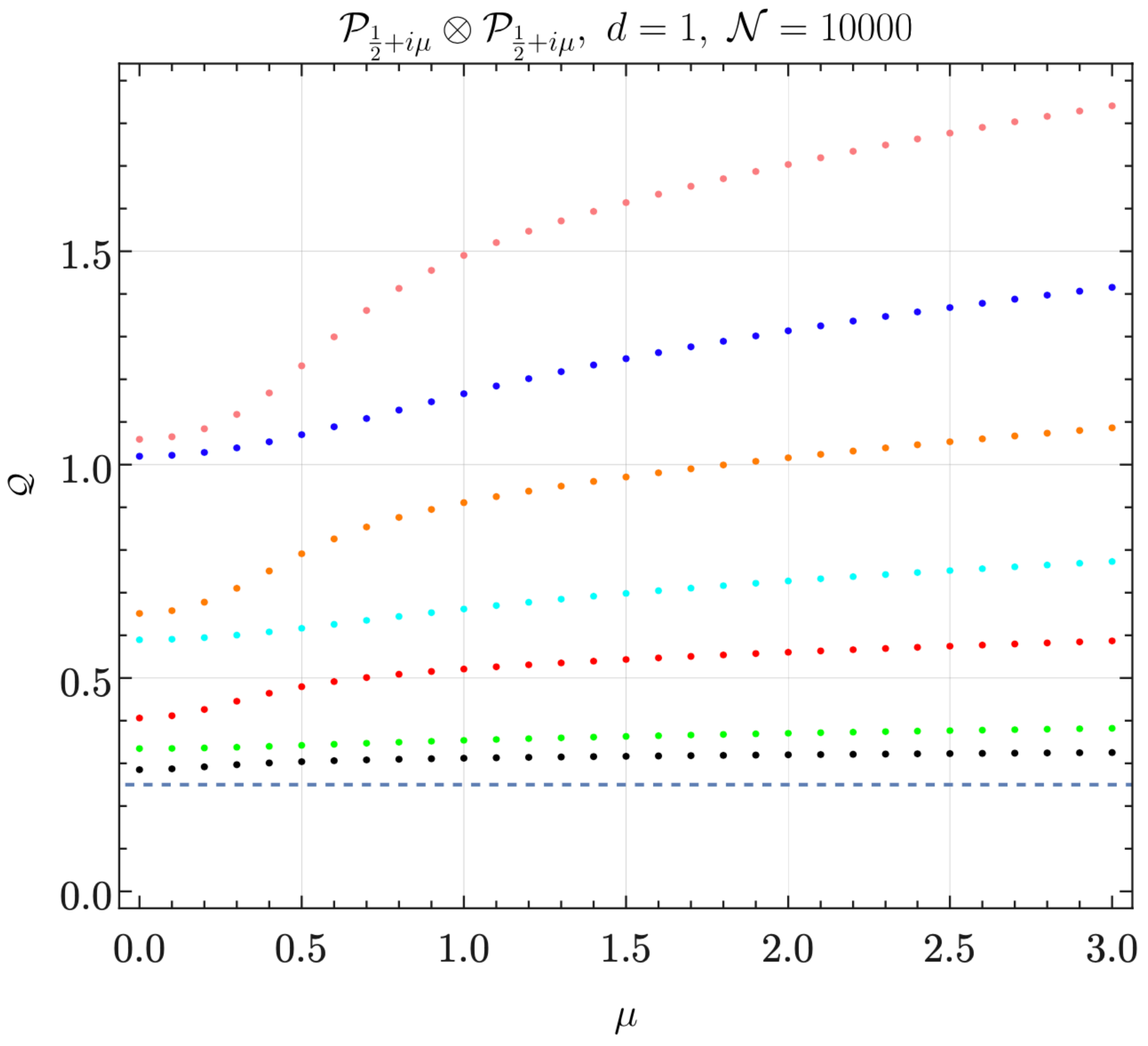
$\otimes$	$\mathcal{P}_{\Delta_1,l_1}$	$\mathcal{C}_{\Delta_1,l_1}$	$\mathcal{V}_{p,l_1}$	$\mathcal{U}_{s,t}$
$\mathcal{P}_{\Delta_2,l_2}$	?	?	?	?
$\mathcal{C}_{\Delta_2,l_2}$	?	?	?	?
$\mathcal{V}_{p,l_2}$	?	?	?	?
$\mathcal{U}_{s,t}$	?	?	?	?

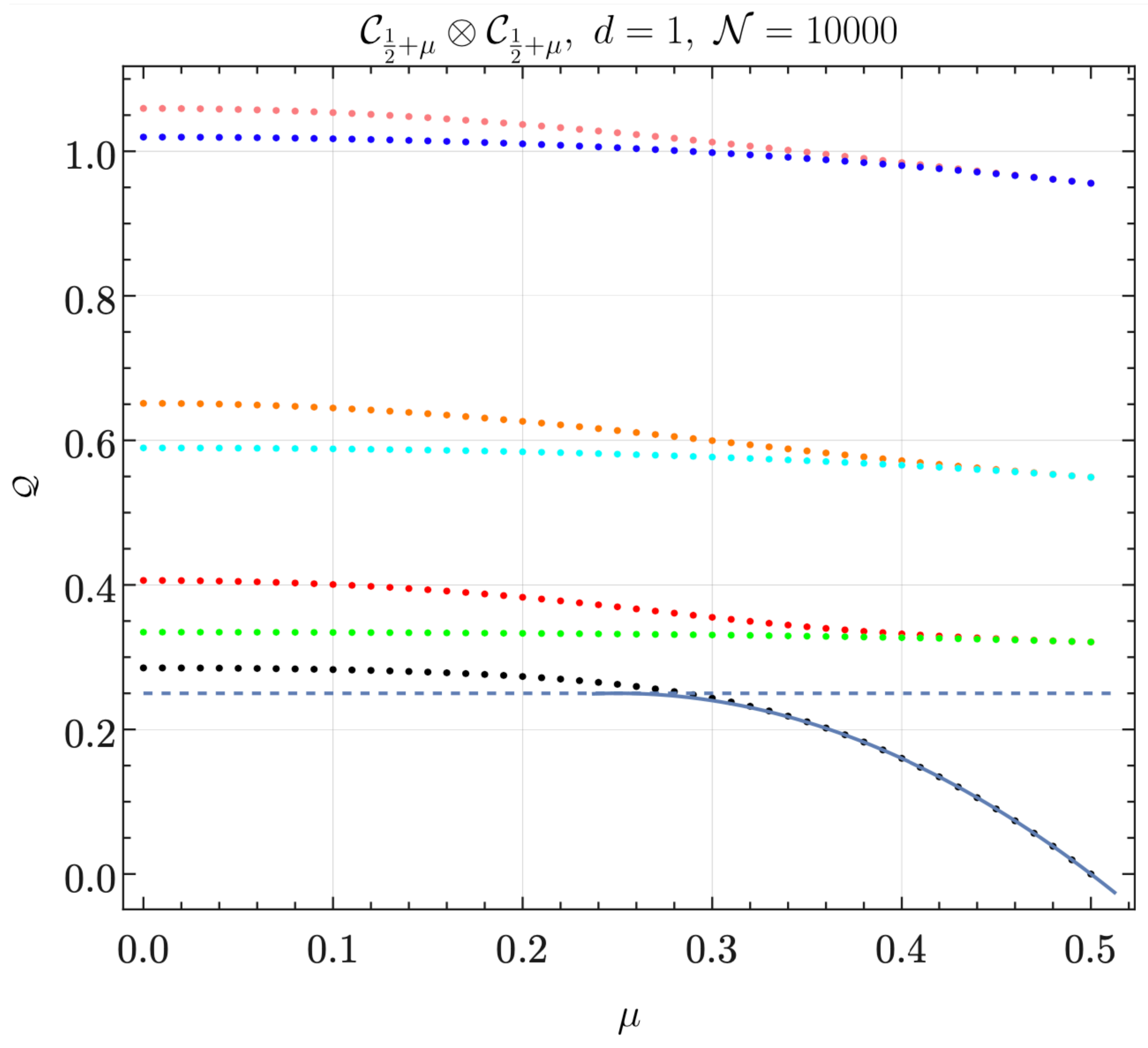
$\mathrm{SO}(2,d+1)\to\mathrm{SO}(1,d+1)$	$\mathcal{R}_{\tilde{\Delta},0}$	$\mathcal{R}_{\tilde{\Delta},\ell}$
$d=1$	?	?
$d=2$	?	?
$d\geq 3$	?	?

- Direct computation:

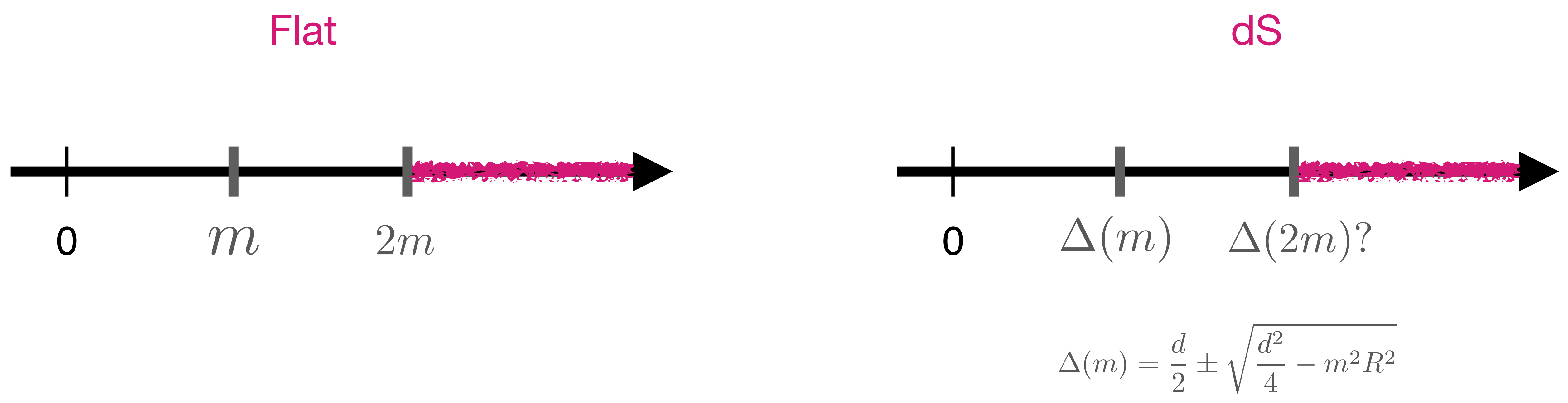
  Building the corresponding state and checking whether it is normalizable or not.
  Works for exceptional/discrete representations

- Numerically diagonalizing truncated SO(1,d+1) Casimir:





- Harish-Chandra character analysis:



- All the principal series?

- Complementary series → Pole crossings when analytically continue

- Harish-Chandra character analysis: $\Theta_R(g) \equiv \mathrm{tr}_{\mathcal{H}}(g) \quad , \quad g \in \mathcal{G}$

Tensor product character in compact groups :

Character of the tensor products is equal to the product of the two characters!

$$R_1 \otimes R_2 = \oplus_a R_a \quad \longrightarrow \quad \Theta_{R_1 \otimes R_2} = \Theta_{R_1} \times \Theta_{R_2} = \sum_a n_a \Theta_{R_a}$$

For example: SO(3) spin-s representaions

$$\chi_{\mathrm{s}=2} \otimes \chi_{\mathrm{s}=1} = \chi_{\mathrm{s}=1} + \chi_{\mathrm{s}=2} + \chi_{\mathrm{s}=3}$$

Generalizing to non-compact groups:

$$\Theta_{R_1 \otimes R_2} = \Theta_{R_1} \Theta_{R_2} = \sum_s \int_\Delta \mathcal{K}_{\Delta,s} \Theta(\mathcal{P}_{\Delta,s}) \ + \ \text{other irreps}$$

An example: principal x principal (d=1)

Explicit expressions of Harish-Chandra character

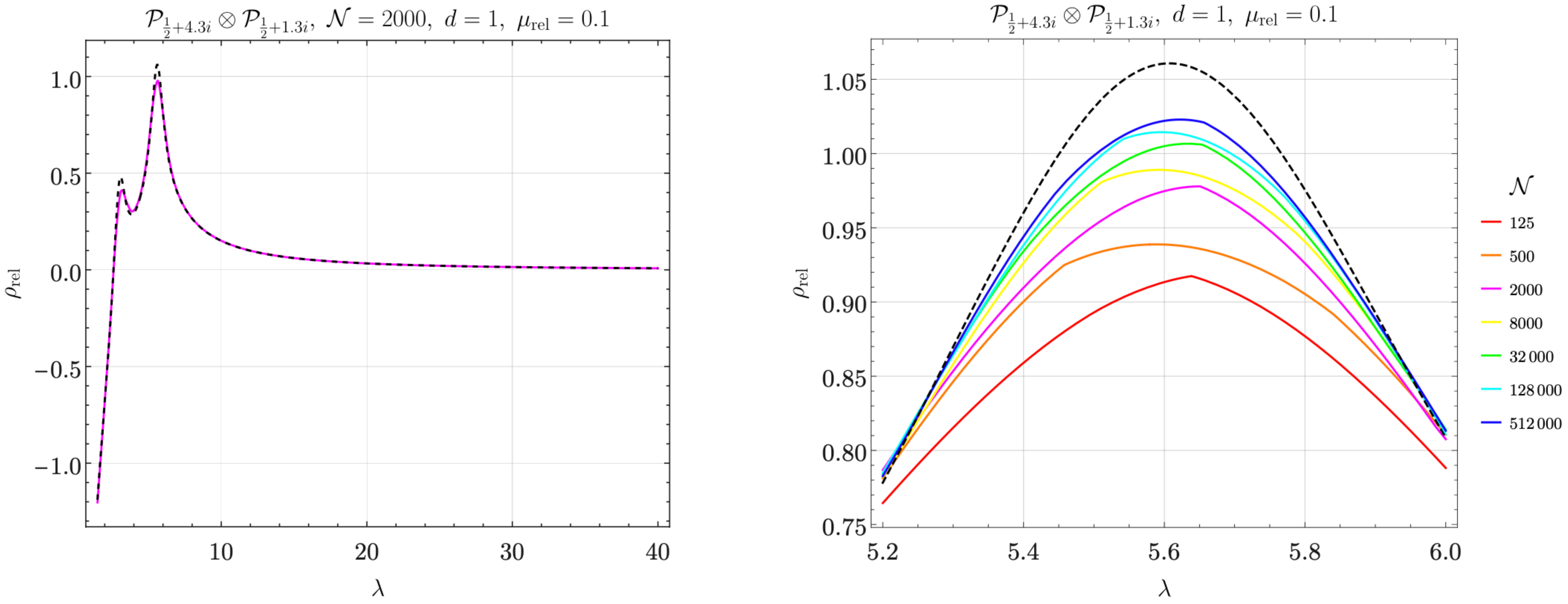
$$\Theta_{\Delta_1}\Theta_{\Delta_2} \sim \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} d\Delta\ \mathcal{K}(\Delta)\ \Theta_\Delta$$

After some regularization procedure :

$$\mathcal{K}(\Delta) = -\frac{1}{2\pi}\sum_{\pm,\pm,\pm}\psi\left(\frac{1}{2} \pm i\mu_1 \pm i\mu_2 \pm i\lambda\right)$$

$$\Delta_i = \frac{1}{2} + i\mu_i\ ,\ \Delta = \frac{1}{2} + i\lambda$$

Matching of the numerics with Character Analysis



The tables:

$\otimes$	$\mathcal{P}_{\Delta_1}$	$\mathcal{C}_{\Delta_1}$	$\mathcal{D}^+_{k_1}$	$\mathcal{D}^-_{k_1}$
$\mathcal{P}_{\Delta_2}$	$\int_\Delta \mathcal{P}_\Delta \oplus \sum_k \mathcal{D}^\pm_k$			
$\mathcal{C}_{\Delta_2}$	$\int_\Delta \mathcal{P}_\Delta \oplus \sum_k \mathcal{D}^\pm_k$	$\int_\Delta \mathcal{P}_\Delta \oplus \sum_k \mathcal{D}^\pm_k \oplus \mathcal{C}_{\Delta_1+\Delta_2-1}$		
$\mathcal{D}^+_{k_2}$	$\int_\Delta \mathcal{P}_\Delta \oplus \sum_k \mathcal{D}^+_k$	$\int_\Delta \mathcal{P}_\Delta \oplus \sum_k \mathcal{D}^+_k$	$\sum_{k\geq k_1+k_2} \mathcal{D}^+_k$	
$\mathcal{D}^-_{k_2}$	$\int_\Delta \mathcal{P}_\Delta \oplus \sum_k \mathcal{D}^-_k$	$\int_\Delta \mathcal{P}_\Delta \oplus \sum_k \mathcal{D}^-_k$	$\int_\Delta \mathcal{P}_\Delta \oplus \sum_{k=1}^{\lvert k_1-k_2\rvert} \mathcal{D}^{\mathrm{sign}(k_1-k_2)}_k$	$\sum_{k\geq k_1+k_2} \mathcal{D}^-_k$

Tensor products d=1

$\otimes$	$\mathcal{P}_{\Delta_1,m_1}$	$\mathcal{C}_{\Delta_1}$
$\mathcal{P}_{\Delta_2,m_2}$	$\sum_m \int_\Delta \mathcal{P}_{\Delta,m}$	$\sum_m \int_\Delta \mathcal{P}_{\Delta,m}$
$\mathcal{C}_{\Delta_2}$	$\sum_m \int_\Delta \mathcal{P}_{\Delta,m}$	$\sum_m \int_\Delta \mathcal{P}_{\Delta,m} \oplus \mathcal{C}_{\Delta_1+\Delta_2-2}$

Tensor products d=2

The tables:

$\otimes$	$\mathcal{P}_{\Delta_1,0}$	$\mathcal{C}_{\Delta_1,0}$	$\mathcal{V}_{1,0}$	$\dots$
$\mathcal{P}_{\Delta_2,0}$	$\sum_s \int_\Delta \mathcal{P}_{\Delta,s}$			
$\mathcal{C}_{\Delta_2,0}$	$\sum_s \int_\Delta \mathcal{P}_{\Delta,s}$	$\sum_s \int_\Delta \mathcal{P}_{\Delta,s} \oplus \sum_{n,s} \mathcal{C}_{\Delta_1+\Delta_2-d-s-2n,s}$		
$\mathcal{V}_{1,0}$	?	?	$\sum_s \int_\Delta \mathcal{P}_{\Delta,s} \oplus \mathcal{U}_{1,0}$	
$\dots$	?	?	?	?

Tensor products higher d

$\mathrm{SO}(2,d+1) \to \mathrm{SO}(1,d+1)$	$\mathcal{R}_{\tilde{\Delta},0}$	$\mathcal{R}_{\tilde{\Delta},\ell}$	$\dots$
$d=1$	$\int_\Delta \mathcal{P}_\Delta \oplus \mathcal{C}_{1-\tilde{\Delta}}$	$\int_\Delta \mathcal{P}_\Delta \oplus \sum_{k=1}^{\lvert\ell\rvert} \mathcal{D}_k^{\mathrm{sign}(\ell)}$	–
$d=2$	$\int_\Delta \mathcal{P}_{\Delta,0} \oplus \mathcal{C}_{2-\tilde{\Delta}}$	$\sum_{\lvert m\rvert\le\ell} \int_\Delta \mathcal{P}_{\Delta,m}$	–
$d\ge 3$	$\int_\Delta \mathcal{P}_{\Delta,0} \oplus \mathcal{C}_{d-\tilde{\Delta},0}$	$\sum_{s=0}^{\ell} \int_\Delta \mathcal{P}_{\Delta,s}$	?

CFT

Fun facts:

- Complementary series come about as analytic continuation of principal series. They are the pole crossings over principal series contour and show up as discrete sum

- Perfect agreement with the examples (Free theory and CFT) of Kallen Lehmann decomposition

- Two massless scalars give photons

Summary and open questions:

Summary

- Deriving the Kallen Lehmann decomposition for spinning two-point function in dS

- Spectral density inversion formula using the analytic continuation to EAdS (harmonic analysis) and the
  The explicit expressions for the free theory and CFT

- Analytic structure of the spectral density. The boundary theory!

- Decomposition of Fock space and CFT multiplets using character analysis

Future direction

- Bounds on EFT coefficient in dS. Role of the Hubble scale?
- Tensor products of all spinning reps in d>2
- Making sense of bulk-to-boundary expansion: What is the boundary doperators definition
- Flat-space limit?
- Bootstraping four-point function in higher dimensions! Where to look at?

Thank You!

KL decomposition in Minkowski:

A spectral decomposition of the two-point function into a sum/integral over free propagators (kinematical functions)

$$\langle\phi(x_1)\phi(x_2)\rangle = \int d\mu^2 \rho(\mu^2)\, G_{\text{free}}(x_{12}, \mu^2)$$

Is it useful? Yes! Some examples:

1. No higher derivative terms in the Lagrangian

This yields our spectral representation:[9]

$$\Delta'(p) = \int_0^\infty \rho(\mu^2) \frac{d\mu^2}{p^2 + \mu^2 - i\epsilon} . \qquad (10.7.16)$$

One immediate consequence of this result and the positivity of $\rho(\mu^2)$ is that $\Delta'(p)$ cannot vanish for $|p^2| \to \infty$ faster** than the bare propagator $1/(p^2 + m^2 - i\epsilon)$. From time to time the suggestion is made to include higher derivative terms in the unperturbed Lagrangian, which would make the propagator vanish faster than $1/p^2$ for $|p^2| \to \infty$, but the spectral representation shows that this would necessarily entail a departure from the positivity postulates of quantum mechanics.

KL decomposition in Minkowski:

A spectral decomposition of the two-point function into a sum/integral over free propagators (kinematical functions)

$$\langle \phi(x_1)\phi(x_2)\rangle = \int d\mu^2 \rho(\mu^2)\, G_{\text{free}}(x_{12}, \mu^2)$$

Is it useful? Yes! Some examples:

1. No higher derivative terms in the Lagrangian
2. Bounds on EFT coefficients

$$\mathcal{L}_{\text{EFT}} = \frac{1}{2}\phi\left[\Box + \lambda_1 \frac{\Box^2}{\Lambda^2} + +\lambda_2 \frac{\Box^4}{\Lambda^4} + \cdots\right]\phi$$

$$\lambda_1 = \Lambda^2 \int_\Lambda^\infty dm^2\, \frac{\rho_\Lambda(m^2)}{m^2} \geq 0$$

$$\lambda_1^2 - \lambda_2 = \Lambda^4 \int_\Lambda^\infty dm^2\, \frac{\rho_\Lambda(m^2)}{m^4} \geq 0$$

$$\lambda_1^3 - 2\lambda_2\lambda_1 + \lambda_3 = \Lambda^6 \int_\Lambda^\infty dm^2\, \frac{\rho_\Lambda(m^2)}{m^6} \geq 0$$

- Harish-Chandra character analysis: $\chi_R(g) \equiv \mathrm{tr}_{\mathcal{H}}(g)\ , \quad g \in \mathcal{G}$

Tensor product character in compact groups :

Weyl character of the tensor products is equal to the product of the two characters!

$$R_1 \otimes R_2 = \oplus_a R_a \quad \longrightarrow \quad \chi_{R_1}\chi_{R_2} = \sum_a n_a \chi_{R_a}$$

For example: SO(3) spin-s representaions

$$\chi_{\mathrm{s}=2} \otimes \chi_{\mathrm{s}=1} = \chi_{\mathrm{s}=1} + \chi_{\mathrm{s}=2} + \chi_{\mathrm{s}=3}$$

Generalizing to non-compact groups:

$$\Theta_{\mathcal{P}_{\Delta_1}} \otimes \Theta_{\mathcal{P}_{\Delta_2}} = \sum_s \int_\Delta \mathcal{K}_{\Delta,s} \Theta_{\mathcal{P}_{\Delta,s}} + \text{other irreps}$$

An example: principal x principal (d=1)

Character for the group element: $g = e^{tD}\ , \quad q = e^{-|t|}$

We have the Harish-Chandra character explicit expression: $\Theta_\Delta(q) = \dfrac{q^\Delta + \bar{q}^\Delta}{1-q}$

Focusing on (regular) relative kernel: $\Theta_{\Delta_1}(q)\,\Theta_{\Delta_2}(q) - \Theta_{\Delta_3}(q)\,\Theta_{\Delta_4}(q) = \int_0^\infty d\lambda\, \mathcal{K}_{\rm rel}(\lambda)\Theta_{\frac{1}{2}+i\lambda}(q)$

$$\mathcal{K}_{\rm rel}(\lambda) = \frac{1}{2\pi}\sum_{\pm,\pm,\pm}\left(\psi\left(\frac{1}{2}\pm i\mu_3 \pm i\mu_4 \pm i\lambda\right) - \psi\left(\frac{1}{2}\pm i\mu_1 \pm i\mu_2 \pm i\lambda\right)\right) \qquad \Delta_i = \frac{1}{2}+i\mu_i\ , \Delta = \frac{1}{2}+i\lambda$$

- All the principal series
- Analytic continuation in λ_i ⟶ Pole crossing and new term ⟶ Complmenetary series

d=1, The good, the bad, the ugly?

Spin indices $T_{\mu_1 \cdots \mu_J}$ $\longrightarrow$ chirality $T_{\underbrace{\pm \cdots \pm}_{J}}$

$$\langle T^{(J)}(Y_1) T^{(J)}(Y_2) \rangle \sim \sum_{\ell=0,1} \int_{\frac{d}{2}-i\infty}^{\frac{d}{2}+i\infty} \rho_\ell(\Delta)\, \nabla_1^{J-\ell} \nabla_2^{J-\ell}\, G_\ell(Y_{12}, \Delta)$$

Scetch of the derivation in Minkowski

$$\mathbb{1} = \sum_n |\psi_n\rangle\langle\psi_n| = \int_{p,\mu} |p,\mu\rangle\langle p,\mu|$$

$|p,\mu\rangle$ single-particle state with mass μ

$$\langle\phi(x_1)\phi(x_2)\rangle$$

Scetch of the derivation in Minkowski

$$\mathbb{1} = \sum_n |\psi_n\rangle\langle\psi_n| = \int_{p,\mu} |p,\mu\rangle\langle p,\mu|$$

$|p,\mu\rangle$ single-particle state with mass μ

Irreps!

$$\langle\phi(x_1)\phi(x_2)\rangle$$

Scetch of the derivation in Minkowski

$$\mathbb{1} = \sum_n |\psi_n\rangle\langle\psi_n| = \int_{p,\mu} |p,\mu\rangle\langle p,\mu|$$

$|p,\mu\rangle$ single-particle state with mass μ

$$\langle\phi(x_1)\phi(x_2)\rangle = \int_{p,\mu} \langle 0|\phi(x_1)|p,\mu\rangle\langle p,\mu|\phi(x_2)|0\rangle$$

Scetch of the derivation in Minkowski

$$\mathbb{1} = \sum_n |\psi_n\rangle\langle\psi_n| = \int_{p,\mu} |p,\mu\rangle\langle p,\mu|$$

$|p,\mu\rangle$ single-particle state with mass μ

$$\langle\phi(x_1)\phi(x_2)\rangle = \int_{p,\mu} \langle 0|\phi(x_1)|p,\mu\rangle\langle p,\mu|\phi(x_2)|0\rangle$$

$$\langle 0|\phi(x)|p,\mu\rangle = e^{ip.x}\langle 0|\phi(0)|p,\mu\rangle$$

Scetch of the derivation in Minkowski

$$\langle \phi(x_1)\phi(x_2)\rangle = \int_{p,\mu} \langle 0|\phi(x_1)|p,\mu\rangle\langle p,\mu|\phi(x_2)|0\rangle$$

$$= \int_{p,\mu} e^{ip.x_{12}} |\langle 0|\phi(0)|p,\mu\rangle|^2$$

Scetch of the derivation in Minkowski

$$\langle \phi(x_1)\phi(x_2)\rangle = \int_{p,\mu} \langle 0|\phi(x_1)|p,\mu\rangle\langle p,\mu|\phi(x_2)|0\rangle$$

$$= \int_{p,\mu} e^{ip.x_{12}} |\langle 0|\phi(0)|p,\mu\rangle|^2$$

$$= \int_0^\infty d\mu^2 \rho(\mu^2)\, G_{\text{free}}(x_{12},\mu)$$

≥ 0

Integration over momentum

Minkowski vs dS

$$\mathbb{1} = \sum_n |\psi_n\rangle\langle\psi_n| = \int_{p,\mu} |p,\mu\rangle\langle p,\mu|$$

$|p,\mu\rangle$ single-particle state with mass μ

$$\langle\phi(x_1)\phi(x_2)\rangle = \int_{p,\mu} \langle 0|\phi(x_1)|p,\mu\rangle\langle p,\mu|\phi(x_2)|0\rangle$$

$$\langle 0|\phi(x)|p,\mu\rangle = e^{ip.x}\langle 0|\phi(0)|p,\mu\rangle$$

$$\mathbb{1} = \sum_n |\psi_n\rangle\langle\psi_n| = \int_{Q,\Delta} |Q,\Delta\rangle\langle Q,\Delta|$$

$$\langle\phi(Y_1)\phi(Y_2)\rangle = \int_{Q,\Delta} \langle 0|\phi(Y_1)|Q,\Delta\rangle\langle Q,\Delta|\phi(Y_2)|0\rangle$$

$$\langle 0|\mathcal{O}(Y)|Q\rangle = c_{\mathcal{O}}(\Delta)\mathcal{K}_\Delta(Y,P)$$