




Integration-By-Part Reduction

Lectures on Feynman Integrals

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Outline

1. Introduction
2. One-loop Vacuum
3. One-loop Bubble
4. IBP Reduction with Computer Programs

1 Introduction

2 One-loop Vacuum

3 One-loop Bubble

4 IBP Reduction with Computer Programs

From the Parameter Representations...

- L-loop scalar integral family with N propagators:

$$I_{a_1 a_2 \ldots a_N} = \int \prod_{\ell=1}^{L} \mathrm{d}^D k_\ell \, \frac{1}{P_1^{a_1} P_2^{a_2} \cdots P_N^{a_N}}, \quad a_i \in \mathbb{Z}.$$

- The Feynman parameter representation:

$$I_{a_1 a_2 \ldots a_N} \propto \frac{\Gamma(A - LD/2)}{\prod_{i=1}^{N} \Gamma(a_i)} \int_0^\infty \prod_{i=1}^{N} \left(\mathrm{d}x_i \, x_i^{a_i - 1} \right) \delta\left(1 - \sum_{i=1}^{N} x_i \right) \frac{\mathcal{U}^{A-(L+1)D/2}}{\mathcal{F}^{A-LD/2}}.$$

- Since $a_i \in \mathbb{Z}$, there are infinitely many such integrals:

Do we need to evaluate every integral?

Relations Among Integrals

- For each integral family:
  1. Integrals share the same propagators but have different powers.
  2. Changing the propagator powers leaves the kinematics unchanged.
- Question:

Can we find relations between those integrals?

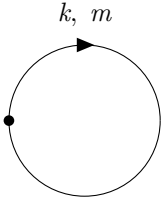
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3 One-loop Bubble

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The Topology and Integral Family



- One propagator of mass m, with no external momentum.

$$I_a(m^2) = \int \frac{\mathrm{d}^D k}{[k^2 - m^2 + i0]^a}, \quad a \in \mathbb{Z}.$$

- The Feynman parameter representation:

$$I_a(m^2) = \frac{\Gamma(a - D/2)}{\Gamma(a)} (m^2 - i0)^{D/2-a}.$$

Goal: Find a relation between integrals I_a.

The Total Derivative

- Consider the identity for each I_a:

$$\int \mathrm{d}^D k \frac{\partial}{\partial k^\mu} \frac{k^\mu}{[k^2 - m^2]^a} = 0.$$

The surface term vanishes in dim. reg., we leave the $+i0$ prescription implicit below.

The proof of this identity is left as an exercise.

- Evaluate the derivative:

$$\int \mathrm{d}^D k \Big[\frac{D}{(k^2 - m^2)^a} - \frac{2ak^2}{(k^2 - m^2)^{a+1}} \Big] = 0. \qquad \frac{\partial k^\mu}{\partial k^\mu} = \delta^\mu_\mu = D$$

- Therefore:

$$\int \mathrm{d}^D k \Big[\frac{D}{(k^2 - m^2)^a} - \frac{2a\cancel{(k^2 - m^2)}}{(k^2 - m^2)^{\cancel{a+1}a}} - \frac{2am^2}{(k^2 - m^2)^{a+1}} \Big] = 0. \qquad k^2 = (k^2 - m^2) + m^2$$

The Relations of Integrals

- We can obtain the relations:

$$(D-2a)I_a - 2am^2 I_{a+1} = 0.$$

- $a \to a-1$:

$$I_a = \frac{D-2a+2}{2(a-1)m^2} I_{a-1}, \quad a \geq 2.$$

The relation between I_a and I_{a-1}.

- For example:

$$I_2 = \frac{D-2}{2m^2} I_1, \quad I_3 = \frac{D-4}{4m^2} I_2 = \frac{(D-4)(D-2)}{8m^4} I_1.$$

- For $a \in \mathbb{Z},\ a > 0$:

$$I_a = \frac{(-1)^a (1-D/2)_{a-1}}{(a-1)!(m^2)^{a-1}} I_1. \qquad (x)_n = x(x+1)(x+2)\cdots(x+n) = \frac{\Gamma(x+n)}{\Gamma(x)}$$

For $a \leq 0$, the integrals become scaleless and vanish.

The Master Integrals and IBP Identity

- Now, all integrals I_a are determined by I_1 alone:

$$I_1 = \Gamma(1 - D/2)(m^2)^{D/2-1}.$$

  We call I_1 as a *master integral*.

- The Integration-By-Part (IBP) identity:

$$\int \prod_{\ell=1}^{L} \mathrm{d}^D k_\ell \frac{\partial}{\partial k_i^\mu} \left[\frac{q_{\mathrm{IBP}}^\mu}{P_1^{a_1} P_2^{a_2} \cdots P_N^{a_N}} \right] = 0. \qquad q_{\mathrm{IBP}}^\mu \in \{k_\ell^\mu, q_i^\mu, k_i \ell^\mu + q_i^\mu, \cdots\}$$

  $q_{\mathrm{IBP}} \to$ IBP momenta: Linear combination of k_ℓ and external momenta q_i.

- The procedure above is a simple example of IBP reduction:

  1. Choose q_{IBP}^μ form loop and external momenta.
  2. Expand the derivative and get the relations between integrals.
  3. Reduce the integrals to master integrals.

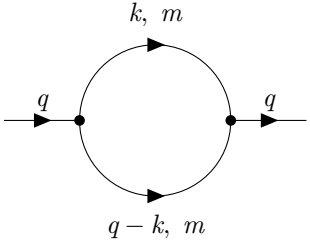
1 Introduction

2 One-loop Vacuum

3 One-loop Bubble

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The Topology and Integral Family



- Two propagator of mass m and external momentum q:

$$s := q^2 \neq 0$$

$$I_{a_1\,a_2}(s, m^2) = \int \frac{\mathrm{d}^D k}{[k^2 - m^2]^{a_1}[(q-k)^2 - m^2]^{a_2}}.$$

Massless Case: IBP Identity

- Set $m = 0$:

$$I_{a_1\,a_2}(s) = \int \frac{\mathrm{d}^D k}{[k^2]^{a_1}[(q-k)^2]^{a_2}}.$$

- IBP identity:

$$\int \mathrm{d}^D k \frac{\partial}{\partial k^\mu} \frac{k^\mu}{[k^2]^{a_1}[(q-k)^2]^{a_2}} = 0. \qquad q^\mu_{\mathrm{IBP}} = k^\mu$$

- Therefore:

$$\int \mathrm{d}^D k \Big[\frac{D}{[k^2]^{a_1}[(q-k)^2]^{a_2}} - \frac{2a_1 k^2}{[k^2]^{a_1+1}[(q-k)^2]^{a_2}} - \frac{2a_2(k^2 - q\cdot k)}{[k^2]^{a_1}[(q-k)^2]^{a_2+1}} \Big] = 0.$$

Using $2k\cdot(k-q) = k^2 + (q-k)^2 - q^2$:

$$\frac{2a_2(k^2 - q\cdot k)}{[k^2]^{a_1}[(q-k)^2]^{a_2+1}} = \frac{2a_2}{[k^2]^{a_1-1}[(q-k)^2]^{a_2+1}} + \frac{a_2\cancel{(q-k)^2}}{[k^2]^{a_1}[(q-k)^2]^{\cancel{a_2+1}a_2}} - \frac{a_2(\cancel{k^2} + s)}{[k^2]^{\cancel{a_1}a_1-1}[(q-k)^2]^{a_2+1}}.$$

- The relation:

$$(D - 2a_1 - a_2) I_{a_1\,a_2} - a_2 I_{(a_1-1)(a_2+1)} + a_2 s I_{a_1(a_2+1)} = 0.$$

Massless Case: Reduction

- $a_2 \to a_2 - 1$:

$$\boxed{I_{a_1 a_2} = \frac{D - 2a_1 - a_2 + 1}{(a_2 - 1)s} I_{a_1 (a_2 - 1)} + \frac{1}{s} I_{(a_1 - 1) a_2}.}$$

- The exchange symmetry: $k \to q - k, \quad I_{ab} = I_{ba}$:

$$I_{a_1 a_2} = -\frac{D - 2a_2 - a_1 + 1}{(a_1 - 1)s} I_{(a_1 - 1) a_2} + \frac{1}{s} I_{a_1 (a_2 - 1)}.$$

- When an index reaches zero, the integral becomes scaleless:

$$I_{a_1 0} = I_{0 a_2} = 0.$$

- When $a_2 = 1$:

$$\boxed{I_{a_1 1} = \frac{(3 - D)_{a_1 - 1}}{(a_1 - 1)! s^{a_1 - 1}} I_{11}.}$$

Reduction Rules

- The master integral: I_{11}.

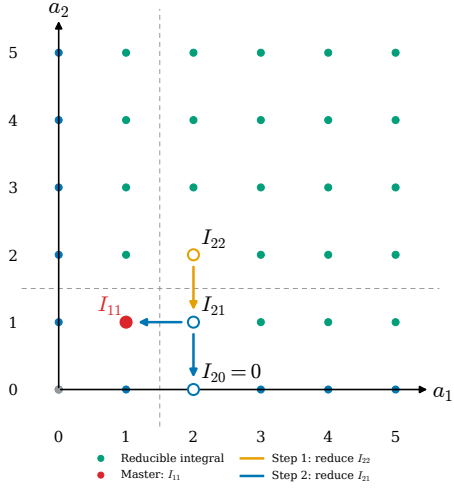
Massless Case: Reduction of I_{22}

- The reduction of I_{22}:

$$
\begin{aligned}
I_{22} =& -\frac{D-5}{s} I_{21} + \frac{1}{s} I_{12}, \\
I_{12} =& I_{21}, \\
I_{21} =& -\frac{D-3}{s} I_{11} + \underbrace{\frac{1}{s} I_{20}}_{=0}.
\end{aligned}
$$

- The result:

$$
I_{22} = \frac{(D-6)(D-3)}{s^2} I_{11}.
$$



Massive Case: IBP Identity

- For massive case:

$$I_{a_1 a_2}(s, m^2) = \int \frac{\mathrm{d}^D k}{[k^2 - m^2]^{a_1}[(q-k)^2 - m^2]^{a_2}}.$$

- IBP identity:

$$\int \mathrm{d}^D k \frac{\partial}{\partial k^\mu} \frac{k^\mu}{[k^2 - m^2]^{a_1}[(q-k)^2 - m^2]^{a_2}} = 0. \qquad q^\mu_{\mathrm{IBP}} = k^\mu$$

- Therefore:

$$(D - 2a_1 - a_2) I_{a_1 a_2} - 2a_1 m^2 I_{(a_1+1)a_2} - a_2 I_{(a_1-1)(a_2+1)} + a_2(s - 2m^2) I_{a_1(a_2+1)} = 0$$

Massive Case: Reduction Rules

- Eliminate $I_{(a_1+1)a_2}$ by symmetry $I_{ab} = I_{ba}$:

$$\begin{aligned} I_{a_1 a_2} = & -\frac{s(D-2a_1-a_2+1)+2m^2(a_1-a_2+1)}{s(a_2-1)(s-4m^2)} I_{a_1,a_2-1} \\ & + \frac{s-2m^2}{s(s-4m^2)} I_{a_1-1,a_2} + \frac{2a_1 m^2}{s(a_2-1)(s-4m^2)} I_{a_1+1,a_2-2}, \quad a_2 \geq 2. \end{aligned}$$

$$\begin{aligned} I_{a_1 a_2} = & -\frac{s(D-a_1-2a_2+1)+2m^2(a_2-a_1+1)}{s(a_1-1)(s-4m^2)} I_{a_1-1,a_2} \\ & + \frac{s-2m^2}{s(s-4m^2)} I_{a_1,a_2-1} + \frac{2a_2 m^2}{s(a_1-1)(s-4m^2)} I_{a_1-2,a_2+1}, \quad a_1 \geq 2. \end{aligned}$$

- When one index is zero:

$$I_{a0} = I_{0a} = \int \frac{\mathrm{d}^D k}{[k^2-m^2]^a} = \frac{(-1)^a (1-D/2)_{a-1}}{(a-1)!(m^2)^{a-1}} I_{01}.$$

- Two master integrals: I_{11}, I_{01}.

Massive Case: Reduction of I_{22}

- Set $(a_1, a_2) = (2, 2)$:

$$I_{22} = -\frac{s(D-5)+2m^2}{s(s-4m^2)} I_{21} + \frac{s-2m^2}{s(s-4m^2)} I_{12} + \frac{4m^2}{s(s-4m^2)} I_{30},$$
$$I_{12} = I_{21}, \quad I_{30} = I_{03} = \frac{D-4}{4m^2} I_{02},$$

- Therefore:

$$I_{22} = \frac{(6-D)s-4m^2}{s(s-4m^2)} I_{12} + \frac{D-4}{s(s-4m^2)} I_{02}.$$

- Set $(a_1, a_2) = (1, 1)$:

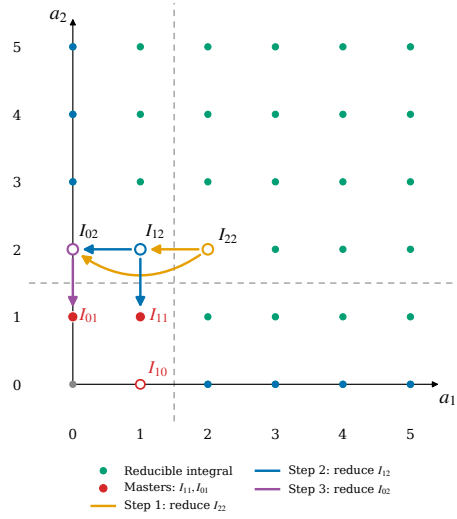
$$I_{12} = -\frac{D-3}{s-4m^2} I_{11} + \frac{1}{s-4m^2} I_{02}, \quad I_{02} = \frac{D-2}{2m^2} I_{01}.$$

- The result:

$$\boxed{I_{22} = \frac{(D-3)\left[(D-6)s+4m^2\right]}{s(s-4m^2)^2} I_{11} + \frac{(D-2)\left[s-2(D-3)m^2\right]}{m^2 s(s-4m^2)^2} I_{01}.} \tag{1}$$

Massive Case: The Reduction Path

- The reduction path:



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4 IBP Reduction with Computer Programs

Why use computer programs?

- With more propagators or higher powers, the number of intermediate integrals grows quickly.

Package	**Role**
`LiteRed` / `LiteRed2`	Reduction rules.

R. N. Lee, J. Phys. Conf. Ser. 523, 012059 (2014).

`FIRE`, `Kira`, `Reduze`	Laporta algorithm.

A. V. Smirnov and F. S. Chukharev, Comput. Phys. Commun. 247, 106877 (2020).
P. Maierhöfer et al., Comput. Phys. Commun. 230, 99–112 (2018).

A. von Manteuffel and C. Studerus, arXiv:1201.4330 (2012).

- Examples: Use `LiteRed2` to verify the massless and massive bubble reductions.