Clocks Algebras and Cosmology

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1.- Time in Quantum Gravity

If in QGr we define physical observables by the condition

$$[H\,\mathcal{O}] = 0 \qquad (1)$$

for H the Hamiltonian and $\mathcal{O}$ local observables then all physical observables are stationary

So how is it that we observe time dependence in the world?

$$(1) \iff \mathcal{A}_{ph} = \mathcal{A}^{H}$$

↳ The algebra of local observables

Page - Wootters 1983

This problem is similar to the problem we find with super-selection changes

$$A_{ph} = \{ \mathcal{O} \in A ; [Q\ \mathcal{O}] = 0 \} \qquad \mathcal{O} \text{ - local observables}$$

$\Rightarrow$ $c_1 |q_1\rangle + c_2 |q_2\rangle$ are unobservable.

The way to address this problem originally suggested by Aharonov-Susskind 1967 is to add a clock / reference frame quantum system

i.e. To extend the algebra as well as the Hilbert space representation

In quantum information super-selection charges are associated with total lack of information about reference frame.
(Bartlett - Rudolph and Spekkens 2007)

How to add a reference frame?

For the case of a clock you formally define a clock Algebra $\mathcal{A}_{clock}$ by

$$[\hat{H}_c, \hat{t}\,] = i\hbar \qquad \begin{array}{l}\hat{H}_c \text{ - clock hamiltonian} \\ \hat{t} \text{ - "clock time" operator}\end{array}$$

commuting with the algebra $\mathcal{A}$ describing the observed system and you define the algebra of physical observables as

$$(\mathcal{A} \otimes \mathcal{A}_{clock})^{\hat{H}+\hat{H}_c} \to a \in \mathcal{A} \times \mathcal{A}_{clock} \,;\ [a, \hat{H}+\hat{H}_c] = 0$$

and the Hilbert space

$$\mathcal{H}_{ph} = \{ |\psi\rangle \in \mathcal{H} \times \mathcal{H}_{clock} \,; (\hat{H}+\hat{H}_c)|\psi\rangle = 0 \}$$

How to define a clock in Q.M is an old problem that goes back to the problem of how to interpret Energy-time uncertainty

$$\Delta E \, \Delta t \geq \hbar$$

This relation will follow from a clock algebra A_{clock}

$$[\hat{H}_c, \hat{t}] = i\hbar \qquad (1)$$

But how to define $\hat{t}$?

Note some problems associated with (1).

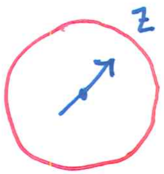
If $\hat{H}_c$ and $\hat{t}$ are selfadjoint operators then both have as spectrum the whole real line $\mathbb{R}$. i.e the "clock hamiltonian" $\hat{H}_c$ is not positive.

Moreover if $\hat{H}_c$ positive the clock will not work.

Unruh - Wald 1989

2.- Clocks

The simplest way to define a clock in QM is to use a quantum system with a "coordinate" observable z that tells us the time.



$$[\hat{z}, \hat{p}_z] = i\hbar \qquad i\hbar\,\dot{\hat{z}} = [\hat{H}_c, \hat{z}]$$

$$\hat{H}_c$$

and to use a "clock quantum state" Φ_{clock} such that

$$\frac{\Delta \dot{\hat{z}}}{|\langle \dot{\hat{z}} \rangle|} \ll 1 \qquad \langle \dot{\hat{z}} \rangle = \langle \Phi_{clock} | \dot{\hat{z}} | \Phi_{clock} \rangle$$

For this clock the time uncertainty is

$$\Delta t = \frac{\Delta \hat{z}}{\langle \dot{\hat{z}} \rangle}$$

But how to define for this clock $\hat{t}$?

Aharonov - Bohm 1961

Imagine we use for $\hat{H}_c$ a free particle: $\hat{H}_c = \dfrac{\hat{p}_z^{\,2}}{2M}$

and we define:

$$\hat{t} = \frac{1}{2} M \left(\hat{z} \frac{1}{\hat{p}_z} + \frac{1}{\hat{p}_z} \hat{z} \right) \qquad \text{formally hermitian}$$

$$[\hat{H}_c , \hat{t}] = i\hbar$$

But $\hat{t}$ is ill defined for $p_z = 0$

Thus you need to reduce the clock Hilbert space to states with $\langle \hat{p}_z \rangle \neq 0$ i.e to those with $\dot{z} = \dfrac{\bar{p}_z}{M} \neq 0$

i.e. those moving forward or backwards in time.

But we want to use the clock as a RF to set the time at which we measure some Local observable O. This means we need to define some local operator $\boxed{\hat{O}(t, z)}$ depending on "coordinate time" t and on "clock time" as indicated by z.

$$i\hbar \frac{d\hat{O}}{dt} = [(\hat{H} + \hat{H}_c), \hat{O}]$$

↳ hamiltonian of the observed system.

Now invariance under time coordinate reparametrizations ⇒

$$\boxed{[(\hat{H} + \hat{H}_c), \hat{O}] = 0} \qquad (1)$$

The algebra of local observables satisfying $[(\hat{H}+\hat{H}_c), \hat{\mathcal{O}}] = 0$

is generated by

$$\{ e^{i\hat{t}\hat{H}} \mathcal{O} e^{-i\hat{t}H}, \hat{H}_c \}$$

for $\hat{t}$ satisfying $[\hat{H}_c, \hat{t}] = i\hbar$

In case we deal with V.N algebras this is the crossed product

Witten 2021

$$(A_0 \times A_{clock})^{\hat{H}+H_c}$$

(* Note that in order to define $\hat{\mathcal{O}}$ we need to define $\hat{t}$ for the clock)

We will refer to operators of the type $e^{i\hat{t}\hat{H}}\mathcal{O}e^{-i\hat{t}\hat{H}}$ as "clock dressed operators".

At the level of the Hilbert space we have states $|\Psi\rangle$ with wave function $\Psi(x, z, t)$ and such that

$z \to$ "clock degree of freedom"

$x \to$ degree of freedom of the observed system

$$i\hbar \frac{d}{dt}\Psi = (\hat{H} + \hat{H}_c)\Psi$$

Let us denote $\Psi_{\hat{\Theta}}$ the wave function of $\hat{\Theta}|\Psi\rangle$. If the clock is in a localized state and not interacting with the observed system we will assume "product states"

$$\Psi_{\hat{\Theta}} = \Phi_{clock}(z, t)\, \psi_{\hat{\Theta}}(x, z, t)$$

with

$$i\hbar \frac{d}{dt}\psi_{\hat{\Theta}} = \left(\frac{1}{2M}(2\bar{P}_z \hat{P}_z + \hat{P}_z^2) + \hat{H}\right)\psi_{\hat{\Theta}}$$

if $i\hbar \frac{d}{dt}\Psi_{\hat{\Theta}} = 0$

$$\boxed{\left(\frac{2\bar{P}_z\hat{P}_z}{2M} + \frac{\hat{P}_z^2}{2M} + \frac{\bar{P}_z^2}{2M} + \hat{H}\right)\psi_{\hat{\Theta}} = 0}$$

In summary:

- clock $\overset{Q}{\vee}$ system $\left\{ \begin{array}{l} [\hat{z}\,\hat{p}_z] = i\hbar \\ \hat{H}_c \end{array} \right\} \Rightarrow [\hat{t}\,\hat{H}_c] = i\hbar$

Projection on states with $\langle \hat{p}_z \rangle \neq 0$

- clock dressed $\hat{\Theta} = e^{i\hat{t}\hat{H}}\,\Theta\,e^{-i\hat{t}\hat{H}}$

- For "product states" $\hat{\Theta}|\Psi\rangle$ wave function

$$\Psi_{\bar{\Theta}} = \Phi_{clock}(z,t)\,\psi_{\bar{\Theta}}(x,z,t)$$

$$\left(\hat{H} + \frac{2\bar{p}_z\hat{p}_z}{2M} + \frac{\hat{p}_z^2}{2M} + \frac{\bar{p}_z^2}{2M}\right)\psi_{\bar{\Theta}}(x,z,t) = 0$$

Schrodinger equation for "gauge invariant"

(coordinate time reparametrization invariance)

"quantum fluctuations"

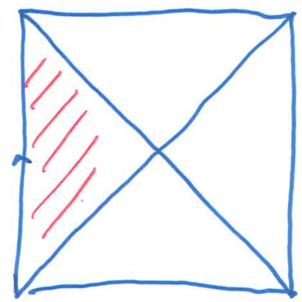
3:- von Neumann algebras and dS-space-time

We will be interested in defining weakly coupled QFT in fixed dS background.

$\mathcal{H}$ full QFT Hilbert space

A_{dS} algebra of local observables with support on the observer static patch.

We will say that A_{dS} is a vN factor if we can define a representation



$$\pi : A_{dS} \longrightarrow \mathcal{B}(\mathcal{H})$$

such that $\boxed{\pi(A_{dS})'' = \pi(A_{dS})}$

and $\pi(A_{dS})$ has trivial center. In these conditions

$$\boxed{\mathcal{B}(\mathcal{H}) = A_{dS} \otimes A'_{dS}}$$

Once we have defined A_{ds} as a vN factor we can define the GNS representation using a linear form $f: A_{ds} \to \mathbb{R}$

$$\left.\begin{array}{l} a \in A_{ds} \longrightarrow |a\rangle \quad ; \quad \langle a\, b\rangle = f(a^+ b) \\ |0\rangle_{GNS} \longrightarrow |\mathbb{1}_d\rangle \end{array}\right\} \; H_{GNS} \approx \mathcal{H}$$

We can identify the "GNS-vacuum" $|0\rangle_{GNS}$ with Bunch-Davis vacuum.

$$\boxed{\text{In LQFT } \quad A_{ds} \text{ is a type } \underline{III_1}\text{-factor}}$$

This means No split property of $\mathcal{H}$ as

$$\mathcal{H} = H_1 \otimes H_2$$

with $\quad B(H_1) = A_{ds} \quad$ and $\quad B(H_2) = A'_{ds}$

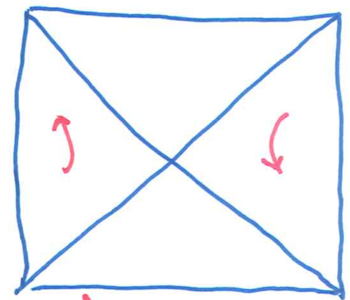
This would be the case for A_{ds} a type I factor. In such case we could describe physics in the static patch using pure states in H_1 or density matrix $\rho = \mathrm{tr}_{H_2} |\psi\rangle\langle\psi|$.

For type III factors neither pure states nor density matrix.

However we can define $\overbrace{\text{an}}^{\text{for a given GNS representation}}$ $\underbrace{\text{outer}}_{\text{state dependent}}$ automorphism generated by T.T modular Hamiltonian $\hat{h}$:

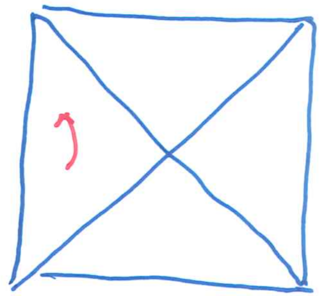
$$\hat{h} |0\rangle_{GNS} = 0$$

$\hat{h}$ is associated to a Killing vector V



(After Euclidean continuation V generates rotations
$\Rightarrow$ G.H thermal interpretation of $|0\rangle_{GNS}$)

Note that for $\mathcal{A}_{dS}$ a type III-factor the Hamiltonian H ~~defined~~ associated to



is not defined : $H|\Psi\rangle \notin \mathcal{H}_{GNS}$ for any $|\Psi\rangle$

i.e $\langle\Psi|H^2|\Psi\rangle = \infty$ (∞-entanglement)

Let us now come back to our original problem:

How to define reparametrization invariant QFT-observables?

The invariant subalgebra $\mathcal{A}_{dS}^{\hat{h}}$ is trivial.

(CLPW) Chandrasekaran, Longo, Penington Witten suggestion:

Add a "clock algebra" $[\hat{H}_c, \hat{t}] = i\hbar$ commuting with $\mathcal{A}_{dS}$

and define $\boxed{(\mathcal{A}_{dS} \times \mathcal{A}_{clock})^{\hat{h}+\hat{H}_c}}$ $\{\hat{a} = e^{i\hat{t}\hat{H}} a\, e^{-i\hat{t}\hat{h}}, \hat{H}_c\}$

But as discussed $\mathrm{Spec}\,\hat{H}_c = \mathbb{R}$ so if we require $\hat{H}_c$ to be positive

$$\boxed{\Pi \left(A_{dS} \times A_{clock} \right)^{\hat{h} + \hat{H}_c} \Pi} = \tilde{A}$$

According to (CLPW) this algebra is a type II_1 factor.

This means that for any $|\Phi\rangle$ in the extended Hilbert space and any clock dressed operator $\hat{a}$

$$\langle \Phi | \hat{a} | \Phi \rangle = \mathrm{tr}\left(\hat{a}\, \rho_{\Phi} \right)$$

for $\rho_I\, \hat{a} \in \tilde{A}$
and with $\mathrm{tr} : \tilde{A} \to \mathbb{R}$
satisfying trace property

$S_{vN}(\rho_{\Phi})$ is a generalized Bekenstein entropy

For the state $|\hat{\Psi}_{dS}\rangle = |0_{GNS}(BD)\rangle \,\Phi_{clock}$ (thermal) $\qquad \rho_{\hat{\Psi}_{dS}} = 1\!\!1/d$.

clock in "thermal equilibrium"

$$S_{VN}(\rho_{\hat{\Psi}_{dS}}) = 0 = \lim_{G_N \to 0} \left(\frac{A}{4G_N} - \infty \right)$$

$|\hat{\Psi}_{dS}\rangle$ is the <u>max entropy state</u>.

- Note that $\nexists\ \rho_{|0_{GNS}(BD)\rangle}$.

For a generic product state $|\hat{\Phi}\rangle = |\Phi\rangle \,\Phi_{clock}$ we get
$\Phi \in \mathcal{H}$

$$S_{VN}(\rho_{\hat{\Phi}}) = \begin{cases} \text{deficit entropy } (SdS) \text{ for } E_{clock} \sim \langle \hat{H}_{clock} \rangle \\ \text{distinguishability between } |\Phi\rangle \text{ and } |0_{GNS}(BD)\rangle. \end{cases}$$

4: Inflationary Cosmology

The key point in Inflationary Cosmology is to identify gauge invariant quantum fluctuations during the primordial exponentially expanding dS-phase.

As we have discussed in pure dS the set of gauge invariant q.f on dS background is trivial i.e non existent.

The lesson of the previous discussion is ADD a clock in order to have a non trivial set of gauge invariant quantum fluctuations.

So instead of adding your favor slow roll inflaton potential let us define gauge invariant primordial quantum fluctuations adding a clock and using the corresponding crossed product algebra

$$\left(A_{dS} \times A_{clock}\right)^{h_{dS}+\hat{H}_c}$$

for some clock hamiltonian $\hat{H}_c$.

In this approach the standard gauge invariant Mukhanov - Sasaki variables should be interpreted as <u>clock dressed operators</u> in the crossed product algebra. i.e

C.G. 2022

$$\boxed{\text{M-S gauge inv observables} \iff \text{clock dressed operators}}$$

Moreover for any M-S variable $\hat{O}$ gauge invariance will imply the corresponding Schrödinger equation for $\psi_{\hat{O}}$ namely:

$$\boxed{\left(\hat{H}_{ds} + \langle \hat{H}_{clock} \rangle \right) \psi_{\hat{O}} = 0}$$

Schrodinger equation for gauge invariant quantum fluctuations.
(Chibisov - Mukhanov equation)

Linear scalar fluctuations:

$$ds^2 = a^2\left((1+2\phi)d\eta^2 - 2B_i dx^i d\eta - ((1-2\Psi)\delta_{ij} + 2E_{,ij})dx^i dx^j\right)$$

η - conformal time.

Let us consider the clock dressing of Ψ i.e. think $\Psi \in A_{ds}$ and look for the dressing $\hat{\Psi} = e^{i\hat{t}H_{ds}}\Psi e^{-i\hat{t}H_{ds}}$ for some added clock algebra $[\hat{H}_c, \hat{t}] = i\hbar$.

Define this clock using φ_0 (inflaton) as coordinate: $[\varphi_0, P_0] = i\hbar$
$\hat{H}_c(P_0, \varphi_0)$; $i\hbar\dot{\varphi}_0 = [\varphi_0, \hat{H}_c]$. As discussed $\hat{t}$ for this clock is determined by $\dfrac{\Delta\varphi_0}{|\langle\dot{\varphi}_0\rangle|}$ on some clock state.

Now you can show that

$$\boxed{\hat{\Psi} = \Psi + \frac{\delta\varphi_0 H}{\dot{\varphi}_0} \equiv \chi}$$

M.S gauge invariant variable.

clock dressed q. fluctuation

Now act with χ (that belongs to the crossed product algebra)

on some product state in the extended Hilbert space

$$\chi\,|\Psi\rangle = \Phi_{clock}\,\psi_x \qquad \psi_x(\eta, \varphi_0)$$

φ_0 ⟶ clock coordinate

η ↳ coordinate time

gauge invariance ⇒

$$\left(\hat{H}_{ds} + \langle H_c\rangle\right)\psi_x = 0$$

We should compare this equation with Ch-M equation

$$\psi_x'' - \left\{\frac{2}{\eta^2} + \left(\frac{z''}{z} - \frac{2}{\eta^2}\right)\right\}\psi_x = 0 \qquad z = a\frac{\dot{\varphi}}{H} = a\sqrt{\varepsilon}$$

$$\langle \hat{H}_c\rangle = \frac{z''}{z} - \frac{2}{\eta^2}$$

⇒ Clock hamiltonian + Clock state ⟺

"slow roll" effects on gauge invariant quantum fluctuations.

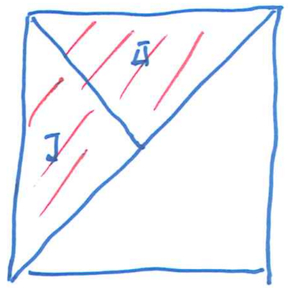
5: A model independent dS - clock

Until this point we are simply relating

$$\boxed{\text{gauge invariant primordial quantum fluctuations} \iff \text{Crossed product algebra for some clock hamiltonian}}$$

Inflaton potential $\overset{\sim}{\iff}$ Clock hamiltonian + Clock state

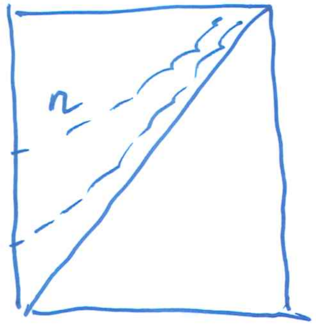
But: $\boxed{\text{Can we select a natural (and model independent) clock Hamiltonian?}}$



Let us work in the planar patch

Associate A_{dS} to I and A'_{dS} to II

Hence you need to introduce two clock algebras A^{I}_{clock} A^{II}_{clock}



These two clocks are expected to be entangled.

We can foliate the planar patch using conformal time η. The entangled two clock state $\Phi_{clock}(\eta)$ can be defined using the Bogolyubov transform in pure dS:

$$\Pi_B(\eta) : A^{I}_{clock} \times A^{II}_{clock} \longrightarrow A^{I}_{clock} \times A^{II}_{clock}$$

Bogolyubov automorphism

The corresponding "clock ground state" at time η

$$|\eta\rangle_{clock} \qquad |\kappa, \eta\rangle_{clock}$$

for clock modes of momentum κ.

This clock state is a two modes squeezed state (in creat anhil basis)

$$\boxed{|\kappa\, n\rangle = \sum e^{i n \phi(\kappa n)}\, c(\kappa n)^{n}\, |n_{\kappa}\, n_{-\kappa}\rangle}$$

The phase ϕ is (in this basis) conjugated to N $[\phi\, N] = -i\hbar$

i.e $\boxed{\phi \text{ play the role of } \hat{t} \text{ for this squeezed dS clock}}$

Thus we can evaluate $(\Delta \hat{t})^2(\kappa\, n)$

For the clock dressed quantum fluctuation χ we get

$$P_{\chi}(\kappa, n) \longleftrightarrow (\Delta \hat{t})^{2}_{\kappa n}$$

This correspondence leads to concrete predictions on inflationary parameters $((1-n_s) = 0.0318$ (C.G, R Jimenez)

In summary:

1) Use a clock dressing to define primordial scalar gauge invariant quantum fluctuations χ

2) Define the clock hamiltonian and the clock state using the Bogolyubov ~~#~~ automorphism on the clock algebra.

3) Map the power spectrum of χ to $(\Delta\hat{t})^2$ on the clock state

Can we make this rule a predictive recipe for Inflationary Cosmology?

Thank you