DGLAP splitting functions at small x and applications to lepton PDFs

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based on 2309.07516, 2603.02312, 26xx.xxxxx
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Phenomenology seminars (UniTo and INFN)
05.11.2026

Small-x regime at high-energy colliders

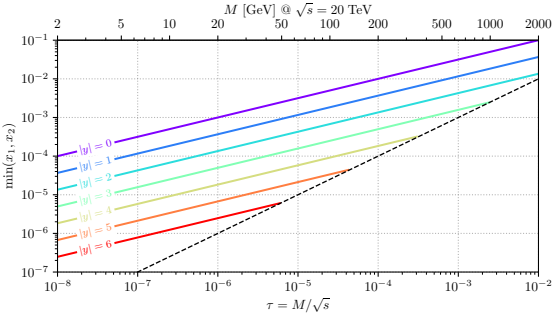
$$d\sigma_{h_1 h_2 \to X}(Q^2, s) = \sum_{ij} \int dx_1\, dx_2\, f_{i/h_1}(x_1, \mu_F^2) f_{j/h_2}(x_2, \mu_F^2)\, d\hat{\sigma}_{ij \to X}(x_1, x_2, Q^2, s, \mu_F^2)$$

e.g. production of an object with invariant mass M

$$x_{1,2} = \frac{M}{\sqrt{s}}\, e^{\pm y} = \sqrt{\tau}\, e^{\pm y}$$

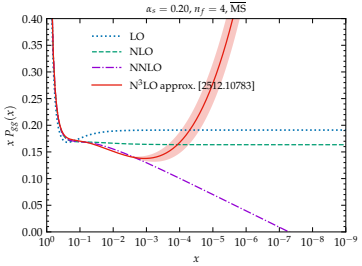
Much of the phase space at small x!

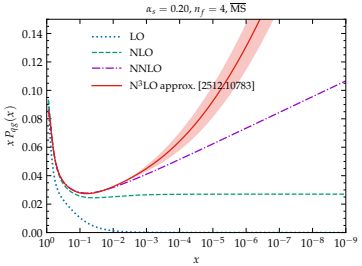
Potentially large $\log x$ terms in PDF evolution and in short-distance $d\hat{\sigma} \to$ resummation!



(The factorization formula applies to a lepton collider as well, with $h_1 = \ell^+$ and $h_2 = \ell^-$, where the PDFs $f_{i/l^\pm}$ with $i = \gamma, \ell, q, g, \ldots$ feature a perturbative initial condition.)

Small-x logarithms in fixed-order splitting functions





$L \equiv \log \frac{1}{x}$

	xP_{gg}	xP_{qg}
LO	$\alpha_s\, c$	0
NLO	$\alpha_s^2(L + c)$	$\alpha_s^2\, c$
NNLO	$\alpha_s^3(L^2 + L + c)$	$\alpha_s^3(L + c)$
N^{3}LO	$\alpha_s^4(L^3 + L^2 + L + c)$	$\alpha_s^4(L^2 + L + c)$

accidental zeros at NLO and NNLO

- Small-x resummation is gluon-driven: P_{gg}, P_{gq} are LL, while P_{qg}, P_{qq} start at NLL
- Logs start to grow already for $x \lesssim 10^{-2}$ (at $Q \sim 5\,\text{GeV}$): very large resummation region
- The perturbative instability worsens with the order: N^{3}LO much more unstable than NNLO

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A (too) short story of small-x resummation

- Small-x (or high-energy) resummation is a longstanding topic in QCD, with dedicated textbooks e.g. [Forshaw, Ross (1997)] Resummation linked to the BFKL equation [Balitsky, Lipatov, Kuraev, Fadin (1976-1978)]
- NLL accuracy allowed by the calculation of the two-loop BFKL kernel [Fadin, Lipatov (1998)] [Ciafaloni, Camici (1998)]. Calculation of the three-loop BFKL kernel in progress [Gardi et al. (2021-)] [Caola et al. (2021-)]
- The fixed-order BFKL kernel suffers from perturbative instability. Approaches to construct solid resummed anomalous dimensions: ABF [Altarelli, Ball, Forte (1995-2008)], CCSS [Ciafaloni, Colferai, Salam, Stasto (1999-2007)], TW [Thorne, White (1999-2007)]
- `HELL` framework [Bonvini, Marzani et al. (2016-2018)] implementing and extending ABF

In this work, motivated by the muon PDF program (but not limited to it),
we improved analytical ingredients (e.g. resummation of P_{qg})
and technical aspects of `HELL` (making it more robust and able to reach large α_s values)

A poor man's introduction to small-x resummation

Leading small-x logs only in the 2×2 (singlet quark, gluon) sector:

$$\gamma_{ij}(\alpha_s, N) = \int_0^1 dx\, x^N P_{ij}(\alpha_s, x), \qquad \tfrac{1}{x}\log^n x \;\leftrightarrow\; \text{poles at } N = 0$$

$$\textbf{LL}: (\alpha_s/N)^n \qquad \textbf{NLL}: \alpha_s(\alpha_s/N)^n$$

Two ingredients suffice: chosen to be eigenvalue γ_+ and γ_{qg} (enters at NLL).
The rest can be obtained as

$$\gamma_{gg}^{\rm NLL} = \gamma_+^{\rm NLL} - \frac{C_F}{C_A}\gamma_{qg}^{\rm NLL}, \quad \gamma_{gq}^{\rm NLL} = \frac{C_F}{C_A}\gamma_{gg}^{\rm NLL}, \quad \gamma_{qq}^{\rm NLL} = \frac{C_F}{C_A}\gamma_{qg}^{\rm NLL}$$

Resummation performed differently for γ_+ and γ_{qg}

- γ_+: duality between DGLAP and BFKL
- γ_{qg}: collinear factorization of the quark Green function in the high-energy limit

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Resummation of γ_+: duality with BFKL

Consistency of DGLAP and BFKL gives an all-order duality ($\chi_{\rm BFKL}$ = BFKL characteristic function). Ignoring running-coupling effects:

$$\chi_{\rm BFKL}(\alpha_s, \gamma_+(\alpha_s, N)) = N$$

At LO, $\chi_{\rm BFKL} = \frac{C_A}{\pi}\alpha_s\,\chi(\gamma)$, with

$$\chi(\gamma) = 2\psi_0(1) - \psi_0(\gamma) - \psi_0(1-\gamma) = \frac{1}{\gamma} + \sum_{k=1}^{\infty} 2\zeta_{2k+1}\gamma^{2k}$$

so the duality becomes

$$1 = \bar{\alpha}_s\,\chi(\gamma_s(\bar{\alpha}_s))\,, \qquad \bar{\alpha}_s \equiv \frac{C_A\alpha_s}{\pi N}$$

$\gamma_s(\bar{\alpha}_s)$ = pure-LL part of γ_+, resums all $(\alpha_s/N)^k$.

Solving $1 = a\,\chi\big(\gamma_s(a)\big)$ for γ_s: new results

- by exploiting the Faa di Bruno formula, one finds the perturbative coefficients to all orders

$$\gamma_s(a) = a\left(1 + \sum_{k=1}^{\infty}\sum_{i=1}^{k}\frac{a^k}{(k-i+1)!}\,B_{k,i}(Z)\right)$$

with $B_{k,i}$ the incomplete Bell polynomials and $Z = (0, 0, 2\zeta_3 3!, 0, 2\zeta_5 5!, \dots)$ with odd ζ's.

- by using the Hankel-path representation of $1/\Gamma$, a Borel transform and the generating function of the incomplete Bell polynomials

$$\exp\left(u\sum_{j=1}^{\infty}x_j\frac{t^j}{j!}\right) = \sum_{n=0}^{\infty}\sum_{k=0}^{n}B_{n,k}(x_1,\dots,x_n)\,\frac{t^n}{n!}\,u^k$$

one obtains an integral representation (with $\hat{\chi}(z) \equiv z\chi(z) - 1$)

$$\gamma_s(a) = a\left(1 \mp \frac{i}{2\pi}\int_0^{\infty} dt\, e^{-t}\int_{\mathcal{H}^{\pm}}\frac{d\omega}{(\pm\omega)^2}\,e^{\pm\omega}\left\{\exp\left[\pm\omega\,\hat{\chi}\left(\frac{at}{\pm\omega}\right)\right] - 1\right\}\right)$$

Intermezzo: some nice math results

Faa di Bruno formula

To compute an arbitrary number of derivatives of a composite function of a single variable

$$\frac{d^q f(g(\beta))}{d\beta^q} = \sum_{i=1}^{q} f^{(i)}(g(\beta))\, B_{q,i}\big(g^{(1)}(\beta), g^{(2)}(\beta), \ldots, g^{(q-i+1)}(\beta)\big)$$

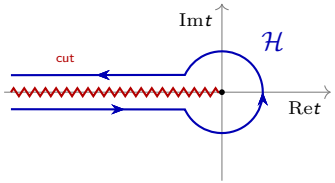
valid for any two differentiable single-variable functions f and g.

Hankel path

Integral representation of the reciprocal Gamma function, valid for all $z \in \mathbb{C}$:

$$\frac{1}{\Gamma(z)} = \frac{1}{2\pi i} \int_{\mathcal{H}} dt\; e^t\, t^{-z}$$

$\mathcal{H}$ comes from $-\infty$ below the branch cut of t^{-z}, encircles $t = 0$ counter-clockwise and goes back to $-\infty$ above the cut



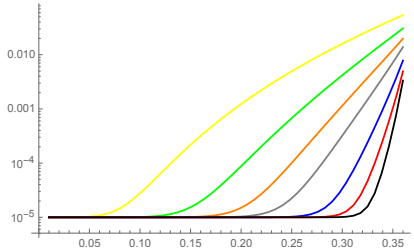
Solving $1 = a\,\chi\big(\gamma_s(a)\big)$ for γ_s: new results

- Introducing $\hat{\chi}(z) \equiv z\chi(z) - 1$ and the sequence

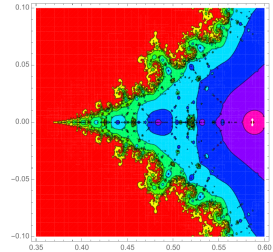
$$\gamma_s^{(0)}(a) = a, \quad \gamma_s^{(n+1)}(a) = a\big[1 + \hat{\chi}\big(\gamma_s^{(n)}(a)\big)\big] = a\big[1 + \hat{\chi}(a[1 + \hat{\chi}(a[1 + \hat{\chi}(\cdots)])])\big]$$

γ_s is the fixed point of the sequence

$$\gamma_s(a) = \Phi_a\big(\gamma_s(a)\big), \qquad \Phi_a(x) = a\big(1 + \hat{\chi}(x)\big) \equiv a\,x\,\chi(x)$$



Study of convergence for real $a < 1/\log 16$



Convergence speed for complex a? *Fractal* structure!

From LL γ_s to a stable $\gamma_+^{\mathrm{NLL}'}$

Beyond LL the duality with the NLO BFKL kernel is *unstable* (NLO corrections completely change the behavior near $\gamma = 0$ and $\gamma = 1$, owing to the novel presence of higher poles with opposite signs)

The strategy adopted in `HELL`:

1. duality "in reverse" $\rightarrow$ resum the $\gamma = 0$ collinear poles of the BFKL kernel;
2. use $\gamma \rightarrow 1 - \gamma$ symmetry to also fix the $\gamma = 1$ pole;
3. momentum conservation + duality again $\rightarrow$ "double-leading" BFKL kernel & anomalous dimension;
4. resum subleading running-coupling terms

At the end the final anomalous dimension is given by

$$\gamma_+^{\mathrm{NLL}'} = \gamma_s(\bar{\alpha}_s) + \alpha_s\, \gamma_{ss}(\bar{\alpha}_s) + \text{subl. log} + \text{subl. power}$$

and an inverse Mellin transform is used to get the result (P_+) in x-space

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Steps that required improvement to reach large values of α_s

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Resummation of γ_{qg}: collinear factorization of the quark Green function

Based on the seminal work [Catani, Hautmann (1994)]

γ_{qg} starts at NLL, i.e. a prefactor α_s times a function of α_s/N

$$\gamma_{qg}(\alpha_s, N)|_{\text{NLL}} = \frac{\alpha_s}{3\pi}\, T_R \left[1 + \frac{5}{3}\bar{\alpha}_s + \frac{14}{9}\bar{\alpha}_s^2 + \left(\frac{82}{81} + 2\zeta_3\right)\bar{\alpha}_s^3 + \dots\right].$$

It had never been resummed (it is possible to determine the perturbative coefficients in an iterative way, but with very large intermediate expressions). The previous implementation in `HELL` used a Borel–Padé procedure based on a limited number (16) of coefficients.

For further manipulation, it is convenient to rewrite it as

$$\gamma_{qg}|_{\text{NLL}} = \frac{\alpha_s}{3\pi}\, T_R\, h_{qg}\big(\gamma_s(\bar{\alpha}_s)\big), \qquad h_{qg}(\gamma) = \sum_{k\geq 0} h_k \gamma^k$$

Make everything depend on $\bar{\alpha}_s$ *through* γ_s, because γ_s captures the complexity of the resummed result

Recap of collinear factorization of the quark Green function

In $d = 4 - 2\epsilon$ dimensions

$$\hat{\mathcal{G}}^{(0)}_{qg} = \hat{\mathcal{G}}_{qg}\,\Gamma_{gg} + \hat{\Gamma}_{qg}$$

with $\hat{\mathcal{G}}^{(0)}_{qg}$ the bare Green function (whose computation is extremely involved) and $\hat{\mathcal{G}}_{qg}$ the finite Green function; Γ_{gg} and $\hat{\Gamma}_{qg}$ the transition functions

$$\hat{\Gamma}_{qg}(\bar{\alpha}_s, \epsilon) = \frac{1}{\epsilon}\int_0^{\bar{\alpha}_s} da\, h_{qg}(\gamma_s(a))\ \Gamma_{gg}(a, \epsilon)|_{\mathrm{LL}}$$

$$\Gamma_{gg}(\bar{\alpha}_s, \epsilon)|_{\mathrm{LL}} = \exp\left[\frac{1}{\epsilon}\,\Omega\left(\gamma_s(\bar{\alpha}_s)\right)\right]$$

$$\Omega(\gamma) = \gamma - \gamma \log\left[1 + \hat{\chi}(\gamma)\right] + \gamma\int_0^1 dw\, \log\left[1 + \hat{\chi}(w\gamma)\right]$$

h_{qg} and $\hat{\mathcal{G}}_{qg}$ both unknown: requiring finiteness of $\hat{\mathcal{G}}_{qg}$ as $\epsilon \to 0$ allows an extraction of both of them

Resummed γ_{qg} and perturbative coefficients

The closed all-order form of h_{qg}, valid for any complex γ

$$h_{qg}(\gamma) = \frac{\gamma\,\chi(\gamma)}{4}\left(3\,e^{2\Omega(\gamma)} + e^{\frac{2}{3}\Omega(\gamma)}\right)$$

Closed form for perturbative coefficients

$$h_{qg}(\gamma) = \sum_{n\geq 0} h_n \gamma^n, \quad h_n = \frac{1}{n!}\left[\tfrac{3}{4}\,B_n\big(\widehat{Z}_2\big) + \tfrac{1}{4}\,B_n\big(\widehat{Z}_{2/3}\big)\right]$$

$$\widehat{Z}_b^{[k]} = b\left(\delta_{k,1} + \frac{1}{b}\sum_{j=1}^{k}(-)^{j+1}(j-1)!B_{k,j}(Z) - (k-1)\sum_{j=1}^{k-1}(-)^{j+1}(j-1)!B_{k-1,j}(Z)\right)$$

Explicit form for the first few coefficients:

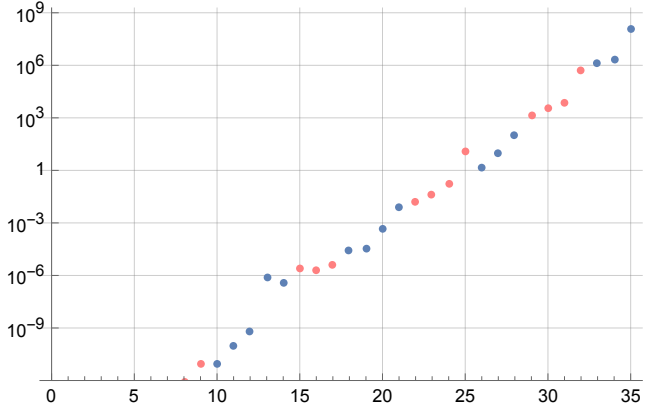
$$h_0 = 1, \quad h_1 = \tfrac{5}{3}, \quad h_2 = \tfrac{14}{9}, \quad h_3 = \tfrac{82}{81} + 2\zeta_3, \quad h_4 = \tfrac{122}{243} + \tfrac{5}{6}\zeta_3, \quad h_5 = \tfrac{146}{729} - \tfrac{14}{9}\zeta_3 + 2\zeta_5, \quad \ldots$$

Comparison to previous determinations of h_n

35 coefficients numerically determined in [Altarelli, Ball, Forte (2008)]

$$R_n = \left| 1 - \frac{h_n(\text{ABF})}{h_n(\text{exact})} \right|$$

Blue points: $h_n(\text{exact}) > 0$
Red points: $h_n(\text{exact}) < 0$



Fine up to $n \sim 20$, then deviations explode for $n > 25$ (large numerical cancellations)

Bonus: finite qg Green function (closed, all-order)

We found a relationship between the finite $\mathcal{G}_{qg}$ and $\mathcal{G}_{gg}$ Green functions:

$$\hat{\mathcal{G}}_{qg}(\epsilon \to 0) = \varphi_0(\Omega(\gamma_s)) + \varphi_1(\gamma_s)\frac{\mathcal{G}_{gg}(\epsilon \to 0)}{\gamma_s \chi(\gamma_s)}$$

with

$$\varphi_0(z) = -\frac{1}{4}\left(3e^{2z} + e^{\frac{2}{3}z}\right), \quad \varphi_1(z) = \frac{1}{4}\left[\frac{3}{1-z} + \frac{12}{2-z} - \frac{15}{3-z}\right]$$

The expression for $\mathcal{G}_{gg}(\epsilon \to 0)$ was given in [Catani, Hautmann (1994)]

$$\mathcal{G}_{gg}(\epsilon \to 0) = \exp\left[\frac{1}{2}\gamma_s \int_0^1 dy\, \frac{\chi_1(y\gamma_s) - \chi'(y\gamma_s)}{\chi(y\gamma_s)} + \gamma_s \psi_0(1)\right.$$
$$\left. - \frac{1}{2}\log\Gamma(1+\gamma_s) + \frac{1}{2}\log\Gamma(1-\gamma_s) + \frac{1}{2}\log\left(\gamma_s \chi(\gamma_s)\right) - \frac{1}{2}\log\left(-\gamma_s^2 \chi'(\gamma_s)\right)\right]$$

with

$$\chi'(z) = d\chi/dz, \quad \chi_1(z) = 2\psi_1(1) - \psi_1(z) - \psi_1(1-z)$$

γ_{qg} resummation with running coupling

The inclusion of running-coupling effects is achieved with

$$\gamma_{qg}^{\mathrm{NLL}'}(\alpha_s(Q^2), N) = \frac{\alpha_s(Q^2)}{3\pi}\, T_R \sum_{k=0}^{\infty} h_k \left[\partial_\nu^k U(N, Q^2 e^\nu, Q^2)\right]_{\nu=0}$$

with the evolution function

$$U(N, \mu^2, \mu_0^2) = \exp\left[\int_{\mu_0^2}^{\mu^2} \frac{dq^2}{q^2}\, \gamma_+\left(\alpha_s(q^2), N\right)\right]$$

In our numerical implementation, we adopt

$$\gamma_+(\alpha_s(q^2), N) \overset{\mathrm{ABF}}{=} \frac{\gamma_+(\alpha_s(\mu_0^2), N)}{1 + r(\alpha_s(\mu_0^2), N) \log \frac{q^2}{\mu_0^2}}, \quad r(\alpha_s, N) = \alpha_s \beta_0$$

By exploiting the Cauchy formula and the integral representation of the Γ function, we can make the all-order result appear

$$\gamma_{qg}^{\mathrm{NLL}'}(\alpha_s(Q^2), N) = \frac{\alpha_s(Q^2)}{3\pi}\, T_R\, \frac{1}{2\pi i} \int_0^\infty dw\, e^{-w} \oint \frac{dz}{z}\, U(N, Q^2 e^{w/z}, Q^2)\, h_{qg}(z)$$

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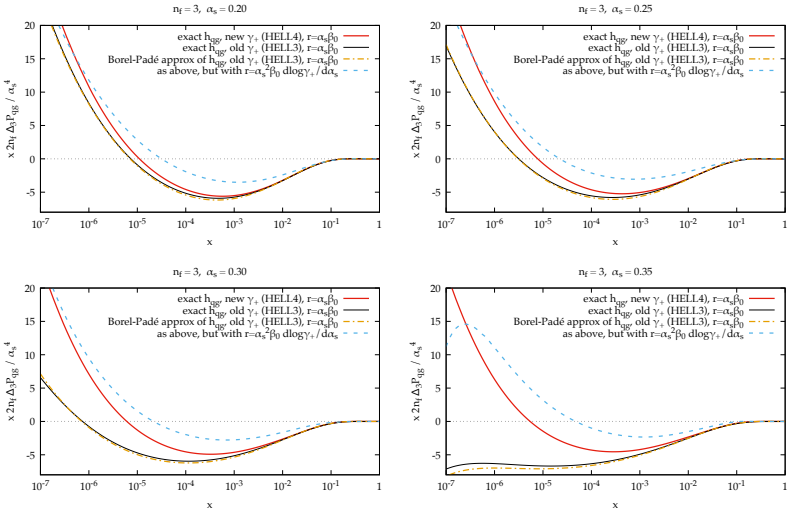
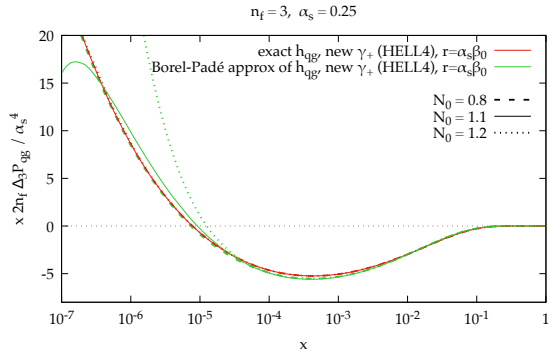
P_{qg}: comparison to previous predictions

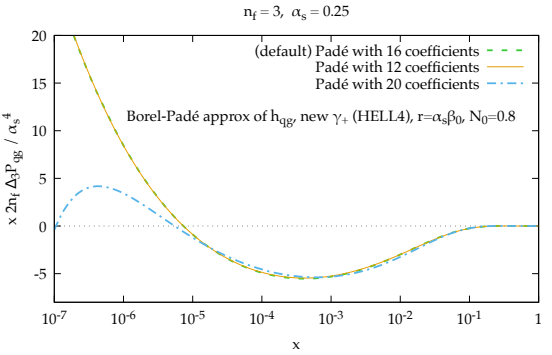
$$\Delta_3 P_{qg} = P_{qg}^{\mathrm{NLL}'} - \sum_{k=0}^{2} \alpha_s^{k+1} P_{qg}^{\mathrm{NLL}'(k)}$$



- New γ_+ similar to old γ_+ at small α_s, but more robust at large α_s
- New γ_+ within uncertainty band of old γ_+ (obtained from two r variations)
- Exact h_{qg} surprisingly similar to its Borel–Padé approximation?

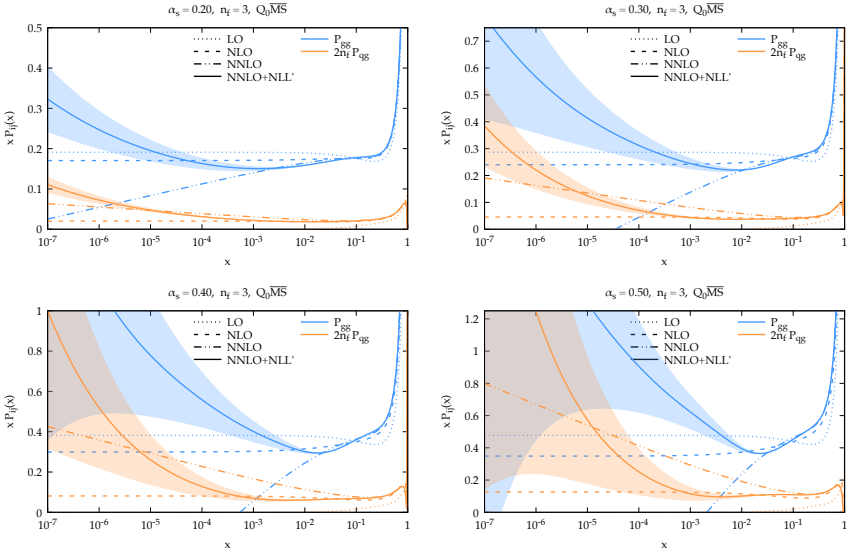
P_{qg}: comparison to previous predictions





Borel–Padé P_{qg} depends strongly on the Mellin inversion path and the Padé order
Exact function is stable under path deformation

Results for singlet splitting matrix



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PDFs at lepton colliders

At lepton colliders, same collinear factorization as for hadrons, but the initial conditions at $\mu_0 \sim m_\ell$ are computable; at leading order, only $f_\ell(x, \mu_0^2) = \delta(1-x)$, while up to $\mathcal{O}(\alpha)$ [Frixione (2019)]

$$f_\ell(x,\mu_0^2) = \delta(1-x) + \frac{\alpha}{2\pi}\left[\frac{1+x^2}{1-x}\left(\log\frac{\mu_0^2}{m_\ell^2(1-x)^2} - 1\right)\right]_+ + K_\ell$$

$$f_\gamma(x,\mu_0^2) = \frac{\alpha}{2\pi}\frac{1+(1-x)^2}{x}\left(\log\frac{\mu_0^2}{m_\ell^2 x^2} - 1\right) + K_\gamma, \qquad f_{\bar{\ell},\ell',q,\bar{q},g}(x,\mu_0^2) = 0$$

- They are factorization-scheme dependent: in $\overline{\text{MS}}$, $K_\ell = K_\gamma = 0$, but alternatives are available (e.g. Δ scheme [Frixione (2021)], where we impose $f_\ell(x, m_\mu^2) = \delta(1-x)$ at all orders)
- The main purpose of introducing PDFs for incoming leptons is to resum large $\log(s/m_\ell^2)$ terms. More generally, they provide a model for initial-state radiation.
- Quarks and gluons are generated by evolution: $\ell \to \gamma(\to q\bar{q})\ell$ is $\mathcal{O}(\alpha^2)$, like $\ell \to \gamma(\to \ell^+\ell^-)\ell$; but then $q \to qg$ starts a QCD shower, which may dominate over its QED analogue at small z.
- At muon colliders, evolution starts at $m_\mu \simeq 106\,\text{MeV} < \Lambda_{\text{QCD}}$: we need α_s in the infrared!

The non-perturbative α_s model

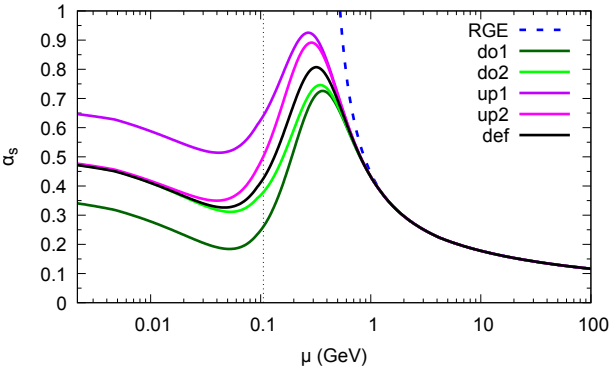
Idea: remove the Landau pole with a non-perturbative term [Webber (1998)]

$$\frac{1}{\beta_0 \log x} \;\to\; \frac{1}{\beta_0}\left[\frac{1}{\log x} + \frac{1}{1-x}\right], \quad x = \frac{\mu^2}{\Lambda^2}$$

Our choice in [Frixione, GS (2023)]:
$\alpha_s = \alpha_s^{\rm pert} + \alpha_s^{\rm np}$ for $\mu < \mu_{\rm pert} = m_c$

$$\alpha_s^{\rm np} = \frac{1}{b_0}\,\frac{x+b}{(1-x)(1+b)}\left(\frac{1+c}{x+c}\right)^p$$

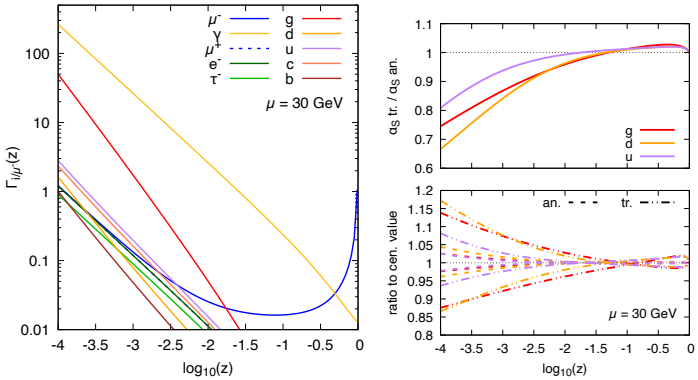
Default choice: $b = \frac{1}{4}$, $c = 4$, $p = 4$.



Alternative *truncated* approach [Han et al. (2021)] [Garosi et al. (2023)]:
QCD switched off below $Q_0 = \mathcal{O}(\Lambda)$, at the price of IR sensitivity to Q_0

The partonic content of the muon

Evolution with LO QCD and LO QED splitting functions [Frixione, GS (2023)]:



- The muon dominates as $x \to 1$; at small x the photon is the largest, the gluon the steepest
- Analytical α_s model gives larger PDFs and narrower bands than the truncated approach

Extending evolution to higher orders in eMELA

The eMELA code [Bertone, GS et al. (2022)], originally developed for electron PDFs at NLO in QED, has now been extended to higher QCD and mixed orders and interfaced to HELL for small-x resummation.

In each n_f region, we determine the evolution operator $\mathbf{E}$ by solving

$$\frac{\partial \mathbf{E}}{\partial \log \mu^2} = \mathbf{M}\,\mathbf{E}, \qquad \mathbf{M} = a\,P^{[0]} + a^2 P^{[1]} + a_s\,P^{(0)} + a_s^2 P^{(1)} + a_s^3 P^{(2)} + a\,a_s\,P^{(1,1)} + \Delta P^{\rm res}$$

with $a = \alpha/2\pi$, $a_s = \alpha_s/2\pi$.

In the alternative schemes, with $f^{(\Delta)} = \mathbb{R}\, f^{\overline{\rm MS}}$, the evolution operator changes as

$$\mathbf{M} \to \mathbb{R}\,\mathbf{M}\,\mathbb{R}^{-1} + \beta_{\rm QED}\,(d\mathbb{R}/da)\mathbb{R}^{-1}$$

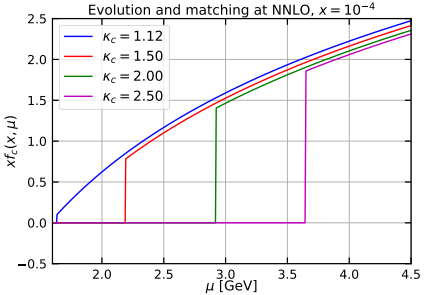
Evolution in Mellin space with a path ordering method; x-space result obtained with a numerical inverse Mellin transform on a Talbot contour.

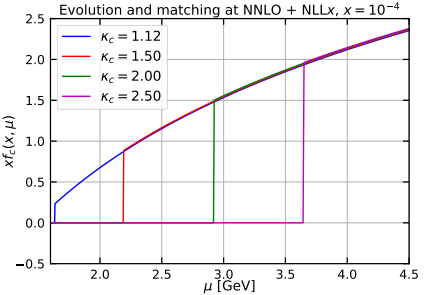
Intermezzo: matching conditions and small-x resummation

At heavy-quark thresholds $\mu = k_Q\, m_Q$, we further impose a matching condition:

$$f_i^{[n_f+1]}(x,\mu^2) = \int_x^1 \frac{dz}{z}\, K_{ij}\big(z,\alpha_s(\mu^2)\big)\, f_j^{[n_f]}\Big(\frac{x}{z},\mu^2\Big), \qquad K_{ij}\big(z,\alpha_s(m_Q^2)\big) \neq 0 \quad \text{from NNLO}$$

Perturbatively generated charm much less dependent on the (unphysical) matching scale once resummation is included [xFitter (2018)]





Outline

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   - Resummation of γ_{qg}
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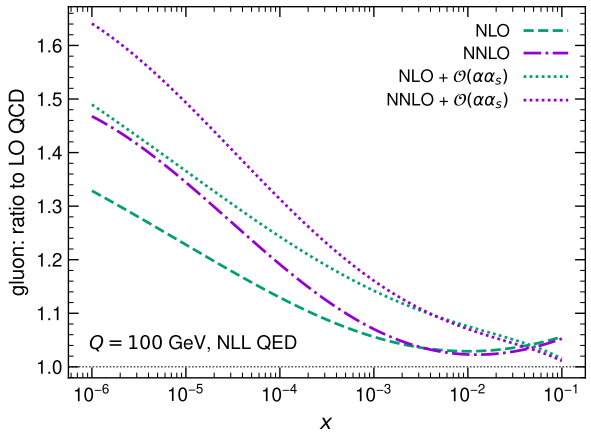
Fixed-order QCD corrections and mixed QED-QCD

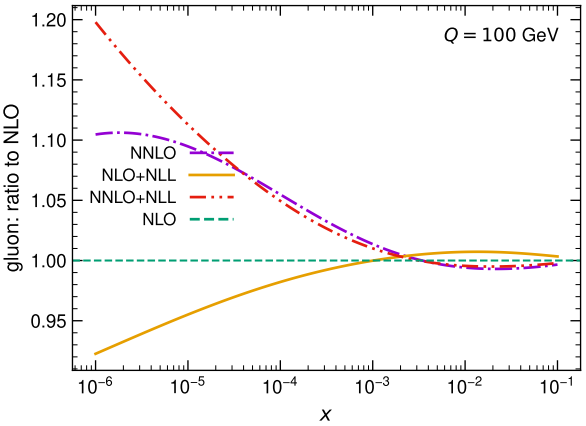
PRELIMINARY



- Large NLO corrections at small x; NNLO smaller but not negligible.
- The mixed $\mathcal{O}(\alpha\alpha_s)$ is as large as NNLO for the gluon.
- Quarks: larger corrections, same pattern; all corrections decrease with the scale; photon and muon unaffected.

Size of the resummation and stability of the expansion

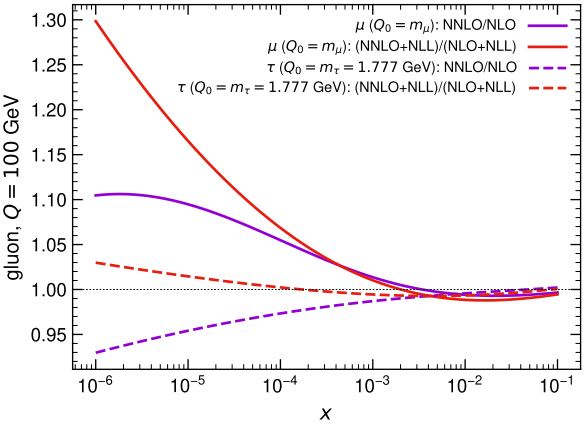
PRELIMINARY



- Matched to NLO, the resummation suppresses the small-x gluon; matched to NNLO it compensates the NNLO undershoot.
- Expected: resummed results *closer* to each other than the fixed-order ones; but for the muon they are further apart.
- Same for quarks; the gap is larger at lower scales.

The role of the starting scale

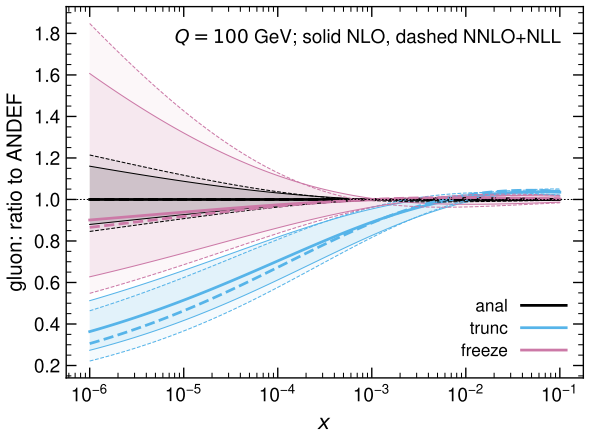
PRELIMINARY



- The muon evolution crosses a non-perturbative region with $\alpha_s \simeq 0.8$.
- Same evolution from a perturbative starting scale, a τ beam ($Q_0 = m_\tau$): the expected hierarchy is recovered.
- For the muon the spread is a genuine uncertainty of the non-perturbative window.

Dependence on the low-scale α_s

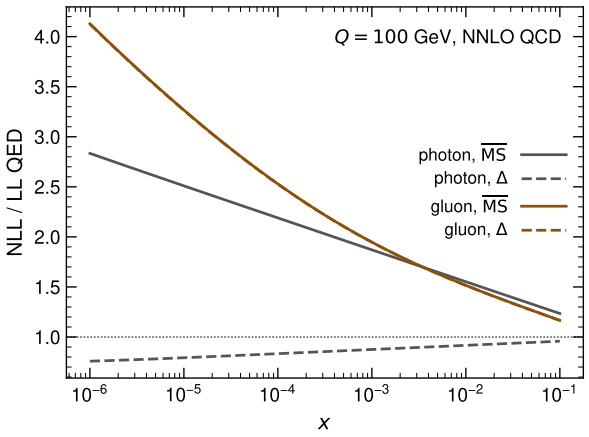
PRELIMINARY



- Three treatments below $1\,\mathrm{GeV}$, each with the envelope of its variations: analytic model (black), cutoff (blue), frozen coupling (pink).
- The cutoff removes the QCD evolution below Q_0: a factor of a few on the small-x gluon; the frozen coupling spans the analytic band on both sides.
- Same bands at NLO and NNLO+NLLx, and the resummation effect is stable across all settings: the uncertainty is independent of the perturbative order.

LL versus NLL QED

PRELIMINARY



- NLL QED is a large effect, also for quarks and gluon, almost independent of the QCD order and larger at low scales.
- Photon very different in $\overline{\text{MS}}$ and in Δ: its origin can be traced back to

$$K^{\Delta}_{\gamma\ell}(z) = \frac{1+(1-z)^2}{z}\left(1+2\log z\right)$$

$$\implies K^{\Delta}_{\gamma\ell,N} \overset{N\to 1}{\longrightarrow} -\frac{4}{(N-1)^2}$$

  a double pole, exponentiated with the evolution (harmless in $\overline{\text{MS}}$)
- Quarks and gluon unaffected by QED factorization scheme

Alternative Δ schemes

PRELIMINARY

However, it is not necessary that the γ initial condition be exactly equal to zero in the Δ scheme! Only requirements:

- $K_{\gamma\ell,N} \to 1/N$ at large N
- momentum conservation: $\int_0^1 dz\, z\, K_{\gamma\ell}(z) = -\frac{17}{9}$

Possible alternatives:

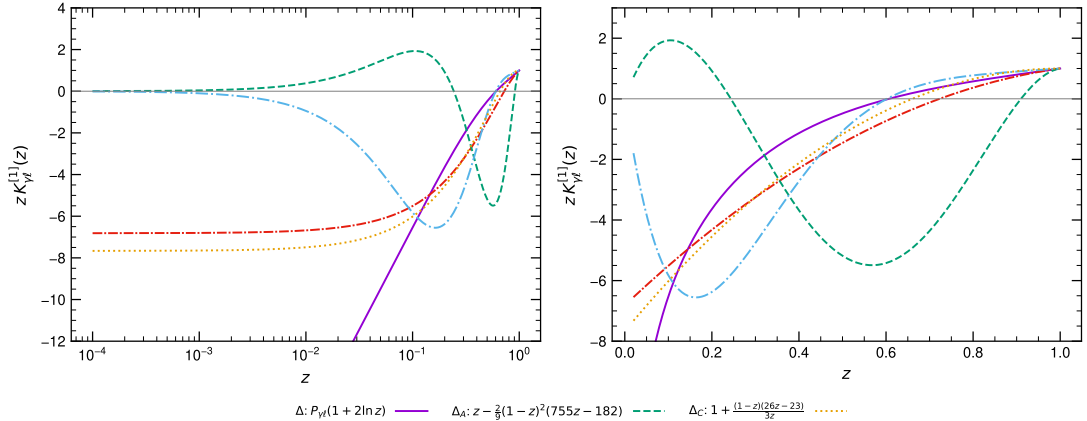
$$K_{\gamma\ell}^{\Delta_A} = 1 + a(1-z) + b(1-z)^2 + c(1-z)^3, \qquad a = -1\,, \quad b = -\frac{382}{3}\,, \quad c = \frac{1510}{9}$$

$$K_{\gamma\ell}^{\Delta_B} = 1 - a(1-z)^b, \qquad a = \frac{43}{18}\left(2 + 3b + b^2\right), \quad b > 0\,(b = 5)$$

$$K_{\gamma\ell}^{\Delta_C} = \frac{1}{z}\Big(1 - a(1-z)^2\Big), \quad a = \frac{26}{3}$$

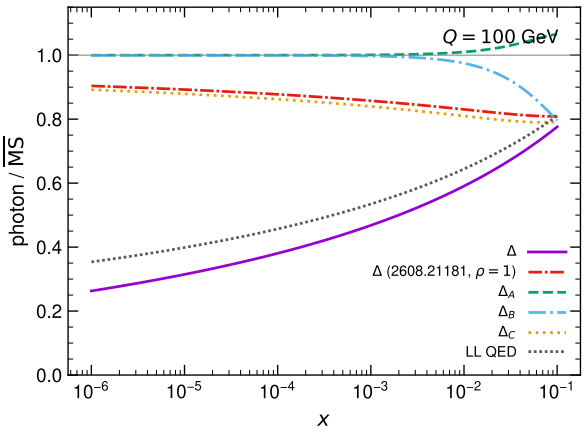
Alternative Δ schemes: comparison of $K_{\gamma\ell}$

PRELIMINARY



Photon in the Δ-family schemes

PRELIMINARY



- Small-x photon set by the small-z form of the restoring term; similar at all scales.
- Gluon (and quarks) scheme independent to 10^{-3}, at every QCD order; beam muon identical in all Δ-family schemes.

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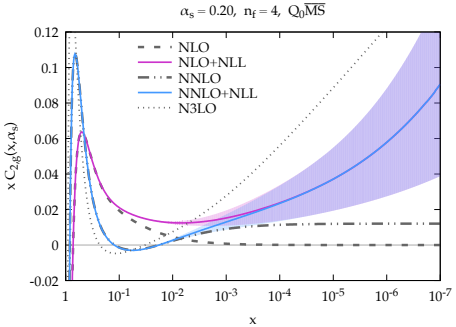
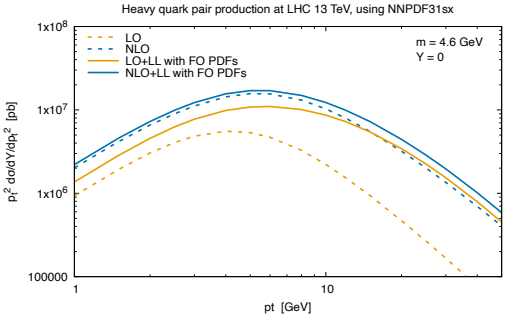
Coefficient functions at small x

In addition to splitting functions and matching conditions, we need to include resummation also in short-distance cross sections. So far done for two processes:

- **DIS** [Bonvini, Marzani, Muselli (2017)]: fixed-order coefficient functions unstable at small x (N^{3}LO: $\alpha_s^3 \log \frac{1}{x}$), while the resummed ones are stable, with a band from subleading logs
- **Heavy-quark production** [Bonvini, Silvetti (2022)]: due to the kinematic constraint $x\, e^{2|y|} \leq 1/(1 + (p_t^2/m^2))$, at large p_t a perturbative instability arises, cured by resummation





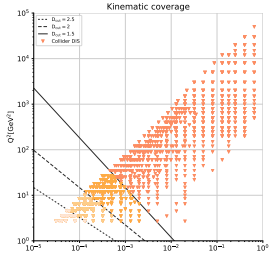
Previous PDF fits with small-x resummation

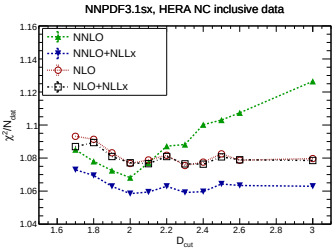
NNPDF3.1sx [Ball et al. (2017)]	xFitter [xFitter (2018)]
neural net, MC uncertainty	polynomial, Hessian uncertainty
DIS+Tevatron+LHC ($\sim$ 4000 pts)*	HERA only ($\sim$ 1200 pts)
fitted charm	perturbative charm
(N)NLO, (N)NLO+NLLx	NNLO, NNLO+NLLx

*since hadronic observables are not resummed, additional cut $\alpha_s(Q^2)\log\frac{1}{x} > H_{\rm cut} = 0.6$ on Tevatron and LHC data

With NLLx resummation the fit quality improves and stays stable as small-x data are added

$\chi^2/N_{\rm dat}$	NLO	NLO+NLLx	NNLO	NNLO+NLLx
NNPDF3.1sx (global)	1.117	1.120	1.130	1.100
xFitter (HERA only)	–	–	1.23	1.17

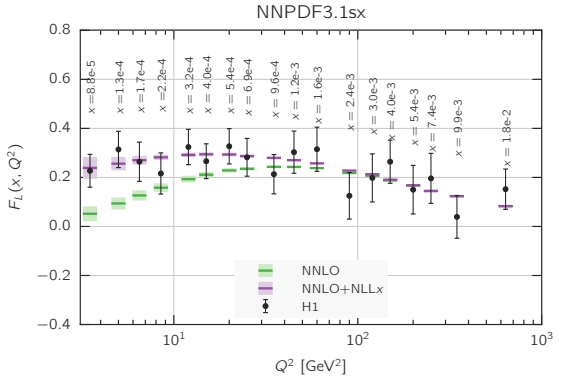




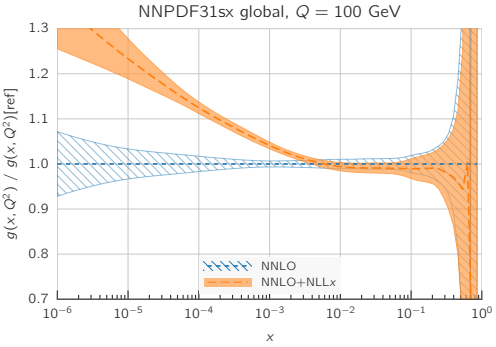
Better description of HERA data

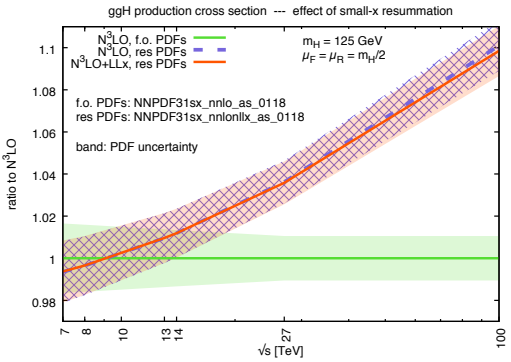
Low-Q^2, low-x HERA data in tension with fixed order:
NNLO describes them worse than NLO [H1, ZEUS (2015)]

Small-x resummation resolves the tension
and improve descriptions of F_L data (not included in NNPDF fit).



Impact on PDFs and on collider phenomenology





- Resummation strongly enhances the gluon at $x \lesssim 10^{-3}$, well outside the NNLO uncertainty band
- Since $x_{\min} = Q^2/s$, effects grow at low invariant masses and higher energies: ggH cross section $+1\%$ at the LHC, $\sim +10\%$ at the FCC-hh w.r.t. NNLO PDFs [Bonvini, Marzani (2018)]
- Mostly a PDF effect: the small-z region of the partonic cross section is weighted by PDFs at large x, which die fast

Towards a new fit with small-x resummation

The improvements on the `HELL` side call for a new PDF fit! In particular, we would like to consider an N^{3}LO+NLLx fit, given that N^{3}LO is much more unstable at small x

Work in progress with members of the NNPDF collaboration:

- interface `EKO` (NNPDF's evolution code) to `HELL`: done
  (the debugging against `eMELA` allowed us to identify a bug in a $\mathcal{O}(\alpha)$ splitting function in `EKO`)
- add resummation of DIS coefficient functions in `yadism` (NNPDF's DIS provider)
- add resummation for other coefficient functions (once available)
- resolve the mismatch between the $Q_0\overline{\rm MS}$ scheme used for resummation and standard $\overline{\rm MS}$ used for everything else (the two coincide up to NNLO)

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Conclusions

- New all-order analytics for γ_s,
  which may provide insights into a (semi-)analytical implementation of $\gamma_+^{\mathrm{NLL}'}$
- First proper P_{qg} resummation,
  which supersedes previously fine-tuned results based on limited knowledge
- Several technical improvements
  to make `HELL` more robust and able to reach large α_s values

New version 4.0 of `HELL` soon to be released

Applications:

- Preliminary results for lepton PDFs with small-x resummation
  A new version of `eMELA` will be released with the paper
- Work in progress towards a new fit of hadronic PDFs with resummed splitting functions

https://indico.cern.ch/event/1733750

Massive Precision: bottom and charm at colliders

Feb 1 – 3, 2027
Genova, Italy
Europe/Rome timezone

Enter your search term

- Overview
- Call for Abstracts
- Registration
- Participant List
- Venue
- How to reach Genova
- Once you are in Genova

This short workshop will discuss recent developments in collider Standard Model phenomenology involving heavy quarks, covering fixed-order and resummed calculations, parton showers, massive-massless scheme matching, parton distribution and fragmentation functions, jet substructure and energy-energy correlators.

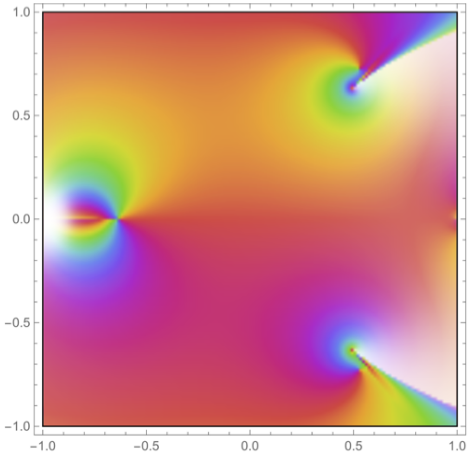
The event is endorsed by Università degli Studi di Genova and INFN, with financial support by the EU Horizon Europe research and innovation programme under the Marie Sklodowska-Curie Action HIPFLAPP, grant agreement 101149076

Confirmed speakers: Christian Biello, Marzia Bordone, Alessandro Broggio, Leticia Cunquiero, Silvia Ferrario Ravasio, Andrea Ghira, Felix Hekhorn, Rene Poncelet, Daniel Reichelt, Kay Schoenwald, Yannick Ulrich

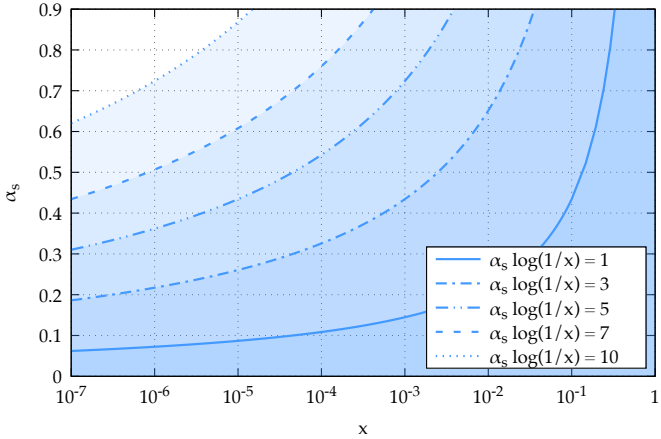
We plan to organize a dedicated session featuring contributions from PhD students and early-career postdocs, see Call for Abstracts

BACKUP

The function $h_{qg}(z)$ in the complex plane



Size of $\alpha_s \log(1/x)$



Reaching large α_s: improvements in HELL

- new approximate fixed-order anomalous dimension with modified asymptotic behaviour at large-N to be used in the first "iteration" of duality (at small α_s it reduces to previous implementation) [details next]
- new approximation of BFKL kernel used in resummation of running coupling effects (more robust even at small α_s) [details next]
- new extrapolation of large-x behaviour in $P_+(x)$ to avoid a change of branch in the solution of duality with large α_s [details next]

Duality "in reverse" resums the collinear ($\gamma = 0$) pole of the BFKL kernel by adding the inverse of a fixed-order anomalous dimension (χ_s = dual of γ^{approx}).

- The kernel has a further subleading singularity at $\gamma = -1$: if kept, after symmetrisation it would dominate the large-N asymptotics, spoiling the physical log-divergent behaviour.
- HELL trick [Bonvini, Marzani, Muselli (2017)]: replace exact γ^{FO} by an approximation with the correct small-N poles but asymptoting to a constant > -1, so its dual dominates over the $\gamma = -1$ pole $\Rightarrow$ resummed γ keeps its input asymptotics (then subtract approx. FO, add back the exact one).

Old approximation (valid at LO and NLO)

$$\gamma^{\text{approx}}(N) = \frac{a_1}{N} + a_0 - (a_1 + a_0)\,\frac{2N}{N+1}, \qquad a_i = \alpha_s a_{1i} + \alpha_s^2 a_{2i}$$

Last term enforces $\gamma^{\text{approx}}(N = 1) = 0$ (momentum cons.), form chosen so that the inverse is analytic. Problem: the asymptotic constant stays > -1 only for $\alpha_s \lesssim 0.5$.

Minimal generalisation: replace $\frac{2N}{N+1} \to \frac{(1+\lambda)N}{N+\lambda}$, with $0 < \lambda \leq 1$ controlling the large-N limit

$$\gamma^{\text{approx}}(N) = \frac{a_1}{N} + a_0 - (a_1 + a_0)\frac{(1+\lambda)N}{N+\lambda} \xrightarrow{N\to\infty} -(1+\lambda)a_1 - \lambda a_0$$

Smaller $\lambda \Rightarrow$ larger (less negative) asymptotic value, as needed at large α_s. We promote λ to a smooth function of α_s:

$$\lambda = \frac{1}{1+6\alpha_s^2} \qquad \left(\lambda \to 1 \text{ at small } \alpha_s, \quad \lambda \sim \tfrac{1}{5} \text{ at } \alpha_s = 0.8\right)$$

The dual is still analytic:

$$\chi_s(M) = \frac{a_1 + \lambda a_0 - \lambda M + \sqrt{(\lambda M + a_1 - \lambda a_0)^2 + 4\lambda(1+\lambda)a_1(a_1+a_0)}}{2\big[(1+\lambda)a_1 + \lambda a_0 + M\big]}$$

Reduces to the old result for $\lambda = 1$; keeps small-α_s behaviour untouched.

Running-coupling effects in the duality are resummed by solving BFKL evolution with an analytically solvable approximation of the kernel [Bonvini, Marzani, Muselli (2017)]. Two ingredients:

- α_s **dependence**: linearised, $\chi(\alpha_s,\gamma) = \chi(\alpha_0,\gamma) + (\alpha_s-\alpha_0)\chi'(\alpha_0,\gamma)$ (a simpler $\chi \simeq \frac{\alpha_s}{\alpha_0}\chi(\alpha_0,\gamma)$ gives the uncertainty band);
- γ **dependence**: two-pole form, symmetric about the minimum $\gamma_{\min}$, matching the double-leading kernel to $\mathcal{O}\big((\gamma-\gamma_{\min})^2\big)$:

$$\chi_{\rm coll} = c + \frac{\gamma_{\min}^3\kappa}{4}\left[\frac{1}{\gamma} + \frac{1}{2\gamma_{\min}-\gamma} - \frac{2}{\gamma_{\min}}\right]$$

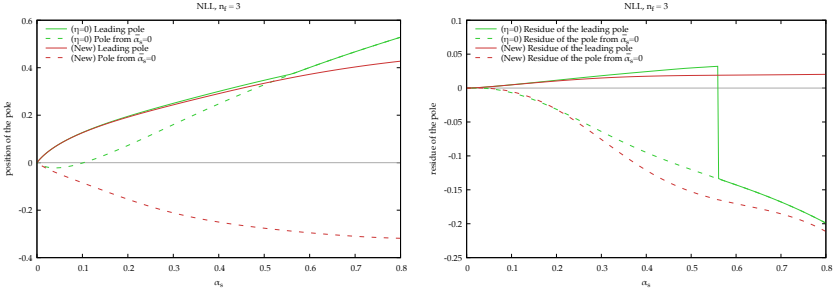
New: shift the two poles by $\pm\eta$ (keeping the symmetry)

$$\chi_{\rm coll,new} = c + \frac{(\gamma_{\min}+\eta)^3\kappa}{4}\left[\frac{1}{\gamma+\eta} + \frac{1}{2\gamma_{\min}-\gamma+\eta} - \frac{2}{\gamma_{\min}+\eta}\right]$$

Parameters re-adapted to match the old expansion around $\gamma_{\min}$ $\Rightarrow$ net effect is just a global shift $\gamma_{\min} \to \gamma_{\min}+\eta$ (trivial to implement). Solution $\gamma_{\rm rc}$ still analytic (Bateman function k_ν).

Besides the physical (Bateman) pole, the solution has a spurious pole from the prefactor $\bar{\alpha}_s = 0$ at $N_- = c - \frac{\kappa}{2}(\gamma_{\min} + \eta)^2$, with a large negative residue. For $\eta = 0$ it becomes the *leading* pole already at $\alpha_s \sim 0.55$, making P_+ decrease at small x (unphysical, against Regge & LL).

Since $\kappa > 0$, a positive η pushes N_- to the left; the natural choice $\eta = (1-\lambda)a_1 + \lambda a_0 = \mathcal{O}(\alpha_s)$ (left pole of the true double-leading kernel) is negligible at small α_s, and at large α_s leaves the Bateman pole always leading.



Leading pole (solid) and $\bar{\alpha}_s = 0$ pole (dashed) [left], and their residues [right], vs. α_s. Green: $\eta = 0$; Red: new $\eta(\alpha_s)$.

Duality is solved with a zero finder along the Mellin contour ($\gamma_+(\alpha_s, \bar{N}) = \overline{\gamma_+(\alpha_s, N)} \Rightarrow$ upper half-plane suffices).

- **Robust start**: at large real N the solution $\rightarrow$ the fixed-order γ used to resum the kernel. Start at $N = 15$, march down to N_0 along the real axis (scan + Regula Falsi, previous point as guess), then follow the contour with the Secant method. The real-axis path is only used to fix the starting point N_0.
- **The problem**: at large $|\mathrm{Im}(N)|$ two nearby α_s can select different branches of γ_+ (branch cuts) $\Rightarrow$ a sudden jump in the large-x behaviour of P_+ when varying α_s. Small x (our region) is unaffected.

Steepening the contour to stay on one branch works only up to a point (instabilities). Instead: keep the contour fixed ($N_0 = 1.1$, $\theta = \frac{2}{3}\pi$) and repair the large-x tail.

Work with the resummed-only piece $\Delta\gamma_+ = \gamma_+ - (\text{FO exp.})$ and its Mellin inverse $\Delta P_+(\alpha_s, x)$:

1. compute ΔP_+ ignoring the branch jump;
2. its large-x tail is unstable *and* uncontrolled (small-x resummation) $\rightarrow$ replace by a cubic extrapolation in $\log x$ for $x > x_{\text{extr}} = 0.02$;
3. extrapolation breaks momentum conservation $\rightarrow$ approximate the whole function on a Chebyshev basis so its Mellin transform is analytic;
4. restore $\int_0^1 dx\, x\, \Delta P_+^{\text{new}} = 0$ and use the parametrised $\Delta\gamma_+$ everywhere else in HELL.

Factoring the known low-x growth $x^{-1-N_B(\alpha_s)}$ (N_B = leading pole), fit $f = x^{1+N_B+p}\Delta P_+^{\text{extr}}$ in fractional powers of x:

$$f^{\text{cheb,pow}} = \sum_{k=0}^{n} c_k\, x^{k/q} \implies \tilde{f}^{\text{cheb,pow}}(N) = \sum_{k=0}^{n} \frac{c_k}{N+1+k/q}$$

($x_{\min} = 10^{-7}$, $q = 6$, $n = 12$, $p = -0.1$). Only 13 coefficients reconstruct γ_+ at any N (vs. ~ 2000 sampled points before).

Small-x structure of splitting functions

slide from M. Bonvini

Singlet:

$$P(x,\alpha_s) = \sum_{n=0}^{\infty} \alpha_s^{n+1} \left[\sum_{k=0}^{n} a_{nk} \frac{\log^k x}{x} + \sum_{k=0}^{2n} b_{nk} \log^k x + \dots \right]$$

$$\int_0^1 dx\, x^N P(x,\alpha_s) \equiv \gamma(N,\alpha_s) = \sum_{n=0}^{\infty} \alpha_s^{n+1} \left[\sum_{k=0}^{n} \frac{a_{nk}}{N^{k+1}} + \sum_{k=0}^{2n} \frac{b_{nk}}{(N+1)^{k+1}} + \dots \right]$$

Single log enhancement at leading small x, in the singlet sector

$$P_{\text{singlet}} = \begin{pmatrix} P_{gg} & P_{gq} \\ P_{qg} & P_{qq} \end{pmatrix} = \begin{pmatrix} \text{LL} & \text{LL} \\ \text{NLL} & \text{NLL} \end{pmatrix}$$

eigenvalues: P_+ is LL, P_- is not enhanced at small x.

Non-singlet:

$$P^{\text{NS}}(x,\alpha_s) = \sum_{n=0}^{\infty} \alpha_s^{n+1} \left[\sum_{k=0}^{2n} b_{nk}^{\text{NS}} \log^k x + \dots \right], \qquad \gamma^{\text{NS}}(N,\alpha_s) = \sum_{n=0}^{\infty} \alpha_s^{n+1} \left[\sum_{k=0}^{2n} \frac{b_{nk}^{\text{NS}}}{(N+1)^{k+1}} + \dots \right]$$

is double log enhanced but subleading.

Ingredients for resummation

Small-x resummation is based on the interplay of

Collinear factorization: $\sigma(x,Q^2) = \int_x^1 \frac{dz}{z}\, C_i\big(z,\alpha_s(Q^2)\big)\, f_i\Big(\frac{x}{z},Q^2\Big)$

DGLAP evolution: $\mu^2 \frac{d}{d\mu^2} f_i(x,\mu^2) = \int_x^1 \frac{dz}{z}\, P_{ij}\big(z,\alpha_s(\mu^2)\big)\, f_j\Big(\frac{x}{z},\mu^2\Big)$

with

k_t factorization: $\sigma(x,Q^2) = \int_x^1 \frac{dz}{z} \int_0^\infty dk_t^2\, \mathcal{C}_i(z,k_t^2,\alpha_s)\, \mathcal{F}_i\Big(\frac{x}{z},k_t^2\Big)$

BFKL evolution: $-x\frac{d}{dx}\mathcal{F}(x,k_t^2) = \int_0^\infty \frac{dq_t^2}{k_t^2}\, \mathcal{K}\Big(\frac{k_t^2}{q_t^2},\alpha_s(\cdot)\Big)\, \mathcal{F}(x,q_t^2)$

- $\mathcal{F}_i(x,k_t^2)$: unintegrated (k_t-dependent) PDF
- $\mathcal{C}_i(z,k_t^2,\alpha_s)$: off-shell coefficient function

Consistency between equations allows to resum small-x logs:

- DGLAP + BFKL eqns $\to$ resum $P(x,\alpha_s)$
- collinear + k_t factorizations $\to$ resum $C(x,\alpha_s)$ and $K(x,\alpha_s)$

The key: relation between unintegrated and integrated PDFs

slide from M. Bonvini

From [Catani,Hautmann NPB472] we know that in N space at LL

$$\mathcal{F}_g(N, k_t^2) = R(N, \alpha_s) \frac{d}{dk_t^2} f_g(N, k_t^2)$$

R is a scheme-dependent function. In $Q_0\overline{\rm MS}$ scheme $R = 1$, so

$$\mathcal{F}_g(x, k_t^2) = \frac{d}{dk_t^2} x f_g(x, k_t^2) \qquad (Q_0\overline{\rm MS})$$

In $\overline{\rm MS}$ we have $R_{\overline{\rm MS}}(N, \alpha_s) = 1 + \mathcal{O}(\alpha_s^3)$, so $Q_0\overline{\rm MS} = \overline{\rm MS}$ up to N^{3}LO

More in general, one has

$$\mathcal{F}_i(N, k_t^2) = \frac{d}{dk_t^2} \left[G_{ij}\left(N, \alpha_s(k_t^2)\right) f_j(N, k_t^2) \right]$$

$$G_{gg} = R \qquad G_{gq} = \frac{C_F}{C_A}[R - 1]$$

$$G_{qg} = \text{no closed form found yet} \qquad G_{qq} = \frac{C_F}{C_A}[G_{qg} + \mathcal{O}(\alpha_s)]$$

Small-x resummation of DGLAP evolution (1)

slide from M. Bonvini

Let's focus on the gluon PDF, which is appropriate at LL. In Mellin space we have

$$\text{DGLAP:} \qquad \mu^2 \frac{d}{d\mu^2} f(N,\mu^2) = \gamma\big(N,\alpha_s(\mu^2)\big)\, f(N,\mu^2)$$

$$\text{BFKL:} \qquad N\mathcal{F}(N,k^2) = \int_0^\infty \frac{dk'^2}{k^2}\, \mathcal{K}\Big(\frac{k^2}{k'^2},\alpha_s(k^2),\alpha_s(k'^2)\Big)\, \mathcal{F}(N,k'^2)$$

DGLAP evolution can be solved explicitly, and using the relation

$$f(N,\mu^2) = f(N,\mu_0^2)\exp\int_{\mu_0^2}^{\mu^2}\frac{d\nu^2}{\nu^2}\,\gamma\big(N,\alpha_s(\nu^2)\big), \qquad \mathcal{F}(N,k^2) = \frac{d}{dk^2}f(N,k^2) = \frac{1}{k^2}\gamma\big(N,\alpha_s(k^2)\big)\, f(N,k^2)$$

we obtain

$$N\gamma\big(N,\alpha_s(k^2)\big)\cancel{f(N,k^2)} = \int_0^\infty \frac{dk'^2}{k'^2}\,\mathcal{K}\Big(\frac{k^2}{k'^2},\alpha_s(k^2),\alpha_s(k'^2)\Big)\,\gamma\big(N,\alpha_s(k'^2)\big)\exp\left[\int_{k^2}^{k'^2}\frac{d\nu^2}{\nu^2}\gamma\big(N,\alpha_s(\nu^2)\big)\right]\cancel{f(N,k^2)}$$

Duality Relation

Small-x resummation of DGLAP evolution (2)

slide from M. Bonvini

If the strong coupling is frozen

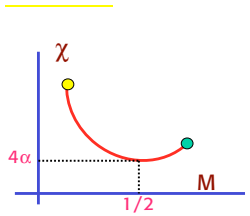
$$N = \int_0^\infty \frac{dk'^2}{k'^2} K\left(\frac{k^2}{k'^2}, \alpha_s\right) \left(\frac{k'^2}{k^2}\right)^{\gamma(N,\alpha_s)} \equiv \chi\big(\gamma(N,\alpha_s),\alpha_s\big)$$

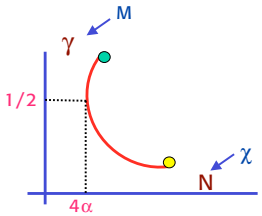
where $\chi(M,\alpha_s) = \int \frac{dk'^2}{k'^2} \left(\frac{k'^2}{k^2}\right)^M K\left(\frac{k^2}{k'^2},\alpha_s\right)$ is the BFKL kernel in M Mellin space
Running coupling corrections can be carried out perturbatively

The duality relation **at fixed coupling** is an inverse-function relation in double-Mellin space

$$\chi\big(\gamma(N,\alpha_s),\alpha_s\big) = N \qquad \leftrightarrow \qquad \gamma\big(\chi(M,\alpha_s),\alpha_s\big) = M$$

For $\chi(M,\alpha_s) = \alpha_s \chi_0(M)$
the dual γ contains all orders in α_s/N
$\alpha_s^k \frac{1}{x} \log^{k-1}\frac{1}{x} \leftrightarrow \left(\frac{\alpha_s}{N}\right)^k$ LL terms
BFKL at NLO $\to$ NLL resummation





Small-x resummation of coefficient functions

slide from M. Bonvini

In the small-x and large-Q^2 limit (in N Mellin space)

$$\sigma(N,Q^2) = C\left(N,\alpha_s(Q^2)\right) f(N,Q^2) \qquad \text{(collinear factorization)}$$

$$= \int_0^\infty dk_t^2\, \mathcal{C}(N,k_t^2,\alpha_s)\, \mathcal{F}(N,k_t^2) \qquad (k_t \text{ factorization})$$

In the $Q_0\overline{\rm MS}$ scheme (in N-Mellin space)

$$\mathcal{F}_i(N,k_t^2) = \frac{d}{dk_t^2}\left[G_{ij}(N,\alpha_s) f_j(N,k_t^2)\right] = \frac{d}{dk_t^2}\left[G_{ij}(N,\alpha_s) U_{jk}\left(N,\frac{k_t^2}{Q^2}\right)\right] f_k(N,Q^2),\ U\left(N,\frac{k_t^2}{Q^2}\right) = \mathrm{P}\exp\int_{Q^2}^{k_t^2}\frac{d\nu^2}{\nu^2}\gamma(N,\alpha_s(\nu^2))$$

with *resummed* anomalous dimension $\gamma(N,\alpha_s)$, so that [Bonvini,Marzani,Peraro 1607.02153]

$$C_k(N,\alpha_s) = \int_0^\infty dk_t^2\, \mathcal{C}_i\left(N,\frac{k_t^2}{Q^2},\alpha_s\right)\frac{d}{dk_t^2}\left[G_{ij}(N,\alpha_s) U_{jk}\left(N,\frac{k_t^2}{Q^2}\right)\right]$$

(collinear singularities are a tricky point but can be handled)

Features:

- formally equivalent to ABF, but more general, easier and numerically stabler
- also more suitable for resummation of new processes (no need to compute M-Mellin transform)
- only known at lowest logarithmic order (LL)

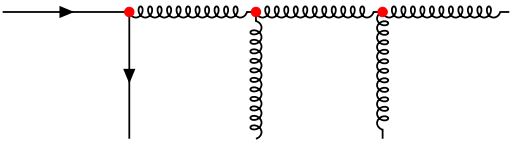
Do we need to worry about small-x logs in QED evolution?

slide from M. Bonvini

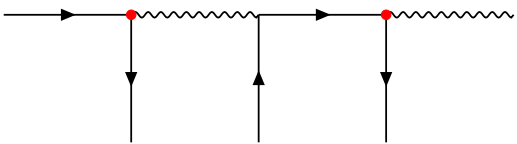
There are small-x logs in QED splitting functions, appearing already at $\mathcal{O}(\alpha)$ in $P_{\gamma f}$
Are they enhanced?

Yes, but much less than in QCD

QCD: Single-log enhancement, due to non-abelian nature of strong interactions



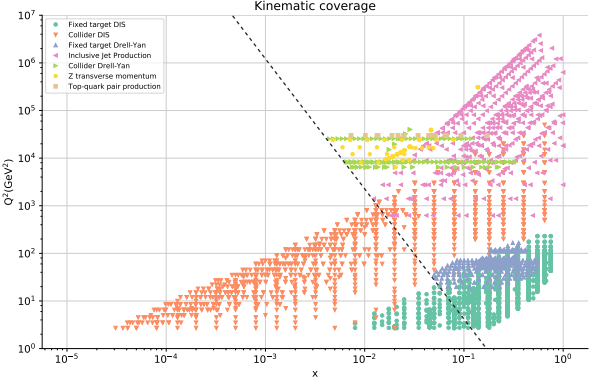
QED: Half-log enhancement (one extra power of the log every two powers of α), due to the abelian nature of EM interactions

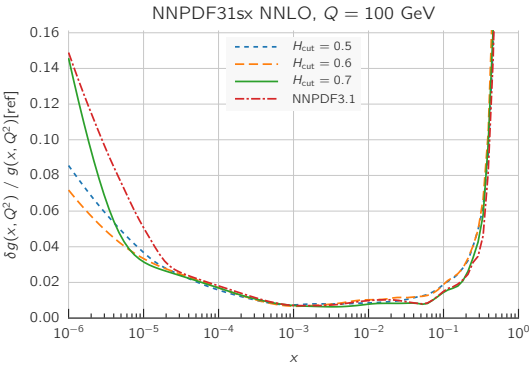


Given also that $\alpha \ll \alpha_s$, their resummation is definitely not needed

Global fit with small-x resummation

$$\alpha_s(Q^2)\log\frac{1}{x} > H_{\rm cut} = 0.6 \quad \text{(default cut)}$$





Low x points at HERA

DIS data from HERA extend down to $\boldsymbol{x \sim 3 \times 10^{-5}}$ in the "perturbative region" $\boldsymbol{Q^2 > 2}\,\mathrm{GeV}^2$

Tension between HERA data at low Q^2 and low x with fixed-order theory

