The Laporta's Algorithm

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The Laporta's algorithm provides a systematic method for reducing Feynman integrals to a finite set of master integrals using integration-by-parts (IBP) identities. By ordering the integrals according to their complexity, it organizes these identities into a linear system that can be solved by Gaussian elimination. We first outline the algorithm and then illustrate its application through two explicit examples.

1 Basic procedure of the Laporta's algorithm

This section introduces the integral representation, the seed cutoffs, and the ordering used in the reduction. These definitions provide the basis for the algorithm workflow presented in Sec. 1.2. Throughout, we follow the notation and conventions of [Laporta(2000)].

1.1 The notation and definitions

Throughout this note, we follow the notation and definitions in [Laporta(2000)].

Integral families Consider an integral family with N_k loop momenta, N_e external momenta and N_d propagators. The loop momenta are $k_i,\ i = 1, \cdots, N_k$, the independent external momenta are $p_i,\ i = 1, \cdots, N_p$, where $N_p = N_e - 1,\ N_e > 0$. Using the Minkowski metric $\eta^{\mu\nu} = \mathrm{diag}(+, -, \cdots, -)$, we write

$$D_j = q_j^2 - m_j^2 + i0, \tag{1.1}$$

where each q_j is a linear combination of loop and external momenta. The number of independent scalar products involving loop momenta is

$$N_{\mathrm{sp}} = N_p N_k + \frac{N_k(N_k+1)}{2}. \tag{1.2}$$

After expressing reducible scalar products through the denominators and cancelling common factors, an integral can be written as

$$I_{n,i,\alpha,\beta} = \int \prod_{r=1}^{N_k} \mathrm{d}^D k_r \, W(n, i, \alpha, \beta), \tag{1.3}$$

with

$$W(n, i, \alpha, \beta) = \frac{\prod_{j=1}^{N_{\mathrm{sp}}-n} S_j^{\beta_j}}{\prod_{j=1}^{n} D_{i_j}^{\alpha_j}}, \qquad \alpha_j \geq 1, \quad \beta_j \geq 0. \tag{1.4}$$

Here n is the number of active denominators, $i = \{i_1, \ldots, i_n\}$ specifies their combination, and S_j are irreducible scalar products (ISPs) relative to that combination. The count $N_{\mathrm{sp}} - n$ assumes that the active denominators are independent as functions of the loop-dependent scalar products.

IBP identities For each *seed* integrand W, the total derivatives takes

$$0 = \int \prod_{r=1}^{N_k} \mathrm{d}^D k_r \, \frac{\partial}{\partial k_\ell^\mu} \left(v^\mu W\right). \tag{1.5}$$

Here v^μ is IBP momenta which can be chosen as $v \in \{k_1, \ldots, k_{N_k}, p_1, \ldots, p_{N_p}\}$. The total derivatives generate $N_k(N_p + N_k)$ identities.

$$\sum_j c_{ej} I_j = 0. \tag{1.6}$$

Seed range Allowing arbitrary exponents α and β gives infinitely many seed integrands W and IBP identities. We therefore introduce cutoffs to select a finite seed range. Following [Laporta(2000)], we define

$$M_p = \sum_{j=1}^{N_{\mathrm{sp}}-n} \beta_j, \quad M_d = \sum_{j=1}^{n} (\alpha_j - 1). \tag{1.7}$$

Thus M_p measures the total ISP numerator degree, whereas M_d counts the total denominator power above unity. For each denominator combination i, choose non-negative cutoffs a_i and b_i and generate identities from seeds satisfying

$$0 \le M_p \le a_i, \qquad 0 \le M_d \le b_i. \tag{1.8}$$

These bounds restrict the seed integrands; the integrals generated by differentiation can lie outside this range.

Ordering Having specified the seed range, we next order the integrals to determine which integral is eliminated in each identity. Following [Laporta(2000)], the elimination priority is determined by comparing

$$\left\{n,\, M_d,\, M_p,\, i_1, \ldots, i_n,\, \alpha_1, \ldots, \alpha_n,\, \beta_1, \ldots, \beta_{N_{\mathrm{sp}}-n}\right\} \tag{1.9}$$

lexicographically: the entries are compared from left to right, and the integral with the larger entry at the first difference has higher elimination priority.

For example, consider the triangle family $I_{a_1 a_2 a_3}$, the relevant ordering data for several scalar integrals are

Integral	n	M_d	M_p	i	α
I_{221}	3	2	0	$(1,2,3)$	$(2,2,1)$
I_{131}	3	2	0	$(1,2,3)$	$(1,3,1)$
I_{211}	3	1	0	$(1,2,3)$	$(2,1,1)$
I_{121}	3	1	0	$(1,2,3)$	$(1,2,1)$
I_{111}	3	0	0	$(1,2,3)$	$(1,1,1)$
I_{210}	2	1	0	$(1,2)$	$(2,1)$

. (1.10)

The resulting elimination priority is

$$I_{221} \succ I_{131} \succ I_{211} \succ I_{121} \succ I_{111} \succ I_{210}. \tag{1.11}$$

where $\succ$ denotes higher elimination priority.

1.2 Algorithm workflow

With the notation, seed range, and integral ordering defined above, we now summarize the workflow of the Laporta's algorithm in Fig.1. The flowchart follows [Laporta(2000)] and provides an overview of the procedure used in the examples below.

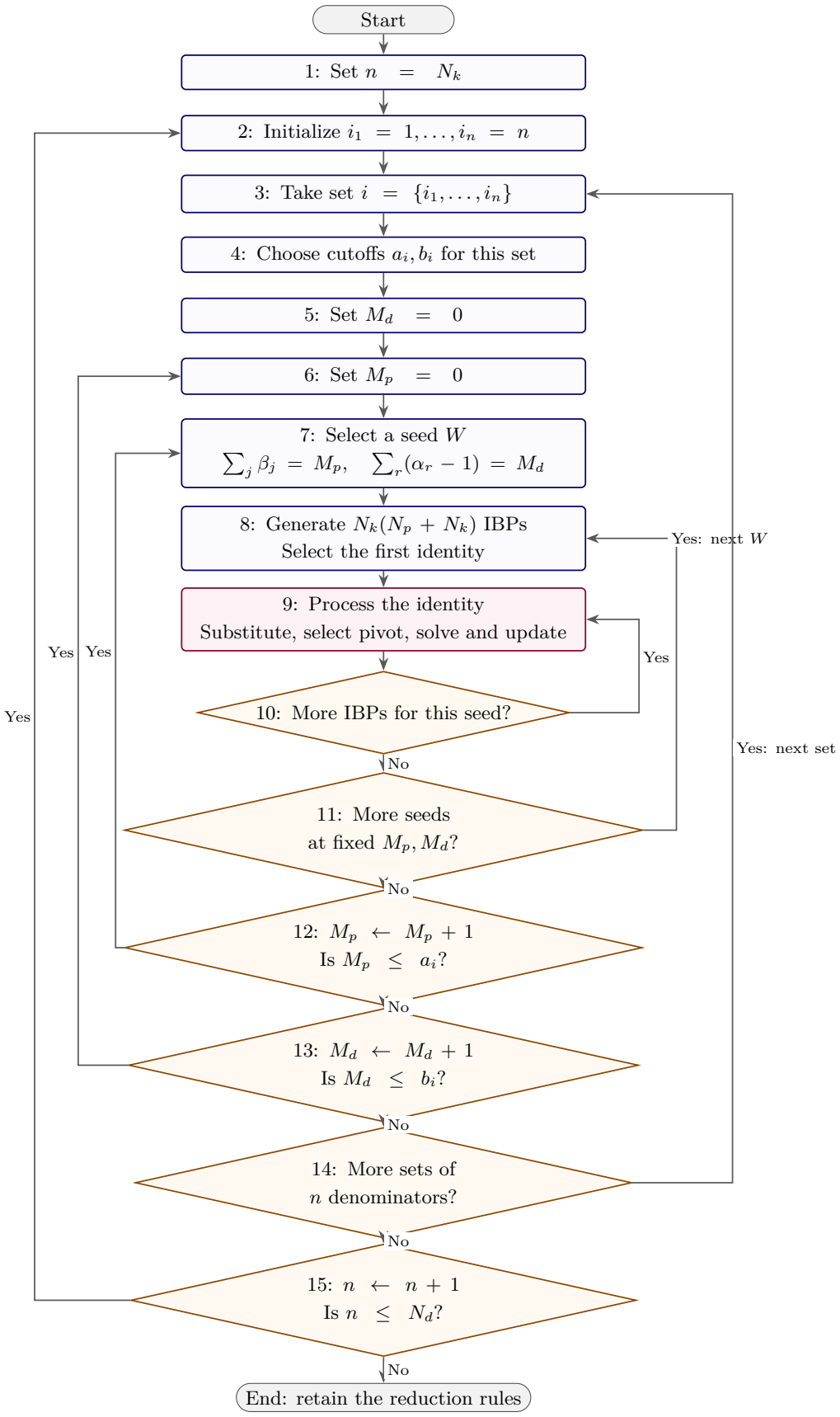


Figure 1: Flowchart of the Laporta's algorithm

We now examine Step 9, where each new IBP identity is processed using the chosen integral ordering. The corresponding elimination procedure is shown in Fig.2.

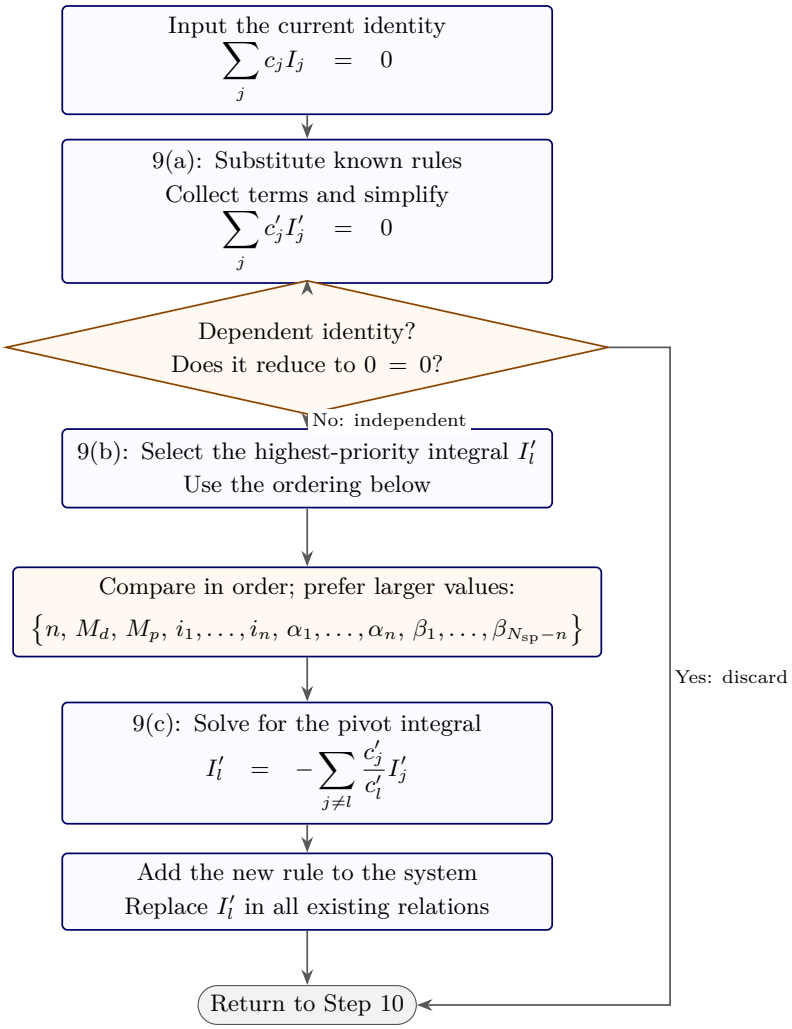


Figure 2: Step 9

Step 9 is essentially Gaussian elimination applied to the linear system of IBP identities, with the integrals treated as unknowns. Substituting known rules eliminates previously determined pivot integrals from the new equation. The highest-priority remaining integral is then chosen as the next pivot, and its coefficient is normalized to unity. Substituting the resulting rule into the existing relations eliminates this pivot from the other equations. The integral ordering therefore determines the column priority in the elimination.

2 Explicit examples

We now apply the procedure described above to explicit integral families. We begin with a one-loop massive bubble, for which the seed generation, integral ordering, and matrix elimination can all be displayed explicitly.

2.1 One-loop massive bubble

Consider the bubble family as Fig.3,

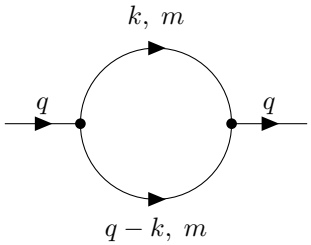


Figure 3: The one-loop bubble diagram

we define

$$D_1 = k^2 - m^2, \quad D_2 = (k+p)^2 - m^2, \quad s = p^2, \tag{2.1}$$

and the integral family takes

$$I_{a_1 a_2} = \int \mathrm{d}^D k \, \frac{1}{D_1^{a_1} D_2^{a_2}}. \tag{2.2}$$

We work at generic dimension and kinematics, with $m^2 \neq 0$, $s \neq 0$, and $s - 4m^2 \neq 0$. Our target is I_{21}.

This family has

$$N_k = 1, \quad N_p = 1, \quad N_d = 2, \quad N_{\mathrm{sp}} = 2. \tag{2.3}$$

Following Fig.1, we generate identities first in the one-denominator sectors ($n = N_k = 1$) and then in the two-denominator sector ($n = 2$).

One-denominator sectors: $n = 1$. The denominator combinations are $i = (1)$ and $i = (2)$. For each combination, choose

$$a_i = b_i = 0, \quad M_p = M_d = 0. \tag{2.4}$$

The corresponding seeds are $1/D_1$ and $1/D_2$. Although these sectors admit an ISP relative to the full external momentum space, the selected seeds have unit numerator.

For the first seed, choose the IBP vector $v^\mu = k^\mu$. Using

$$\frac{\partial}{\partial k^\mu} k^\mu = D, \quad k^\mu \frac{\partial D_1}{\partial k^\mu} = 2k^2 = 2(D_1 + m^2), \tag{2.5}$$

we obtain the first identity

$$\begin{aligned} 0 &= \int \mathrm{d}^D k \, \frac{\partial}{\partial k^\mu} \left(\frac{k^\mu}{D_1} \right) = D I_{10} - 2 \int \mathrm{d}^D k \, \frac{D_1 + m^2}{D_1^2} \\ &= (D-2) I_{10} - 2m^2 I_{20}. \end{aligned} \tag{2.6}$$

The ordering gives $I_{20} \succ I_{10}$, since they have the same denominator combination but $M_d = 1$ and 0, respectively. Thus Step 9 gives the first reduction rule

$$I_{20} = \frac{D-2}{2m^2} I_{10}. \tag{2.7}$$

For the second IBP momenta $v^\mu = k^\mu + p^\mu$, it gives

$$(k+p)^\mu \frac{\partial}{\partial k^\mu} D_2 = 2(D_2 + m^2), \tag{2.8}$$

the same calculation gives

$$(D-2) I_{01} - 2m^2 I_{02} = 0. \tag{2.9}$$

A shift of the loop momentum also gives

$$I_{01} = I_{10}, \quad I_{02} = I_{20}. \tag{2.10}$$

These equalities identify the two equal-mass tadpole sectors.

For the IBP momenta $v^\mu = p^\mu$, it gives

$$0 = \int \mathrm{d}^D k \, \frac{\partial}{\partial k^\mu} \left(\frac{p^\mu}{D_1} \right) = -2 \int \mathrm{d}^D k \, \frac{k \cdot p}{D_1^2}. \tag{2.11}$$

The vanishing can also be understood from the transformation $k \to -k$: the denominator $(k^2 - m^2 + i0)^2$ is unchanged, whereas the numerator $k \cdot p$ changes sign. For the seed $1/D_2$, we instead have

$$0 = \int \mathrm{d}^D k \, \frac{\partial}{\partial k^\mu} \left(\frac{p^\mu}{D_2} \right) = -2 \int \mathrm{d}^D k \, \frac{k \cdot p + s}{D_2^2}. \tag{2.12}$$

In this case, the oddness becomes explicit after the shift $\ell = k+p$. Using $s = p^2$, the numerator becomes $k\cdot p+s = \ell\cdot p$, and hence

$$\int \mathrm{d}^D k \, \frac{k \cdot p + s}{D_2^2} = \int \mathrm{d}^D \ell \, \frac{\ell \cdot p}{(\ell^2 - m^2)^2} = 0. \tag{2.13}$$

The relations above do not provide a new relation between the scalar tadpole integrals I_{20} and I_{10}.

Two-denominator sector: $n = 2$. There is only one denominator combination, $i = (1, 2)$. Again choose

$$a_{(1,2)} = b_{(1,2)} = 0. \tag{2.14}$$

The seed is therefore

$$W = \frac{1}{D_1 D_2}, \quad \alpha = (1, 1), \quad M_p = M_d = 0. \tag{2.15}$$

There is no ISP in this sector because

$$k^2 = D_1 + m^2, \quad 2k \cdot p = D_2 - D_1 - s. \tag{2.16}$$

The two IBP vectors are k^μ and p^μ. Their contracted derivatives are

v	$v \cdot \partial_k D_1$	$v \cdot \partial_k D_2$
k	$2(D_1 + m^2)$	$D_1 + D_2 - s + 2m^2$
p	$D_2 - D_1 - s$	$D_2 - D_1 + s$

. (2.17)

Where ∂_k correspond to $\partial/\partial k^\mu$. For $v^\mu = k^\mu$, expanding the total derivative gives

$$\begin{aligned} 0 &= DI_{11} - \int \mathrm{d}^D k \Big[\frac{2(D_1 + m^2)}{D_1^2 D_2} - \frac{D_1 + D_2 - s + 2m^2}{D_1 D_2^2} \Big] \\ &= (D-3)I_{11} - 2m^2 I_{21} + (s - 2m^2)I_{12} - I_{02}. \end{aligned} \tag{2.18}$$

For $v^\mu = p^\mu$, the divergence of the vector vanishes, giving

$$\begin{aligned} 0 &= -\int \mathrm{d}^D k \Big[\frac{D_2 - D_1 - s}{D_1^2 D_2} - \frac{D_2 - D_1 + s}{D_1 D_2^2} \Big] \\ &= sI_{21} - sI_{12} - I_{20} + I_{02}. \end{aligned} \tag{2.19}$$

Using $I_{02} = I_{20}$ and the known tadpole rule, these become

$$0 = -2m^2 I_{21} + (s - 2m^2)I_{12} + (D-3)I_{11} - \frac{D-2}{2m^2} I_{10}, \tag{2.20}$$

$$0 = sI_{21} - sI_{12}. \tag{2.21}$$

Ordering and Step 9. For the integrals in these equations, the ordering is

$$I_{21} \succ I_{12} \succ I_{11} \succ I_{10}. \tag{2.22}$$

In particular, I_{21} precedes I_{12} because their n, M_d, M_p, and denominator labels agree, while $\alpha_1 = 2$ for I_{21} and $\alpha_1 = 1$ for I_{12}.

Processing Eq. (2.20) first, we isolate its highest-priority integral:

$$I_{21} = \frac{s - 2m^2}{2m^2} I_{12} + \frac{D-3}{2m^2} I_{11} - \frac{D-2}{4m^4} I_{10}. \tag{2.23}$$

Substituting this rule into Eq. (2.21) gives

$$0 = \frac{s(s - 4m^2)}{2m^2} I_{12} + \frac{s(D-3)}{2m^2} I_{11} - \frac{s(D-2)}{4m^4} I_{10}. \tag{2.24}$$

The next pivot is I_{12}, so

$$I_{12} = -\frac{D-3}{s - 4m^2} I_{11} + \frac{D-2}{2m^2(s - 4m^2)} I_{10}. \tag{2.25}$$

Substitution into the previous rule completes the reduction of I_{21}.

Matrix elimination. Using the shift relation $I_{02} = I_{20}$, the relevant identities are

$$0 = (D-2)I_{10} - 2m^2 I_{20}, \tag{2.26}$$

$$0 = (D-3)I_{11} - 2m^2 I_{21} + (s - 2m^2)I_{12} - I_{20}, \tag{2.27}$$

$$0 = sI_{21} - sI_{12}. \tag{2.28}$$

The integral ordering gives

$$I_{21} \succ I_{12} \succ I_{11} \succ I_{20} \succ I_{10}. \tag{2.29}$$

In particular, I_{11} precedes I_{20} because the number of active denominators is compared before M_d.

With the columns arranged in this order, define

$$\mathbf{I} = (I_{21}, I_{12}, I_{11}, I_{20}, I_{10})^{\mathrm{T}}. \tag{2.30}$$

The three identities form the homogeneous system

$$A\mathbf{I} = 0, \quad A = \begin{pmatrix} 0 & 0 & 0 & -2m^2 & D-2 \\ -2m^2 & s - 2m^2 & D-3 & -1 & 0 \\ s & -s & 0 & 0 & 0 \end{pmatrix}. \tag{2.31}$$

The rows follow the seed-generation order: the tadpole identity is included before the two bubble identities.

First normalize the tadpole pivot:

$$R_1 \leftarrow -\frac{1}{2m^2} R_1. \tag{2.32}$$

This row represents $I_{20} - (D-2)I_{10}/(2m^2) = 0$. Eliminate I_{20} from the second row by

$$R_2 \leftarrow R_2 + R_1. \tag{2.33}$$

The resulting matrix is

$$\begin{pmatrix} 0 & 0 & 0 & 1 & -\dfrac{D-2}{2m^2} \\ -2m^2 & s-2m^2 & D-3 & 0 & -\dfrac{D-2}{2m^2} \\ s & -s & 0 & 0 & 0 \end{pmatrix}. \tag{2.34}$$

This is precisely the substitution of the known tadpole rule into the new bubble identity in Step 9(a).

Next select I_{21} as the pivot of the second row. Perform

$$R_2 \leftarrow -\frac{1}{2m^2} R_2, \qquad R_3 \leftarrow R_3 - sR_2. \tag{2.35}$$

We obtain

$$\begin{pmatrix} 0 & 0 & 0 & 1 & -\dfrac{D-2}{2m^2} \\ 1 & -\dfrac{s-2m^2}{2m^2} & -\dfrac{D-3}{2m^2} & 0 & \dfrac{D-2}{4m^4} \\ 0 & \dfrac{s(s-4m^2)}{2m^2} & \dfrac{s(D-3)}{2m^2} & 0 & -\dfrac{s(D-2)}{4m^4} \end{pmatrix}. \tag{2.36}$$

The third row now has I_{12} as its highest-priority remaining integral. Normalize this pivot and eliminate I_{12} from the second row:

$$R_3 \leftarrow \frac{2m^2}{s(s-4m^2)} R_3, \qquad R_2 \leftarrow R_2 + \frac{s-2m^2}{2m^2} R_3. \tag{2.37}$$

Finally, reorder the rows as (R_2, R_3, R_1). The reduced row-echelon form is

$$\mathrm{RREF}(A) = \begin{pmatrix} 1 & 0 & \dfrac{D-3}{s-4m^2} & 0 & -\dfrac{D-2}{2m^2(s-4m^2)} \\ 0 & 1 & \dfrac{D-3}{s-4m^2} & 0 & -\dfrac{D-2}{2m^2(s-4m^2)} \\ 0 & 0 & 0 & 1 & -\dfrac{D-2}{2m^2} \end{pmatrix}. \tag{2.38}$$

The pivot columns correspond to I_{21}, I_{12}, and I_{20}. Reading off the three rows gives

$$\boxed{\begin{aligned} I_{21} = I_{12} &= -\frac{D-3}{s-4m^2} I_{11} + \frac{D-2}{2m^2(s-4m^2)} I_{10}, \\ I_{20} &= \frac{D-2}{2m^2} I_{10}. \end{aligned}} \tag{2.39}$$

Together with $I_{02} = I_{20}$ and $I_{01} = I_{10}$, these rules complete the reduction of all scalar integrals appearing in the displayed identities. The non-pivot columns correspond to I_{11} and I_{10}, which form the master basis of the generic equal-mass family. Thus the same matrix contains both the lower-sector tadpole reduction and the top-sector bubble reduction.

Seed cutoffs. This reduction uses only seeds with $M_p = M_d = 0$ in every sector. Nevertheless, differentiating these seeds generates I_{20}, I_{21}, and I_{12}, which have $M_d = 1$. Thus a target need not itself belong to the seed range to be reducible by the generated identities. No increase of a_i or b_i is required for the present target.

2.2 Two-loop massless kite

We next consider a two-loop massless kite family as Fig.4.

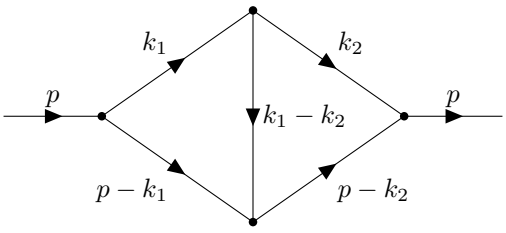


Figure 4: The two-loop massless kite diagram.

Unlike the bubble example, we display only the main stages of the reduction, rather than carrying the elimination through to the final master-integral representation.

We define

$$D_1 = k_1^2, \quad D_2 = (k_1 - p)^2, \quad D_3 = k_2^2, \quad D_4 = (k_2 - p)^2, \quad D_5 = (k_1 - k_2)^2, \quad s = p^2 \neq 0, \tag{2.40}$$

with the Feynman prescription implicit, and write

$$I_{a_1 a_2 a_3 a_4 a_5} = \int \mathrm{d}^D k_1 \mathrm{d}^D k_2 \, \frac{1}{D_1^{a_1} D_2^{a_2} D_3^{a_3} D_4^{a_4} D_5^{a_5}}. \tag{2.41}$$

We take I_{21111} as a representative target. The family has

$$N_k = 2, \quad N_p = 1, \quad N_d = 5, \quad N_{\mathrm{sp}} = 5. \tag{2.42}$$

Seed generation and sector organization. Following Algorithm 1, the seed identities are generated in the order

$$n = 2 \longrightarrow 3 \longrightarrow 4 \longrightarrow 5. \tag{2.43}$$

For each denominator combination, the exponents are enumerated within the chosen bounds

$$M_p \leq a_i, \quad M_d \leq b_i. \tag{2.44}$$

Scaleless sectors are set to zero, and equivalent integrals may be identified by loop-momentum transformations. The resulting lower-sector rules are then available when higher-sector identities are processed.

For clarity, the equations below are presented starting from the target sector and following the lower-sector integrals that it generates. This presentation should be distinguished from the ascending seed-generation order.

The five-denominator sector. All loop-dependent scalar products can be expressed through the five denominators:

$$k_1^2 = D_1, \quad k_2^2 = D_3, \quad 2k_1 \cdot p = D_1 + s - D_2, \quad 2k_2 \cdot p = D_3 + s - D_4, \quad 2k_1 \cdot k_2 = D_1 + D_3 - D_5. \tag{2.45}$$

Thus the top sector contains no ISP. Start with the scalar seed

$$W = \frac{1}{D_1 D_2 D_3 D_4 D_5}, \quad M_p = M_d = 0. \tag{2.46}$$

For each seed there are

$$N_k(N_p + N_k) = 6 \tag{2.47}$$

IBP identities, obtained by differentiating with respect to k_1 or k_2 and choosing $v^\mu = k_1^\mu, k_2^\mu, p^\mu$. Linear combinations of these vectors may also be used.

For example, choose the derivative with respect to k_1 and the vector $v^\mu = k_1^\mu - p^\mu$. The nonzero contracted

denominator derivatives are

$$\begin{aligned}(k_1-p)^\mu\frac{\partial}{\partial k_1^\mu}D_1 &= D_1+D_2-s,\\ (k_1-p)^\mu\frac{\partial}{\partial k_1^\mu}D_2 &= 2D_2,\\ (k_1-p)^\mu\frac{\partial}{\partial k_1^\mu}D_5 &= D_2+D_5-D_4.\end{aligned} \tag{2.48}$$

Since D_3 and D_4 are independent of k_1, their derivatives vanish. Expanding the total derivative gives

$$\begin{aligned}0 &= \int \mathrm{d}^D k_1 \mathrm{d}^D k_2 \,\frac{\partial}{\partial k_1^\mu}\left((k_1-p)^\mu W\right)\\ &= DI_{11111} - \int \mathrm{d}^D k_1 \mathrm{d}^D k_2 \,\frac{D_1+D_2-s}{D_1^2 D_2 D_3 D_4 D_5} - 2I_{11111} - \int \mathrm{d}^D k_1 \mathrm{d}^D k_2 \,\frac{D_2+D_5-D_4}{D_1 D_2 D_3 D_4 D_5^2}\\ &= (D-4)I_{11111} - I_{20111} - I_{10112} + I_{11102} + sI_{21111}.\end{aligned} \tag{2.49}$$

The ordering selects I_{21111} as the highest-priority integral in this equation. Therefore,

$$I_{21111} = -\frac{D-4}{s}I_{11111} + \frac{1}{s}\left(I_{20111} + I_{10112} - I_{11102}\right). \tag{2.50}$$

This is a first reduction rule, not the final result: the four-denominator integrals on the right must still be processed.

Another identity follows from $v^\mu = k_1^\mu$:

$$\begin{aligned}k_1^\mu\frac{\partial}{\partial k_1^\mu}D_1 &= 2D_1,\\ k_1^\mu\frac{\partial}{\partial k_1^\mu}D_2 &= D_1+D_2-s,\\ k_1^\mu\frac{\partial}{\partial k_1^\mu}D_5 &= D_1+D_5-D_3.\end{aligned} \tag{2.51}$$

It gives

$$0 = (D-4)I_{11111} - I_{02111} - I_{01112} + I_{11012} + sI_{12111}. \tag{2.52}$$

Matrix organization. To illustrate the coefficient-matrix construction, retain the two displayed identities and order their integrals as

$$\begin{aligned}\mathbf{I} = \big(&I_{21111}, I_{12111}, I_{11111}, I_{20111}, I_{11102},\\ &I_{11012}, I_{10112}, I_{02111}, I_{01112}\big)^{\mathrm{T}}.\end{aligned} \tag{2.53}$$

The four-denominator columns are also ordered according to their denominator labels and exponents. Equations (2.49) and (2.52) form the block

$$A_{\text{shown}}\mathbf{I} = 0, \tag{2.54}$$

where

$$A_{\text{shown}} = \begin{pmatrix} s & 0 & D-4 & -1 & 1 & 0 & -1 & 0 & 0\\ 0 & s & D-4 & 0 & 0 & 1 & 0 & -1 & -1\end{pmatrix}. \tag{2.55}$$

Normalizing the two rows by s gives pivots in the I_{21111} and I_{12111} columns. The first row reproduces Eq. (2.50).

This block contains only two selected identities. In the full calculation, the other seed identities and the lower-sector equations are included as additional rows. Every distinct integral that appears must have its own column, even if it lies outside the original seed range.

Four-denominator sectors and ISP numerators. Consider the sector with active denominators $\{D_1, D_3, D_4, D_5\}$, which appears through I_{10112}. These denominators determine k_1^2, k_2^2, $k_2\cdot p$, and $k_1\cdot k_2$, but do not determine $k_1\cdot p$.

We may therefore choose

$$S = k_1 \cdot p \tag{2.56}$$

as the ISP in this sector.

Introduce the notation

$$\begin{aligned} I_{abcd} &= \int \mathrm{d}^D k_1 \mathrm{d}^D k_2 \, \frac{1}{D_1^a D_3^b D_4^c D_5^d}, \\ J_{abcd} &= \int \mathrm{d}^D k_1 \mathrm{d}^D k_2 \, \frac{S}{D_1^a D_3^b D_4^c D_5^d}. \end{aligned} \tag{2.57}$$

For the scalar seed $1/(D_1 D_3 D_4 D_5)$, choose ∂_{k_1} and $v^\mu = p^\mu$. The required derivatives are

$$p^\mu \frac{\partial}{\partial k_1^\mu} D_1 = 2S, \quad p^\mu \frac{\partial}{\partial k_1^\mu} D_5 = 2S - D_3 + D_4 - s. \tag{2.58}$$

Consequently,

$$0 = -2J_{2111} - 2J_{1112} + I_{1012} - I_{1102} + sI_{1112}. \tag{2.59}$$

Thus numerator integrals can appear even when the seed has $M_p = 0$. The two J integrals have $M_p = 1$ and $M_d = 1$.

When to enlarge the numerator cutoff. Equation (2.59) alone constrains a combination of numerator integrals. It does not determine each of them separately. The first step is to combine it with the other available identities and apply elimination. The appearance of an ISP does not by itself imply that the cutoff must be increased.

If the required integrals remain unresolved, one possible extension is to increase the local cutoff to $a_i = 1$ and include the numerator seed

$$W_S = \frac{S}{D_1 D_3 D_4 D_5}, \quad M_p = 1, \quad M_d = 0. \tag{2.60}$$

The new identities must include the derivative of the numerator as well:

$$\frac{\partial}{\partial k_1^\mu}(v^\mu S F) = (\frac{\partial}{\partial k_1^\mu} v^\mu) W_S + (v^\mu \frac{\partial}{\partial k_1^\mu} S) \frac{1}{D_1 D_3 D_4 D_5} + S v^\mu \frac{\partial}{\partial k_1^\mu} \frac{S}{D_1 D_3 D_4 D_5}, \tag{2.61}$$

For example,

$$k_1^\mu \frac{\partial}{\partial k_1^\mu} S = S, \quad p^\mu \frac{\partial}{\partial k_1^\mu} S = s. \tag{2.62}$$

The resulting relations are appended to the coefficient matrix and processed using the same ordering. Whether this extension is sufficient is determined by the enlarged system, rather than by the numerator degree of the target alone.

Remaining reduction. The same procedure is applied to the three- and two-denominator sectors. Scaleless integrals vanish, while nonzero lower-sector integrals are reduced using their own IBP systems. Substituting these rules into Eq. (2.50) eventually expresses the target in terms of a master basis:

$$I_{21111} = \sum_\lambda r_\lambda(D, s) B_\lambda. \tag{2.63}$$

We stop here, having displayed the top-sector derivative calculation, the first reduction rule, the matrix construction, and the possible enlargement of the seed range in a sector with an ISP.

References

[Laporta(2000)] S. Laporta, Int. J. Mod. Phys. A **15**, 5087 (2000), arXiv:hep-ph/0102033 .