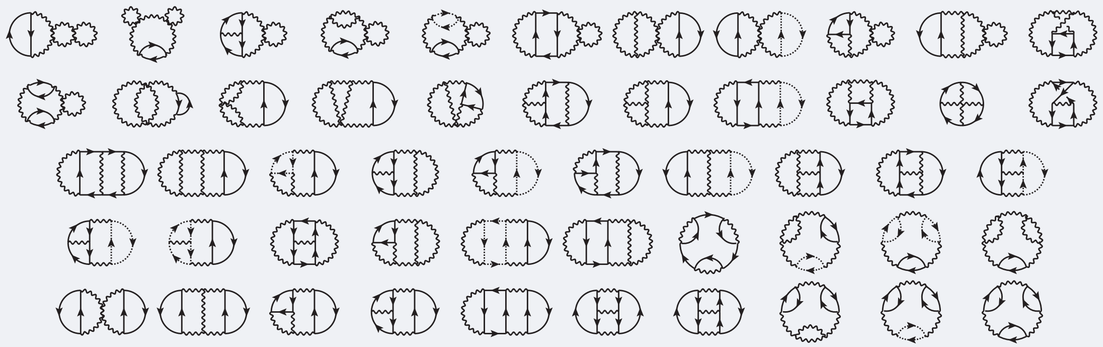
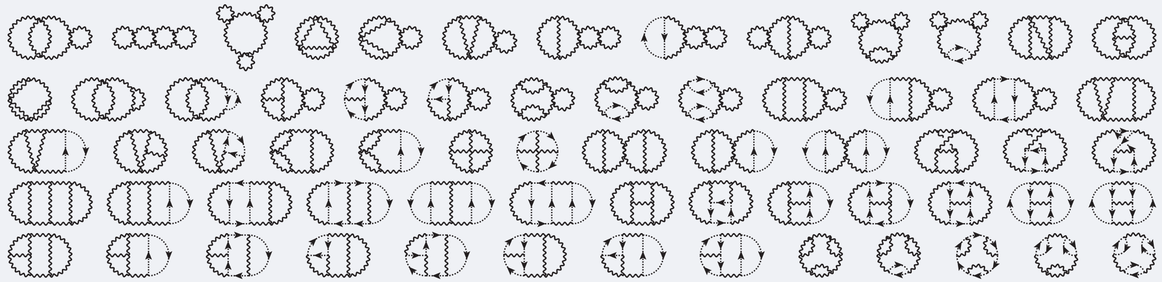
The Four-loop Frontier of Dense QCD and Beyond





Particle Physics Day 2026, Nov 11, Helsinki

Risto Paatelainen

UNIVERSITY OF TURKU

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European Research Council
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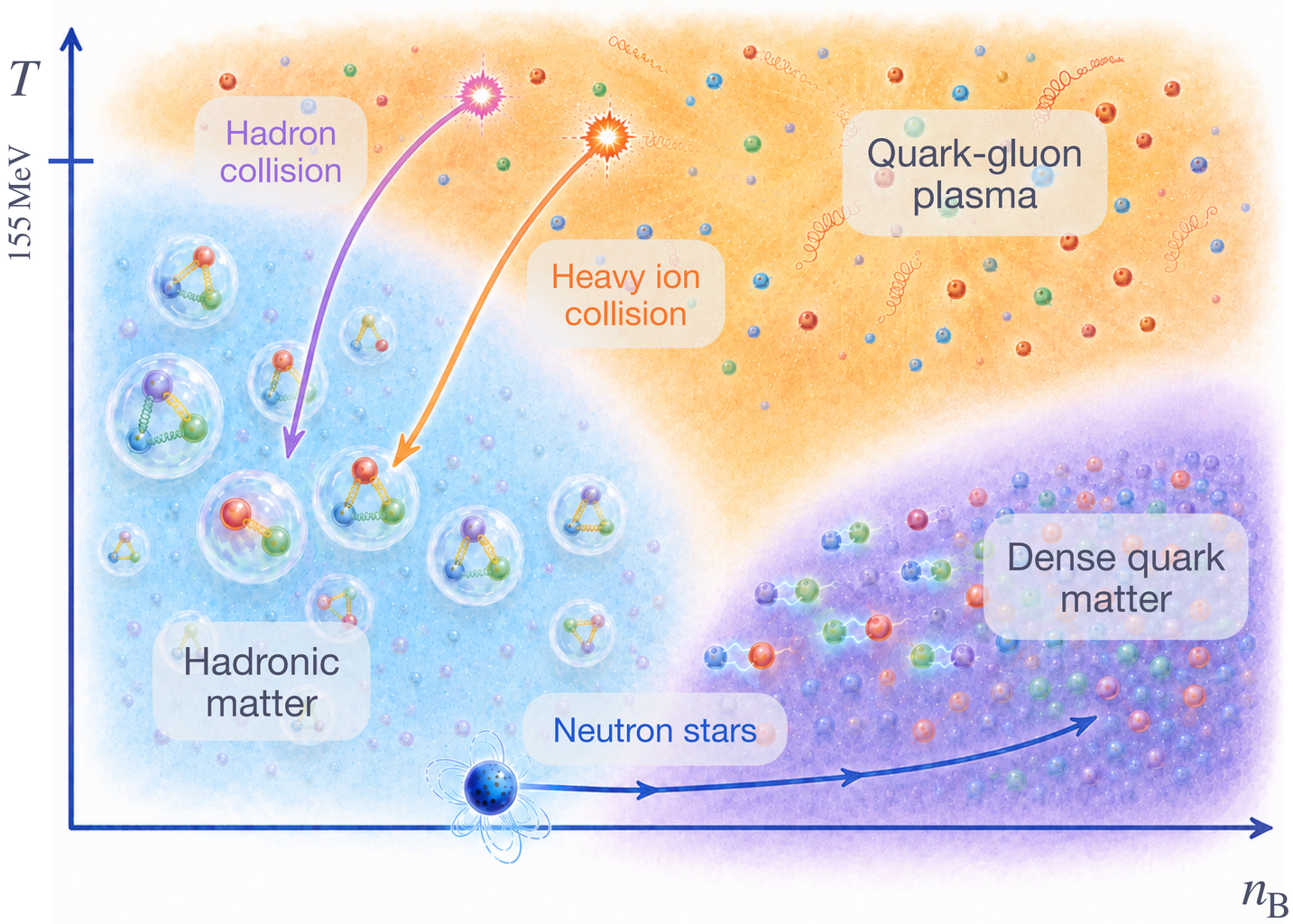
Centre of Excellence neutronstars.fi

QCD matter under extreme conditions



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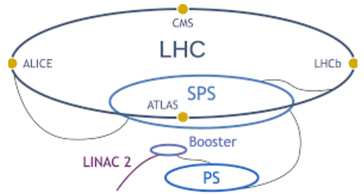
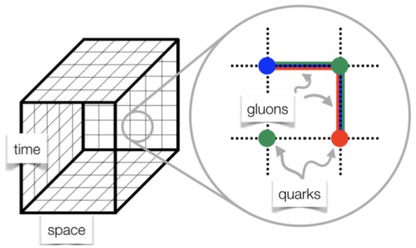
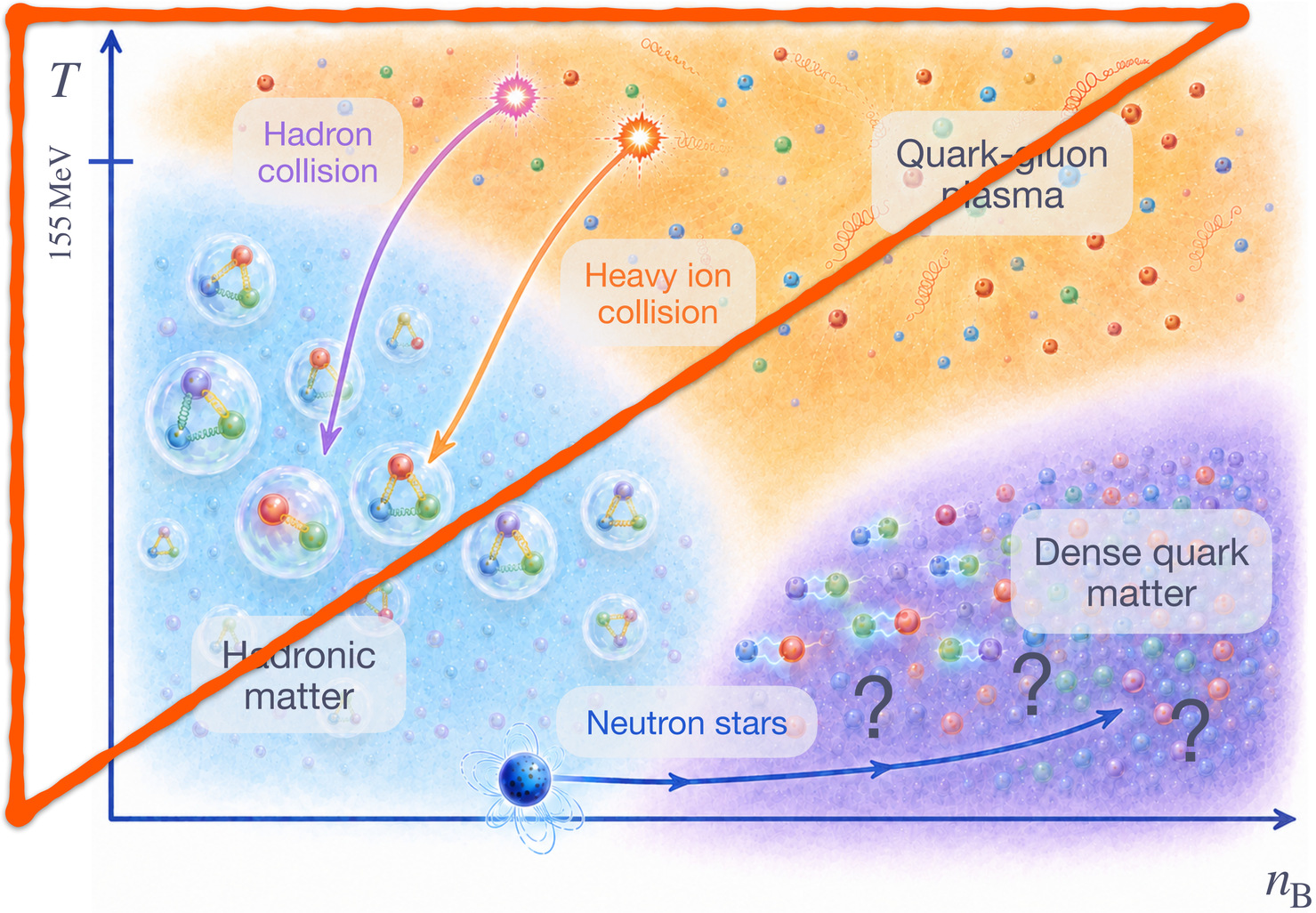
Hot QCD





Lattice QCD



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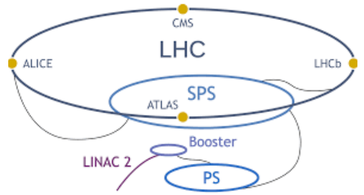

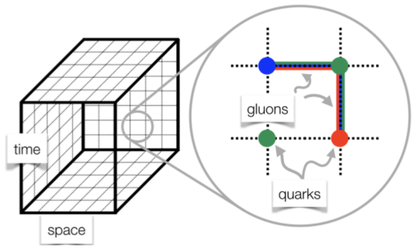

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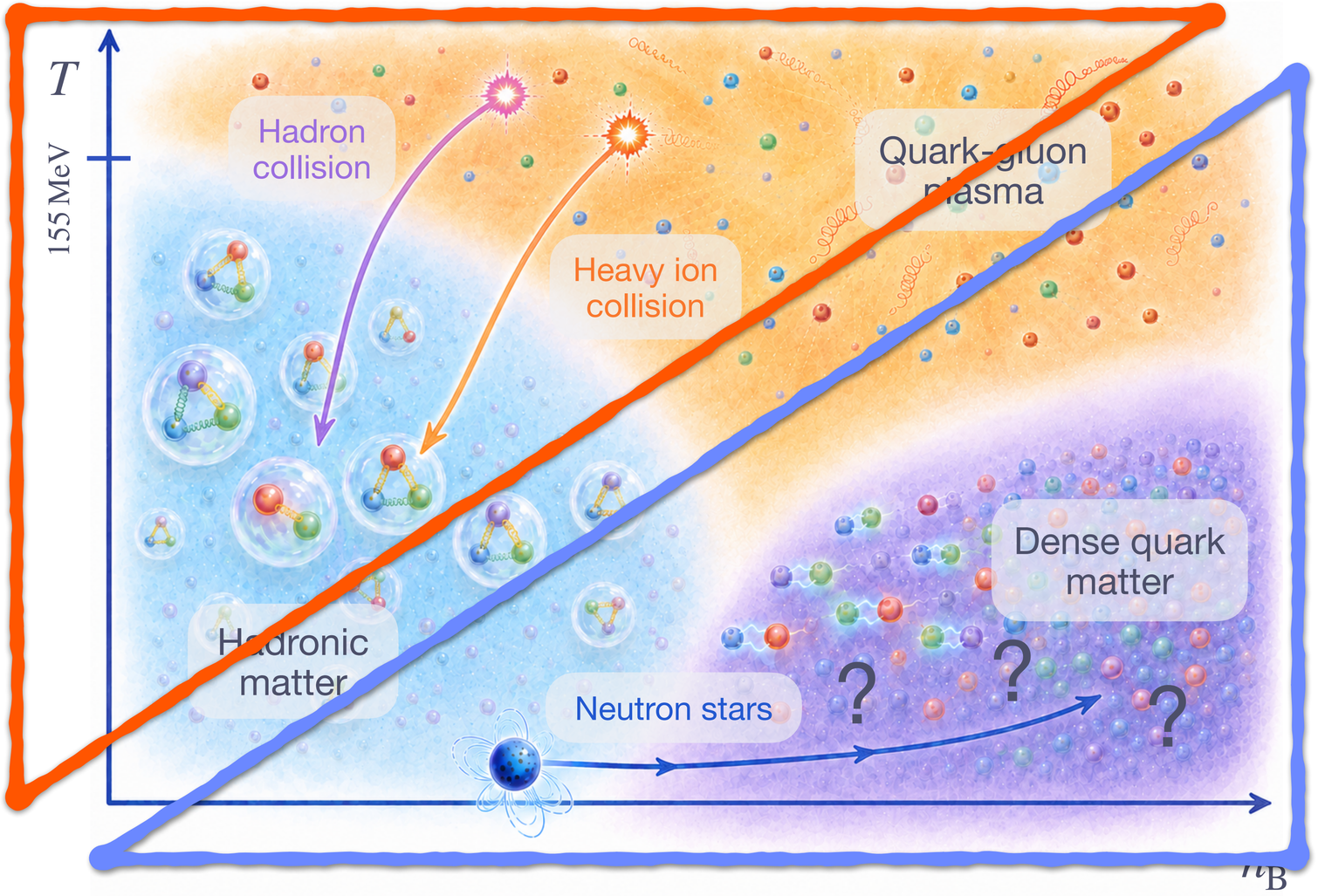
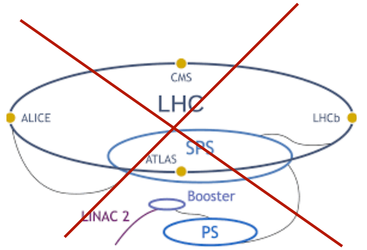


Dense QCD

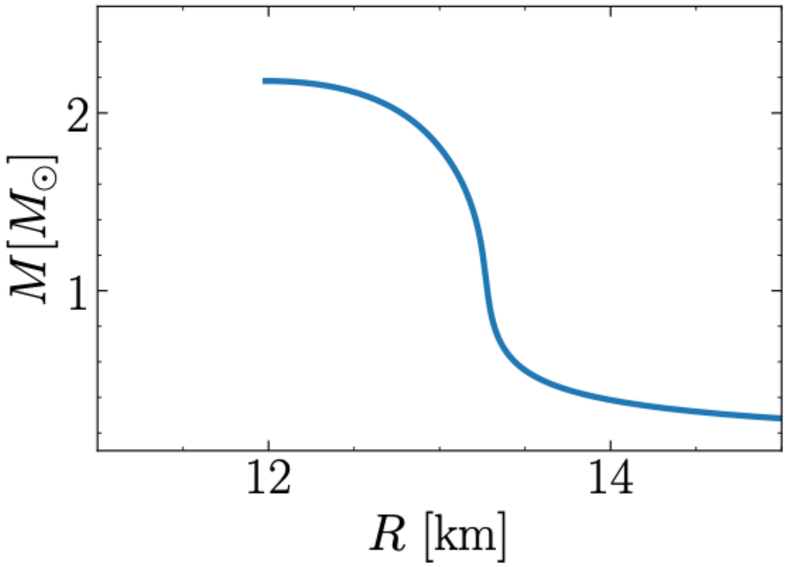


Lattice QCD

Neutron stars

Laboratories of dense matter

Macroscopic properties of the star

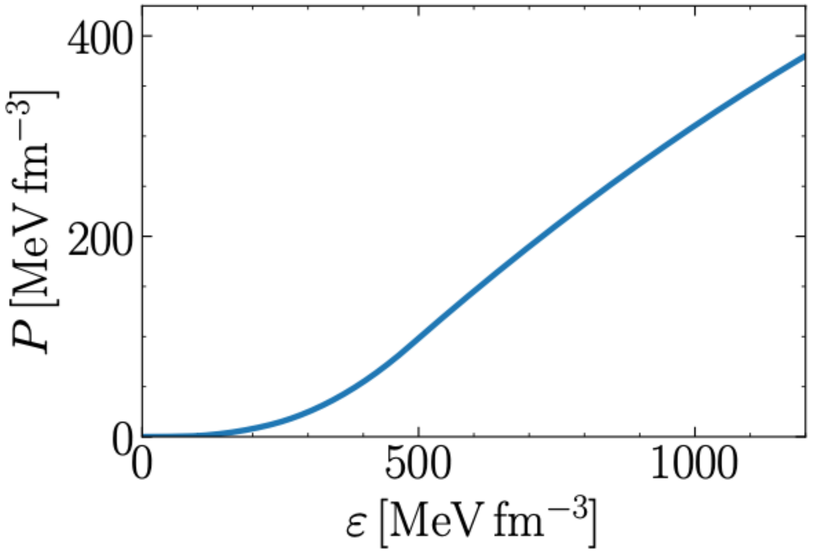


Mass-radius relation

TOV

$\longleftrightarrow$

Microscopic properties of matter



Equation of State (EoS)

Neutron star EoS

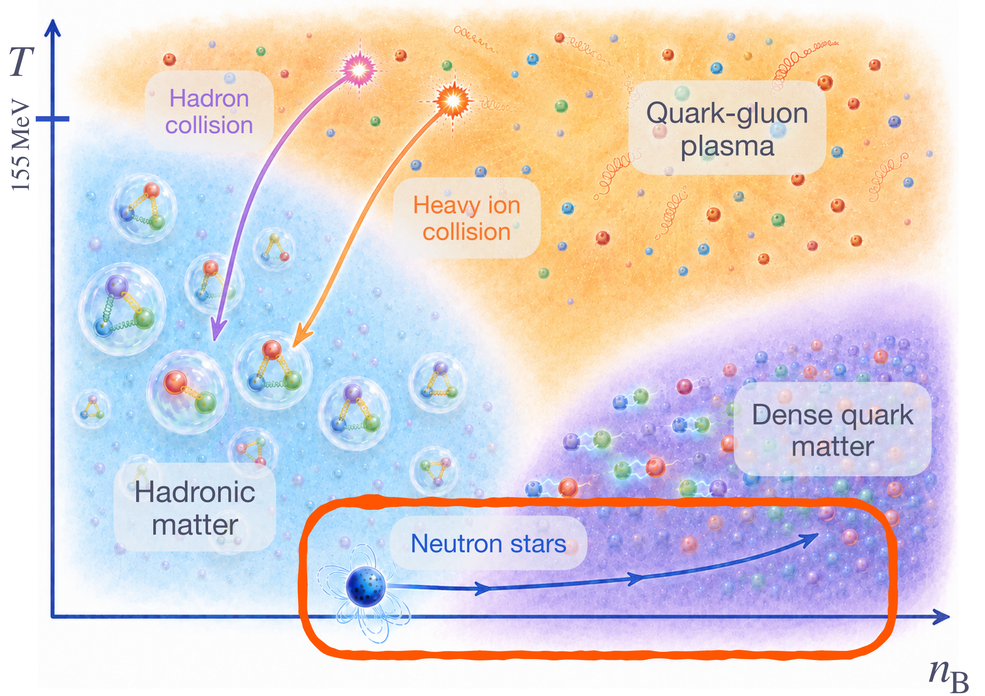
What we know

Neutron stars are strongly interacting cold and dense systems

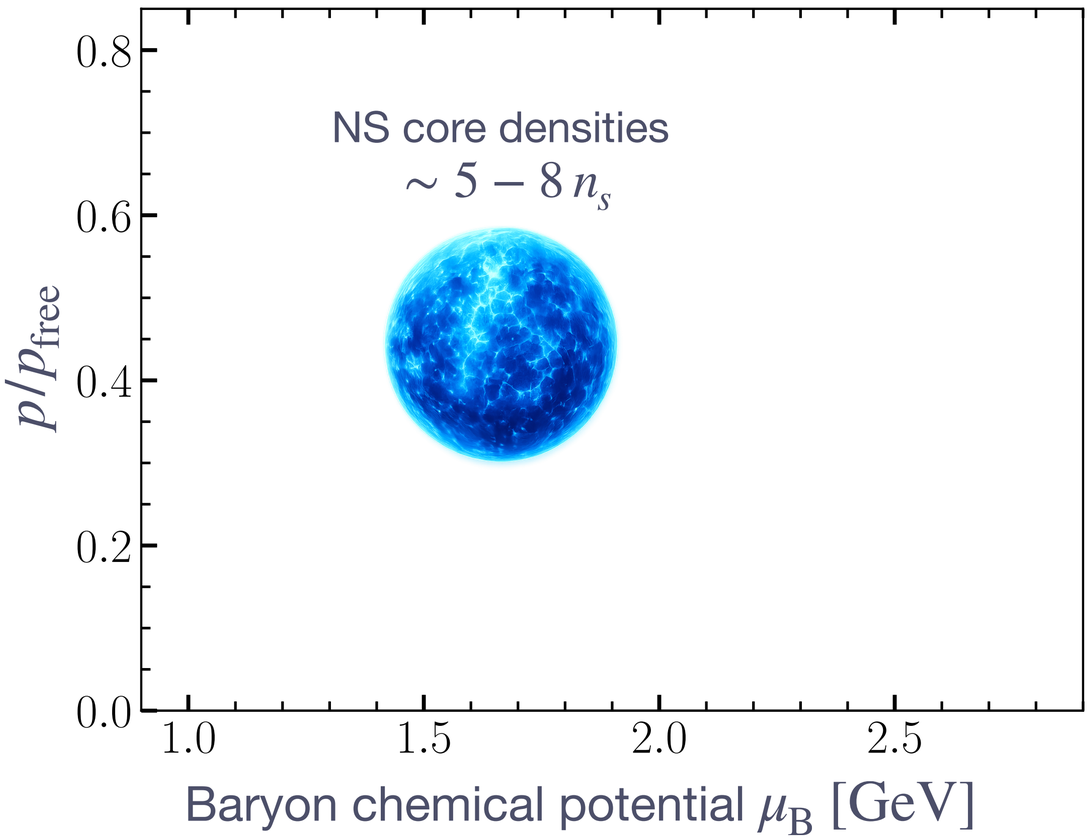
Neutron star EoS ≈ dense QCD matter EoS

with

- $T \simeq 0, \mu_{\mathrm{B}} > 0$
- β-equilibrium and charge neutrality
- d.o.f. from hadrons to 3-flavor quark matter



Dense QCD matter EoS

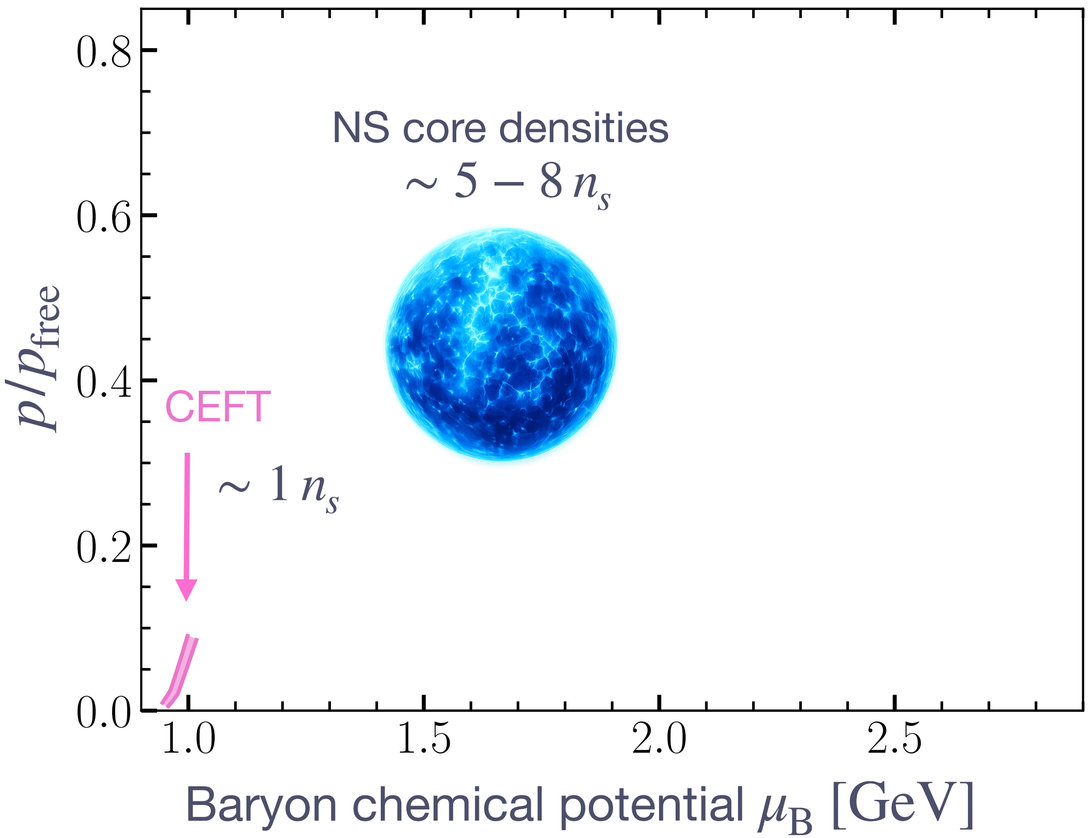


Dense QCD matter EoS

First-principles QCD information only at low and high densities

Chiral effective field theory (CEFT)

nucleons and pions



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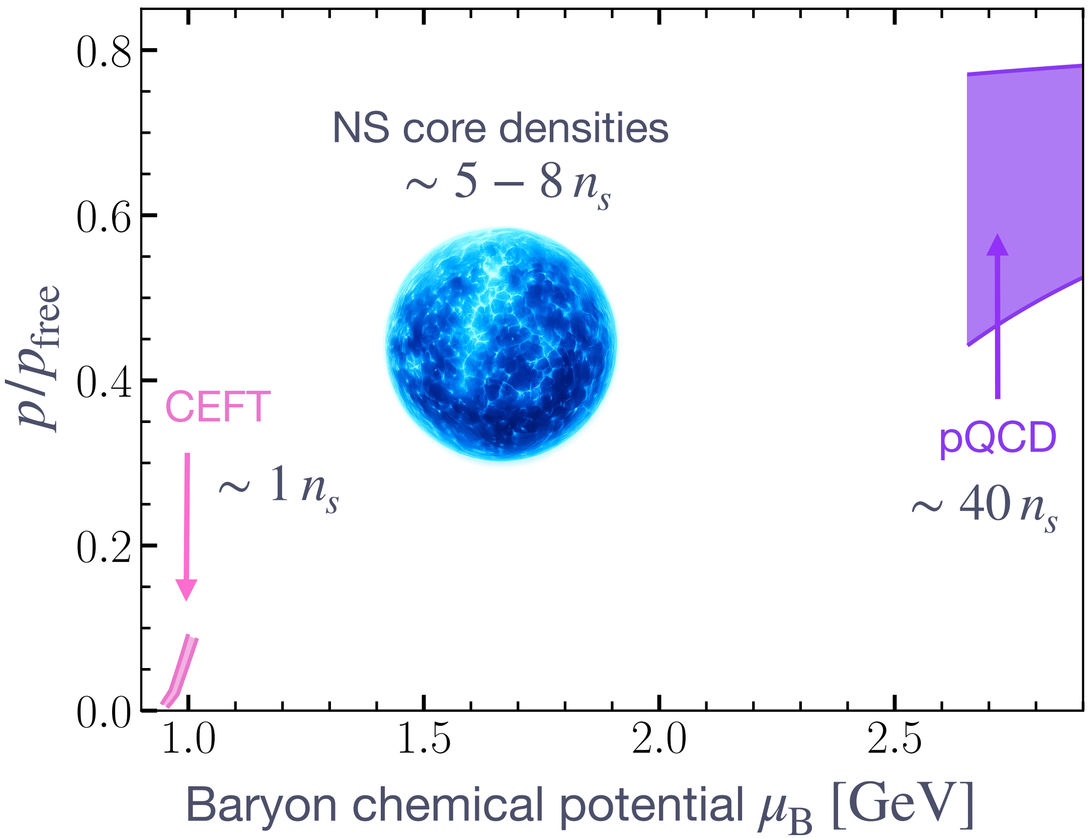

Perturbative QCD (pQCD)

quarks and gluons

Chiral effective field theory (CEFT)

nucleons and pions

Dense QCD matter EoS

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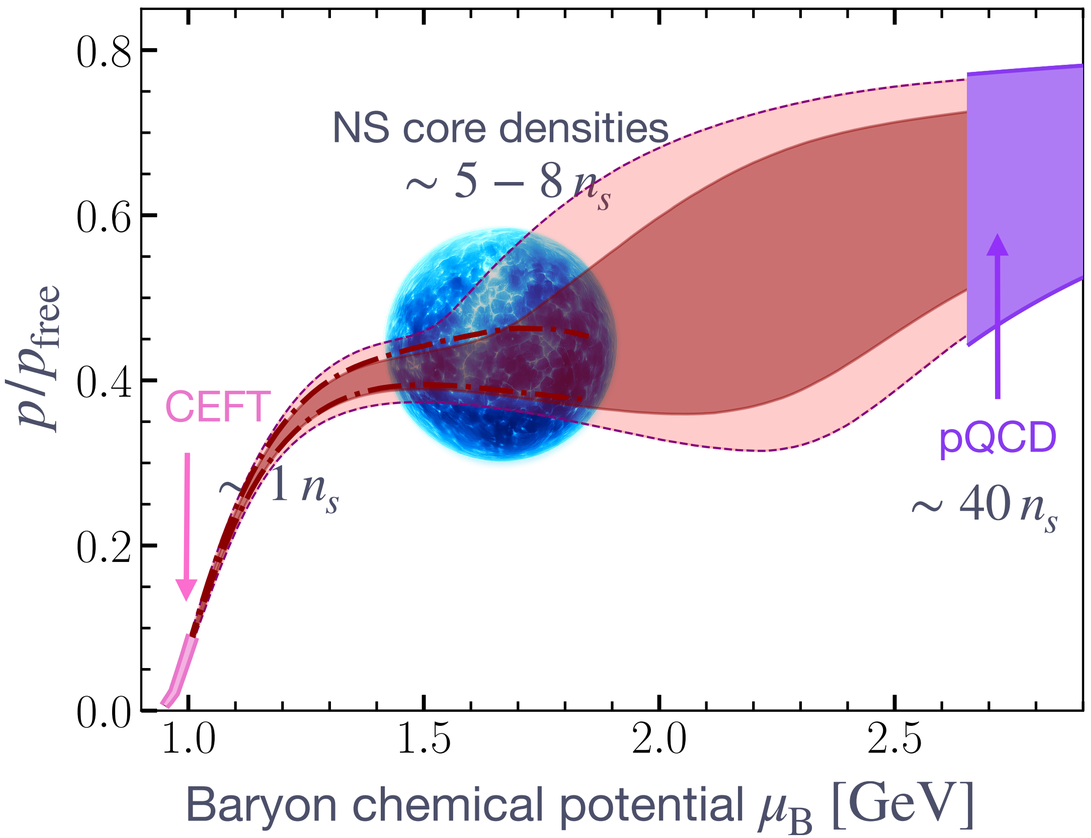

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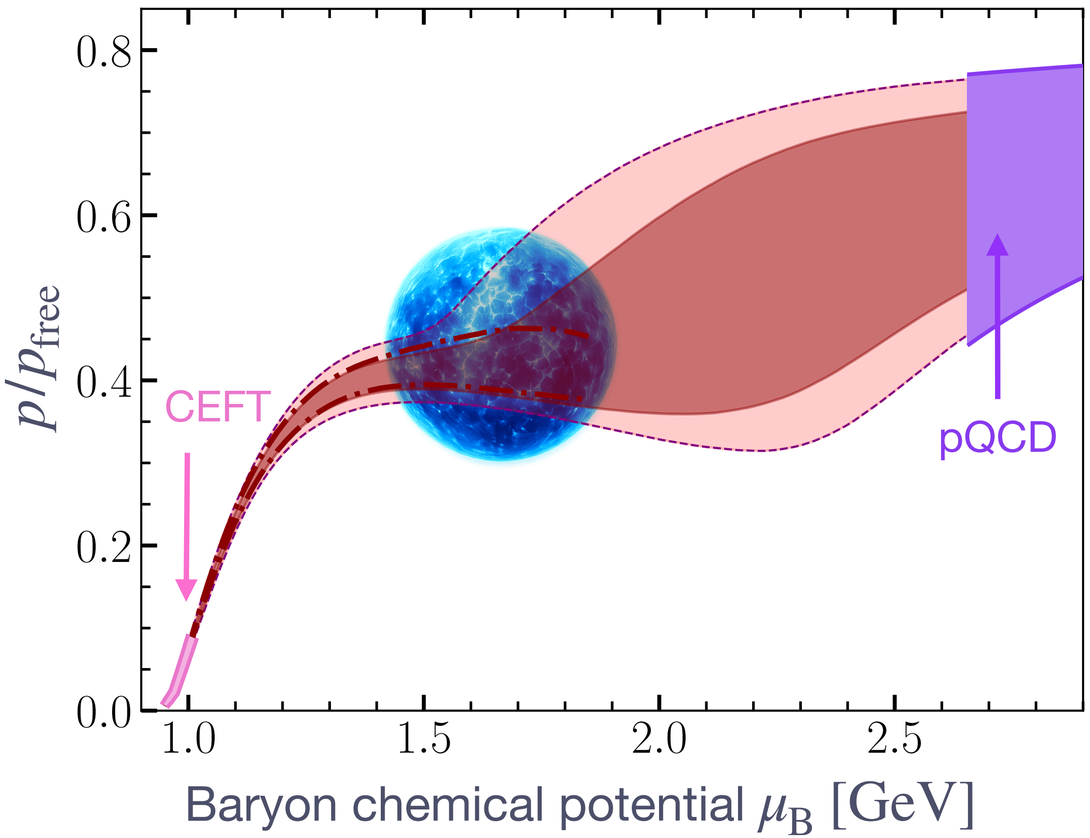

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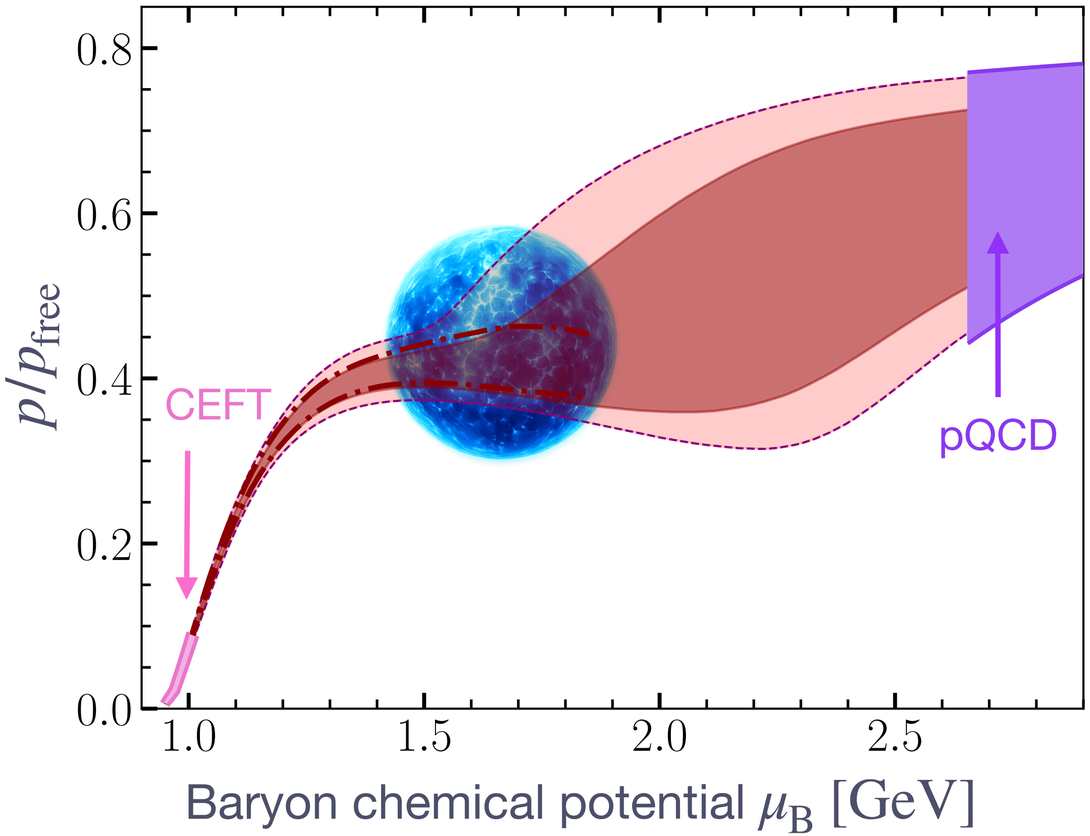


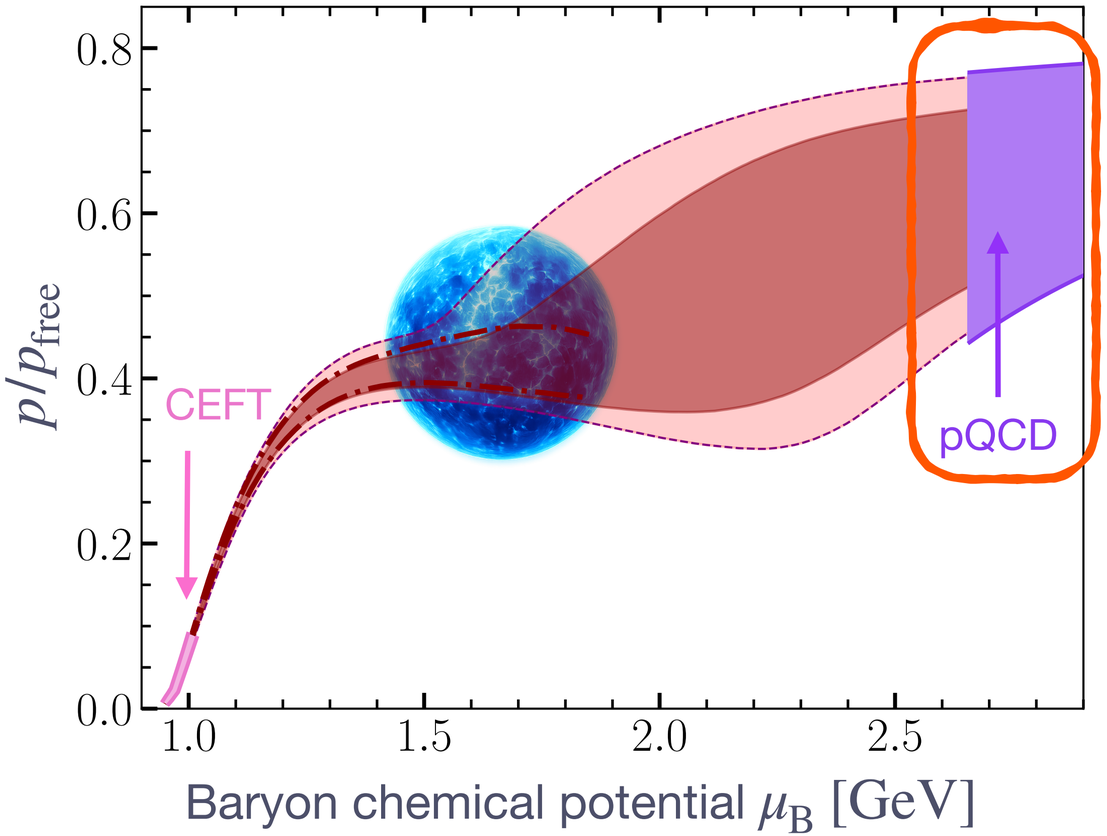
- Model-independent Bayesian inference:

  Annala, Gorda, Hirvonen, Komoltsev, Kurkela, Nättilä, Vuorinen Nat. Comm. 14 (2023)

  (and many more refs…)

Dense QCD matter EoS

This talk



- Model-independent Bayesian inference:

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  (and many more refs…)

- **Current bottleneck:** pQCD constraint has a large uncertainty

⇒ Push pQCD result to higher order to decrease the uncertainty

Quark matter EoS from pQCD

Quark matter EoS from pQCD

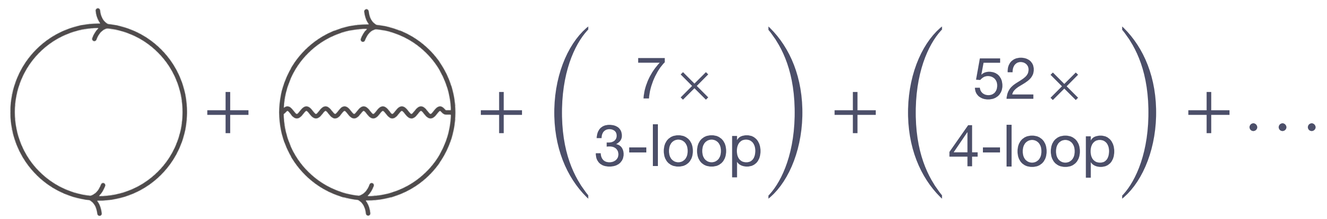
- **Quantity:** massless unpaired 3-flavor QCD pressure at $T = 0$ and $\mu_{\mathrm{B}} > 0$

Quark matter EoS from pQCD

- **Quantity:** massless unpaired 3-flavor QCD pressure at $T = 0$ and $\mu_{\mathrm{B}} > 0$
- **Framework:** small-α_s expansion of QCD partition function

$$p(\mu_{\mathrm{B}}) \sim \ln Z = \ln \int \mathscr{D}\overline{\psi}\, \mathscr{D}\psi\, \mathscr{D}A\, e^{-S_{\mathrm{QCD}}[\overline{\psi},\psi,A]}$$

$$= \quad p_{\mathrm{free}} \quad + \quad p_1\alpha_s \quad + \quad p_2\alpha_s^2 \quad + \quad p_3\alpha_s^3 \quad + \ldots$$



“LO” “NLO” “N2LO” “N3LO”

Chemical potential μ shifts fermionic loop energies

$$k_i^0 \to k_i^0 + i\mu$$

Soft resummations

Caveat starting at 3-loops α_s^2

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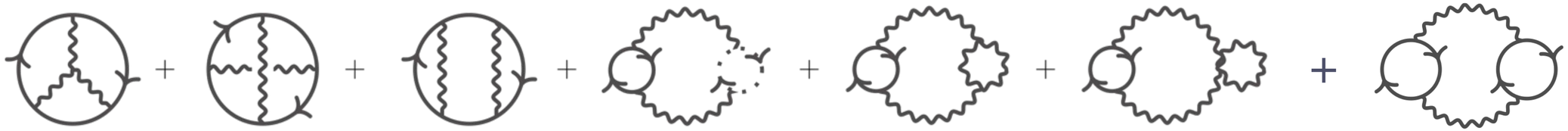
Caveat starting at 3-loops α_s^2

1. **Hard** scale $\sim \mu_{\mathrm{B}}$: Full theory diagrams within naive α_s expansion



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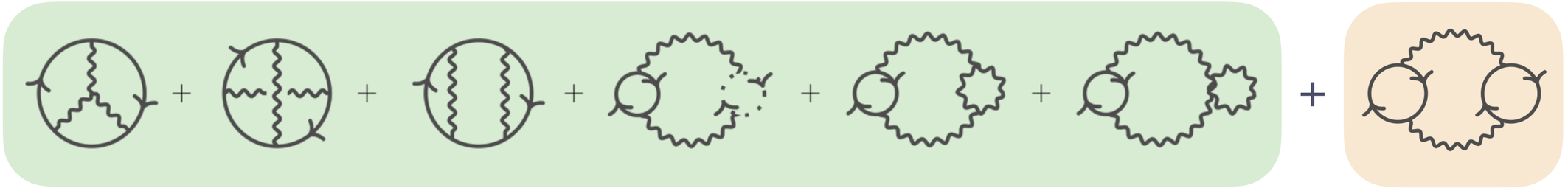

IR finite

IR divergent

Soft resummations

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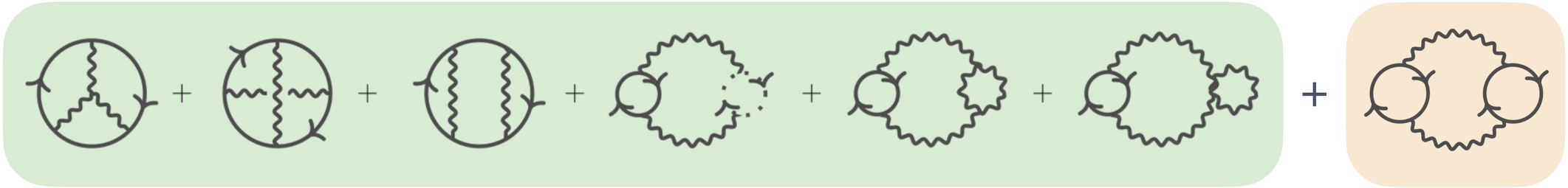

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2. **Soft** scale $\sim \sqrt{\alpha_s}\mu_{\mathrm{B}}$: Hard Thermal Loop (HTL) effective theory **Braaten, Pisarski** NPB 337 (1990)

Soft resummations

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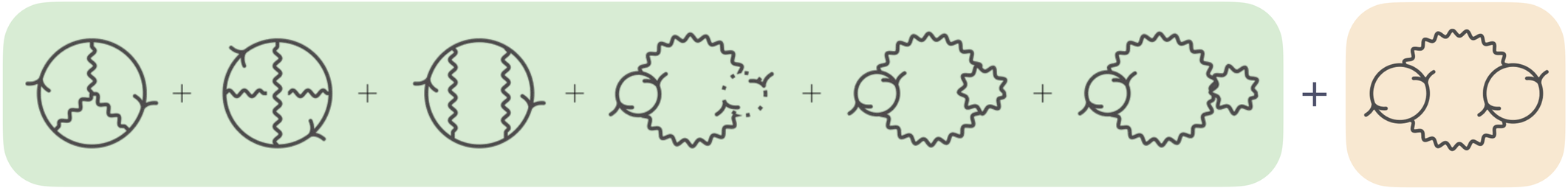

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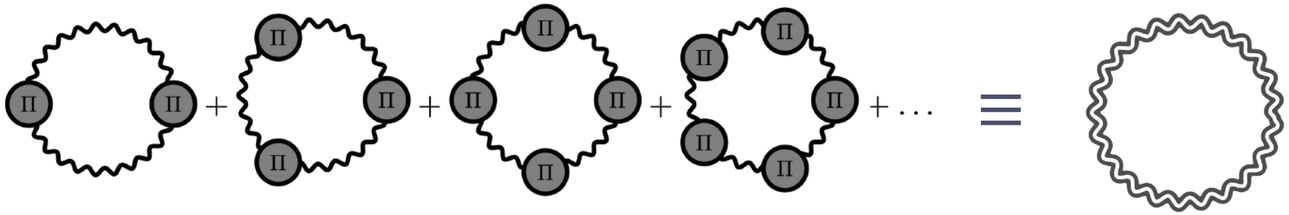
medium-induced gluon mass $\sim \sqrt{\alpha_s}\mu_\mathrm{B}$



∞ number of diagrams contribute at α_s^2

Soft resummations

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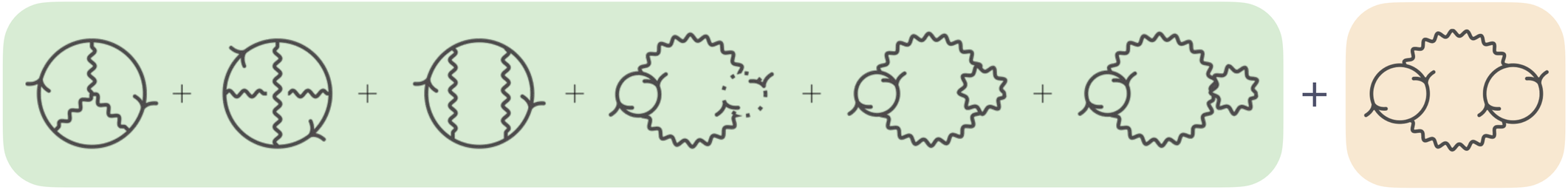

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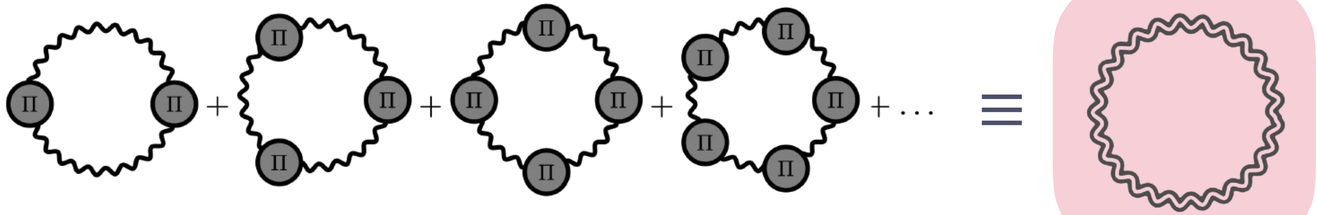

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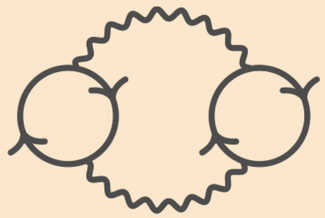
UV divergent

Soft resummations

Caveat starting at 3-loops α_s^2

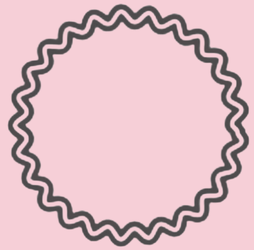
Hard



IR divergent

+

Soft



UV divergent

$$\sim \ln \frac{\sqrt{\alpha_s}\mu_{\mathrm{B}}}{\mu_{\mathrm{B}}} \sim \ln \alpha_s$$

Divergences cancel but non-analytic terms remain!

Quark matter EoS from pQCD

History and progress

$$\frac{p(\mu_{\mathrm{B}})}{p_{\mathrm{free}}} = 1 + p_{1,1}\alpha_s + (p_{2,1}\ln\alpha_s + p_{2,0})\alpha_s^2 + (p_{3,2}\ln^2\alpha_s + p_{3,1}\ln\alpha_s + p_{3,0})\alpha_s^3 + \ldots$$

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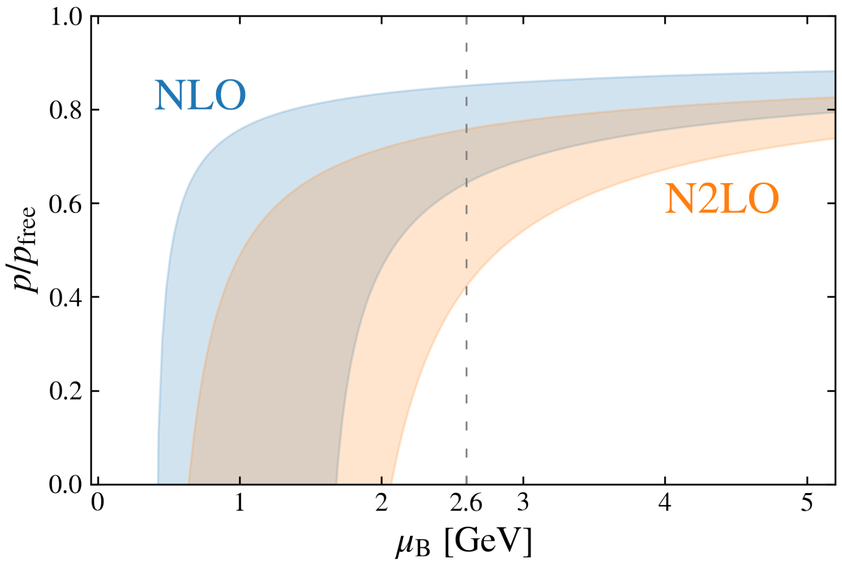
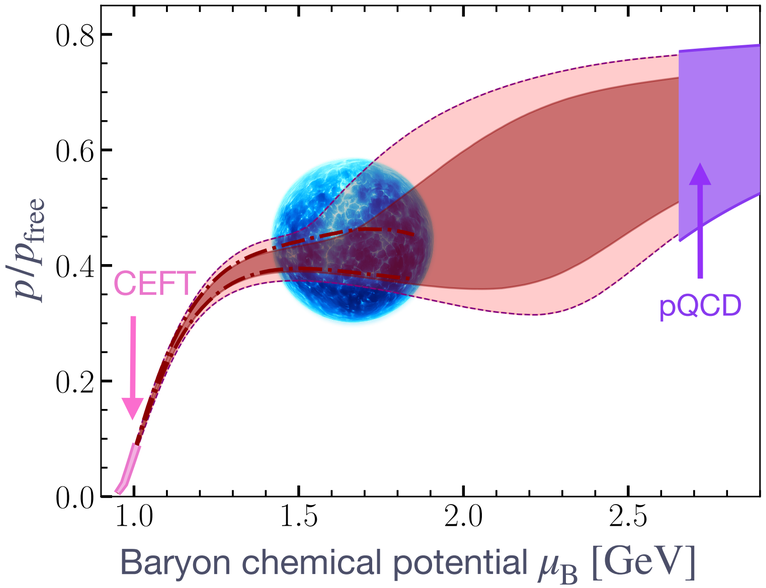
N2LO (3-loop): **Freedman, McLerran** PRD 16 (1977)





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N2LO (3-loop): **Freedman, McLerran** PRD 16 (1977)

Improvements:

- $\overline{\mathrm{MS}}$ scheme and analytic result: **Vuorinen** PRD 68 (2003)
- Quark mass: **Kurkela, Romatschke, Vuorinen** PRD 81 (2010)
  Gorda, Säppi PRD 105 (2022)
  Fernandez, Kneur PRD 111 (2025)
- Temperature: **Kurkela, Vuorinen** PRL 117 (2016)

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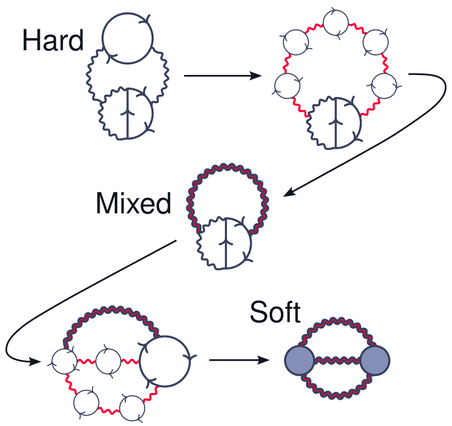
N3LO (4-loop):



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N3LO (4-loop):

- HTL-resummed **soft** and **mixed** contributions:
  - $p_{3,2}$: **Gorda, Kurkela, Säppi, Romatschke, Vuorinen** PRL 121 (2018)
    Gorda, Kurkela, Paatelainen, Säppi, Vuorinen PRL 127 & PRD 104 (2021)
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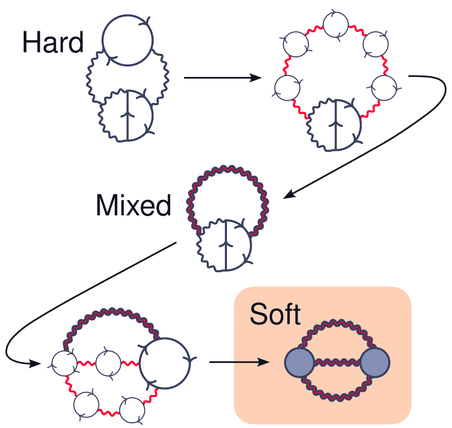

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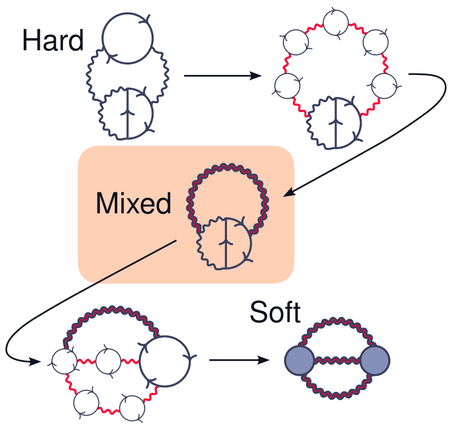
  - $p_{3,1}$: **Gorda, Paatelainen, Säppi, KS** PRL 131 (2023)



Brings result on par with its high-T counterpart:
Kajantie et al. PRD 67 (2003)

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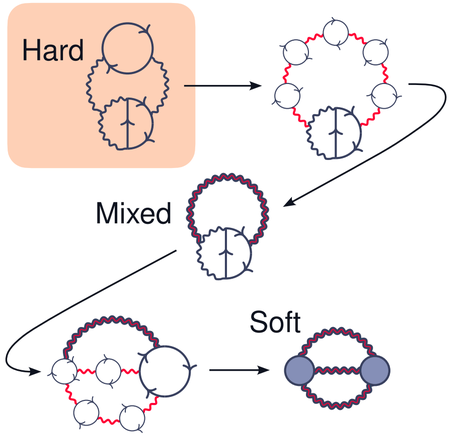
  - $p_{3,1}$: **Gorda, Paatelainen, Säppi, KS** PRL 131 (2023)
- **Hard** contributions:
  - $p_{3,0}$: known up to a single number c_0



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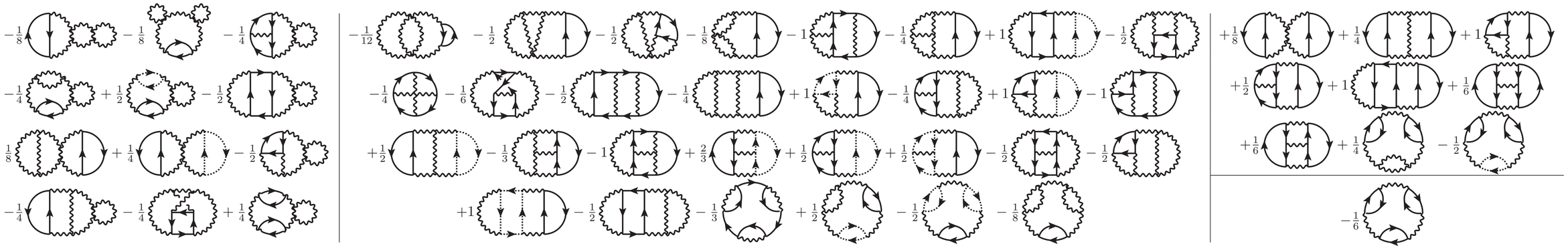
Four-loop computation of cold QM EoS

Huge challenge

Evaluation of 52 four-loop QCD Feynman diagrams with nonzero μ_{B}



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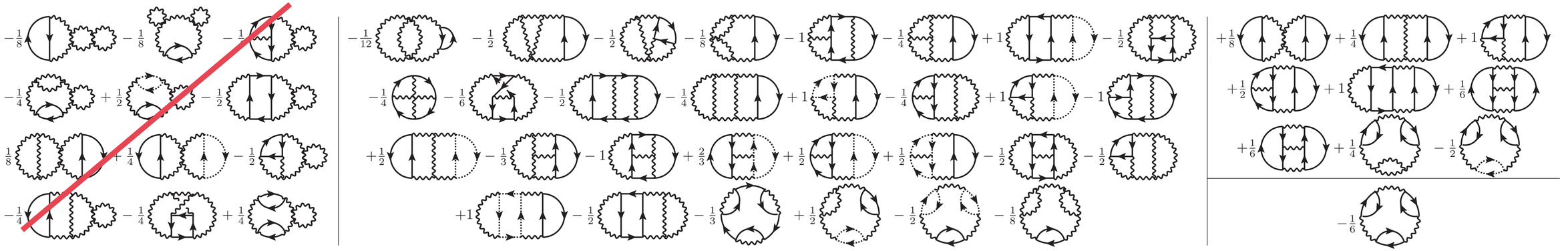

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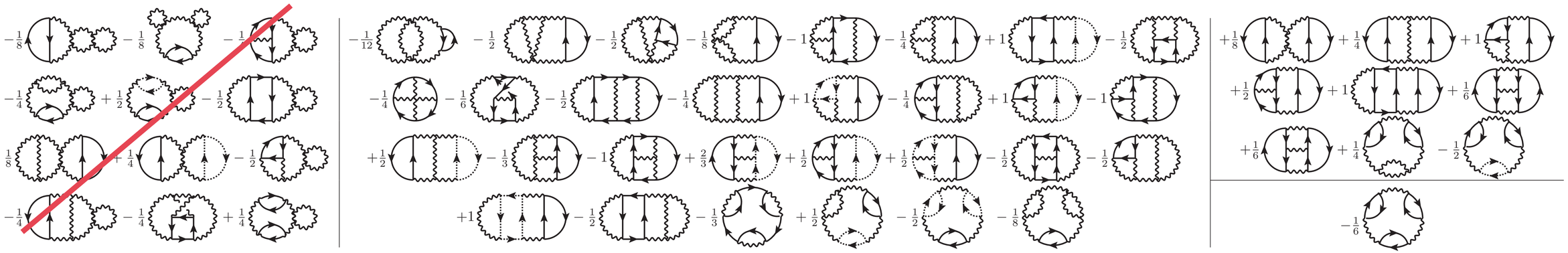


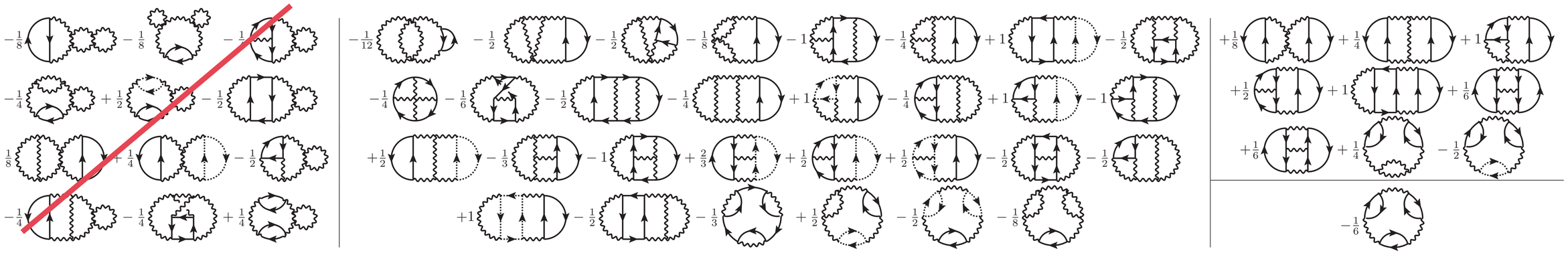
IR finite, UV div

IR and UV div

$$c_0 \sim \text{(30 diagrams)}\, N_f + \text{(9 diagrams)}\, N_f^2 + \text{(1 diagram)}\, N_f^3$$

Huge challenge

Evaluation of 52 four-loop QCD Feynman diagrams with nonzero μ_{B}



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UV div's: renormalization of α_s

IR div's: should cancel against soft HTL parts

Organization

Kärkkäinen, Navarrete, Nurmela, Paatelainen, KS, Vuorinen PRL 135 (2025)



EDITORS' SUGGESTION

Quark Matter at Four Loops: Hardships and How to Overcome Them

Aapeli Kärkkäinen, Pablo Navarrete, Mika Nurmela, Risto Paatelainen, Kaapo Seppänen, and Aleksi Vuorinen

Phys. Rev. Lett. **135**, 021901 (2025) - Published 7 July, 2025

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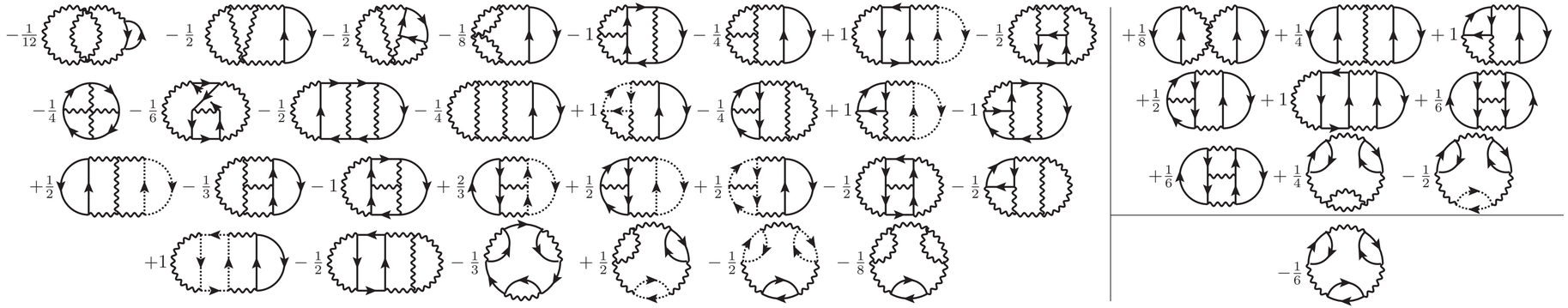

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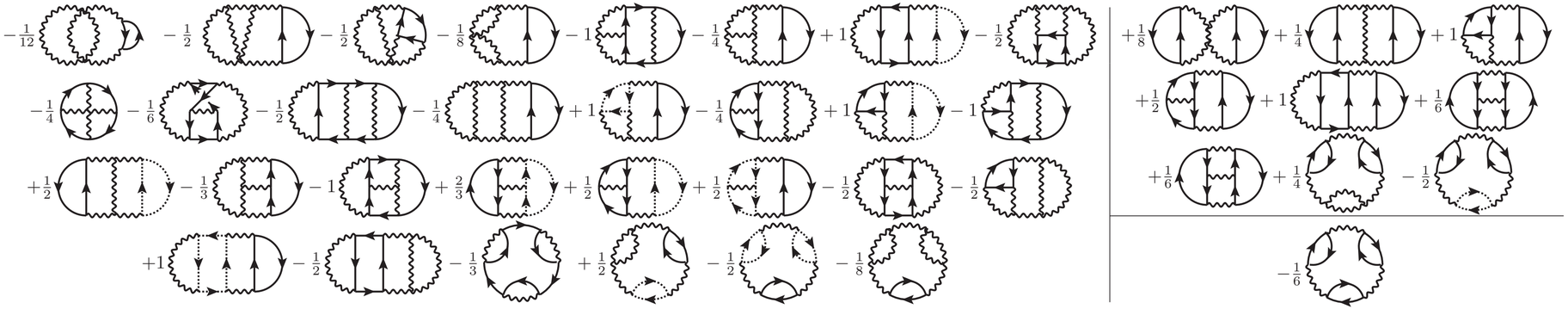

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Organization

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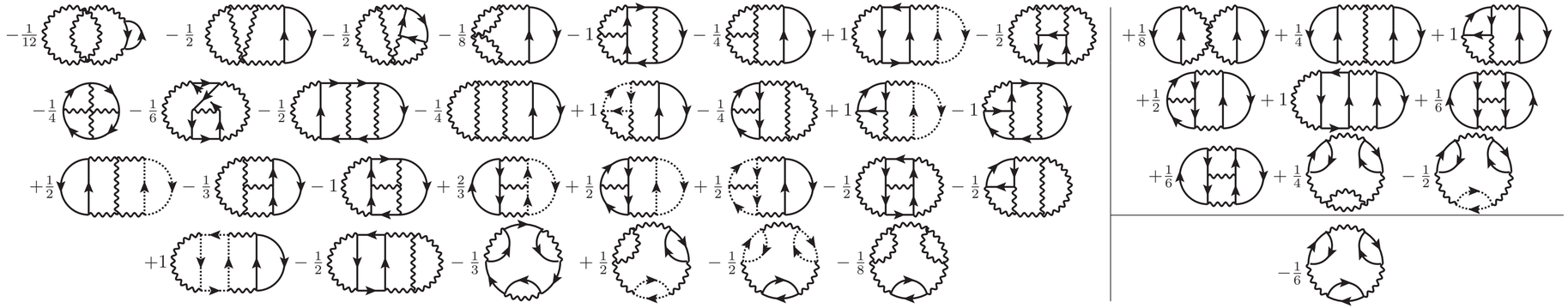

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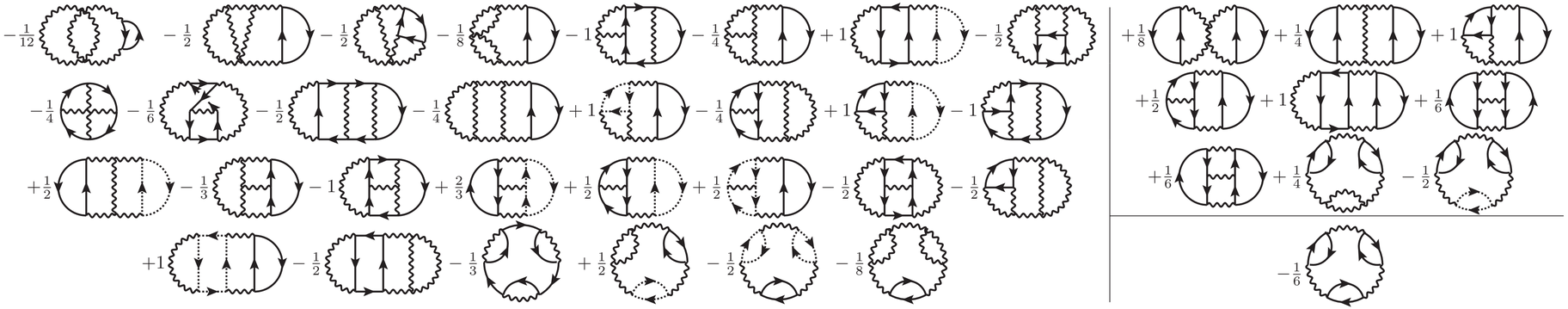

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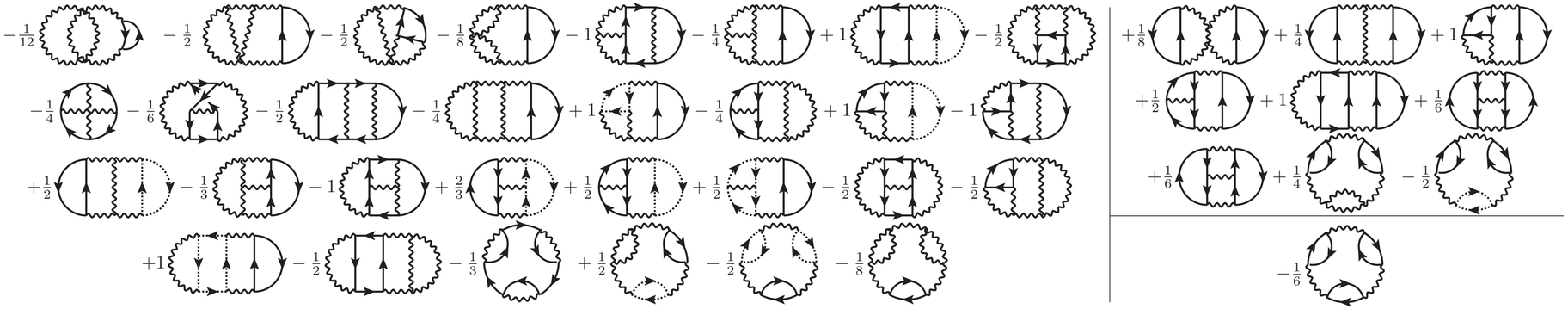

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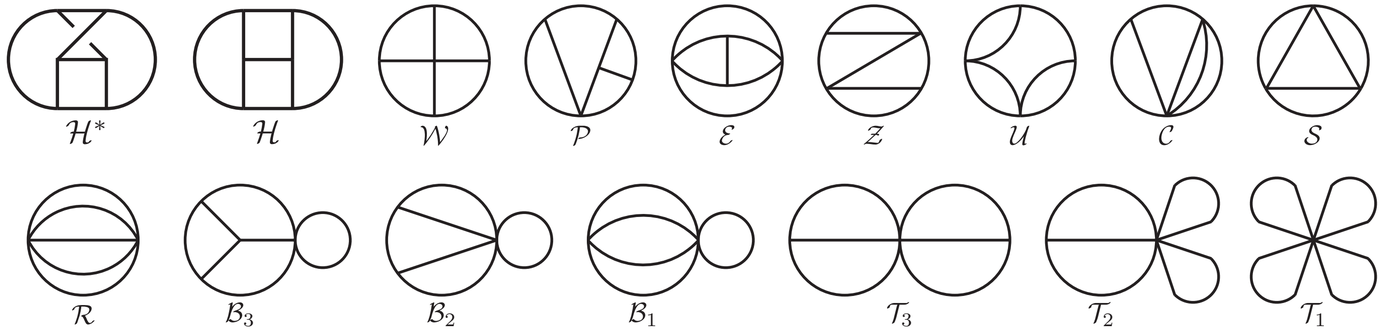
Remaining computational barrier

Computation of $O(100)$ 4-loop master integrals



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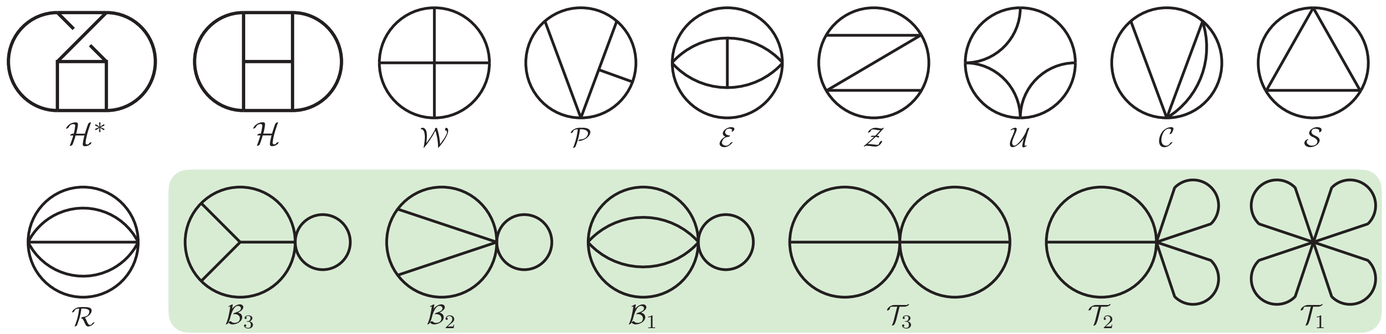

"Difficulty":

- **"Easy":** factorized topologies

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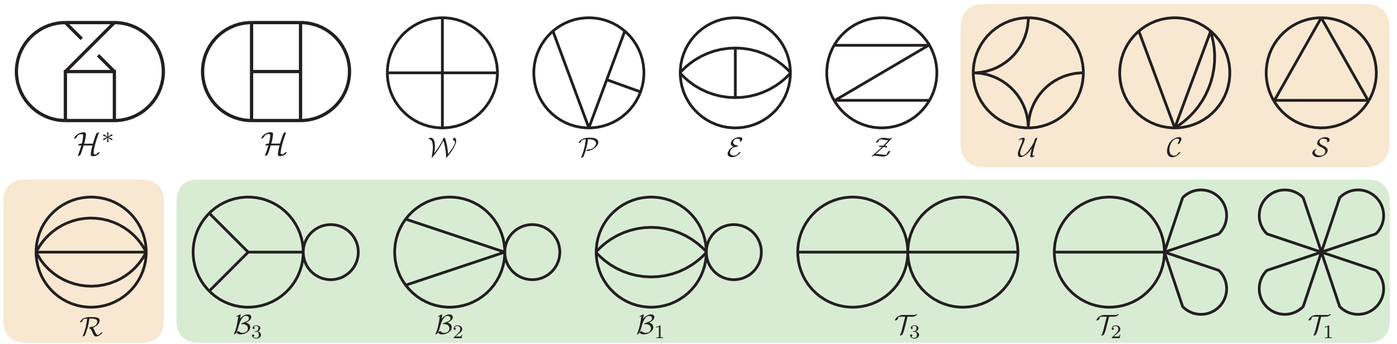

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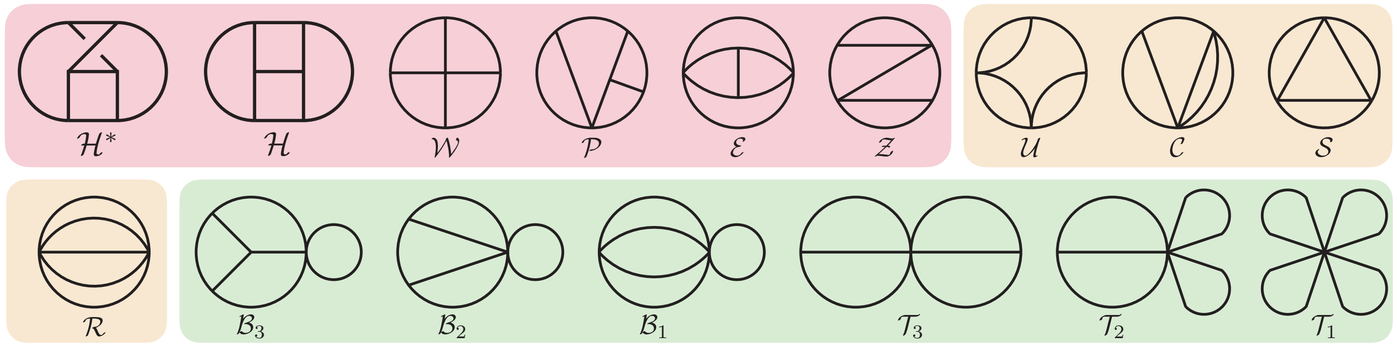

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- **Hard:** traditional analytic methods struggle ? ? ?

Limited analytic tools at $\mu > 0$

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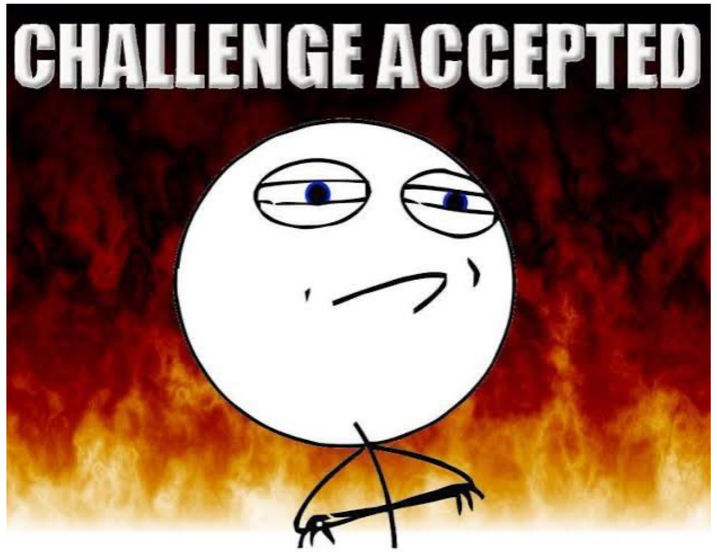

Numerical Loop-Tree Duality (LTD)

Inspiration from collider physics

Manifest the cancellation of divergences early on at integrand level
$\Rightarrow$ direct numerical integration in momentum space

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1. Subtract divergences to make integrand locally finite in $D = 4$

$$\int_K f(K) = \int_K \underbrace{[f(K) - \mathrm{CT}(K)]}_{\equiv f_{\mathrm{fin}}(K)} + \int_K \mathrm{CT}(K)$$

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3. Using Monte Carlo integration, evaluate $\vec{k}$ integrals numerically

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$$\int \frac{\mathrm{d}^3\vec{k}}{(2\pi)^3} f^{3\mathrm{d}}_{\mathrm{fin}}(\vec{k}) \approx \frac{1}{N} \sum_{i=1}^{N} J_i f^{3\mathrm{d}}_{\mathrm{fin}}(\vec{k}_i)$$

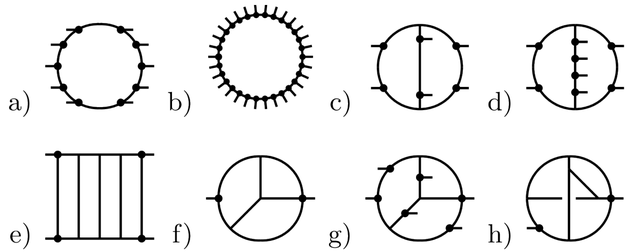
Inspiration from collider physics

Loop-Tree Duality

In collider context:

- Highly developed tool for numerical evaluation of multi-loop multi-scale scattering amplitudes

Capatti, Hirschi, Kermanschah, Ruijl PRL 123 (2019)



G	Reference		Numerical LTD		N $[10^6]$	$[\mu s]$
a)*	[33]	$\mathrm{i}\,4.31638 \cdot 10^{-7}$	$\mathrm{i}\,4.31637(19) \cdot 10^{-7}$		110	1.1
b)	[33]	$\mathrm{i}\,0.358640$	$\mathrm{i}\,0.358646(29)$		210	5.9
c)	[7]	$1.1339(5) \cdot 10^{-4}$	$1.133719(58) \cdot 10^{-4}$		5500	2.5
c)*	[7]	$4.398(1) \cdot 10^{-8}$	$4.39825(17) \cdot 10^{-8}$		5500	2.5
d)*	[7]	$2.409(1) \cdot 10^{-8}$	$2.40869(27) \cdot 10^{-8}$		5500	3.5
e)	[34]	$-1.433521 \cdot 10^{-6}$	$-1.4338(18) \cdot 10^{-6}$		1500	27.4
f)	[35]	$\mathrm{i}\,5.26647 \cdot 10^{-6}$	$\mathrm{i}\,5.236(38) \cdot 10^{-6}$		7000	3.3
g)*	[7]	$\mathrm{i}\,1.7790(6) \cdot 10^{-10}$	$\mathrm{i}\,1.77648(48) \cdot 10^{-10}$		22000	11
h)	[35]	$-8.36515 \cdot 10^{-8}$	$-8.309(31) \cdot 10^{-8}$		7000	15.8

Dense Loop-Tree Duality

Natural generalization



Kaapo Seppänen



Pablo Navarrete

Extends to nonzero μ naturally

$$\int \frac{\mathrm{d}^3\vec{k}}{(2\pi)^3} \int \frac{\mathrm{d}k^0}{2\pi} f(k^0, \vec{k}) \longrightarrow \int \frac{\mathrm{d}^3\vec{k}}{(2\pi)^3} \int \frac{\mathrm{d}k^0}{2\pi} f(k^0 + i\mu, \vec{k})$$

LTD dLTD

$\Rightarrow$ dense Loop-Tree Duality (dLTD)

Navarrete, Paatelainen, KS PRD 110 (2024)

Dense Loop-Tree Duality

Fully automated pipeline

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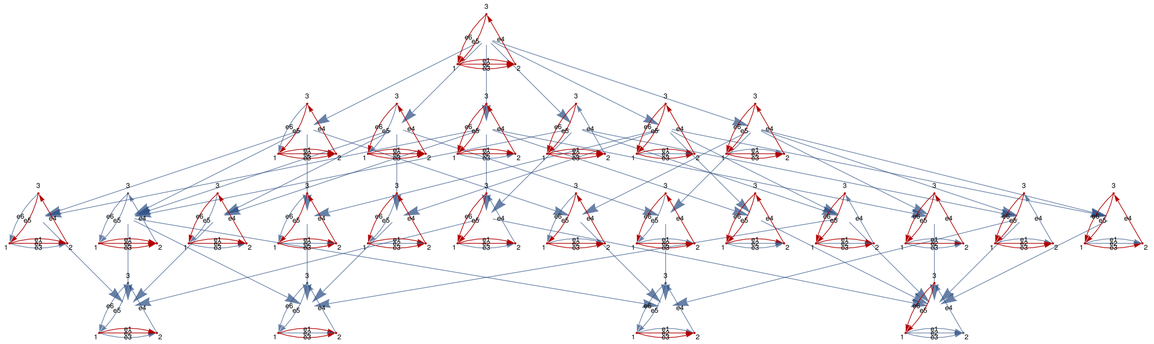
Fully automated pipeline

1. All-order local UV subtraction via BPHZ R-operation

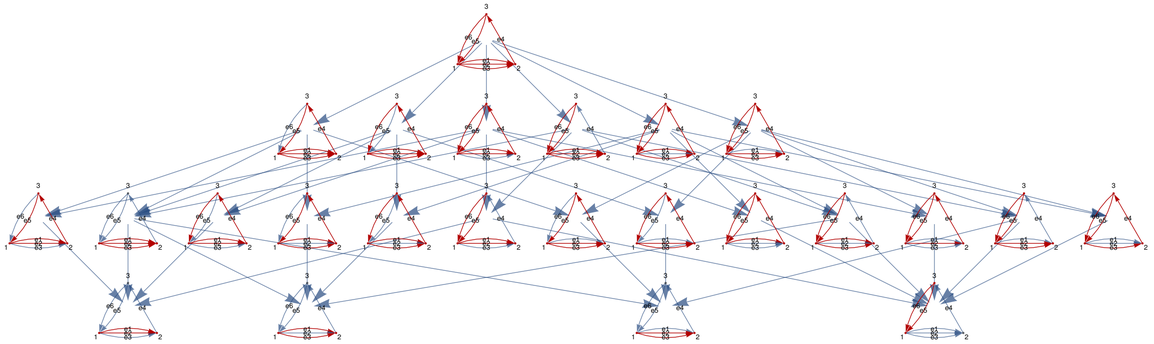


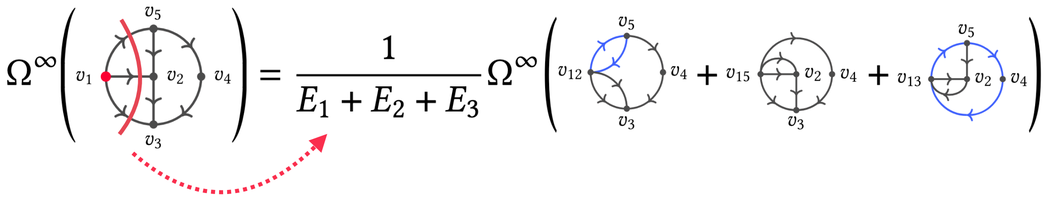
Dense Loop-Tree Duality

Fully automated pipeline

1. All-order local UV subtraction via BPHZ R-operation

2. All-order automated analytic energy integration (=LTD)

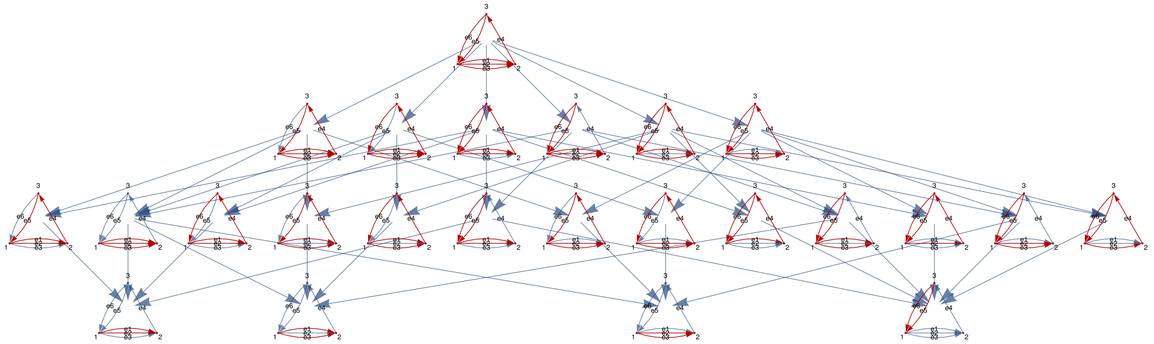




Dense Loop-Tree Duality

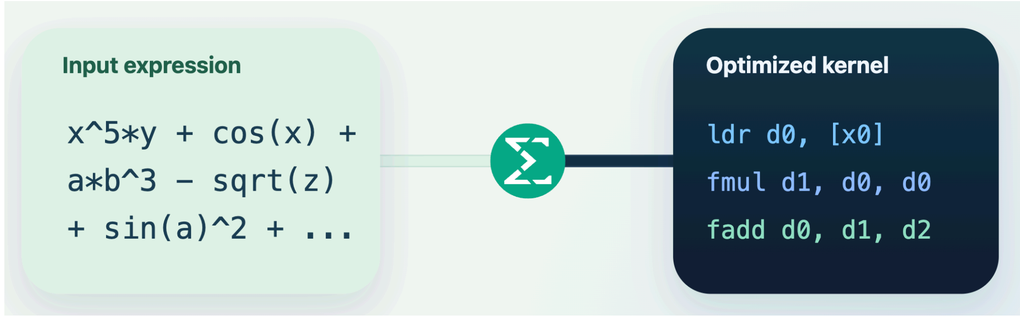
Fully automated pipeline

1. All-order local UV subtraction via BPHZ R-operation
2. All-order automated analytic energy integration (=LTD)
3. Monte Carlo sampling using efficient compiled Symbolica evaluators



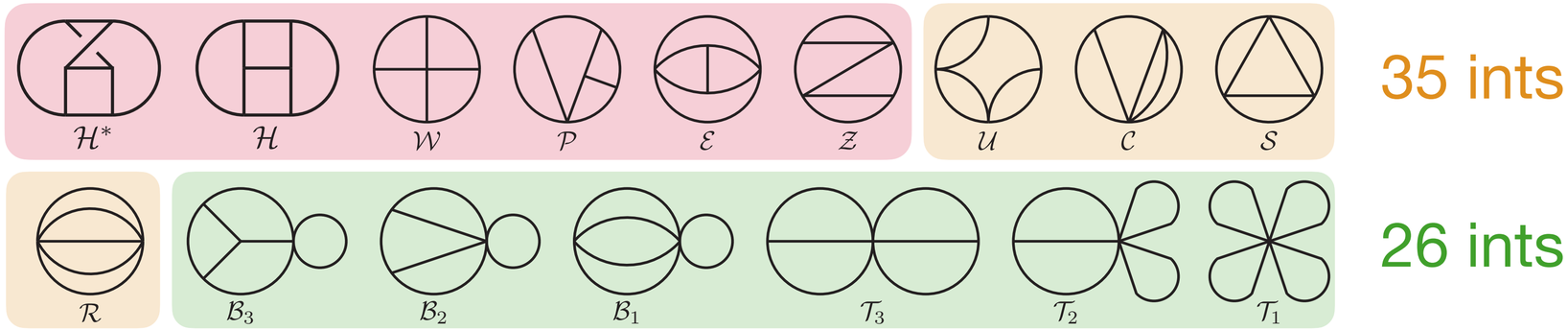
$$\Omega^{\infty}\left(\begin{array}{c} v_5 \\ v_1 \quad v_2 \quad v_4 \\ v_3 \end{array} \right) = \frac{1}{E_1 + E_2 + E_3} \Omega^{\infty}\left(\begin{array}{c} v_5 \\ v_{12} \quad v_4 \\ v_3 \end{array} + \begin{array}{c} v_5 \\ v_{15} \quad v_2 \quad v_4 \\ v_3 \end{array} + \begin{array}{c} v_5 \\ v_{13} \quad v_2 \quad v_4 \end{array} \right)$$

Symbolica



Evaluation of 4-loop masters

56 ints



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Evaluation of 4-loop masters

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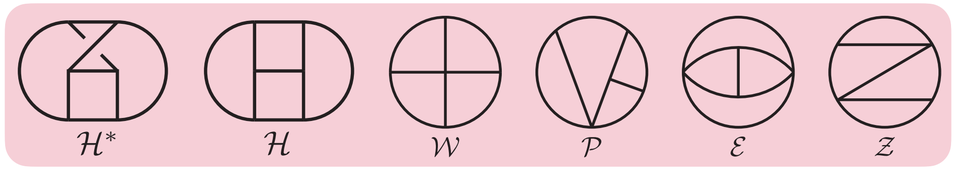

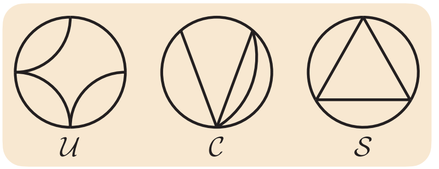

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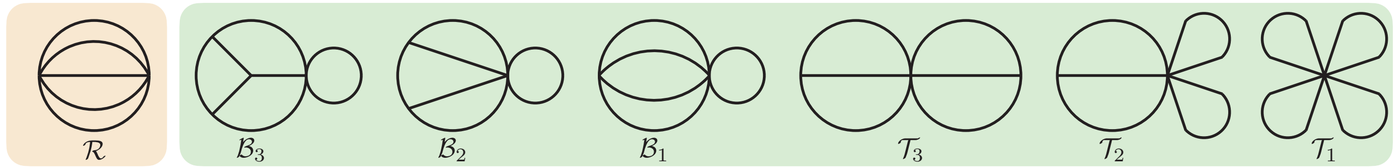

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- All
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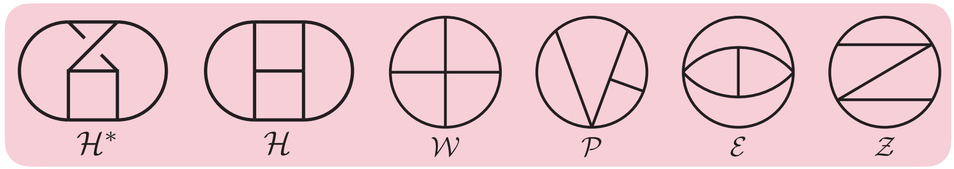

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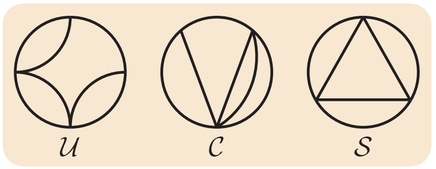

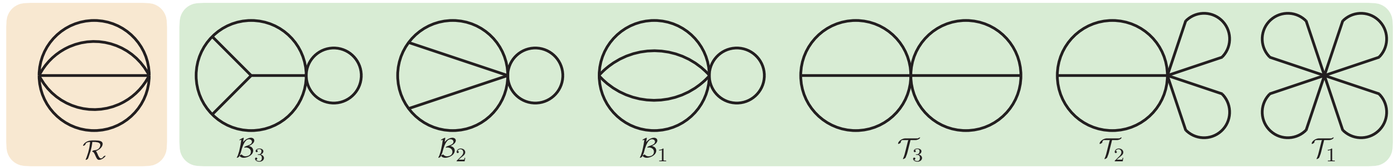

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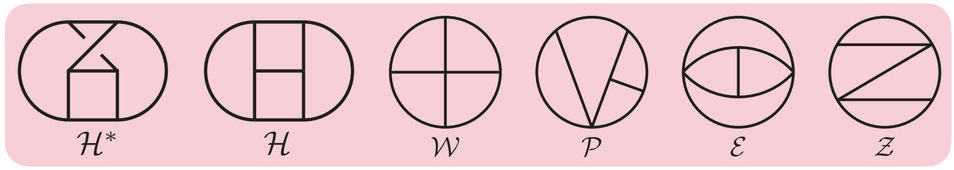

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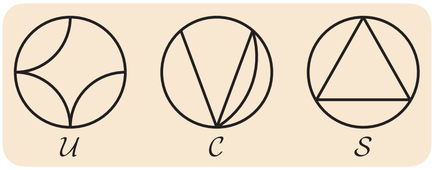

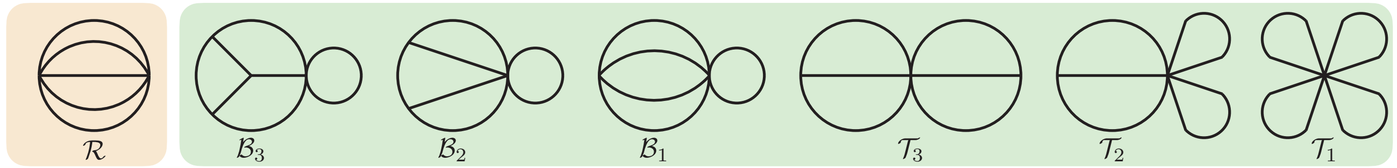

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CPU-hours: $\sim 10^6$

MC samples: $\sim 10^{11}$



CSC

Evaluation of 4-loop masters

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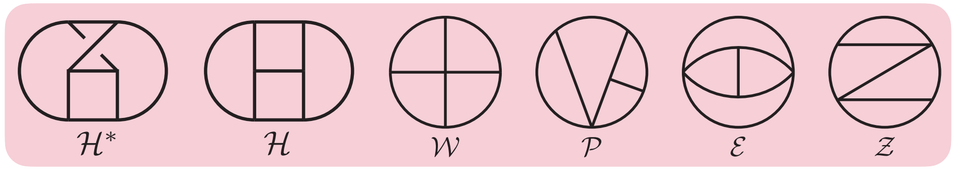

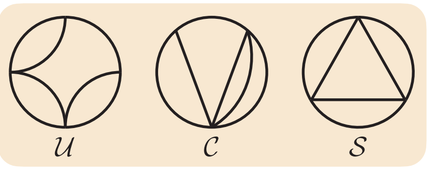

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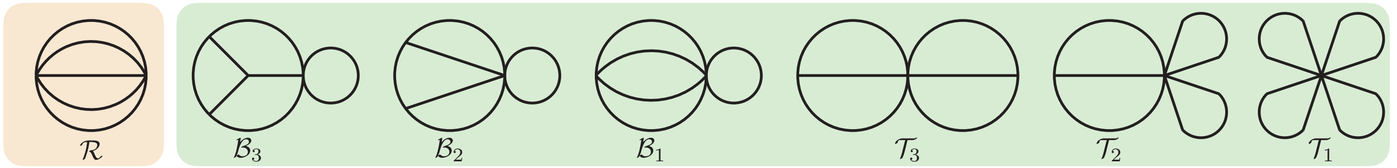

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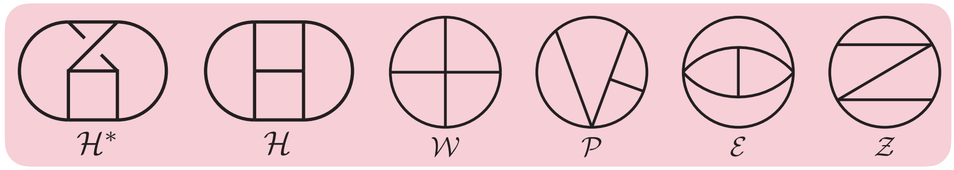
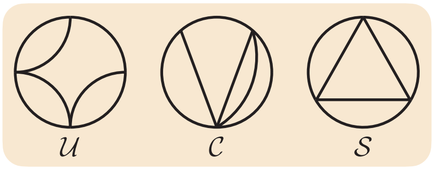
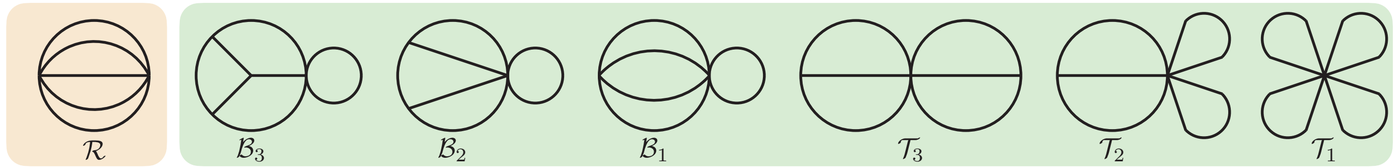
56 ints





35 ints



26 ints

Cutting rules:

- All
- IR div. of

dLTD:

- All
- IR fin. of

Cross-checks (IBP, RG inv., etc.)

CPU-hours: $\sim 10^6$
MC samples: $\sim 10^{11}$

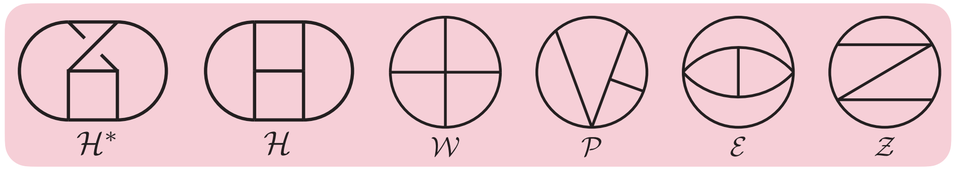


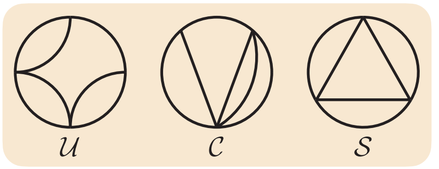
C S C

c_0 computed
up to $\sim 1\,\%\,\sigma$

Evaluation of 4-loop masters

56 ints





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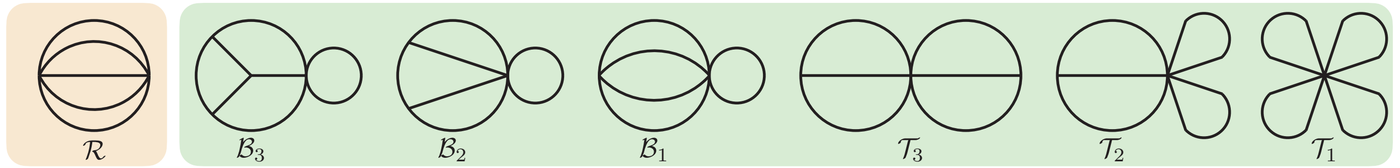

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CSC



QED limit

$$\frac{p_{\mathrm{QED}}(\mu_e)}{p_{\mathrm{free}}} = 1 + p_{1,1}\left(\frac{\alpha_e}{\pi}\right) + \left[p_{2,1}\ln\left(N_f\frac{\alpha_e}{\pi}\right) + p_{2,0}\right]\left(\frac{\alpha_e}{\pi}\right)^2 + \left[p_{3,2}\ln^2\left(N_f\frac{\alpha_e}{\pi}\right) + p_{3,1}\ln\left(N_f\frac{\alpha_e}{\pi}\right) + p_{3,0}\right]\left(\frac{\alpha_e}{\pi}\right)^3 + \ldots$$

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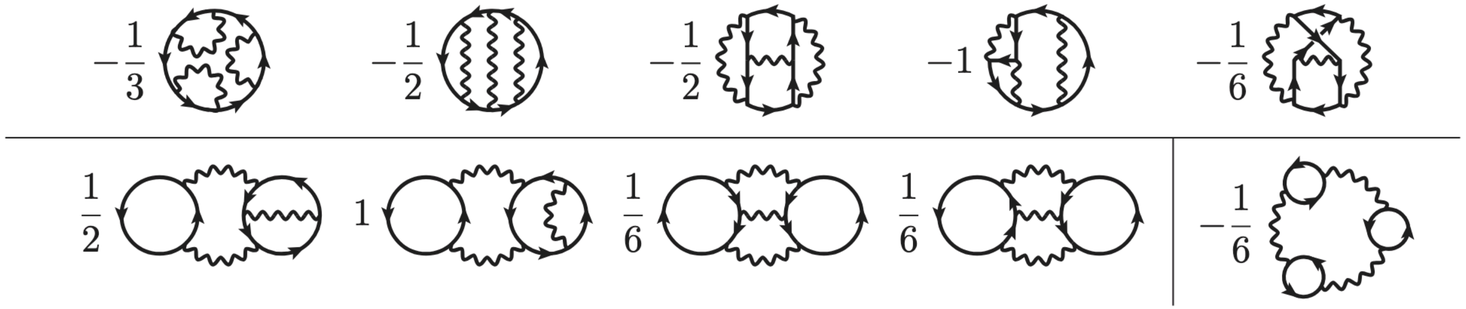

- $p_{3,0}$: hard contribution reduces to 75 4-loop masters

  Kärkkäinen, Navarrete, Nurmela, KS, Abreu, Delto, Kurkela, Paatelainen, Vuorinen 2610.13873

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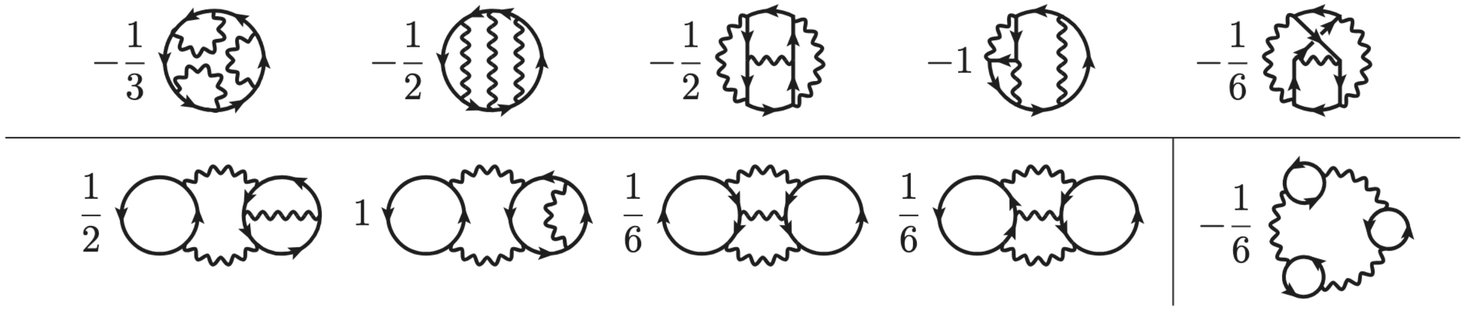

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  Gorda, Kurkela, Österman, Paatelainen, Säppi, Schicho, KS, Vuorinen PRD 107 (2023)

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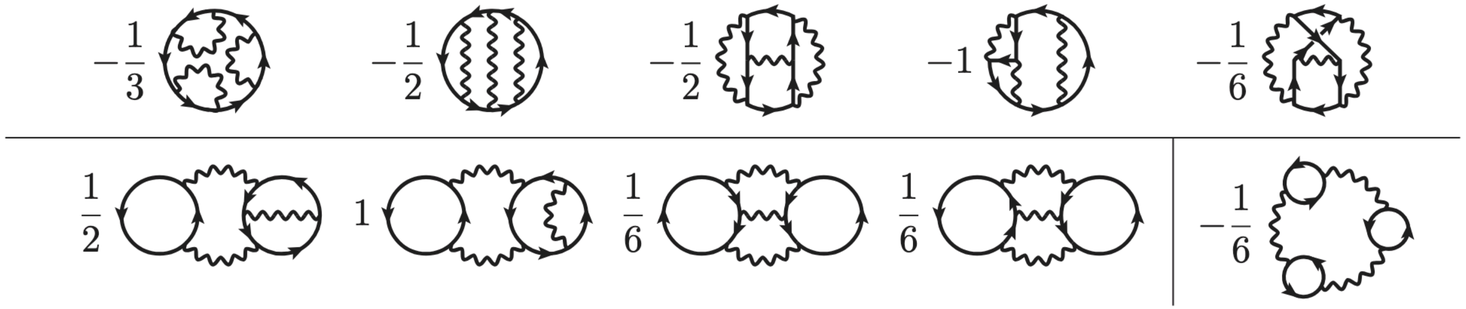

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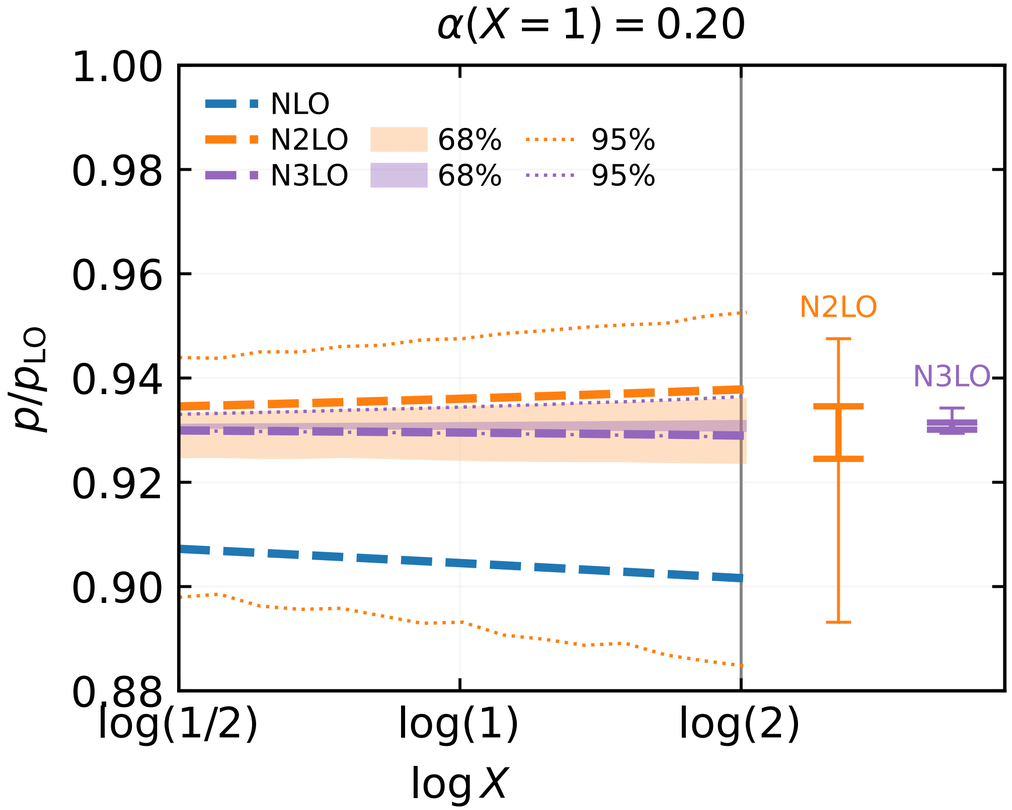
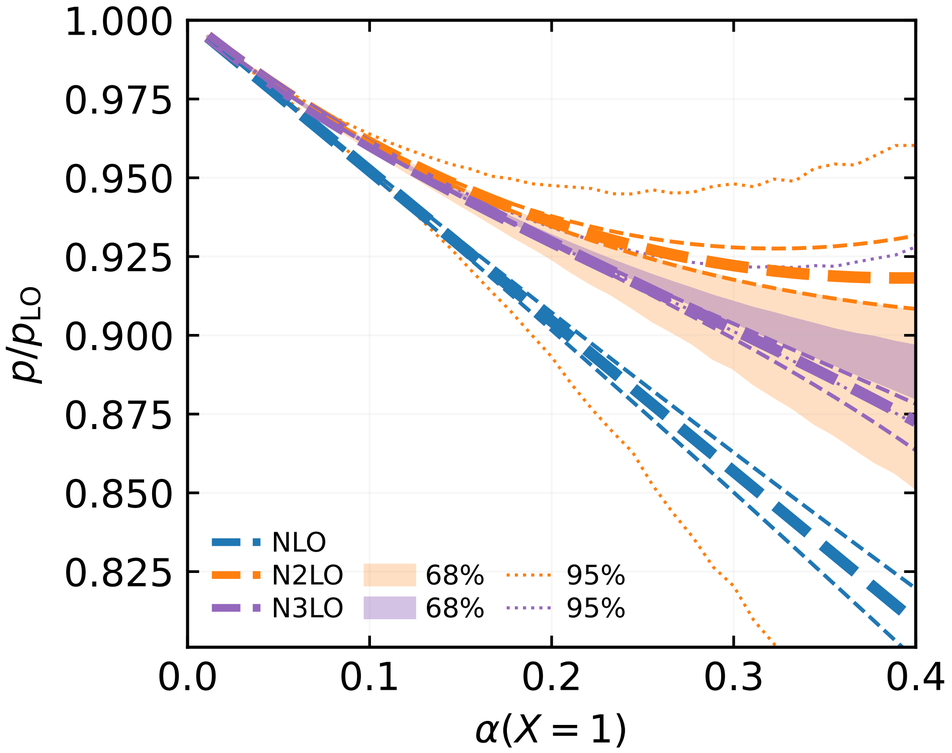
- $p_{3,2}$: vanishes in QED

QED limit





- Bayesian uncertainty estimation of missing higher orders and renormalization scale $X \equiv \overline{\Lambda}/(2\mu)$ dependence à la **Duhr et al.** JHEP 09 (2021)

Thermal Loop-Tree Duality

New precision era in thermal perturbation theory

Thermal LTD

- dLTD extends naturally to finite temperature: $\int \rightarrow \sum\!\!\!\!\!\!\int$



K. Seppänen PhD Thesis (2025)

Thermal LTD

- dLTD extends naturally to finite temperature: $\int \rightarrow \sum\!\!\!\!\!\!\int$
- **NEW**: All-order local UV and IR subtraction at finite T



K. Seppänen PhD Thesis (2025)



Leon Sandbote

Thermal LTD

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The pressure of hot Yang-Mills: QCD without quarks — a theory of interacting gluons

$$\frac{p_{\mathrm{YM}}(T)}{p_{\mathrm{SB}}} = 1 + c_1 g^2 + \cdots + c_6' g^6 \ln g + c_6 g^6 + \cdots$$

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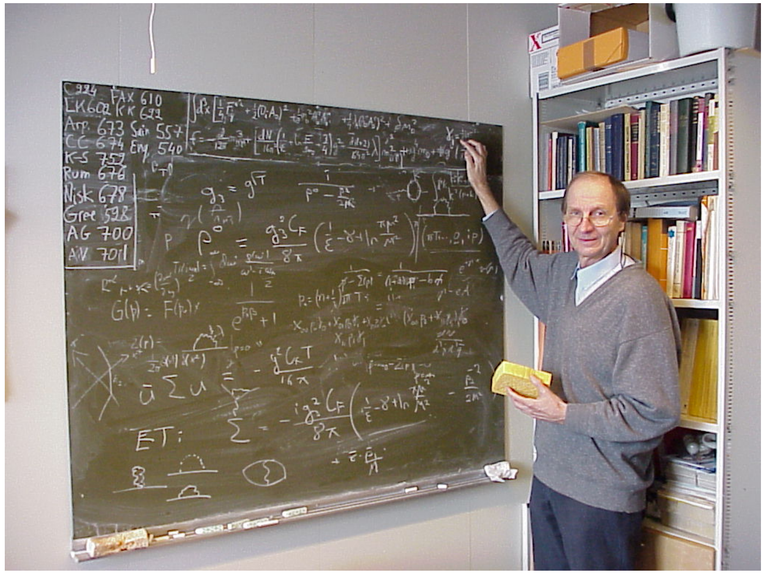

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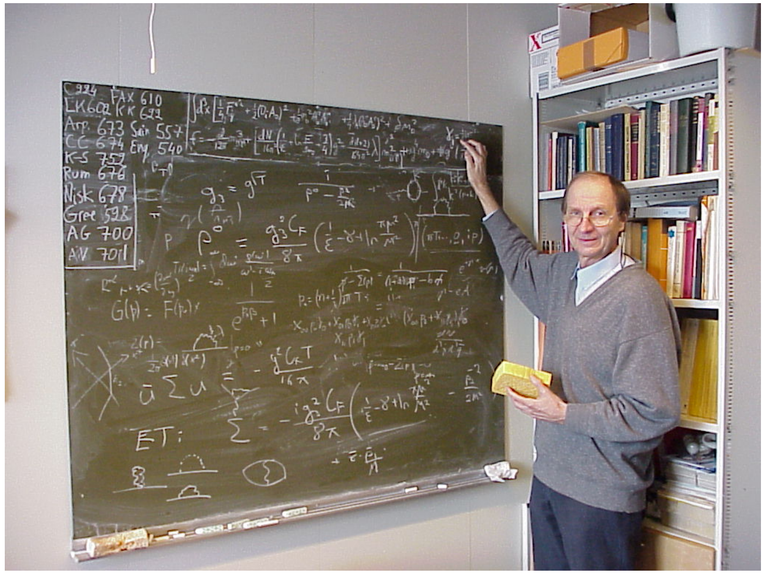
Kajantie, Laine, Rummukainen, Schröder PRD 67 (2003)



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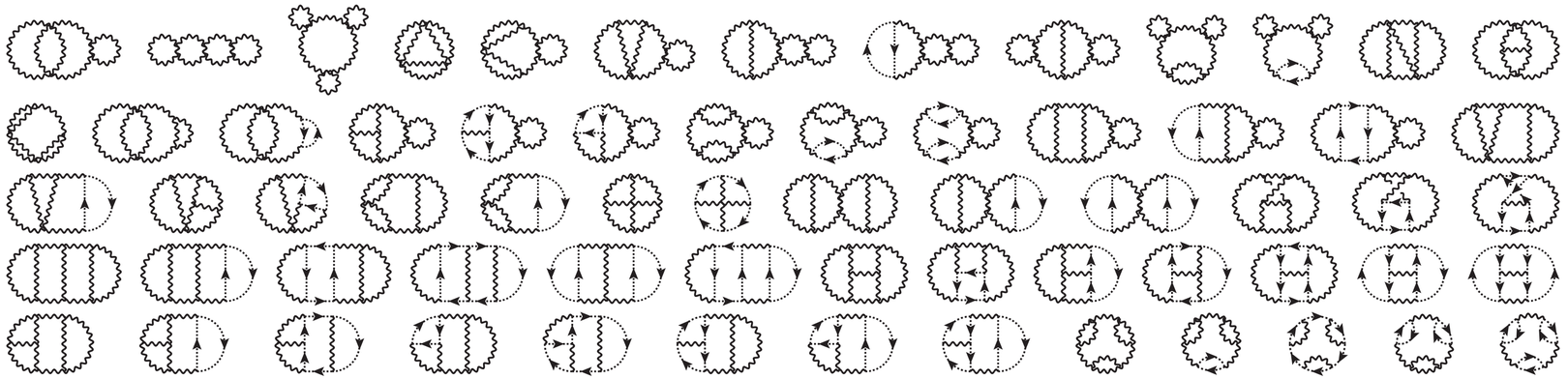
$$\frac{p_{\mathrm{YM}}(T)}{p_{\mathrm{SB}}} = 1 + c_1 g^2 + \cdots + c_6' g^6 \ln g + c_6 g^6 + \cdots$$

Unknown: three decades old open problem

Complete g^6 pressure of hot Yang-Mills

20-year-old problem



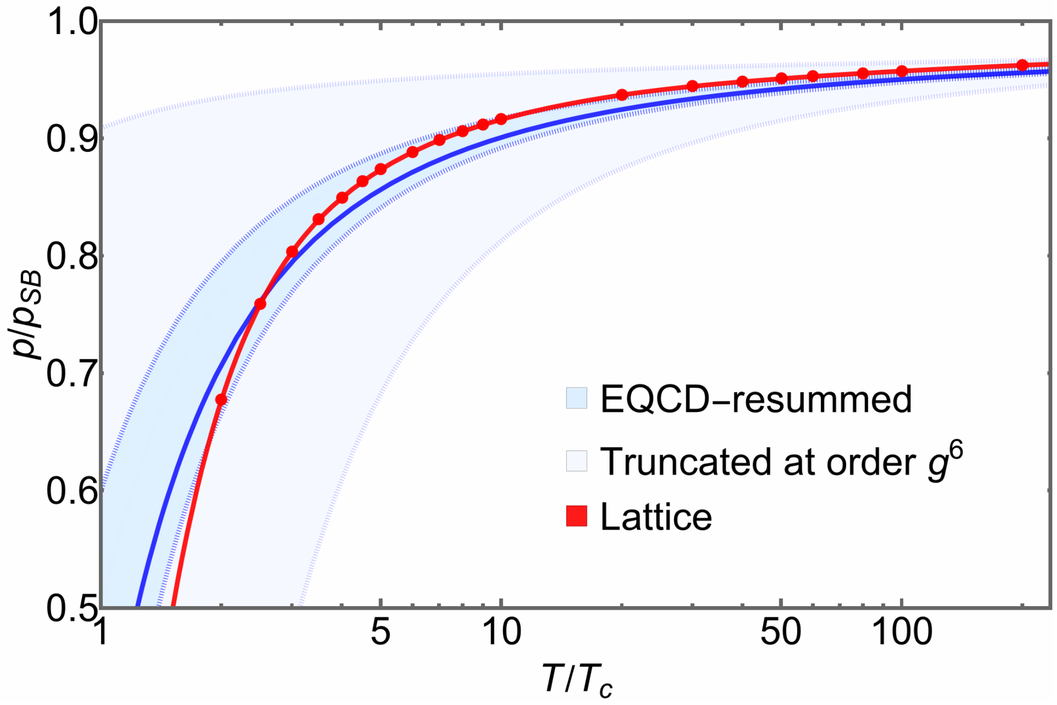
Previously missing hard four-loop contribution at g^6 now computed with LTD:

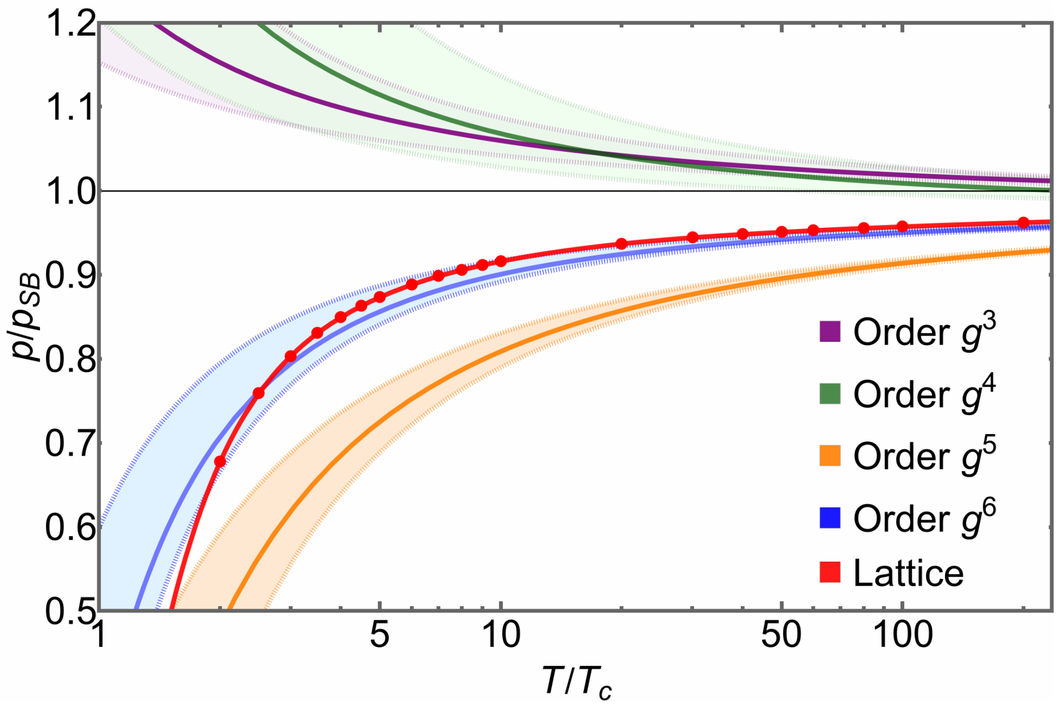
$$\left.\frac{p_{\rm E}}{p_{\rm SB}}\right|_{\hat{g}^6} = \hat{g}^6\left[\frac{c_{-1}(L)}{\epsilon} + c_2 L^2 + c_1 L + \tilde{\beta}_{\rm E1}\right]$$

$$\tilde{\beta}_{\rm E1} = 6236 \pm 40$$

Kajantie, Navarrete, Paatelainen, Sandbote, Schröder, KS, Vuorinen 2610.23675

Complete g^6 pressure of hot Yang-Mills





‘Laine conjecture’:
Expansion reliable only at $O(g^6)$
where all momentum scales enter

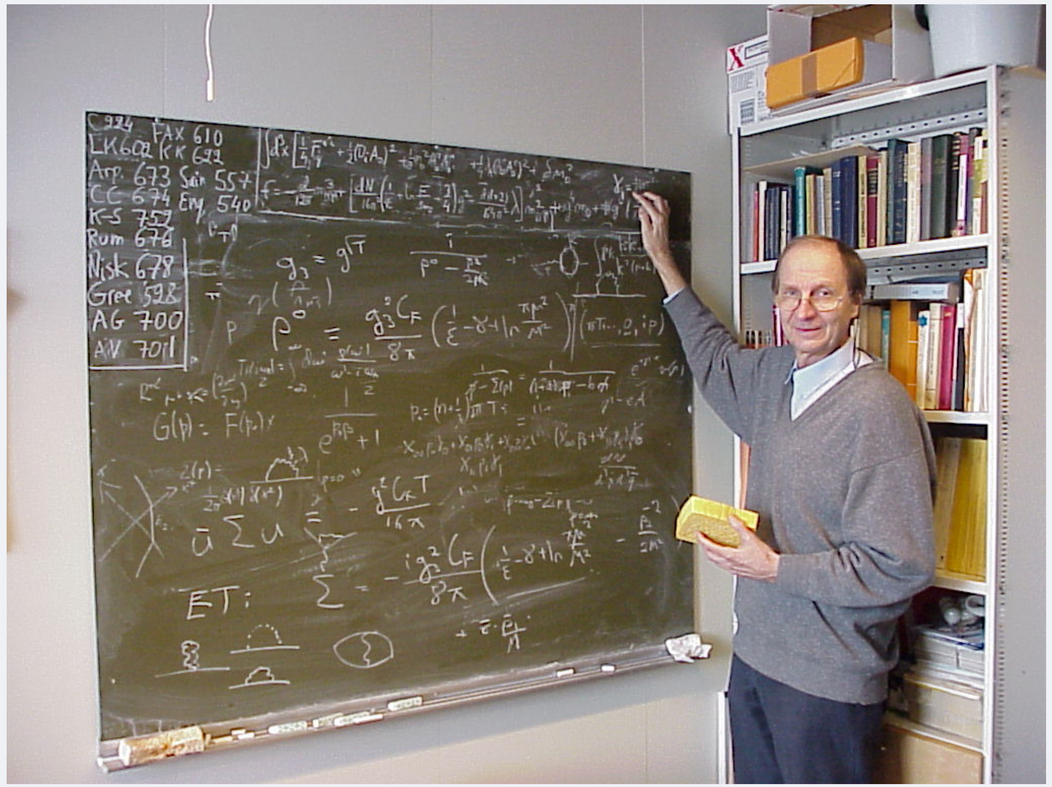
3. Conjecture

Motivated by this physical picture, we propose as an organizing principle for the remainder of this presentation the following conjecture:

> If one starts from the dimensionally reduced theory and determines physical observables up to and including the leading non-perturbative “magnetic” contribution, such that effects from all the scales have made their entrances, then one should already have accounted for most of the dynamics at the scales $\sim gT, g^2T$.



What a journey in Finnish particle physics!



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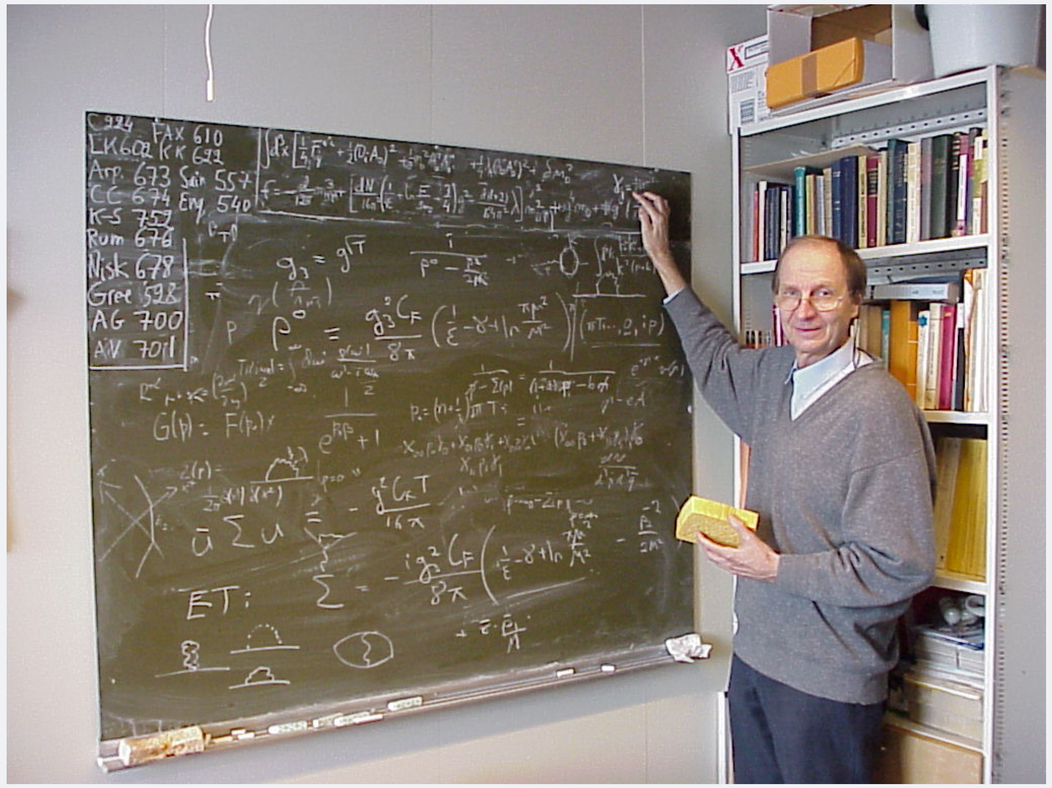

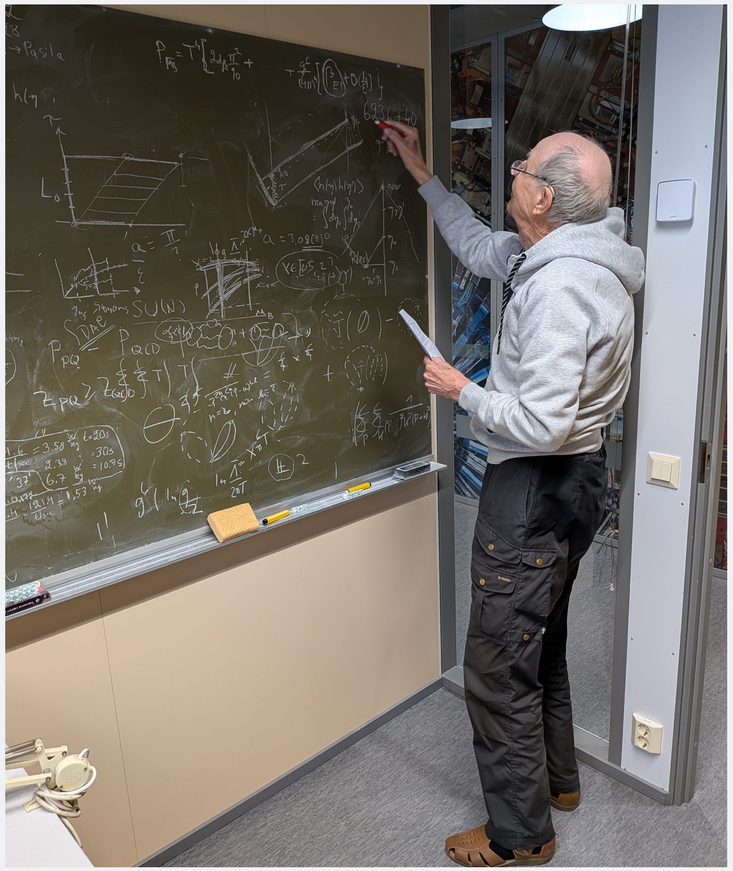

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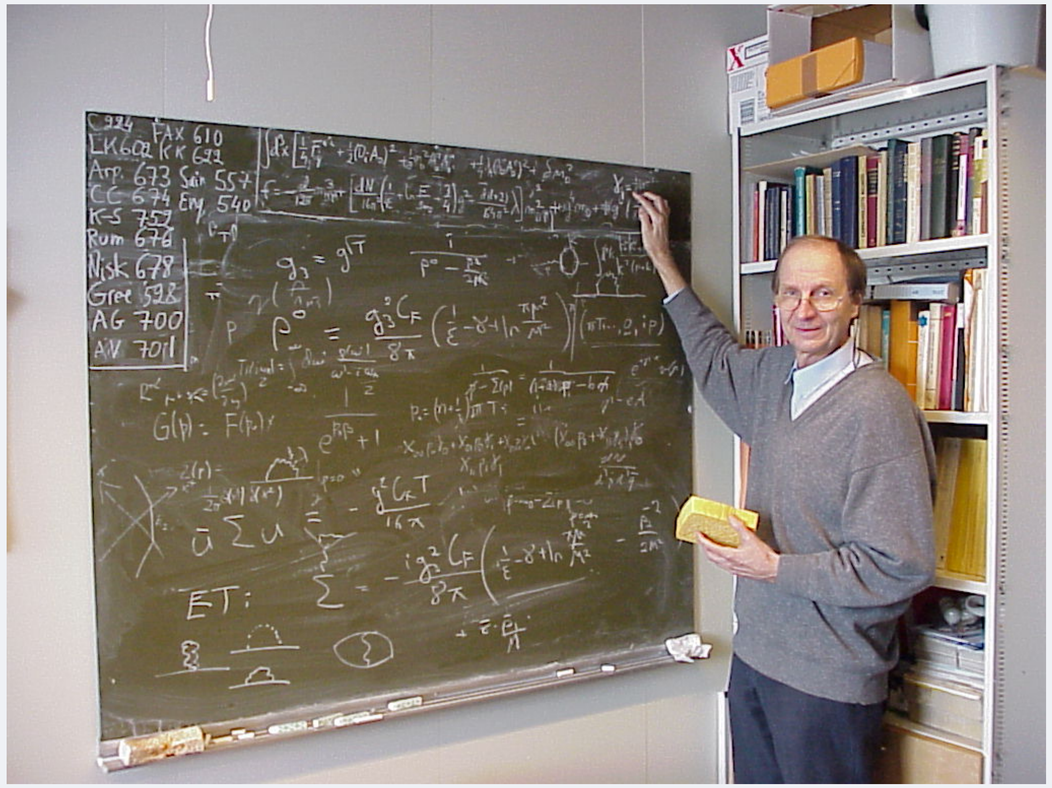
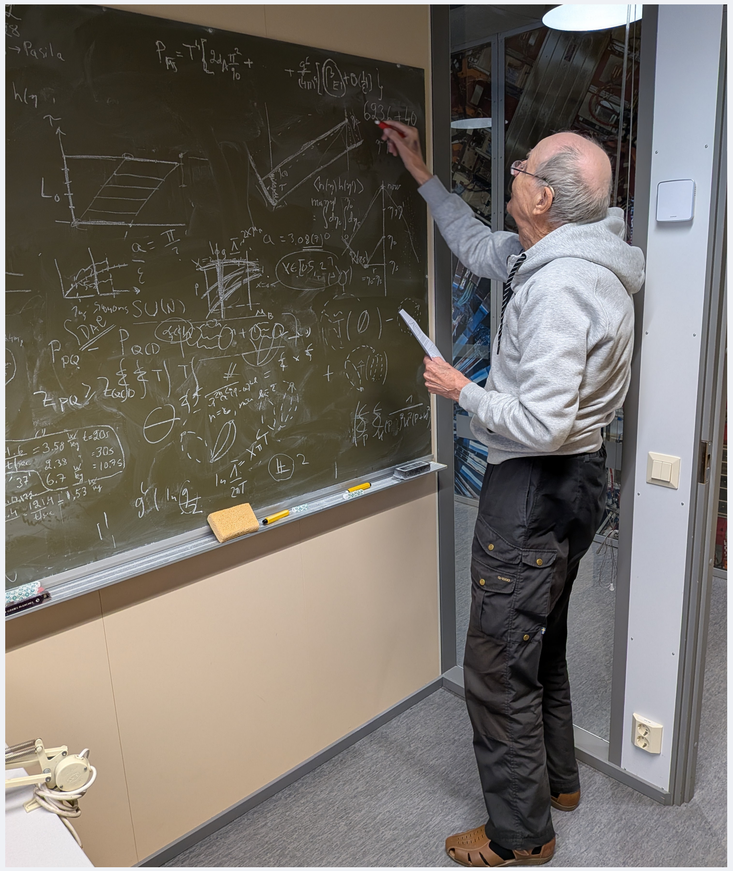
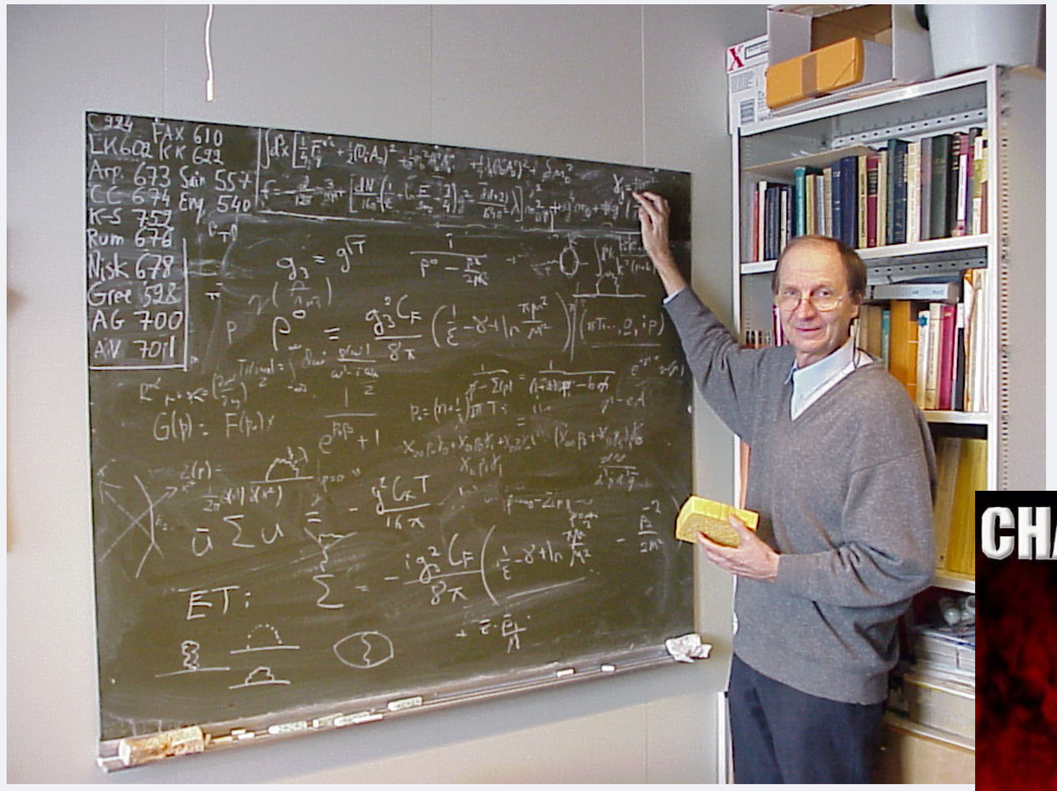
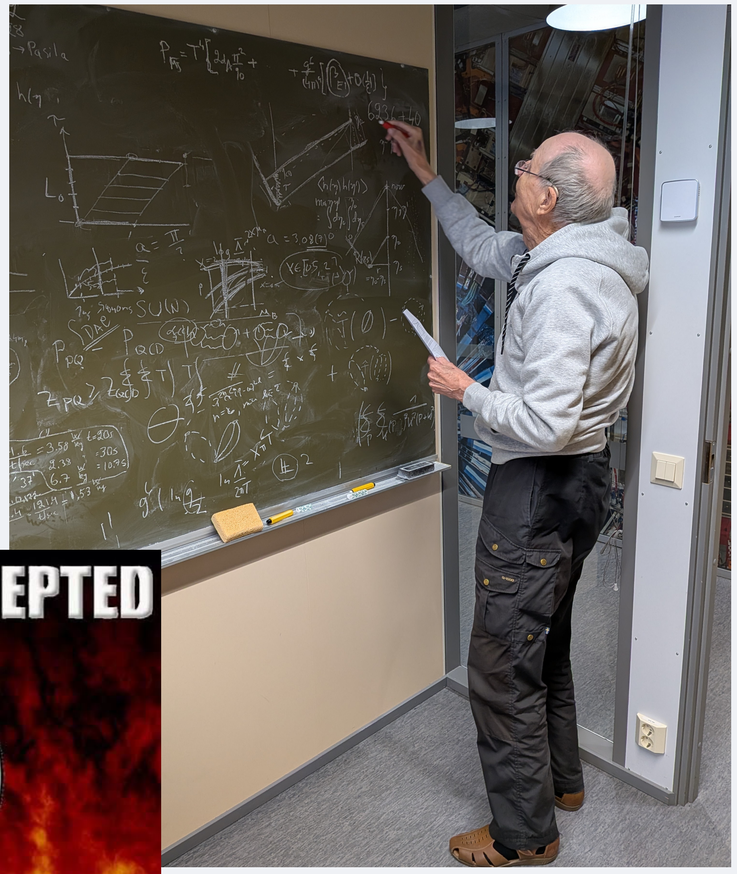




Should we try 5-loops…?

What a journey in Finnish particle physics!







Should we try 5-loops…?

Master sum-integrals for $\tilde{\beta}_{\mathrm{E1}}$

Navarrete, Schröder JHEP 11 (2024)

Reduction to 21 masters of which 10 are nonfactorizing (new)

$$I_n = \frac{T^{4-8\epsilon}\bar{\Lambda}^{8\epsilon}}{16(4\pi)^{4+8\epsilon}}\left[\frac{A_n}{\epsilon^2} + \frac{B_n}{\epsilon} + C_n + O(\epsilon)\right]$$

I_n	$(s_1,\ldots,s_{10})$	A_n	B_n	C_n
I_6	$(1,1,1,0,1,1,1,0,0,0)$	1	$12Z_1 - 4Z_3 + \frac{44}{5} - \frac{\pi^2}{2}$	$(-130.278 \pm 0.004) + 48b$
I_8	$(1,1,1,1,1,0,0,0,0,1)$	$1/3$	$4Z_1 - \frac{4Z_3}{3} + \frac{71}{60}$	$(-45.486 \pm 0.008) + 16b$
I_{14}	$(1,1,1,1,1,1,0,0,-1,1)$	$1/12$	$\frac{11Z_1}{3} - \frac{5Z_3}{3} + \frac{37}{72} - \frac{4\gamma_{\mathrm{E}}}{3}$	$(-94.420 \pm 0.012) + 20b$
I_{15}	$(1,1,1,2,1,1,1,0,0,-2)$	$37/18$	$14Z_1 - \frac{26Z_3}{9} + \frac{1961}{270} + \frac{16\gamma_{\mathrm{E}}}{3} - \frac{5\pi^2}{16}$	$(6.84 \pm 0.09) + 8b + 192s$
I_{16}	$(1,1,1,3,1,1,1,0,0,-3)$	$-499/180$	$-19Z_1 + \frac{178Z_3}{45} - \frac{157}{36} + \frac{13}{225}\zeta(3) - \frac{107\gamma_{\mathrm{E}}}{15} + \frac{7\pi^2}{16}$	$(123.7 \pm 3.8) + 12b - 480s$
I_{17}	$(1,1,1,1,1,1,-1,-1,1,1)$	$-5/12$	$-\frac{7Z_1}{3} + \frac{Z_3}{3} - \frac{5389}{360} - \frac{4\gamma_{\mathrm{E}}}{3} + \frac{4\pi^2}{3}$	$(-36.865 \pm 0.022) - 4b$
I_{18}	$(1,1,1,1,1,1,0,-2,1,1)$	$1/6$	$\frac{22Z_1}{3} - \frac{10Z_3}{3} + \frac{31}{36} - \frac{8\gamma_{\mathrm{E}}}{3}$	$(-191.151 \pm 0.028) + 40b$
I_{19}	$(1,1,1,1,1,1,1,1,-2,0)$	$-5/3$	$-\frac{32Z_1}{3} + 2Z_3 - \frac{937}{180} - \frac{14\gamma_{\mathrm{E}}}{3}$	$(-69.752 \pm 0.012) - 24b$
I_{20}	$(1,1,1,1,1,1,1,1,0,-2)$	$-43/18$	$-\frac{38Z_1}{3} + \frac{14Z_3}{9} - \frac{3229}{270} - 8\gamma_{\mathrm{E}} + \frac{\pi^2}{2}$	$(-181.273 \pm 0.011) - \frac{56b}{3}$
I_{21}	$(1,1,1,1,1,1,1,1,1,-3)$	$-16/3$	$-24Z_1 + \frac{4Z_3}{3} - \frac{2972}{45} - 20\gamma_{\mathrm{E}} + \frac{11\pi^2}{2}$	$(-501.63 \pm 0.11) - 16b$