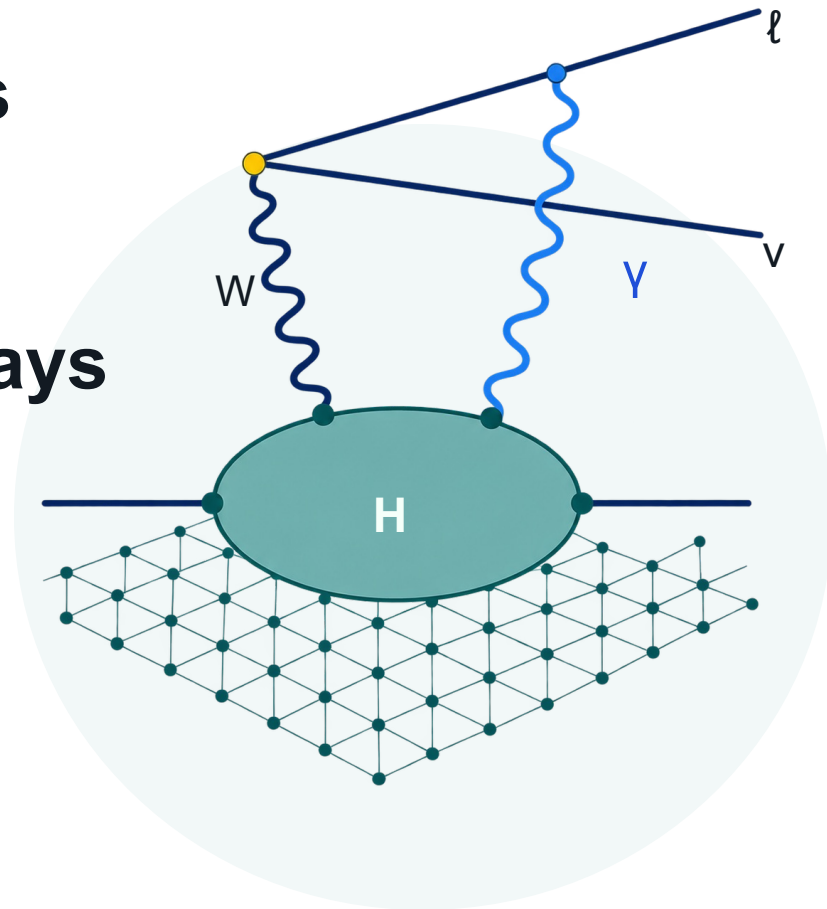


Toward Lattice Calculations of Radiative Corrections in Beta and Semileptonic Decays

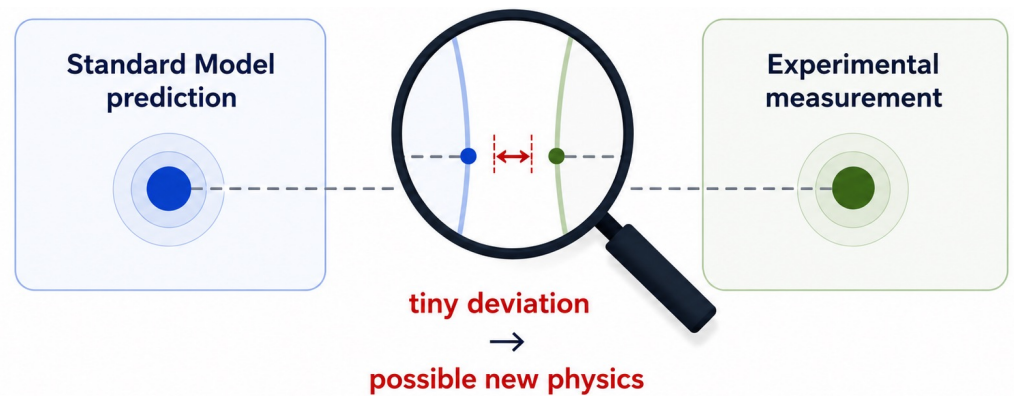
Xin-Yu Tuo



October 13, 2026

Why radiative corrections matter

Precision frontier: indirect
probe of new physics



Key challenge: radiative corrections

small

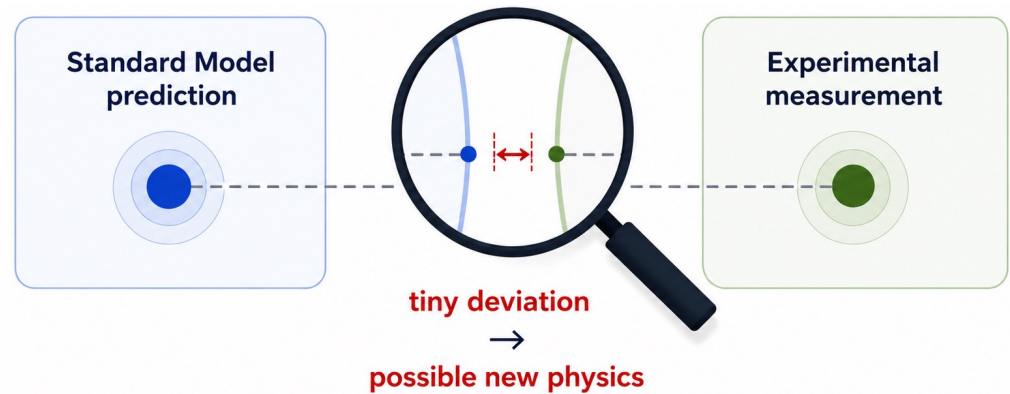
$$\alpha_{QED} \approx \frac{1}{137} \approx 0.7\%$$

but important

precision frontier enters
the sub-percent era

Why radiative corrections matter

Precision frontier: indirect probe of new physics



Key challenge: radiative corrections

small

$$\alpha_{QED} \approx \frac{1}{137} \approx 0.7\%$$

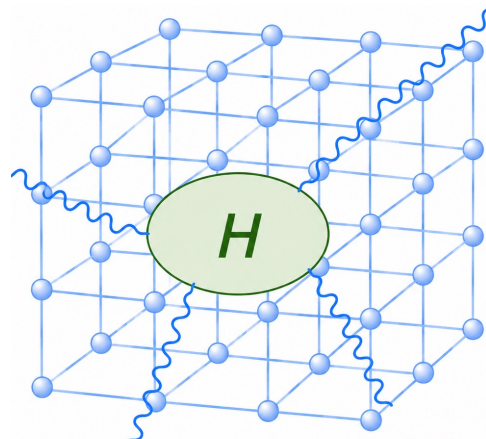
but important

precision frontier enters the sub-percent era

QED coupled to
nonperturbative QCD



major source of
theory uncertainty



**first-principles
lattice QCD+QED**

Precision tests of the CKM unitarity

first row's CKM unitarity: **2~3 σ tension**

superaligned
 $0^+ \rightarrow 0^+$ nuclear β
decays

neutron β decay

precision

$\sim \mathbf{0(0.1\%)}$

Radiative correction

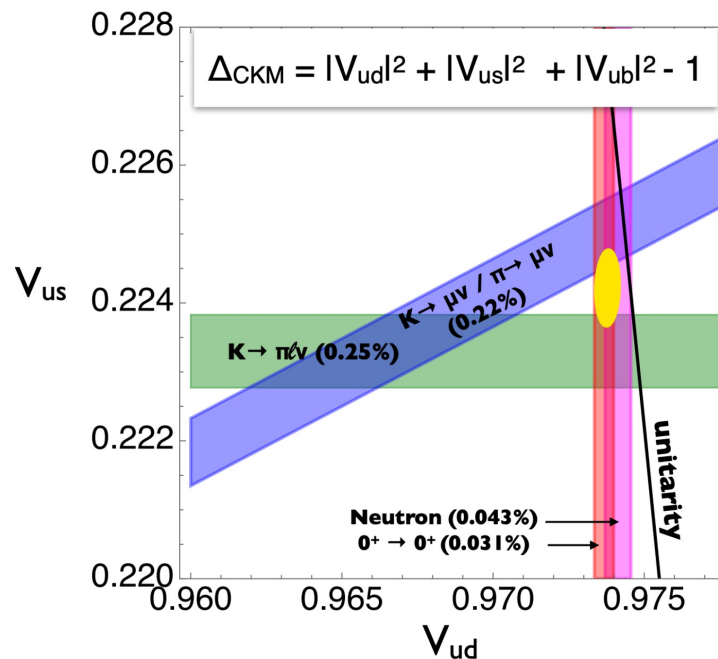
$\sim \mathbf{0(1\%)}$

$\pi^+ \rightarrow \pi^0 e^+ \nu$

$$|V_{ud}|^2 + |V_{us}|^2 + |V_{ub}|^2 = 0.9983(6)V_{ud}(4)V_{us}$$

$\bigcirc K \rightarrow \mu \nu / \pi \rightarrow \mu \nu$

$\bigcirc K \rightarrow \pi \ell \nu (K_{\ell 3})$



Non-perturbative RCs in semileptonic decays

$$\boxed{|V_{ud}|^2} + \boxed{|V_{us}|^2} + |V_{ub}|^2 = 0.9983(6)_{V_{ud}}(4)_{V_{us}}$$

superaligned
 $0^+ \rightarrow 0^+$ nuclear β
decays

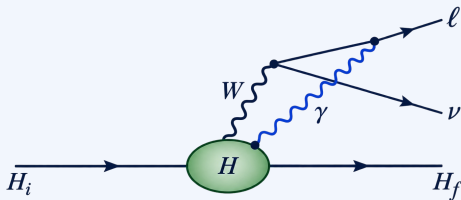
neutron β decay

precision
 $\sim \mathbf{O(0.1\%)}$
Radiative correction
 $\sim \mathbf{O(1\%)}$

$\pi^+ \rightarrow \pi^0 e^+ \nu$

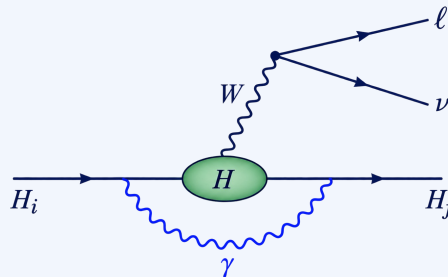
$\odot K \rightarrow \pi \ell \nu (K_{\ell 3})$

γW box



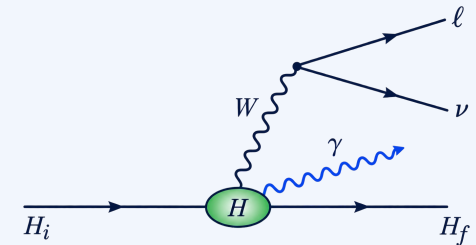
photon between
hadron and lepton

vertex correction



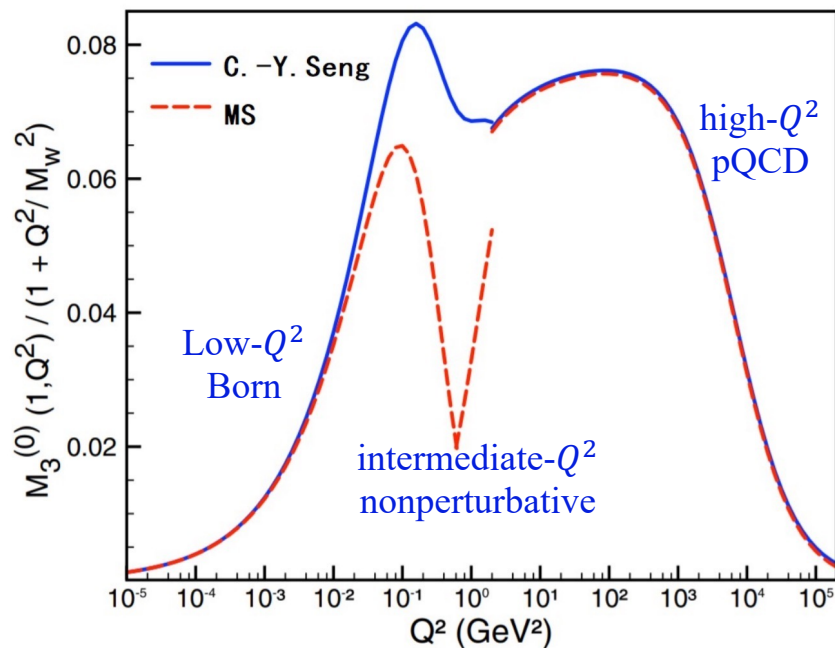
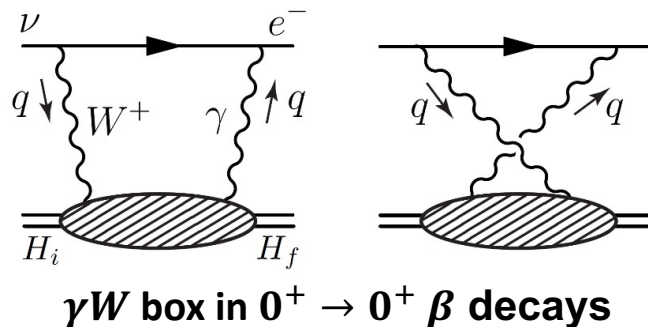
photon on both
hadron legs

real emission



structure-dependent
emission

A lesson: one hadronic input moved V_{ud}



2006: piecewise interpolation

Hadronic regions interpolated
High Q^2 : OPE + pQCD



2018: dispersive analysis

Hadronic input constrained by data
High Q^2 : pQCD

Marciano & Sirlin, PRL96, 032002 (2006)

Seng et.al. PRL 121, 241804 (2018)

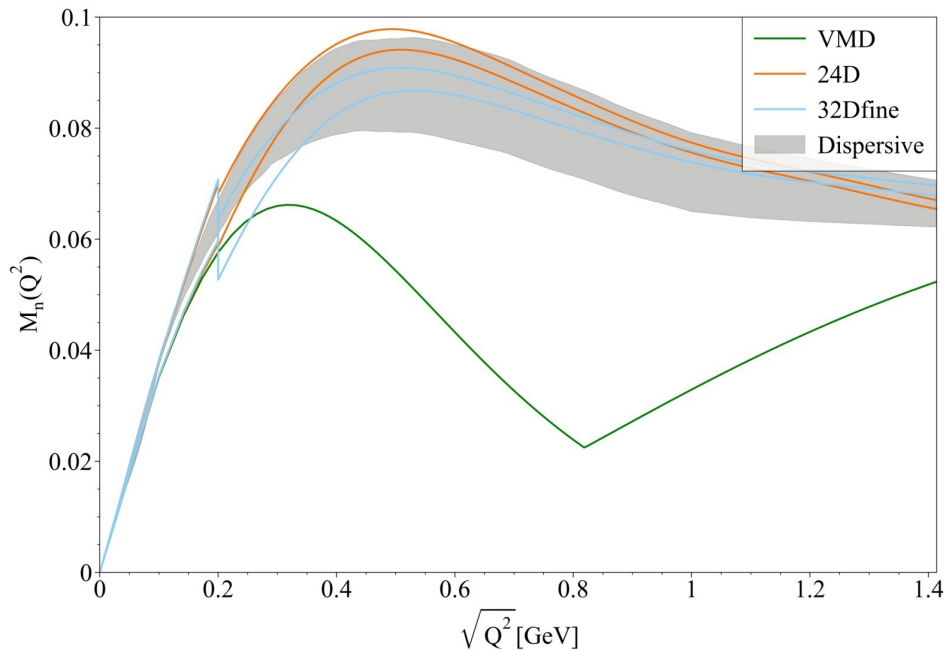
V_{ud}	PDG 2019 0.97420(21)	→	PDG 2020 0.97370(14)
Δ_{CKM}	PDG 2019 -0.00061(47)	→	PDG 2020 -0.00149(45)

Non-perturbative inputs must be confirmed from first principles

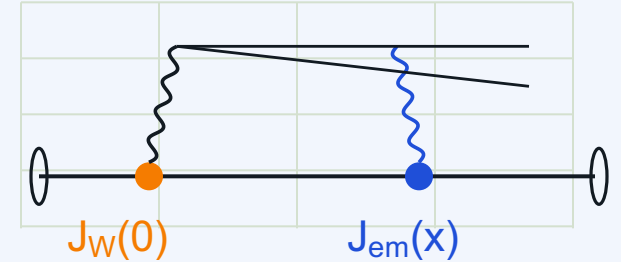
Lattice γW boxes so far — all in the forward limit

lattice QCD: $Q^2 \lesssim 2 \text{ GeV}^2$

pQCD: $Q^2 \gtrsim 2 \text{ GeV}^2$



P. Ma, et al. PRL 132 (2024) 191901



$$H_{\mu\nu}(x) = \langle H_f | T [J_\mu^{\text{em}}(x) J_\nu^W(0)] | H_i \rangle$$

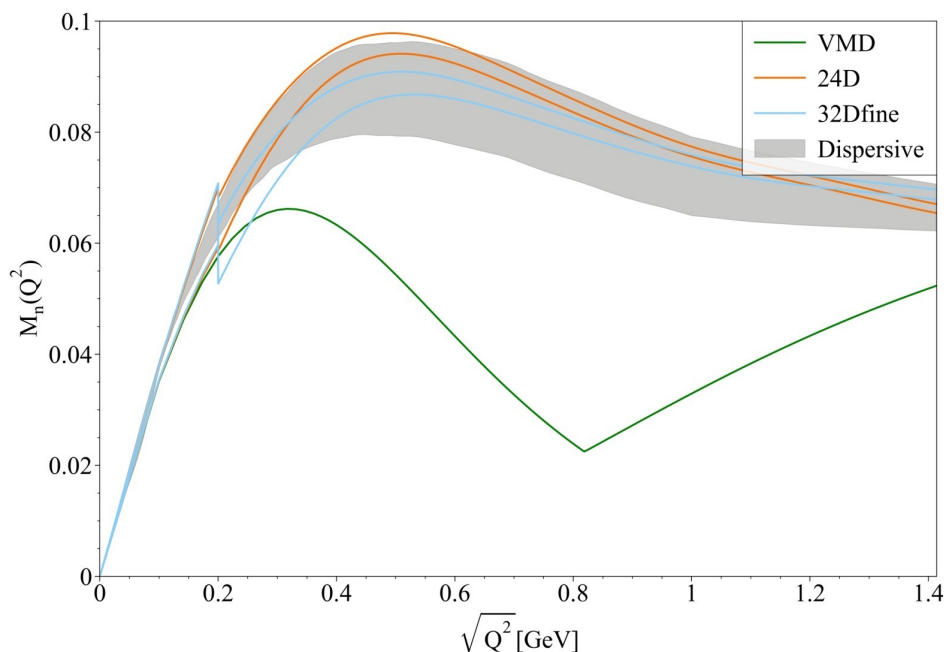


$$\Box_{\gamma W} = \frac{3\alpha}{2\pi} \int \frac{dQ^2}{Q^2} \frac{m_W^2}{m_W^2 + Q^2} M_H(Q^2)$$

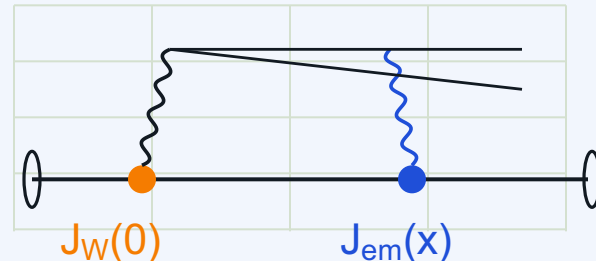
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all three: hadron at rest, equal masses

π · 2020



$$2.830(11)(26) \times 10^{-3}$$

K (SU(3) limit) · 2021



$$2.437(44) \times 10^{-3}$$

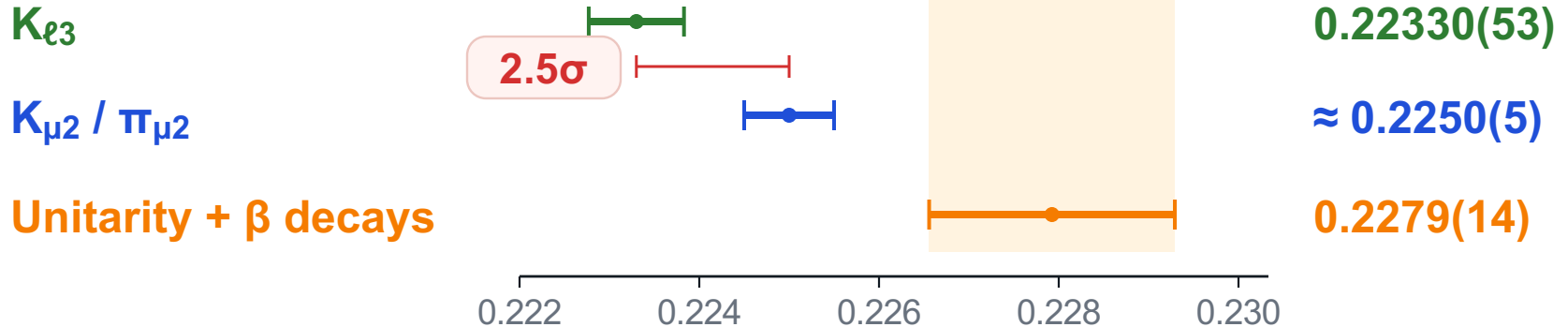
nucleon · 2024



$$3.65(8)(1) \times 10^{-3}$$

$K_{\ell 3}$: tensions with $K_{\ell 2}$ and unitarity

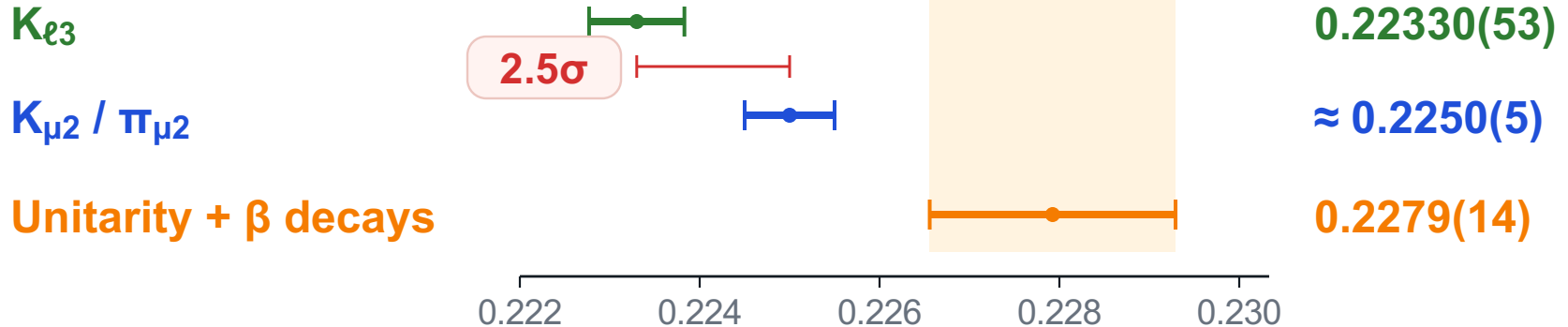
Three determinations of $|V_{us}|$



Unitarity band: $|V_{ud}| = 0.97367(32)$, from superallowed β decays

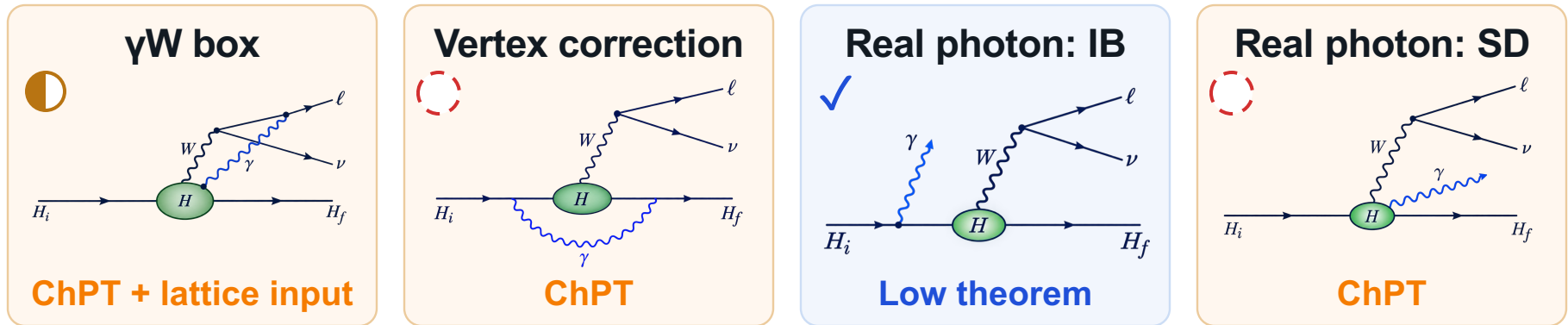
$K_{\ell 3}$: tensions with $K_{\ell 2}$ and unitarity

Three determinations of $|V_{us}|$



Unitarity band: $|V_{ud}| = 0.97367(32)$, from superallowed β decays

Hadronic radiative corrections in $K_{\ell 3}$



Box: lattice input in the $SU(3)$ limit; physical non-forward correction remains.

Both tensions motivate a complete first-principles RC calculation.

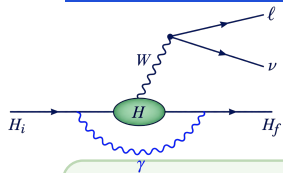
Neutron g_A vertex correction: exp. vs. lattice

Neutron g_A free from nuclear-structure uncertainties

Experiment

$$\lambda = g_A / g_V \\ = 1.2756(13)$$

PDG 2024 precision $\approx 0.1\%$



↕ QED correction
to $g_A = ?$

Lattice QCD (pure QCD)

$$g_A^{\text{QCD}} = 1.263(10)$$

FLAG 2024 precision $\approx 1\%$

Neutron g_A vertex correction: exp. vs. lattice

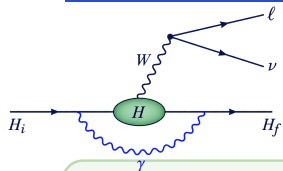
Neutron g_A free from nuclear-structure uncertainties

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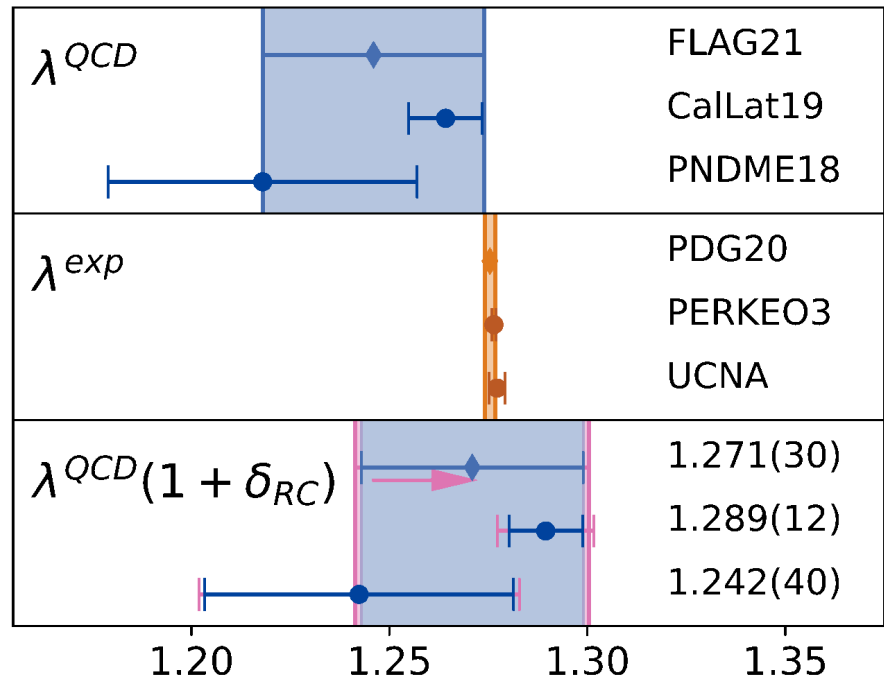
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2022: the pion-induced vertex correction



Cirigliano et al., PRL 129, 121801 (2022)

Neutron g_A vertex correction: exp. vs. lattice

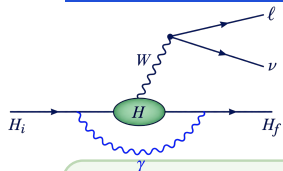
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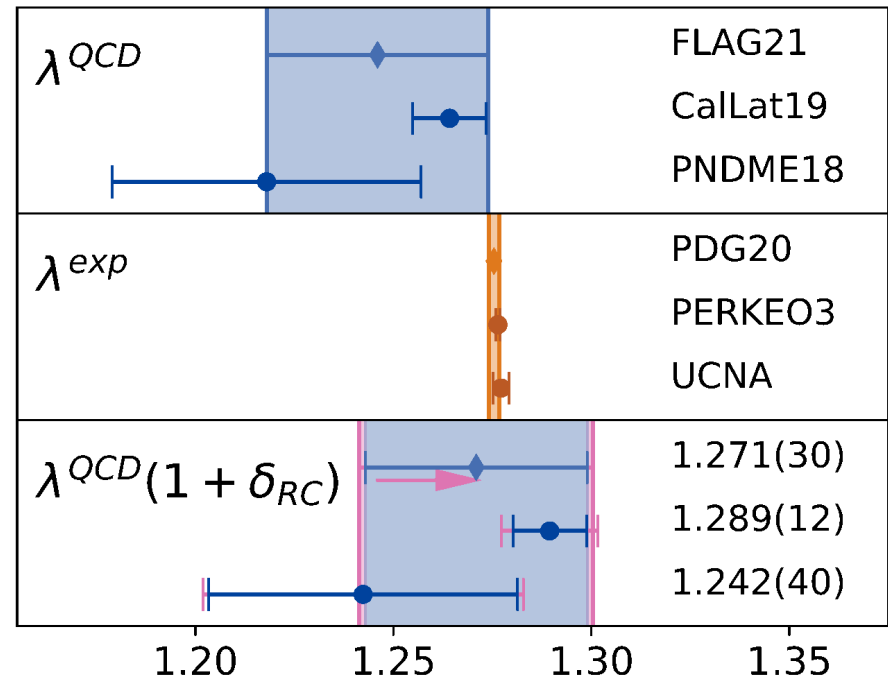
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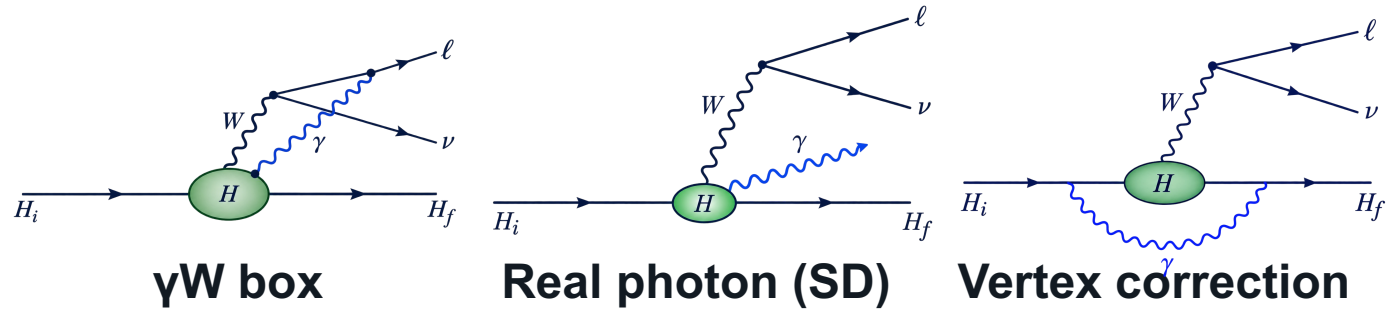
This correction motivates a first-principles calculation

$$\langle p | T \{ J_{\text{em}}^\mu(x) J_A^\rho(0) J_{\text{em}}^\nu(y) \} | n \rangle$$

2 EM + 1 axial current
Five-point lattice correlator

The map: what is needed vs. what the lattice has done

- ✓ Lattice result
- 🕒 Limited lattice input
- ⋯ No lattice result



$\pi^+ \rightarrow \pi^0 e^+ \nu$



2020, 2023



IB dominates¹



ChPT estimate

$0^+ \rightarrow 0^+ \beta$



Nucleon: 2024



IB dominates¹



Protected (CVC)

$K \rightarrow \pi \ell \nu$
($K_{\ell 3}$)



SU(3) only (2021)
Physical: missing



ChPT estimate



ChPT estimate

Neutron g_A



2024



IB dominates¹



ChPT estimate

¹ Real radiation exists in all β decays; this column tracks SD only.

Research sequence: 1 this talk; 2–5 outlook

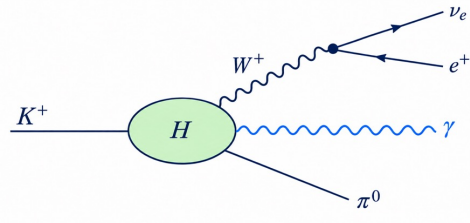
Physical RCs require lattice inputs beyond the existing γW boxes.

Real-photon decay: a first step to RC in $K_{\ell 3}$

$$H_{\mu\nu}(p, q) = \int d^4x e^{iq \cdot x} \langle \pi(p) | T [J_\mu^{\text{em}}(x) J_\nu^W(0)] | K(P) \rangle$$

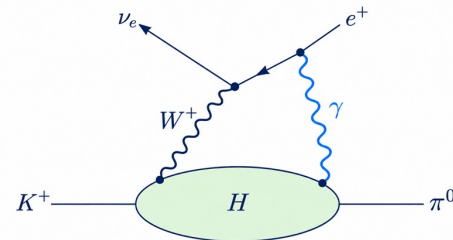
One hadronic input, two processes

Real photon · $q^2 = 0$



$K^+ \rightarrow \pi^0 e^+ \nu \gamma$

Virtual photon · integrate over q



γW box in $K_{\ell 3}$

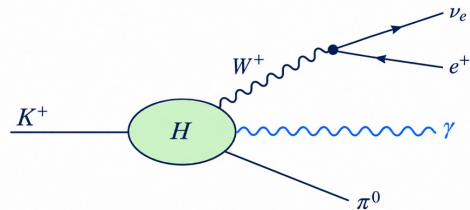
Same difficulty:
moving pion

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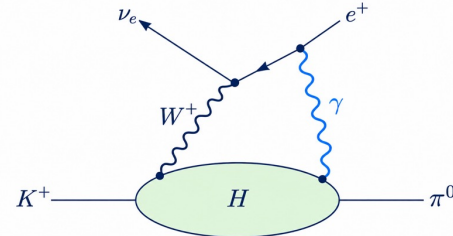
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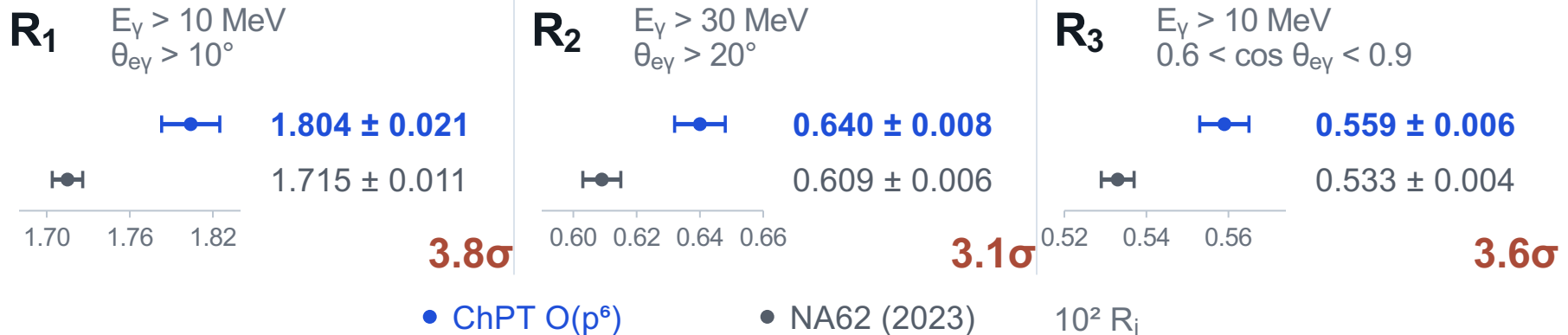
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γW box in $K_{\ell 3}$

Same difficulty:
moving pion

Tensions: ChPT vs. NA62



New strategy 1: preconditioning

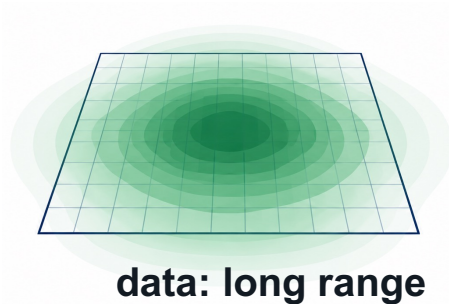
General target integral

$$I^\infty = \int \frac{d^3 k}{(2\pi)^3} \tilde{W}_\infty(\vec{k}) \tilde{H}_\infty(\vec{k}),$$

$$I^{(L)} = \int_V d^3 x W_\infty(\vec{x}) H_L(\vec{x}).$$

$$W_\infty(\vec{x}) = \int \frac{d^3 k}{(2\pi)^3} e^{-i\vec{k}\cdot\vec{x}} \tilde{W}_\infty(\vec{k}).$$

$$H_L(\vec{x}) = \frac{1}{L^3} \sum_{\vec{k}'} e^{i\vec{k}'\cdot\vec{x}} \tilde{H}_\infty(\vec{k}') + O(e^{-mL})$$



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Preconditioned integral

$$I^{(L)}[v] = \int_V d^3 x W_\infty[v](\vec{x}) H_L[v](\vec{x}),$$

$$W_\infty[v](\vec{x}) = \int \frac{d^3 k}{(2\pi)^3} e^{-i\vec{k}\cdot\vec{x}} \tilde{W}_\infty(\vec{k}) v(\vec{k}),$$

$$H_L[v](\vec{x}) = \frac{1}{L^3} \sum_{\vec{k}'} e^{i\vec{k}'\cdot\vec{x}} \frac{\tilde{H}_\infty(\vec{k}')}{v(\vec{k}')}.$$

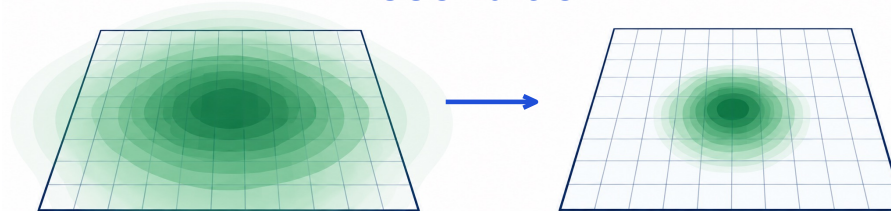
$$v(\vec{k}) \sim \tilde{H}_\infty(\vec{k})$$

$$H_L[v](\vec{x}) \sim \delta_{\vec{x}}$$

Localize the hadronic function
to suppress finite-volume effects

$v(p)$ cancels exactly in infinite-volume limit, unbiased for any $v \neq 0$

Precondition



data: long range

C : fits in the box

Transfer long-distance information into known weights

Application 1: $K \rightarrow \pi \ell \nu$ (without QED)

Semileptonic form factors at arbitrary momentum

$$C_\nu(\mathbf{r}, T) = \langle 0 | J_4^W(\mathbf{r}, T) J_\nu^W(0) | K \rangle,$$

Infinite-volume relation

$$F_\nu(t_p) = h_\nu(\mathbf{p}) = \frac{e^{E_\pi(\mathbf{p})T}}{A} \int d^3r e^{-i\mathbf{p}\cdot\mathbf{r}} C_\nu(\mathbf{r}, T)$$
$$A = \frac{\langle 0 | J_4^W(0) | \pi(\mathbf{k}) \rangle}{2E_\pi(\mathbf{k})} = \frac{if_\pi}{2}, \quad E_\pi(\mathbf{k}) = \sqrt{m_\pi^2 + \mathbf{k}^2}$$

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Direct Fourier transform: lattice momenta

What if we choose non-lattice momentum?

$$\hat{h}_\nu^{\text{M1}}(\mathbf{p}) = \frac{1}{A} \int_V d^3x e^{E_p T - i\mathbf{p}\cdot\mathbf{x}} C_\nu(\mathbf{x}, T)$$

Large FV errors!

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Preconditioned Fourier transform

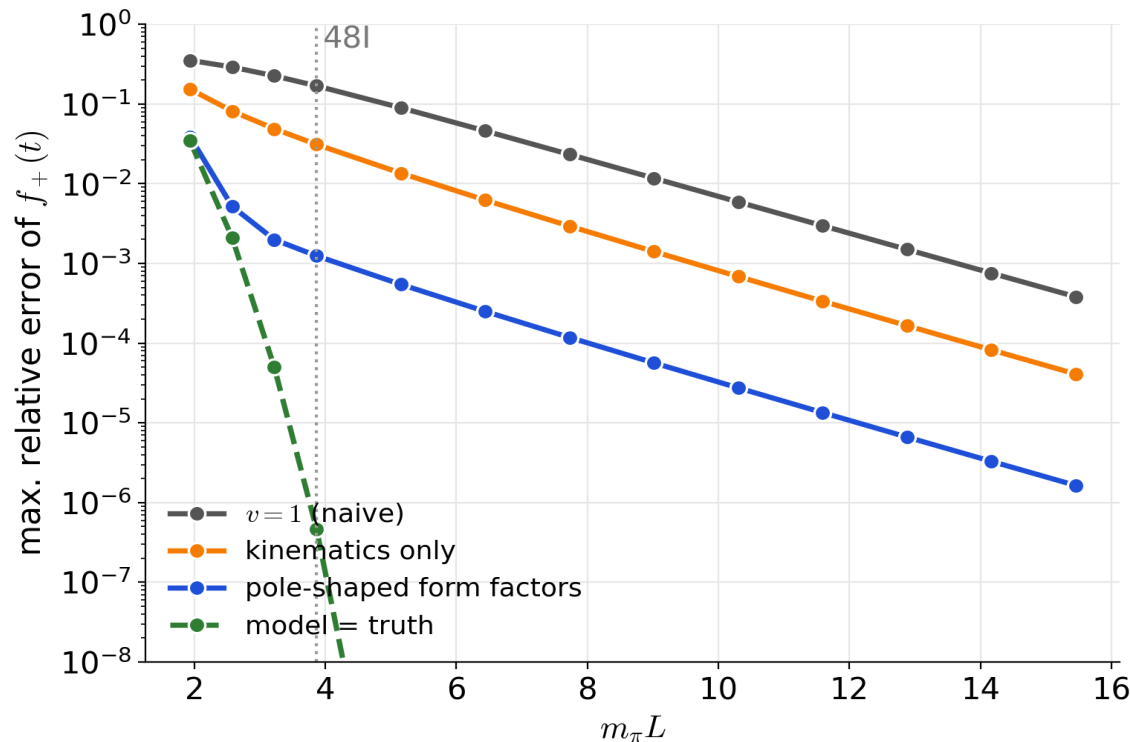
$$v(\mathbf{k}) = \frac{h_{\text{model}}(\mathbf{k}) e^{-E_k T}}{w(\mathbf{k})}$$
$$\hat{h}_\nu^{\text{M2}}(\mathbf{p}) = \frac{h_{\text{model}}(\mathbf{p})}{w(\mathbf{p})} \int_V d^3x e^{-i\mathbf{p}\cdot\mathbf{x}} \tilde{C}[v](\mathbf{x}), \quad \tilde{C}[v](\mathbf{x}) = \text{FFT}^{-1} [\text{FFT}[C(\mathbf{x})]/v]$$

Physical meaning: instead of calculating form factors, we calculate the correction to a guess $h_{\text{model}}(\mathbf{p})$.

Any guess is unbiased. Better guess = smaller FV effects.

Application 1: $K \rightarrow \pi \ell \nu$ (without QED)

Maximum finite-volume error across the phase space (non-lattice p)



at the 48l volume
($m_\pi L \approx 3.9$)

17 %

no preconditioner

Direct method: discrete lattice momenta

0.12 %

pole-shaped model

5×10^{-7}

model = truth
(closure test)

**error falls exponentially
with the volume**

Novel method for semileptonic decays at arbitrary momentum

Application 2: $K \rightarrow \pi \ell \nu \gamma$

$$C_{\mu\nu}(\mathbf{r}, T; \mathbf{x}, \tau) = \langle 0 | J_4^W(\mathbf{r}, T) T [J_\mu^{\text{em}}(\mathbf{x}, \tau) J_\nu^W(0)] | K \rangle.$$

Calculation in infinite volume:

$$H_{\mu\nu}(p, q) = \int_{-\infty}^{\infty} d\tau e^{\omega\tau} \frac{e^{E_\pi(\mathbf{p})T(\tau)}}{A} \int d^3r d^3x e^{-i\mathbf{p}\cdot\mathbf{r} - i\mathbf{q}\cdot\mathbf{x}} C_{\mu\nu}(\mathbf{r}, T(\tau); \mathbf{x}, \tau)$$

Trick 1: preconditioner

$$\hat{H}_{\mu\nu, L}^{\text{short}, [\rho]}(p, q; t_s) = \int_{-\infty}^{t_s} d\tau \frac{e^{\omega\tau + E_\pi(\mathbf{p})T}}{A v_\rho(\mathbf{p}, \mathbf{q}; \tau)} \sum_{\mathbf{r}, \mathbf{x} \in \text{box}} e^{-i\mathbf{p}\cdot\mathbf{r} - i\mathbf{q}\cdot\mathbf{x}} C_{\mu\nu, v_\rho, L}(\mathbf{r}, T; \mathbf{x}, \tau)$$

Trick 2: IB/SD split

$$H(q) - H(0)$$

Soft limit $q = 0$: fixed by Low's theorem

Application 2: $K \rightarrow \pi \ell \nu$

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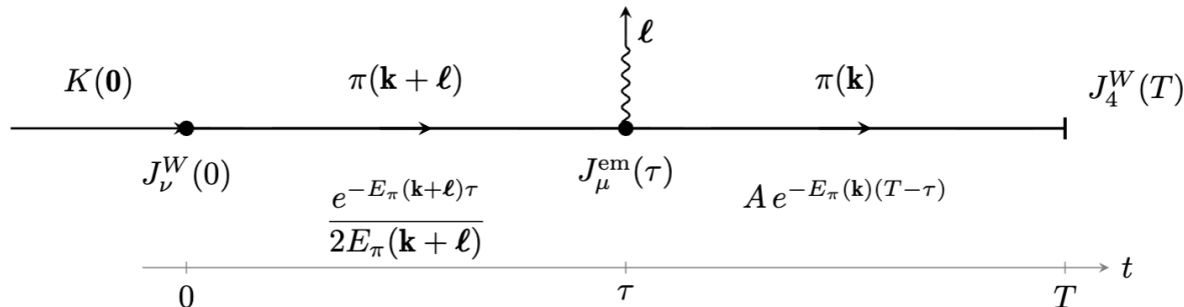
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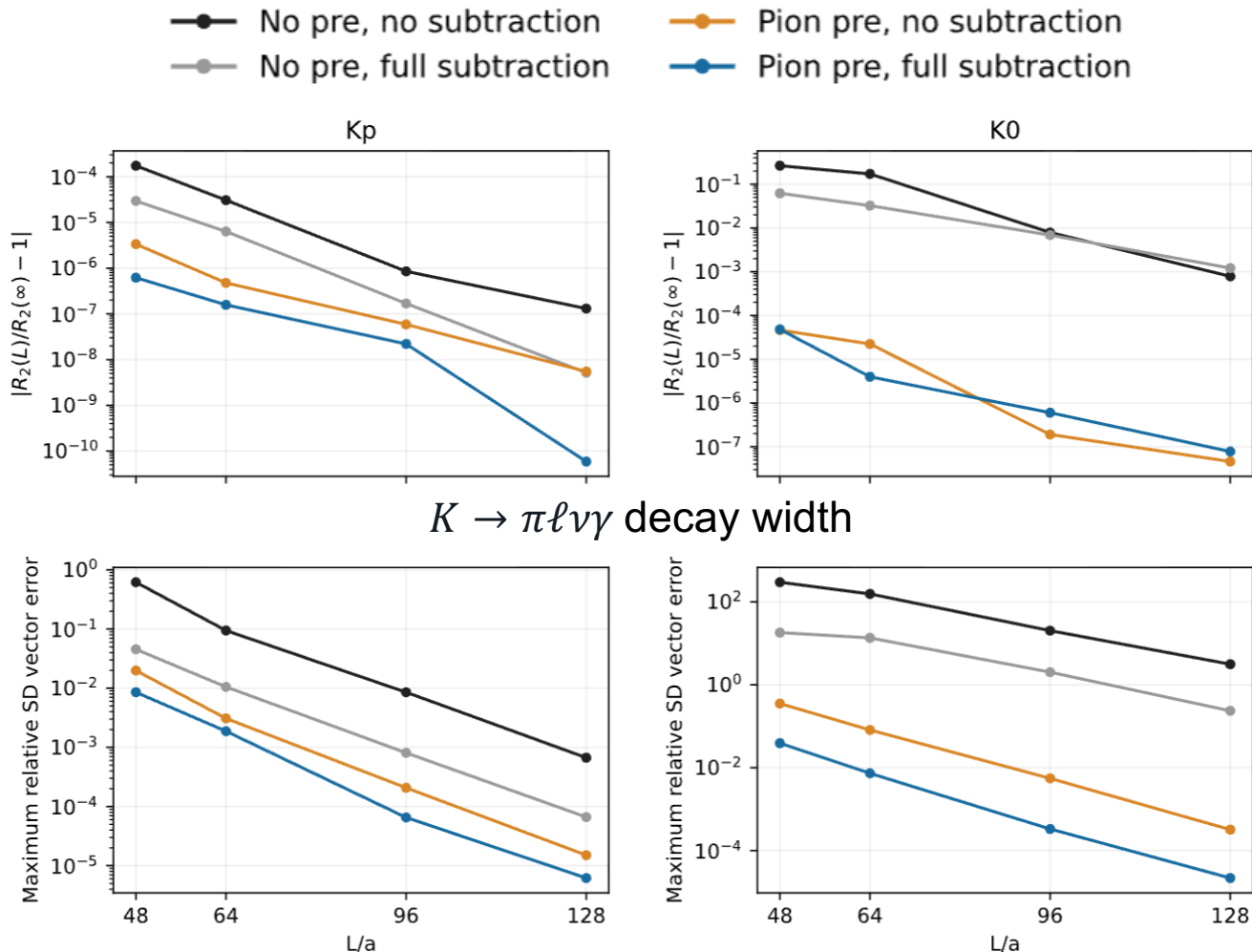
Soft limit $q = 0$: fixed by Low's theorem

Simple choice: constant form factors; precondition the pion propagation



Application 2: $K \rightarrow \pi \ell \nu \gamma$

Finite-volume tests on model data



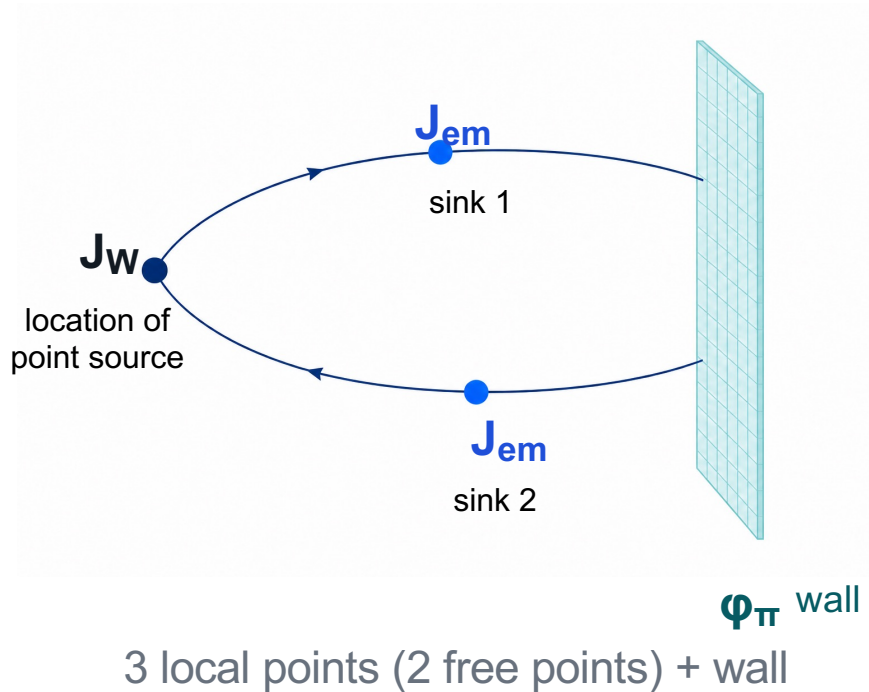
A simple preconditioner strongly suppresses FV errors.

Maximum SD error across the phase space

New strategy 2: importance-sampled contractions

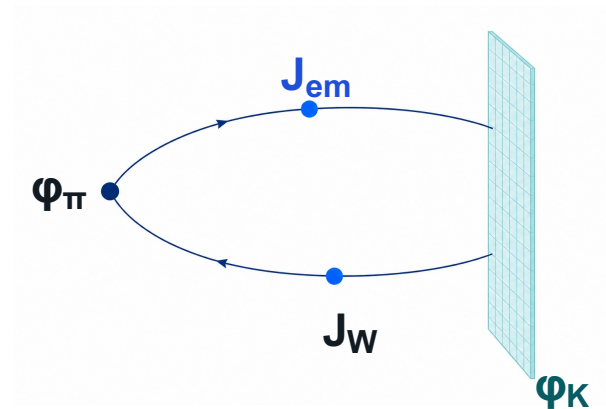
$K_{\ell 2}$ project

$$\langle J_{em} J_{em} J_W \phi_\pi \rangle$$

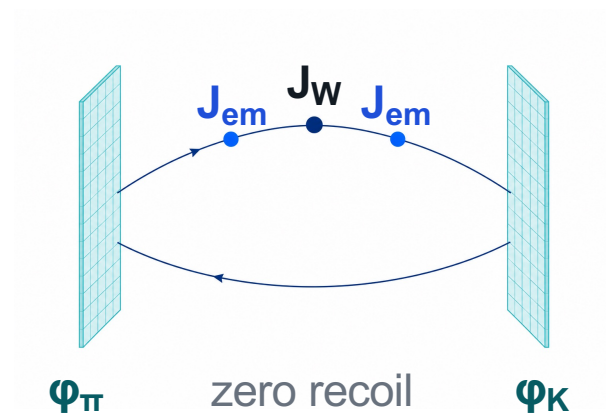


=

γW box in $K_{\ell 3}$



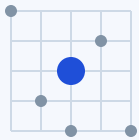
Vertex correction



Same local contraction structure: a sink pair

New strategy 2: importance-sampled contractions

1 Propagator storage

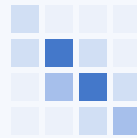


Sink points live on sparse sites

Reuse stored propagators

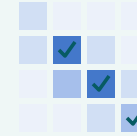
2 Score pairs cheaply

$$s(a, b) = \mathcal{K}(a - b) \mathcal{N}(a) \mathcal{N}'(b)$$



Photon kernel $\times$
Wilson-matrix
norms

3 Sample & reweight



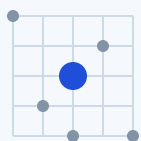
<0.1%

of pairs selected

Much faster contractions

New strategy 2: importance-sampled contractions

1 Propagator storage



Sink points live on sparse sites

Reuse stored propagators

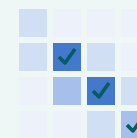
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Photon kernel $\times$
Wilson-matrix
norms

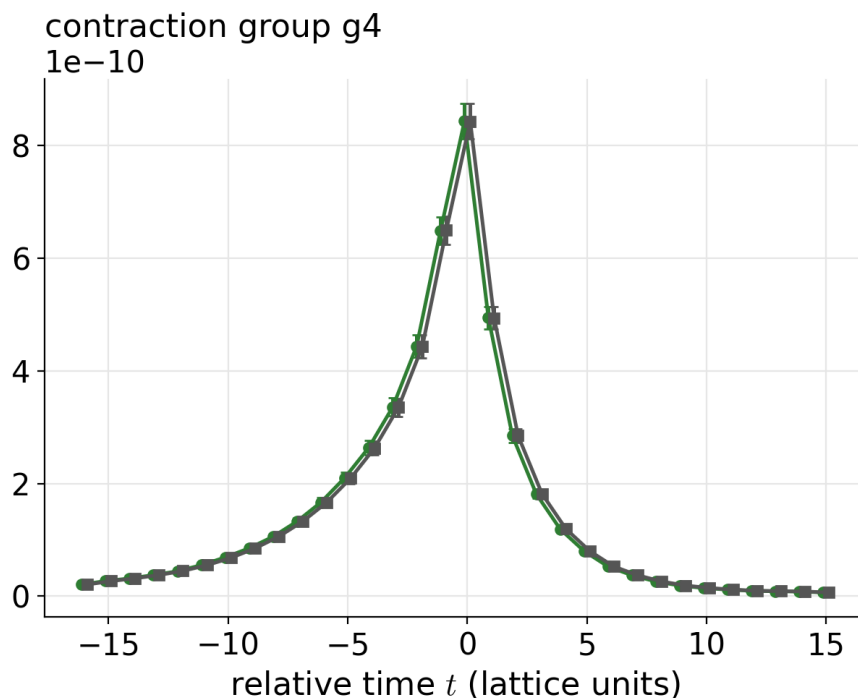
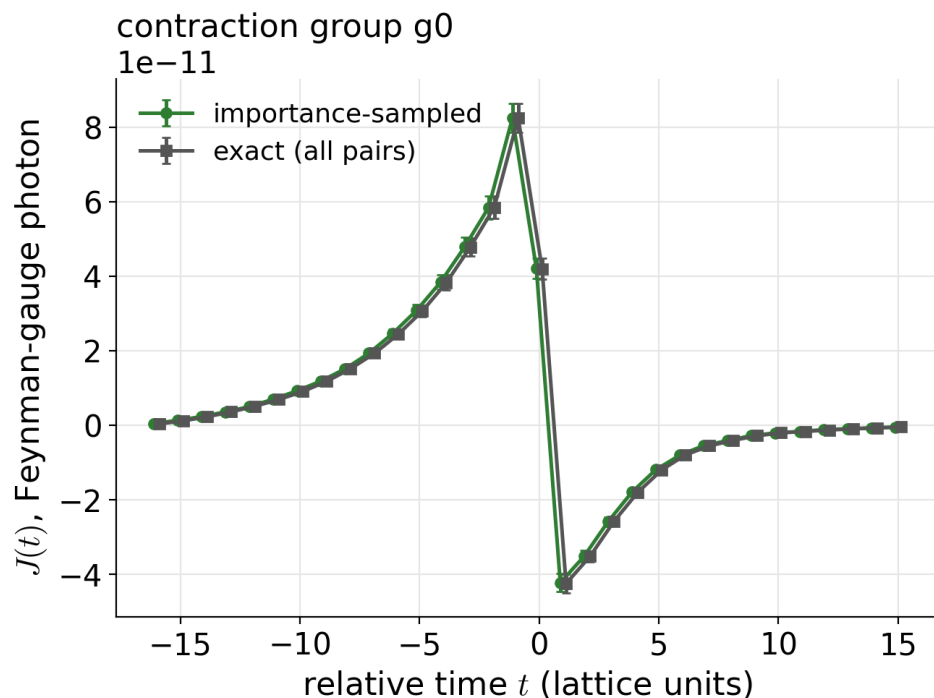
3 Sample & reweight



<0.1%

of pairs selected

Much faster contractions



Outlook: the γW box with a moving pion

$$\delta\Gamma_{\gamma W} = 2 \operatorname{Re} \int d^4 X \int d^3 r \mathcal{W}^{\mu\nu}(\mathbf{r}, X) C_{\mu\nu}(\mathbf{r}, T; X)$$

known weight: QED loop $\times$ lepton phase
space $\times$ pion extraction

already in hand from $K \rightarrow \pi e \nu \gamma$ $\longrightarrow$ new for the γW box

■ moving pion: preconditioning

[] IB / SD split from the Low theorem

✓ Efficient contraction method

∫ Renormalization: lattice at long distance, pQCD at short distance

↔ infrared divergence cancels against real emission

Many technical difficulties are already solved in $K \rightarrow \pi e \nu \gamma$.

Summary



Hadronic radiative corrections limit precision in tests of first-row CKM unitarity.

Key contributions still lack first-principles lattice inputs.



$K \rightarrow \pi \nu \gamma$: a first step. Motivation: NA62–ChPT tension.

Preconditioning suppresses finite-volume effects; the pipeline is validated on synthetic data.



The same tools lead to the γW box and to vertex corrections.

Unbiased pair sampling makes four- and five-point contractions tractable.