Infrared Finite Cross Sections and S-matrix Elements

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arXiv: 1810.10022 with C. Frye[1], N. Paul[1], M. Schwartz[1], and K. Yan[1]
arXiv: 1906.03271 with M. Schwartz[1]
arXiv: 1911.06821 with M. Schwartz[1]

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The S-matrix

- **Fundamental object** of interest in physics.
- Leads to predictions for colliders.
  - Standard model observables computed to NNLO.
- Thought to be more fundamental than QFT in the 1950s.
  - S-matrix bootstrap program having a resurgence.
- Properties extensively studied.
  - What is the best way to **encode its content** (spinors, twistors, amplituhedron)?
  - What are its **symmetries** (dual conformal invariance, Steinmann relations)?
- Despite this interest, **the S-matrix does not exist**.
  - S-matrix elements are divergent in perturbation theory and zero non-perturbatively in theories with massless particles.
  - Why are calculations that have already been done still valuable?

Outline

- Introduction - What is the S-matrix?
  - Traditional definition of S-matrix
  - Reason why the S-matrix does not exist: infrared (IR) divergences
- Ideas for IR finiteness
  1. **Cross section method**
     → IR finite cross section $\sigma \propto \int |\langle f| S |i\rangle|^2 d\Pi_f$
  2. **Modify S-matrix**
     → IR finite S operator
  3. **Modify scattering states (Coherent states)**
     → IR finite S-matrix elements, $S_{fi} = \langle f| S |i\rangle$

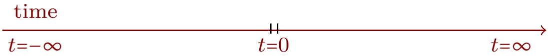
Introduction - What is the S-matrix?

- Cross sections, decay rates, and other observables are proportional to squares of S-matrix elements:

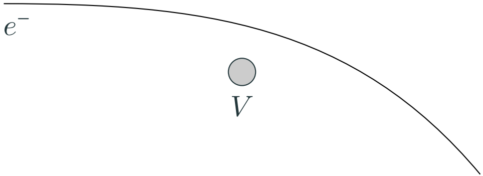
$$\sigma \propto \int |\langle f| S |i\rangle|^2 d\Pi_f, \quad \Gamma \propto \int |\langle f| S |i\rangle|^2 d\Pi_f$$

- Intuitively: S-matrix gives the **probability amplitude** for an initial state $|i\rangle$ at $t = -\infty$ to transform into a final state $|f\rangle$ at $t = +\infty$.
  - Idea: $S = \lim_{t\to\infty} e^{-iHt}$? Gives infinitely oscillating phases when acting on energy eigenstates.
  - Resolution: Project onto **free states** at $t = \pm\infty$.

Introduction - What is scattering?



$e^{-iH_0 t}\,|i\rangle$	e^{-iHt}	$e^{-iH_0 t}\,|f\rangle$
↓	↓	↓
freely evolving state in far past	scattering, evolution with H	freely evolving state in far future

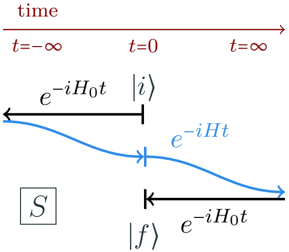


$$S_{fi} = \lim_{t_\pm \to \pm\infty} \langle f|\, e^{iH_0 t_+} e^{-iHt_+} e^{iHt_-} e^{-iH_0 t_-}\,|i\rangle$$

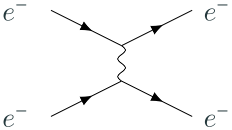
Introduction - What is scattering?

S-matrix: **probability amplitude** of measuring $|f\rangle$ given $|i\rangle$

$$S_{fi} = \lim_{t_\pm \to \pm\infty} \langle f| e^{iH_0 t_+} e^{-iHt_+} e^{iHt_-} e^{-iH_0 t_-} |i\rangle$$



In QFT: Calculate using **Feynman diagrams** in perturbation theory.



Introduction - Traditional definition of S-matrix

$$S_{fi} = \lim_{t_\pm \to \pm\infty} \langle f | e^{iH_0 t_+} e^{-iHt_+} e^{iHt_-} e^{-iH_0 t_-} |i\rangle$$

Free Theory: $S = \mathbb{1}$ $\quad S_{fi} = \langle f|i\rangle$ ✓

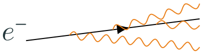
QM, short range potential: ✓

Const. potential $\quad H = H_0 + V_0$: $\quad S_{fi} = \langle f|i\rangle \lim_{T\to\infty} e^{-2iV_0 T}$?

QED: $S = \mathbb{1} - \frac{\alpha}{\epsilon^2} + \cdots = -\infty$?

$S = \exp\left\{-\frac{\alpha}{\epsilon^2}\right\} = 0$?

Interactions do not vanish as $t \to \pm\infty$ in QED



Problems with S-matrix in Quantum Field Theory

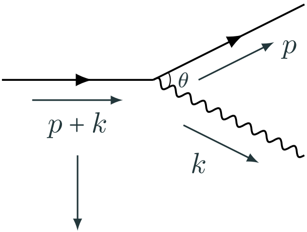
Calculations of probability amplitudes $\langle f|S|i\rangle$ sometimes give infinity: $\langle f|S|i\rangle = \infty$

- Problematic since probabilities $p_{fi} \propto |\langle f|S|i\rangle|^2$ should be less than 1.
- **UV divergences** occur at high energies.
  - Arise since we assume theory still holds at very high energies.
  - Remedy using **renormalization**.
  - S-matrix well-defined in theories with **mass gap**.
- **IR divergences** occur at low energies in theories with massless particles.
  - **No proof of LSZ**.
  - Despite these problems, use S-matrix to make predictions.

Motivations for studying IR finiteness

- Understand **analytic properties** and **symmetries** of the S-matrix.
  - Firmer theoretical foundation when S-matrix is finite.
  - May be advantages to studying a well-defined S-matrix.
- Define **finite**
  - S-matrix elements: $\langle f|S|i\rangle$
  - Cross sections: $\sigma = \int d\Pi_f |\langle f|S|i\rangle|^2$
  - Observables
- Connect to asymptotic states and **asymptotic symmetries**.

IR divergences in QFT



Propagator: $\frac{1}{(p+k)^2} \sim \frac{1}{|k|(1-\cos\theta)}$

Singularities: $|k| \to 0$ *soft* ; $\theta \to 0$ *collinear* } IR divergences

Ideas on how to get IR finite quantities

1. **Cross section method**

   → IR finite cross section $\sigma \propto \int |\langle f| S |i\rangle|^2 d\Pi_f$

   - Bloch-Nordsieck theorem
   - KLN theorem

2. **Modify S-matrix**

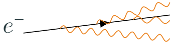
   → IR finite S operator

3. **Modify scattering states (Coherent states)**

   → IR finite S-matrix elements, $S_{fi} = \langle f| S |i\rangle$

1. Cross section method

- Idea: Cross section is **measurable** and needs to be finite.
- Detecting an electron, perhaps a photon with little energy or one close to the electron **escaped detector**.
    - All physical detectors have a finite resolution.

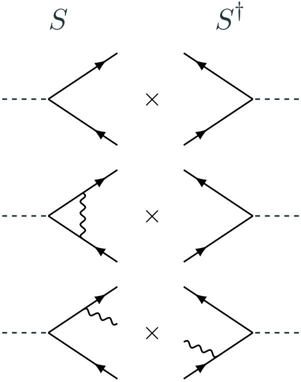


- A sum over all processes **consistent with detector measurement** should give a finite quantity.
- Easiest framework to start with → theorems tell us how to proceed.
    - Bloch-Nordsieck theorem
    - KLN theorem

Theorems on IR divergences

Bloch-Nordsieck (1937): Soft IR divergences cancel in QED when summing over final state photons with finite energy resolution.

Example: $Z \to e^+e^-$ + final state photons



$\Gamma \propto$ finite

$$\Gamma \propto -\frac{1}{\epsilon^2} - \frac{3}{2\epsilon} + \text{finite}$$

$$\Gamma \propto \frac{1}{\epsilon^2} + \frac{3}{2\epsilon} + \text{finite}$$

m_e=0
Dim reg
CM frame

Bloch-Nordsieck (1937): Soft IR divergences cancel in QED when summing over final state photons with finite energy resolution.

Doria, Frenkel, Taylor (1980): Counterexample in QCD: $qq \to \mu\mu qq$ + final state gluons is soft IR divergent at 2-loops.

KLN Theorem (1962-64): S-matrix elements squared are IR finite when summing over **final states and initial states** within some energy window:

$$\sum_{f,i\in[E-E_0,E+E_0]} |\langle f|S|i\rangle|^2 < \infty$$

Stronger KLN Theorem (2018): S-matrix elements squared are IR finite when summing over **final states or initial states:**

$$\sum_f |\langle f|S|i\rangle|^2 < \infty, \qquad \sum_i |\langle f|S|i\rangle|^2 < \infty$$

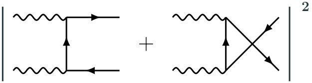
KLN theorem requires forward scattering

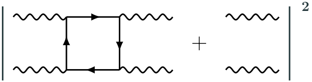
Stronger KLN Theorem (2018): S-matrix elements squared are IR finite when summing over **final states or initial states:**

$$\sum_f |\langle f|S|i\rangle|^2 = \langle i|S^\dagger \sum_f |f\rangle\langle f|S|i\rangle = \langle i|i\rangle \propto 1 < \infty$$

Unitarity: all probabilities sum to one

Example: $\gamma\gamma \to e^+e^-$

$\int$  $d\Pi_f \propto \frac{\alpha^2}{E_{\mathrm{CM}}^2}\left(-\frac{1}{\epsilon}+1\right)$

$\int$  $d\Pi_f \propto \frac{\alpha^2}{E_{\mathrm{CM}}^2}\left(\frac{1}{\epsilon}-1\right)+\dots$

Unitarity: all diagrams (to any non-zero order α) sum to zero

Conclusion of cross section method

Conclusion: KLN theorem = unitarity.

If we sum over **all possible diagrams** we get 1 by unitarity, and 1 is IR finite.

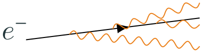
Not closer to finding the **minimal set of diagrams** needed for IR finiteness.

Need new ideas beyond the cross section method.

2. Modify S-matrix to S_H

2. Modify S-matrix to S_H

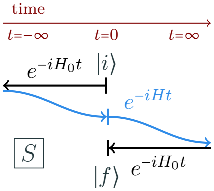
Recall: Interactions do not vanish as $t \to \pm\infty$ in QED.

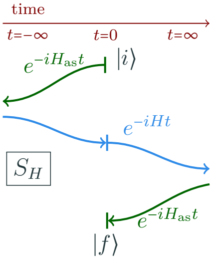


Redefine S-matrix in theories with long range interactions:

$$S_{fi} = \lim_{t_\pm \to \pm\infty} \langle f| e^{iH_0 t_+} e^{-iHt_+} e^{iHt_-} e^{-iH_0 t_-} |i\rangle$$

$$\to S^H_{fi} = \lim_{t_\pm \to \pm\infty} \langle f| e^{iH_{\text{as}} t_+} e^{-iHt_+} e^{iHt_-} e^{-iH_{\text{as}} t_-} |i\rangle$$





Questions

$$S^H_{fi} = \lim_{t_\pm \to \pm\infty} \langle f|\, e^{iH_{\text{as}}t_+} e^{-iHt_+} e^{iHt_-} e^{-iH_{\text{as}}t_-} \,|i\rangle$$

(i) How to pick H_{as}?
- Criteria: IR finite, easy to calculate, useful in practice, consistent with every measurement to date.

(ii) How to calculate matrix elements of S_H?
- Not useful if too complicated to calculate.

(iii) How to interpret S_H?
- Not sufficient to have an IR finite, calculable quantity; must be useful in practice.

(iv) Directions forward

Choice of H_{as}

Answers:

(i) *How to pick H_{as}?*

Use **factorization**, and techniques from Soft-Collinear Effective Theory (SCET):

$$H_{\text{as}} = H_{SCET}$$

- IR finite by construction: Possible because of **universality of IR divergences** in gauge theories.
- States **evolve independently of how they scatter**.
- New UV divergences dealt with using renormalization.
- No scales, most integrals are zero in dimensional regularization.

(ii) *How to calculate matrix elements of S_H?*

$$
\begin{aligned}
S^H_{fi} &= \lim_{t_\pm \to \pm\infty} \langle f | \, e^{iH_{\text{as}} t_+} e^{-iHt_+} e^{iHt_-} e^{-iH_{\text{as}} t_-} \, | i \rangle \\
&= \lim_{t_\pm \to \pm\infty} \sum_{f'} \sum_{i'} \langle f | \underbrace{e^{iH_{\text{as}} t_+} e^{-iH_0 t_+}}_{\Omega^{\text{as}}_+} | f' \rangle \\
&\quad \times \langle f' | \underbrace{e^{iH_0 t_+} e^{-iHt_+} e^{iHt_-} e^{-iH_0 t_-}}_{S} | i' \rangle \langle i' | \underbrace{e^{iH_0 t_-} e^{-iH_{\text{as}} t_-}}_{\Omega^{\text{as}}_-} | i \rangle \\
&= \sum_{f'} \sum_{i'} \underbrace{\langle f | \, \Omega^{\text{as}}_+ \, | f' \rangle}_{\text{TOPT rules}} \underbrace{\langle f' | \, S \, | i' \rangle}_{\text{usual Feynman rules}} \underbrace{\langle i' | \, \Omega^{\text{as}}_- \, | i \rangle}_{\text{TOPT rules}}
\end{aligned}
$$

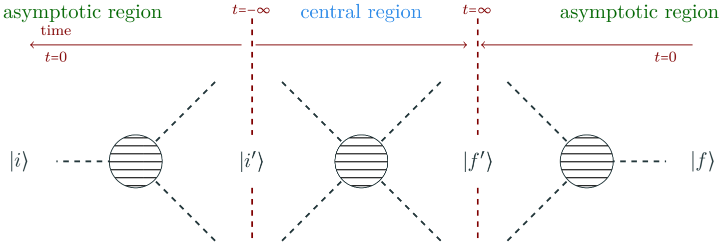
TOPT: Time-ordered perturbation theory

(Old-fashioned perturbation theory)

Calculation trick in perturbation theory:

$$S_{fi}^{H} = \int d\Pi'_f \int d\Pi'_i \underbrace{\langle f|\,\Omega_+^{\mathrm{as}}\,|f'\rangle}_{\text{TOPT rules}} \underbrace{\langle f'|\,S\,|i'\rangle}_{\text{usual Feynman rules}} \underbrace{\langle i'|\,\Omega_+^{\mathrm{as}}\,|i\rangle}_{\text{TOPT rules}}$$

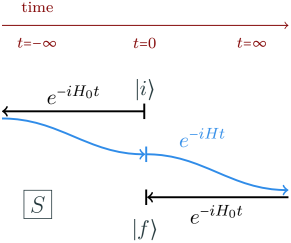
Calculations split into three parts:

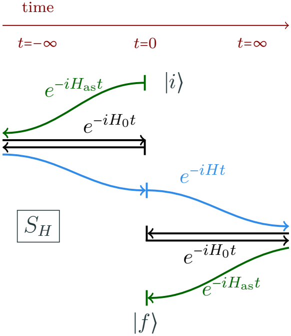


Comparing S and S_H

$$S_{fi} = \langle f | S | i \rangle$$

$$S^H_{fi} = \sum_{f'} \sum_{i'} \underbrace{\langle f | \Omega_+^{\text{as}} | f' \rangle}_{\text{TOPT rules}} \underbrace{\langle f' | S | i' \rangle}_{\text{usual Feynman rules}} \underbrace{\langle i' | \Omega_-^{\text{as}} | i \rangle}_{\text{TOPT rules}}$$



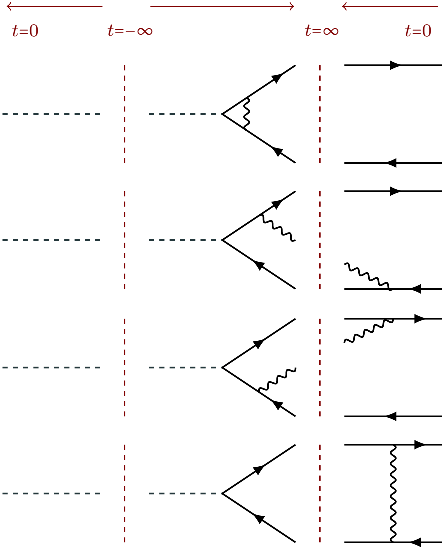


Example: $Z \to e^+e^-$ for $H_{\text{as}} = H_{SCET}$

m_e=0, $L=\ln\frac{-E_{CM}^2}{\mu^2}$

M_0: LO matrix element

Dim reg, CM frame



$$=M_0\,\frac{\alpha}{4\pi}\left[\frac{1}{\epsilon_{\text{UV}}}-\frac{2}{\epsilon_{\text{IR}}^2}-\frac{4+2L}{\epsilon_{\text{IR}}} -8+\frac{\pi^2}{6}-L^2+3L\right]$$

$$=M_0\,\frac{\alpha}{4\pi}\left[\frac{2}{\epsilon_{\text{IR}}^2}+\frac{4+2L}{\epsilon_{\text{IR}}}-\frac{2}{\epsilon_{\text{UV}}^2}-\frac{4+2L}{\epsilon_{\text{UV}}}\right]$$

$$\langle e^+e^-|S_H|Z\rangle^{\overline{\text{MS}}}$$
$$=M_0+M_0\,\frac{\alpha}{4\pi}\left[-8+\frac{\pi^2}{6}-L^2+3L\right]$$

(iii) *How to interpret S_H?*

a. Wilson coefficients in Soft-Collinear Effective Theory (SCET)
   - Encode hard dynamics.

b. Remainder functions in $N = 4$ Supersymmetric Yang-Mills theory (SYM)
   - UV counterterm fixed by 1-loop S_H amplitude makes 2-loop S_H amplitude take a simple form in $N = 4$.

c. Dressed states / Coherent states

3. Coherent States

Coherent States

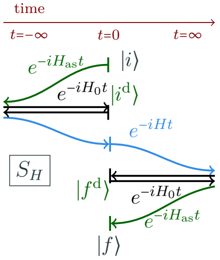
- Arise as intermediate steps in S_H calculations:

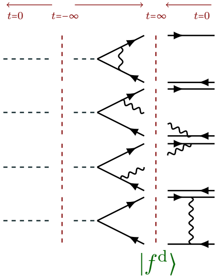
$$S^H_{fi} = \sum_{f'} \sum_{i'} \underbrace{\langle f| \Omega_+^{\text{as}} |f'\rangle}_{\langle f^{\text{d}}|} \langle f'| S \underbrace{|i'\rangle \langle i'| \Omega_+^{\text{as}} |i\rangle}_{|i^{\text{d}}\rangle}$$

- IR divergent states → IR problem moved from S to states:

$$|e^{-\,\text{d}}\rangle = |e^-\rangle + \frac{1}{\epsilon} |e^-\gamma\rangle + \cdots$$

- Explicit cutoffs on energy make calculations difficult.





(iv) *Directions forward*

- What are the analytic properties of S_H?
- What can we learn from bootstrapping S_H?
- What are non-perturbative properties of S_H?
- What are the symmetry properties of S_H?
- What is the relation to remainder functions in $N = 4$ SYM?

Conclusions

Three methods for obtaining IR finite quantities:

1. **Cross section method:** Unitarity.
    - Content of KLN theorem: All probabilities sum to 1.
    - Not closer to finding the minimal set of Feynman diagrams needed for IR finiteness.
2. **Modify S:** Natural to redefine S to S_H in theories with long-range interactions.
    - Choosing S_H based on factorization gives an IR finite, easily calculable and useful quantity.
    - S_H exists.
3. **Coherent states:** Intermediate step in S_H calculations.
    - Infinite linear combinations of Fock states.
    - Hard to do calculations with cutoffs.