Holography for de Sitter 4-point correlators

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Holography

- Gravity is expected to be holographic with any asymptotics.
- However, we only understand in detail the holographic dictionary with special asymptotics (AdS and power-law asymptotics).
- Both cases come from explicit realization in string theory via a near-horizon limit of D-branes.
- In these cases the dual theory is a local QFT.

Holography for de Sitter

- dS/CFT correspondence has been proposed more than 20 years ago [Strominger (2001)]
- The status has remained controversial.
- Different formulations/versions have appeared through the years
  - Wavefunction of the universe [Maldacena (2002)] ...
  - Domain-wall/cosmology correspondence [KS, Townsend (2006)] [Bzowski, McFadden, KS (2009-2013)] ...
  - Cosmological bootstrap [Arkani-Hamed etal (2018)] ....
  - ....
- There is a useful and working version of dS/CFT perturbatively in $1/N$.
- It is unclear that such dualities exist non-perturbatively in $1/N$.
- There is no known embedding in/derivation from string theory.

Issues with de Sitter

- Boundary is spacelike, at future and past infinity
- ➠ Emergent direction is time.
- Which observables?
- In this work: in-in cosmological correlators at future infinity
- Main question:

  Can the in-in correlators be expressed in terms of correlators of a local CFT?

Reference

Based on work with Adam Bzowski and Paul McFadden,

- Holography for de Sitter 4-point correlators, 2305.xxxxx

- There is a long list of relevant work, which includes

[McFadden, KS](2010)(2011) [Bzowski, McFadden, KS (2011)(2012)] [Pimentel, Maldacena (2011)][Hartle, Hawking, Hertog (2012)][Anninos, Hartman, Strominger (2012)][Mata, Raju,Trivedi (2012)] [Kundu, Shukla,Trivedi (2014)][Arkani-Hamed, Maldacena (2015)] .... [Sleight, Toronna (2018-2022)] ... [Arkani-Hamed, Baumann, Lee, Pimentel (2018)] [Baumann et al (2019)(2020)(2021).... [Di Petro, Gorbenko, Komatsu (2021)... [Raju et al (2023)]

Outline

Outline

From EAdS to dS

Start from AdS,

$$ds^2 = \frac{\ell_{AdS}^2}{z^2}\left(dz^2 + d\mathbf{x}^2\right)$$

and analytically continue:

$$\ell_{AdS}^2 \to -\ell_{dS}^2, \qquad z^2 \to -\tau^2$$

to obtain dS

$$ds^2 = \frac{\ell_{dS}^2}{\tau^2}\left(-d\tau^2 + d\mathbf{x}^2\right)$$

Outline

Domain-wall/Cosmology correspondence

- EAdS/dS is an example of a more general **Domain-wall / Cosmology correspondence** [KS, Tonwsend (2006)]

FRW solutions of the theory with potential $V(\Phi)$	$\leftrightarrow$	Domain-wall solutions of the theory with potential $-V(\Phi)$.

Domain-wall solutions

$$
\begin{aligned}
ds^2 &= dr^2 + e^{2A(r)} dx^i dx^i \\
\Phi &= \Phi(r)
\end{aligned}
$$

FRW spacetimes

$$
\begin{aligned}
ds^2 &= -dt^2 + a^2(t) dx^i dx^i \\
\Phi &= \Phi(t)
\end{aligned}
$$

- This correspondence can be understood as analytic continuation.

Inflation/holographic RG correspondence

- A special case of the correspondence is that between inflationary backgrounds and holographic RG flow spacetimes.
- Inflationary spacetimes are mapped to
  - asympotically Anti-de Sitter spacetime,

$$ds^2 \to ds^2 = dr^2 + e^{2r} dx^i dx^i, \qquad \text{as} \quad r \to \infty$$

  - power-law scaling solutions,

$$ds^2 \to ds^2 = dr^2 + r^{2n} dx^i dx^i, \quad (n > 1) \qquad \text{as} \quad r \to \infty$$

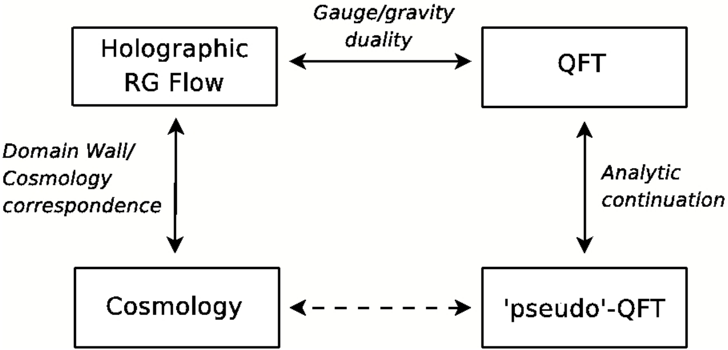
  For special values of n these backgrounds are related to non-conformal branes.
- For these backgrounds there is an established holographic dictionary.

Holographic formulae for cosmology [McFadden, KS (2009)]

- Given an FRW, compute cosmological observables power spectra, non-Gausianities using standard cosmological perturbation theory.
- Corresponding to this FRW there is a domain-wall.
- Use holography to compute energy-momentum tensor correlators for the QFT dual to the domain-wall.

➠ Comparing the two results leads to holographic formulae →

Summary


Holographic RG Flow
Gauge/gravity duality
QFT
Domain Wall/ Cosmology correspondence
Analytic continuation
Cosmology
'pseudo'-QFT

Outline

Wavefunction of the Universe

- The partition function of the dual QFT computes the wavefunction of the Universe [Maldacena (2002)]:

$$\psi[\varphi_{(0)};\ell_{dS}] = Z_{EAdS}[\varphi_{(0)};\ell_{AdS}]\Big|_{\mathrm{a.c.}} = Z_{CFT}[\varphi_{(0)}]\Big|_{\mathrm{a.c.}}$$

- Cosmological observables are computed as

$$\langle\varphi_{(0)}(x_1)\cdots\varphi_{(0)}(x_n)\rangle = \int D\varphi_{(0)}|\psi|^2\varphi_{(0)}(x_1)\cdots\varphi_{(0)}(x_n)$$

- The partition has an expansion in correlation functions:

$$Z_{QFT}[\varphi_{(0)}] = \exp\left(\sum_n \langle O(x_1)\cdots O(x_n)\rangle\varphi_{(0)}(x_1)\cdots\varphi_{(0)}(x_n)\right)$$

$Z_{EAdS}[\varphi_{(0)}; \ell_{AdS}]$

- Action:

$$S_{EAdS} = \ell_P^{1-d} \int d^{d+1}x \sqrt{g} \left(\frac{1}{2}(\partial\varphi)^2 + \frac{1}{2} m_{AdS}^2 \varphi^2 + \ell_P^{-2} V_{int}(\varphi) \right),$$

- Asymptotic solutions:

$$\varphi(z, \vec{x}) = z^{d-\Delta} \varphi_{(0)}(\vec{x}) + \ldots + z^{\Delta} \varphi_{(\Delta)}(\vec{x}) + \ldots, \qquad z \to 0^+$$

- Partition function:

$$Z_{EAdS}[\varphi_{(0)}, \ell_{AdS}] = \int \mathcal{D}\varphi \, e^{-S_{AdS}}$$

$$= \exp\left(\sum_{n=2}^{\infty} \frac{(-1)^n}{n!} \int [d\boldsymbol{q}_n] \langle \mathcal{O}(\boldsymbol{q}_1) \ldots \mathcal{O}(\boldsymbol{q}_n) \rangle \varphi_{(0)}(-\boldsymbol{q}_1) \ldots \varphi_{(0)}(-\boldsymbol{q}_n) \right).$$

- CFT correlators are computed on spacetime with metric:

$$ds^2_{\mathrm{Bdry}} = \ell^2_{AdS} d\mathbf{x}^2$$

$\psi[\varphi_{(0)}; \ell_{dS}]$

- Action:

$$S_{dS} = -\ell_P^{1-d} \int d^{d+1}x \sqrt{-g} \left(\frac{1}{2} (\partial\varphi)^2 + \frac{1}{2} m_{dS}^2 \varphi^2 + \ell_P^{-2} V_{int}(\varphi) \right)$$

- Asymptotic solution:

$$\varphi(\tau, \vec{x}) = (-\tau)^{d-\Delta} \varphi_{(0)}(\vec{x}) + \ldots + (-\tau)^{\Delta} \varphi_{(\Delta)}(\vec{x}) + \ldots, \qquad \tau \to 0^-$$

  We are consider light fields with Δ real and positive.

- Wavefunction:

$$\Psi_{dS}[\varphi_{(0)}; \ell_{dS}] = \langle \varphi_{(0)}(\vec{x}) | 0 \rangle = \int_{\tau=0} \mathcal{D}\varphi \, e^{iS_{dS}}$$

$$= \exp\left(\sum_{n=2}^{\infty} \frac{(-1)^n}{n!} \int [d\boldsymbol{q}_n] \psi_n(\boldsymbol{q}_1, \ldots, \boldsymbol{q}_n) \varphi_{(0)}(-\boldsymbol{q}_1) \ldots \varphi_{(0)}(-\boldsymbol{q}_n) \right)$$

  where $|0\rangle$ is the Bunch-Davies state.

Analytic continuation 1: Planck units

- Work with Planck units $\ell_P = 1$:

$$\ell_{AdS} = i\ell_{dS}, \qquad \varphi_{(0)}^{AdS} = (-i)^{d-\Delta}\varphi_{(0)}^{dS}$$

- Then to all orders in perturbation theory:

$$\psi_n(\boldsymbol{q}_1, \ldots, \boldsymbol{q}_n) = (-i)^{n(d-\Delta)} \langle \mathcal{O}(\boldsymbol{q}_1) \ldots \mathcal{O}(\boldsymbol{q}_n) \rangle \Big|_{\ell_{AdS} \to i\ell_{dS}}$$

- Note that CFT correlators contain factors of ℓ_{AdS} because the boundary metric is $ds^2_{\mathrm{Bdry}} = \ell^2_{AdS} d\mathbf{x}^2$.
- This is the form of the continuation used in [Maldacena (2002)]
- This is natural from the bulk perspective when the r.h.s. is computed via Witten diagrams.

Analytic continuation 2

- Work with AdS/dS units, with $\ell_{AdS} = \ell_{dS} = 1$.
- This can be obtained by a Weyl transformation that removes ℓ_{AdS} from the boundary metric [Garriga, KS, Urakawa (2014)]. This results in

$$\ell_P^{AdS} \to -i\ell_P^{dS}, \quad q_{AdS} \to iq_{dS}, \quad \varphi_{(0)}^{AdS} = \varphi_{(0)}^{dS}$$

- Then

$$\psi_n(\boldsymbol{q}_1, \ldots, \boldsymbol{q}_n) = (-i)^d \langle \mathcal{O}(\boldsymbol{q}_1) \ldots \mathcal{O}(\boldsymbol{q}_n) \rangle \Big|_{\ell_P^{AdS} \to -i\ell_P^{dS},\, q_{AdS} \to iq_{dS}}$$

- Now that CFT correlators are standard correlators in flat space.
- This is the form of continuation that resulted using the domain-wall/cosmology correspondence [McFadden, KS (2009)].
- The continuation of ℓ_P can be expressed in CFT language as $N^2 \to -N^2$ (when the gauge group in $SU(N)$)).
- This is natural from the boundary perspective as the continuation $(q \to iq, N^2 \to -N^2)$ refers only to CFT variables.

In-in dS correlators

In-in correlators can now be computed by:

- doing the final integration over $\varphi_{(0)}$, or
- (equivalently) by using the Schwinger-Keldysh formalism.

$$\langle \varphi_{(0)}(\boldsymbol{q})\varphi_{(0)}(-\boldsymbol{q})\rangle = \frac{1}{2}\frac{1}{\mathrm{Im}\langle \mathcal{O}(\boldsymbol{q})\mathcal{O}(-\boldsymbol{q})\rangle},$$

$$\langle \varphi_{(0)}(\boldsymbol{q}_1)\varphi_{(0)}(\boldsymbol{q}_2)\varphi_{(0)}(\boldsymbol{q}_3)\rangle = -\frac{1}{4}\frac{\mathrm{Im}\langle \mathcal{O}(\boldsymbol{q}_1)\mathcal{O}(\boldsymbol{q}_2)\mathcal{O}(\boldsymbol{q}_3)\rangle}{\prod_{i=1}^{3}\mathrm{Im}\langle \mathcal{O}(\boldsymbol{q}_i)\mathcal{O}(-\boldsymbol{q}_i)\rangle},$$

$$\langle \varphi_{(0)}(\boldsymbol{q}_1)\varphi_{(0)}(\boldsymbol{q}_2)\varphi_{(0)}(\boldsymbol{q}_3)\varphi_{(0)}(\boldsymbol{q}_4)\rangle = -\frac{1}{8}\Bigg[\frac{\mathrm{Im}\langle \mathcal{O}(\boldsymbol{q}_1)\mathcal{O}(\boldsymbol{q}_2)\mathcal{O}(\boldsymbol{q}_3)\mathcal{O}(\boldsymbol{q}_3)\rangle}{\prod_{i=1}^{4}\mathrm{Im}\langle \mathcal{O}(\boldsymbol{q}_i)\mathcal{O}(-\boldsymbol{q}_i)\rangle}$$

$$-\left(\frac{\mathrm{Im}\langle \mathcal{O}(\boldsymbol{q}_1)\mathcal{O}(\boldsymbol{q}_2)\mathcal{O}(\boldsymbol{q}_{12})\rangle\,\mathrm{Im}\langle \mathcal{O}(-\boldsymbol{q}_{12})\mathcal{O}(\boldsymbol{q}_3)\mathcal{O}(\boldsymbol{q}_4)\rangle}{\mathrm{Im}\langle \mathcal{O}(\boldsymbol{q}_{12})\mathcal{O}(-\boldsymbol{q}_{12})\rangle\prod_{i=1}^{4}\mathrm{Im}\langle O(\boldsymbol{q}_i)\mathcal{O}(-\boldsymbol{q}_i)\rangle} + (2\leftrightarrow 3) + (2\leftrightarrow 4)\right)\Bigg]$$

where $\boldsymbol{q}_{ij} = \boldsymbol{q}_i + \boldsymbol{q}_j$, and the imaginary part is taken after the analytic continuation $N^2 \to -N^2, \quad q \to iq$

Renormalization

- These formulas continue to make sense also when the l.h.s. diverges as $\tau \to 0$.
- These infinities are non-local from the perspective of the bulk.
- They are local from the perspective of the dual CFT, and can be removed by standard CFT renormalization.
- From the bulk perspective one can think of them as counterterms required for the wavefunction to be normalizable, or
- as counterterms in the Schwinger-Keldysh path integral at $\tau = 0$.

Remarks

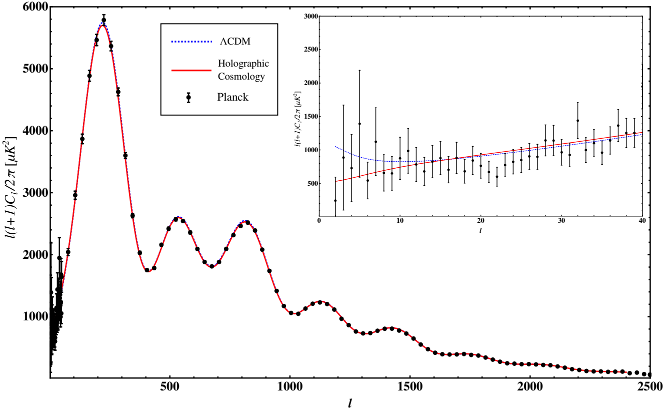
- The formulae take universal form independent of the order of the perturbation theory.
- Using the same formulae in the regime where the QFT (and not the bulk) is perturbative yields sensible results.

  [McFadden, KS][Bzowski, KS (2009–2013)][Nastase, KS (2020)]
- This provides non-geometric holographic models for the very early universe that are competitive with ΛCDM.

  [Easther, Flauger, McFadden, KS (2011)][Afshordi, Coriano, Delle Rose, Gould, KS (2017)]
- It even allows to explicitly investigate the resolution of the Big Bang singularity, which in this context is mapped to the IR finiteness of the dual QFT.

  [LatCos collaboration] (2021)]

Planck 2015 vs ΛCDM vs holographic model (TT)



[Afshordi, Coriano, Delle Rose, Gould, KS, PRL2017] [Afshordi, Gould, KS, PRD2017]

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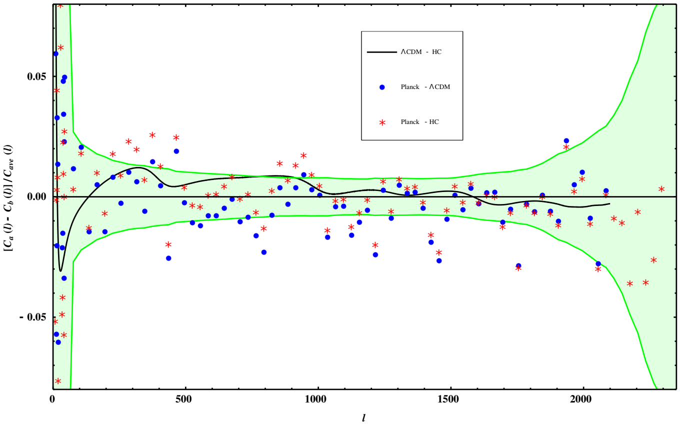

[Afshordi, Coriano, Delle Rose, Gould, KS, PRL2017] [Afshordi, Gould, KS, PRD2017]

Tree-level

$$ds^{\text{ren}}_{[\Delta\Delta]}(q) = -\frac{1}{2}\frac{1}{\text{Im}\, i^{\text{ren}}_{[\Delta\Delta]}(\text{i}q; -\mu)},$$

$$ds^{\text{ren}}_{[\Delta_1\Delta_2\Delta_3]}(q_i;\mu) = -\frac{1}{4}\frac{\text{Im}\, i^{\text{ren}}_{[\Delta_1\Delta_2\Delta_3]}(\text{i}q_i; -\mu)}{\prod_{j=1}^{3}\text{Im}\, i^{\text{ren}}_{[\Delta_j\Delta_j]}(\text{i}q_j, -\mu)},$$

$$ds^{\text{ren}}_{[\Delta_1\Delta_2\Delta_3\Delta_4]}(q_i;\mu) = \frac{1}{8}\frac{\text{Im}\, i^{\text{ren}}_{[\Delta_1\Delta_2\Delta_3\Delta_4]}(\text{i}q_i; -\mu)}{\prod_{j=1}^{4}\text{Im}\, i^{\text{ren}}_{[\Delta_j\Delta_j]}(\text{i}q_j; -\mu)},$$

$$ds^{\text{ren}}_{[\Delta_1\Delta_2;\Delta_3\Delta_4 x\Delta_x]}(q_i, s;\mu) = \frac{1}{8}\prod_{j=1}^{4}\frac{1}{\text{Im}\, i^{\text{ren}}_{[\Delta_j\Delta_j]}(\text{i}q_j; -\mu)}\times$$

$$\left[\text{Im}\, i^{\text{ren}}_{[\Delta_1\Delta_2;\Delta_3\Delta_4 x\Delta_x]}(\text{i}q_i, \text{i}s; -\mu) - \frac{\text{Im}\, i^{\text{ren}}_{[\Delta_1\Delta_2\Delta_x]}(\text{i}q_1, \text{i}q_2, \text{i}s; -\mu)\,\text{Im}\, i^{\text{ren}}_{[\Delta_x\Delta_3\Delta_4]}(\text{i}s, \text{i}q_3, \text{i}q_4; -\mu)}{\text{Im}\, i^{\text{ren}}_{[\Delta_x\Delta_x]}(\text{i}s; -\mu)}\right]$$

Notation: $ds^{\text{ren}}_{[\Delta_1\Delta_2;\Delta_3\Delta_4 x\Delta_x]}$: dS in-in exchange diagram, etc.
$i^{\text{ren}}_{[\Delta_1\Delta_2;\Delta_3\Delta_4 x\Delta_x]}$: AdS exchange diagram, etc.

Outline

dS isometries and CFT Ward identities

- The bulk de Sitter isometries give rise to Ward identities for cosmological correlators.
- Under a diffeomorphism $x^\mu \to x^\mu - \xi^\mu$ by a Killing vector ξ^μ:

$$0 = \sum_{i=1}^{n} \xi^\mu(x_i) \frac{\partial}{\partial x_i^\mu} \langle \varphi(x_1) \dots \varphi(x_n) \rangle.$$

- For ξ^μ the KV for dilatations and special conformal transformations:

$$0 = \sum \Big(\bar{\Delta}_i + \vec{x}_i \cdot \boldsymbol{\partial}_i \Big) \langle \varphi_{(0)}(\vec{x}_1) \dots \varphi_{(0)}(\vec{x}_n) \rangle,$$

$$0 = \sum \Big(-2\bar{\Delta}_i \boldsymbol{b} \cdot \vec{x}_i + \big(x_i^2\, \boldsymbol{b} - 2(\boldsymbol{b} \cdot \vec{x}_i)\, \vec{x}_i \big) \cdot \boldsymbol{\partial}_i \Big) \langle \varphi_{(0)}(\vec{x}_1) \dots \varphi_{(0)}(\vec{x}_n) \rangle.$$

- These are exactly the same as the conformal WIs in d dimensions with $\varphi_{(0)}$ a conformal field of dimension $\bar{\Delta}_i = d - \Delta_i$, the shadow dimension.

In-in correlators and shadow fields

- This suggests that the cosmological correlators *are* CFT correlators of fields with shadow dimension $\bar{\Delta} = d - \Delta$, without any analytic continuation. Explicit computation at tree-level yields:

$$ds_{[\Delta\Delta]} = -\frac{1}{8\bar{\beta}^2 \sin(\pi\bar{\beta})} i^{\text{fin}}_{[\bar{\Delta}\bar{\Delta}]},$$

$$ds_{[\Delta_1,\Delta_2,\Delta_3]} = \frac{2\sin\left[\frac{\pi}{2}(\bar{\beta}_t + \frac{d}{2})\right]}{\prod_{j=1}^3 4\bar{\beta}_j \sin(\pi\bar{\beta}_j)} i^{\text{fin}}_{[\bar{\Delta}_1,\bar{\Delta}_2,\bar{\Delta}_3]},$$

$$ds_{[\Delta_1\Delta_2,\Delta_3\Delta_4]} = -\frac{2\sin\left[\frac{\pi}{2}(\bar{\beta}_T + d)\right]}{\prod_{j=1}^4 4\bar{\beta}_j \sin(\pi\bar{\beta}_j)} i^{\text{fin}}_{[\bar{\Delta}_1,\bar{\Delta}_2,\bar{\Delta}_3,\bar{\Delta}_4]},$$

$$ds_{[\Delta_1\Delta_2;\Delta_3\Delta_4 x\Delta_x]} = \mathcal{B}(\beta_x) i^{\text{fin}}_{[\bar{\Delta}_1\bar{\Delta}_2;\bar{\Delta}_3\bar{\Delta}_4 x\Delta_x]} + \mathcal{B}(\bar{\beta}_x) i^{\text{fin}}_{[\bar{\Delta}_1\bar{\Delta}_2;\bar{\Delta}_3\bar{\Delta}_4 x\bar{\Delta}_x]},$$

here $\bar{\beta}_i = \bar{\Delta}_i - d/2$ and

$$\mathcal{B}(\bar{\beta}_x) = -\frac{2\sin\left(\frac{\pi}{2}(\bar{\beta}_1 + \bar{\beta}_2 + \bar{\beta}_x + \frac{d}{2})\right)\sin\left(\frac{\pi}{2}(\bar{\beta}_3 + \bar{\beta}_4 + \bar{\beta}_x + \frac{d}{2})\right)}{\sin(\pi\bar{\beta}_x)\prod_{j=1}^4 4\bar{\beta}_j \sin(\pi\bar{\beta}_j)}.$$

Remark

- These relation however are only valid when renormalization is not needed.
- When the amplitudes are finite one can show that they agree with the ones we derived earlier:
  - Finite correlators of operators of shadow dimensions, $\bar{\Delta} = d - \Delta$, can be obtained from the original ones by Legendre transform:

$$\langle O_{\bar{\Delta}}(\boldsymbol{q}) O_{\bar{\Delta}}(-\boldsymbol{q})\rangle = \frac{1}{\langle O_{\Delta}(\boldsymbol{q}) O_{\Delta}(-\boldsymbol{q})\rangle}$$

$$\langle O_{\bar{\Delta}_1}(\boldsymbol{q_1}) O_{\bar{\Delta}_2}(\boldsymbol{q_2}) O_{\bar{\Delta}_3}(\boldsymbol{q_3})\rangle = \frac{\langle O_{\Delta_1}(\boldsymbol{q_1}) O_{\Delta_2}(\boldsymbol{q_2}) O_{\Delta_3}(\boldsymbol{q_3})\rangle}{\prod_i \langle O_{\Delta_i}(\boldsymbol{q}_i) O_{\Delta_i}(-\boldsymbol{q}_i)\rangle}$$

$$\langle O_{\bar{\Delta}_1}(\boldsymbol{q_1}) O_{\bar{\Delta}_2}(\boldsymbol{q_2}) O_{\bar{\Delta}_3}(\boldsymbol{q_3}) O_{\bar{\Delta}_4}(\boldsymbol{q_4})\rangle = \ldots.$$

  - Taking the imaginary parts generates the coefficients with sines.

Renormalisation

- There is no renormalised version of the relations involving CFT fields with shadow dimensions.
- This is easiest to see with a counterexample.
- We will consider the case, $ds^{\text{ren}}_{[322]}$, the 3-point function of one massless scalars and two conformal scalars.

Outline

Conformal Ward identities

- The conformal Ward identities in momentum space for 3-point functions boil down to [Bzowski, McFadden, KS] (2013).:

$$0 = \mathrm{K}_{12}\langle \mathcal{O}_1(\boldsymbol{p}_1)\mathcal{O}_2(\boldsymbol{p}_2)\mathcal{O}_3(\boldsymbol{p}_3)\rangle = \mathrm{K}_{23}\langle \mathcal{O}_1(\boldsymbol{p}_1)\mathcal{O}_2(\boldsymbol{p}_2)\mathcal{O}_3(\boldsymbol{p}_3)\rangle,$$

where $(i,j=1,2,3)$

$$\mathrm{K}_{ij} = \mathrm{K}_i - \mathrm{K}_j, \qquad \mathrm{K}_j = \frac{\partial^2}{\partial p_j^2} + \frac{d+1-2\Delta_j}{p_j}\frac{\partial}{\partial p_j}\ .$$

- This system of differential equations is precisely that defining Appell's F_4 generalised hypergeometric function of two variables. [Coriano, Rose, Mottola, Serino][Bzowski, McFadden, KS] (2013).

General solution of CWI for 3-point functions

- Using separation of variables, one obtains a useful integral representation in terms of a *triple-K integral* integral:

$$\langle \mathcal{O}_{\Delta_1}(\boldsymbol{p}_1)\mathcal{O}_{\Delta_2}(\boldsymbol{p}_2)\mathcal{O}_{\Delta_3}(\boldsymbol{p}_3)\rangle = C_{123} p_1^{\Delta_1-\frac{d}{2}} p_2^{\Delta_2-\frac{d}{2}} p_3^{\Delta_3-\frac{d}{2}}$$
$$\int_0^\infty dx\, x^{\frac{d}{2}-1} K_{\Delta_1-\frac{d}{2}}(p_1 x) K_{\Delta_2-\frac{d}{2}}(p_2 x) K_{\Delta_3-\frac{d}{2}}(p_3 x),$$

  where $K_\nu(p)$ is a Bessel function and C_{123} is a constant.
- The integral converges provided

$$\frac{d}{2} - 1 > \sum_{j=1}^{3} |\beta_j| - 1, \qquad \beta_i = \Delta_i - \frac{d}{2}$$

- The integral can be defined by analytic continuation when

$$\frac{d}{2} \pm \beta_1 \pm \beta_2 \pm \beta_3 \neq -2k,$$

  where k is any non-negative integer.

Renormalization and anomalies [Bzowski, McFadden, KS (2015)]

- If the equality holds,

$$d/2 \pm \beta_1 \pm \beta_2 \pm \beta_3 = -2k,$$

  the integral cannot be defined by analytic continuation.
- Non-trivial subtractions and renormalization may be required and this may result in conformal anomalies and beta functions.
- Physically when this equality holds, there are new terms of dimension d that one can add to the action (counterterms) and/or new terms that can appear in T^μ_μ (conformal anomalies).
- We use dimensional regularisation to regulate the theory

$$d \mapsto d + 2u\epsilon, \qquad \Delta_j \mapsto \Delta_j + (u + v_j)\epsilon$$

  u, v_j are constants that characterize the scheme.

Singularities

- There are four different type of singularities, depending on the choice signs.
- Two of them, $(- - -)$ and $(+ + +)$ are projected out by the holographic formulae.
- The other two $(- - +)$ and $(+ + -)$ (and permutations) are important in this context.

$(- - +)$ singularities: beta functions

- $(- - +)$ case: $\Delta_1 + \Delta_2 - \Delta_3 = d + 2k$
- Counterterm: $\int d^d x \Box^{k_1} \phi_1 \Box^{k_2} \phi_2 O_3$
  - The source ϕ_3 of O_3 renormalizes and this results in beta functions.
  - The conformal Ward Identity is anomalous.
  - In the context of dS/CFT, the dS correlator requires renormalization and the beta functions survive the analytic continuation.

Example: $\Delta_1 = 3, \Delta_2 = \Delta_3 = 2, d = 3$

- It satisfies the condition $\Delta_1 + \Delta_2 - \Delta_3 = d + 2k$ with $k = 0$.
- By direct evaluation:

$$i^{\text{ren}}_{[322]}(q_1, q_2, q_3) = (q_2 + q_3) \log\left(\frac{q_t}{\mu}\right) - q_1$$

It satisfies an anomalous CWI.

- The cosmological correlator $ds^{\text{ren}}_{[322]}$ can be explicitly computed:

$$ds^{\text{ren}}_{[322]}(q_1, q_2, q_3) = -\frac{(q_2 + q_3)\log\left(\frac{q_t}{\mu}\right) - q_1}{16 q_1^3 q_2 q_3}$$

- It satisfies an anomalous CWI, consistent with that of $i^{\text{ren}}_{[322]}(q_1, q_2, q_3)$ and the holographic formula that relates them.

$(++-)$ singularities

- $\Delta_1 + \Delta_2 - \Delta_3 = -2k$
- The coefficient of leading divergence is non-local and is the actual correlator. For example, with $k = 0$:

$$\langle O_{\Delta_1}(\boldsymbol{q}_1) O_{\Delta_2}(\boldsymbol{q}_2) O_{\Delta_3}(\boldsymbol{q}_3) \rangle = c \frac{1}{\epsilon} q_1^{(2\Delta_1 - d)} q_2^{(2\Delta_2 - d)}$$

The infinity is absorbed in c.
- This is an example of an extremal correlator. In position space:

$$\langle O_{\Delta_1}(\boldsymbol{x}_1) O_{\Delta_2}(\boldsymbol{x}_2) O_{\Delta_3}(\boldsymbol{x}_3) \rangle = \frac{c_{123}}{|\boldsymbol{x}_1 - \boldsymbol{x}_3|^{2\Delta_1} |\boldsymbol{x}_2 - \boldsymbol{x}_3|^{2\Delta_2}}$$

- It satisfies the non-anomalous Ward identity.
- In the context of dS/CFT, the infinity survives the analytic continuation.

Example: $\bar{\Delta}_1 = 0, \bar{\Delta}_2 = \bar{\Delta}_3 = 1$

- It satisfied $\bar{\Delta}_1 + \bar{\Delta}_2 - \bar{\Delta}_3 = -2k$ with $k = 0$.
- These dimensions are the shadow dimensions of $\Delta_1 = 3, \Delta_2 = \Delta_3 = 2$ in $d = 3$: $\bar{3} = 0, \bar{2} = 1, \bar{2} = 1$, and

$$i^{\text{div}}_{[\bar{3}\bar{2}\bar{2}]} = \frac{1}{\epsilon} \frac{1}{q_1^3 q_2}$$

This is a non-local infinity and cannot be removed by a counterterm in a local CFT.

➠ The relation

$$ds_{[\Delta_1,\Delta_2,\Delta_3]} = \frac{2\sin\left[\frac{\pi}{2}(\bar{\beta}_t + \frac{d}{2})\right]}{\prod_{j=1}^{3} 4\bar{\beta}_j \sin(\pi\bar{\beta}_j)} i_{[\bar{\Delta}_1,\bar{\Delta}_2,\bar{\Delta}_3]}$$

cannot hold when $\Delta_1 = 3, \Delta_2 = \Delta_3 = 2$ in $d = 3$.

Outline

Summary of results

- We provided holographic formula for 4-point functions in dS and different looking analytic continuations are in fact equivalent.
- We worked out explicit results for tree-level dS 4-point functions:
  - external and exchange fields are massless and conformal scalars,
  - general interactions including derivative interaction
- Many of these cases require renormalisation.
- Holographic formulas coming from analytic continuation are always valid, and are consistent with the dual CFT being local.
- Formulation with CFT with shadow dimensions breaks down when renormalization is needed.

Outlook

- Extend the results to different external operators, including spinning and heavy massive fields.
- For the cosmological bootstrap programme, one would need the form of the anomalous dS Ward identities.
- It is important to check at least in an example whether the continuation $N^2 \to -N^2$ makes sense non-perturbatively.

Power spectra

- Power spectra

$$\Delta^2_{\mathcal{R}}(q) = -\frac{q^3}{16\pi^2}\frac{1}{\mathrm{Im}B(q,N)}, \quad \Delta^2_{\mathcal{T}}(q) = -\frac{q^3}{2\pi^2}\frac{1}{\mathrm{Im}A(q,N)}$$

where

$$\langle T_{ij}(q)T_{kl}(-q)\rangle = A(q)\Pi_{ijkl} + B(q)\pi_{ij}\pi_{kl}$$

where Π_{ijkl} is a projector to the transverse-traceless part and π_{ij} is a transverse projector.

Holographic formulae: 3-point functions

- $\langle \zeta(q_1)\zeta(q_2)\zeta(q_3)\rangle$

$$= -\frac{1}{256}\Big(\prod_i \mathrm{Im}[B(\bar{q}_i)]\Big)^{-1} \times \mathrm{Im}\Big[\langle T(\bar{q}_1)T(\bar{q}_2)T(\bar{q}_3)\rangle + (\text{semi}-\text{local terms})\Big]$$

- $\langle \zeta(q_1)\zeta(q_2)\hat{\gamma}^{(s_3)}(q_3)\rangle$

$$= -\frac{1}{32}\Big(\mathrm{Im}[B(\bar{q}_1)]\mathrm{Im}[B(\bar{q}_2)]\mathrm{Im}[A(\bar{q}_3)]\Big)^{-1}$$

$$\times\, \mathrm{Im}\Big[\langle T(\bar{q}_1)T(\bar{q}_2)T^{(s_3)}(\bar{q}_3)\rangle + (\text{semi}-\text{local terms})\Big],$$

[McFadden, KS (2010), (2011)]

Holographic formulae: 3-point functions

- $\langle \zeta(q_1)\hat{\gamma}^{(s_2)}(q_2)\hat{\gamma}^{(s_3)}(q_3)\rangle$

$$= -\frac{1}{4}\Big(\mathrm{Im}[B(\bar{q}_1)]\mathrm{Im}[A(\bar{q}_2)]\mathrm{Im}[A(\bar{q}_3)]\Big)^{-1}$$

$$\times \mathrm{Im}\Big[\langle T(\bar{q}_1)T^{(s_2)}(\bar{q}_2)T^{(s_3)}(\bar{q}_3)\rangle + (\mathrm{semi{-}local\ terms})\Big],$$

- $\langle \hat{\gamma}^{(s_1)}(q_1)\hat{\gamma}^{(s_2)}(q_2)\hat{\gamma}^{(s_3)}(q_3)\rangle$

$$= -\Big(\prod_i \mathrm{Im}[A(\bar{q}_i)]\Big)^{-1} \times \mathrm{Im}\Big[2\langle T^{(s_1)}(\bar{q}_1)T^{(s_2)}(\bar{q}_2)T^{(s_3)}(\bar{q}_3)\rangle + (\mathrm{semi{-}local\ tern}$$

[McFadden, KS (2011)]

Back

$(- - -)$ singularities: conformal anomalies

- $(- - -)$ case: $\Delta_1 + \Delta_2 + \Delta_3 = 2d + 2k$.
- Counterterm: $\int d^d\boldsymbol{x}\, \Box^{k_1}\phi_1 \Box^{k_2}\phi_2 \Box^{k_3}\phi_3$,
  where ϕ_i is the source for O_{Δ_i} and $k_1 + k_2 + k_3 = k$.
- This leads to conformal anomalies
- In the context of dS/CFT, the conformal anomalies are projected out by the analytic continuation and the dS correlator is finite.

  Equivalently, the sine functions in the formulas with shadow fields provide a zero that cancels the singularity.

Back

$(+++)$ singularities

- $(+++)$ case: $\Delta_1 + \Delta_2 - \Delta_3 = d + 2k$
- The coefficient of leading divergence is non-local and is the actual correlator. For example, with $k = 0$:

$$\langle O_{\Delta_1}(\boldsymbol{q}_1) O_{\Delta_2}(\boldsymbol{q}_2) O_{\Delta_3}(\boldsymbol{q}_3)\rangle = c\frac{1}{\epsilon} q_1^{(\Delta_1-\Delta_2-\Delta_3)} q_2^{(\Delta_2-\Delta_1-\Delta_3)} q_3^{(\Delta_3-\Delta_1-\Delta_2)}$$

  The infinity is absorbed in c.
- It satisfies the non-anomalous Ward identity.
- In the context of dS/CFT, the dS correlator is finite and the analytic continuation provides the zero to cancel the infinity.

  Equivalently, the sine functions in the formulas with shadow fields provide a zero that cancels the singularity.

Back