Baikov Representation

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These notes introduce Baikov variables through a one-loop bubble and then derive the general scalar representation directly on a real Minkowski integration domain. Appendix A explains Gram determinants, area elements and wedge products; Appendix B derives the angular measure. The discussion follows the framework of Weinzierl [**?**], with real-volume Jacobians and the Minkowski normalization kept explicit.

Notation

We use

$$\eta_{\mu\nu} = \mathrm{diag}(1, -1, \ldots, -1), \qquad u \cdot v = \eta_{\mu\nu} u^\mu v^\nu, \qquad G(q_1, \ldots, q_r)_{ab} = -q_a \cdot q_b.$$

The loop number is l; the external momenta span $P = \mathrm{span}\{p_1, \ldots, p_e\}$, where the displayed p_a form an independent basis. The number of independent scalar-product coordinates involving loop momenta is

$$N_V = \frac{l(l+1)}{2} + el.$$

They are $-k_i^2$, $-k_i \cdot k_j$ for $i < j$, and $-k_i \cdot p_a$. We use n_{ext} and $n_{\mathrm{int}}$to denote the numbers of external and internal lines.

> **Convention.** The direct spherical-measure proof assumes that P contains a timelike direction. Start at integer $D \geq e + l$ and generic full-rank configurations. The resulting formula is subsequently continued in D and propagator powers. Purely spacelike or degenerate-null external spans, and vacuum integrals, are not covered by this real-spherical proof without further analysis.

All $\mathrm{d}^m x$ below are ordinary real volume measures; their changes of variables use absolute Jacobian determinants. The denominator prescription is $(z_s - i0)^{-a_s}$; the z_s themselves are unshifted real inverse propagators. Angular reductions are understood inside integrals of scalar-product-dependent functions, with a suitable regulator or analytic continuation where needed.

1 A one-loop bubble

1.1 Integral, variables and orthogonal decomposition

Assume that p is future-directed and timelike. Consider

$$I_{a_1,a_2}(s) = \int \frac{\mathrm{d}^D k}{i\pi^{D/2}} \frac{1}{(-k^2+m_1^2-i0)^{a_1}(-(k-p)^2+m_2^2-i0)^{a_2}}, \qquad s=p^2>0. \tag{1.1}$$

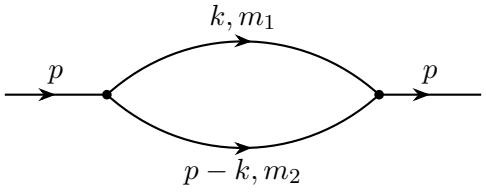


Figure 1: One-loop bubble.

Define

$$z_1 = -k^2+m_1^2, \qquad z_2 = -(k-p)^2+m_2^2. \tag{1.2}$$

Expanding the second square and solving gives

$$k^2 = m_1^2 - z_1, \qquad k\cdot p = \frac{z_2 - z_1 + s + m_1^2 - m_2^2}{2}. \tag{1.3}$$

Thus the integrand depends on k only through these two scalar products. The remaining freedom is the transverse direction, which will be integrated out.

Let $\hat{p} = p/\sqrt{s}$ and write

$$k = x\hat{p} + k_\perp, \qquad x = \frac{k\cdot p}{\sqrt{s}}, \qquad \hat{p}\cdot k_\perp = 0, \qquad \hat{p}^2 = 1. \tag{1.4}$$

In the rest frame of p,

$$p = (\sqrt{s}, \mathbf{0}), \qquad k = (x, \mathbf{y}), \qquad k_\perp = (0, \mathbf{y}), \qquad r = |\mathbf{y}| = \sqrt{-k_\perp^2}. \tag{1.5}$$

Consequently

$$k^2 = x^2 - r^2, \qquad (k-p)^2 = (x-\sqrt{s})^2 - r^2. \tag{1.6}$$

The condition $p\cdot k_\perp = 0$ forces the transverse time component to vanish. On this transverse subspace, $-v\cdot w = \sum_a v^a w^a$ is a positive-definite Euclidean inner product. This is the precise reason spatial spherical coordinates can be used.

1.2 Transverse spherical measure and the squared radius

In the chosen frame the real coordinate volume factorizes as

$$\mathrm{d}^D k = \mathrm{d}x\ \mathrm{d}^{D-1}\mathbf{y} = \mathrm{d}x\, r^{D-2}\,\mathrm{d}r\ \mathrm{d}\Omega_{D-2}, \qquad \int \mathrm{d}\Omega_{D-2} = \frac{2\pi^{(D-1)/2}}{\Gamma((D-1)/2)}. \tag{1.7}$$

Here the angular subscript counts the dimension of the sphere: $\mathbf{y}/r \in S^{D-2}$. The metric sign affects k^2, not the real-coordinate volume $\mathrm{d}x \prod_a \mathrm{d}y^a$. No Wick rotation has been performed.

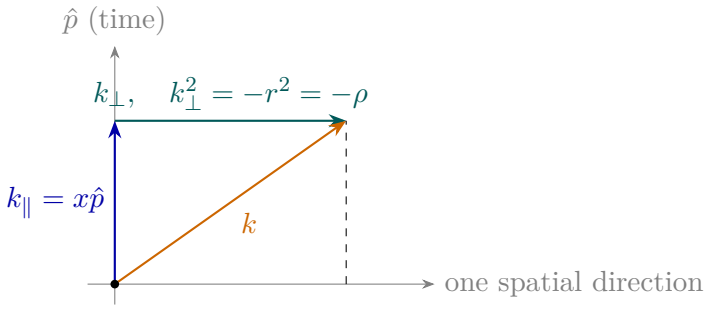


Figure 2: Component decomposition, drawn on a page with Euclidean plotting coordinates. Minkowski lengths are computed with η; the drawing does not assign a Euclidean angle or length to the time direction.

Introduce the squared transverse radius

$$\rho = r^2 = -k_\perp^2 \geq 0, \qquad r^{D-2}\,\mathrm{d}r = \frac{1}{2}\rho^{(D-3)/2}\,\mathrm{d}\rho. \tag{1.8}$$

The transformation $r \mapsto \rho$ is increasing on $r > 0$. Substituting $k^2 = x^2 - \rho$ into the inverse propagators gives

$$\boxed{z_1 = -x^2 + \rho + m_1^2, \qquad z_2 = -(x - \sqrt{s})^2 + \rho + m_2^2.} \tag{1.9}$$

In particular, the plus sign of ρ in z_1 comes from $-k_\perp^2 = \rho$. Since the denominators are independent of transverse angles, their angular integral can now be completed:

$$I_{a_1,a_2}(s) = \frac{1}{i\sqrt{\pi}\,\Gamma((D-1)/2)} \int_{-\infty}^{\infty} \mathrm{d}x \int_0^{\infty} \mathrm{d}\rho\, \frac{\rho^{(D-3)/2}}{(z_1 - i0)^{a_1}(z_2 - i0)^{a_2}}. \tag{1.10}$$

Changing to spherical coordinates is a coordinate transformation. Replacing the angular integral by its total solid angle additionally uses rotational invariance.

1.3 Jacobian and the Baikov polynomial

The Jacobian matrix following from (1.9) is

$$\frac{\partial(z_1,z_2)}{\partial(x,\rho)} = \begin{pmatrix} -2x & 1 \\ -2(x-\sqrt{s}) & 1 \end{pmatrix}, \qquad \det\frac{\partial(z_1,z_2)}{\partial(x,\rho)} = -2\sqrt{s}. \tag{1.11}$$

For real volume measures this gives

$$\mathrm{d}x\ \mathrm{d}\rho = \frac{\mathrm{d}z_1\ \mathrm{d}z_2}{2\sqrt{s}}. \tag{1.12}$$

The negative determinant records an orientation reversal; the volume measure uses its absolute value. The inverse map is unique:

$$x = \frac{z_2 - z_1 + s + m_1^2 - m_2^2}{2\sqrt{s}}, \qquad \rho = z_1 - m_1^2 + x^2.$$

Keep the same polynomial definition as in the general case:

$$\begin{aligned} \mathcal{B}(z_1,z_2) = \det G(k,p) &= \det\begin{pmatrix} -k^2 & -k\cdot p \\ -k\cdot p & -s \end{pmatrix} \\ &= s(m_1^2 - z_1) - \frac{1}{4}(z_2 - z_1 + s + m_1^2 - m_2^2)^2. \end{aligned} \tag{1.13}$$

The orthogonal decomposition immediately gives

$$\boxed{\mathcal{B} = sk^2 - (k\cdot p)^2 = -s\rho, \qquad \rho = -\frac{\mathcal{B}}{s}.} \tag{1.14}$$

Thus the real integration domain is

$$\mathcal{C}_{\mathrm{M}} = \{(z_1,z_2)\in\mathbb{R}^2 : \mathcal{B}(z_1,z_2)\le 0\}. \tag{1.15}$$

Combining (1.10)–(1.14),

$$\boxed{I_{a_1,a_2}(s) = \frac{s^{(2-D)/2}}{2i\sqrt{\pi}\,\Gamma((D-1)/2)}\int_{\mathcal{C}_{\mathrm{M}}} \mathrm{d}z_1\,\mathrm{d}z_2\,\frac{[-\mathcal{B}(z_1,z_2)]^{(D-3)/2}}{(z_1-i0)^{a_1}(z_2-i0)^{a_2}}.} \tag{1.16}$$

2 The general scalar Baikov representation

2.1 Complete variables and assumptions

Let $\sigma \in \mathbb{R}^{N_V}$ list the independent scalar products involving the l loop momenta. Choose a complete affine set of variables

$$z_s = C_{st}\sigma_t + f_s, \qquad \det C \neq 0, \qquad \sigma = C^{-1}(z - f). \tag{2.1}$$

For physical propagators $z_s = -q_s^2 + m_s^2$. If there are fewer independent denominators than N_V, add auxiliary variables and assign their powers zero. If there are more denominators, choose independent coordinates and express the remaining denominators as affine functions of z. Equality of the physical line count with N_V is not a general requirement.

The real-volume Jacobian is

$$\frac{\partial z_s}{\partial \sigma_t} = C_{st}, \qquad \mathrm{d}^{N_V}\sigma = \frac{\mathrm{d}^{N_V} z}{|\det C|}. \tag{2.2}$$

For a complete family with denominator powers $\mathbf{a} = (a_1, \ldots, a_{N_V})$,

$$I_{\mathbf{a}} = \int \prod_{i=1}^{l} \frac{\mathrm{d}^D k_i}{i\pi^{D/2}} \prod_{s=1}^{N_V} (z_s - i0)^{-a_s}. \tag{2.3}$$

Define

$$E = G(p_1, \ldots, p_e), \qquad \Delta = -\det E > 0, \qquad \mathcal{B}(z) = \det G(k_1, \ldots, k_l, p_1, \ldots, p_e)\big|_{\sigma(z)}. \tag{2.4}$$

Because P contains a timelike direction, the restricted Minkowski metric on P has signature $(1, e-1)$, while E has one negative eigenvalue. Hence $\det E < 0$.

Proof roadmap. First establish a Gram-ratio identity. Next derive the measure transformation for one loop momentum, keeping the others fixed. Repeat it for each loop; intermediate Gram determinants cancel. Finally apply (2.2) and describe the domain.

Why the external-space assumption matters. The orthogonal complement of a subspace containing a timelike vector is spacelike. Every nested parallel space below contains P, so every transverse angular integration takes place in a positive-definite spatial space after reversing the metric sign. This is not true for a purely spacelike or a degenerate-null P. Individual external momenta may be null; the assumption concerns their entire span. For example, two noncollinear future-directed null momenta have a timelike sum.

2.2 Nested parallel spaces and Gram ratios

Process $k_1, k_2, \ldots, k_l$ in that order. At stage i, hold the remaining loop momenta fixed and set

$$V_i = \mathrm{span}\{k_{i+1}, \ldots, k_l, p_1, \ldots, p_e\}, \qquad \alpha_i = e + l - i, \qquad n_i = D - \alpha_i. \tag{2.5}$$

Thus

$$P = V_l \subseteq V_{l-1} \subseteq \cdots \subseteq V_1. \tag{2.6}$$

These are the parallel spaces for the successive decompositions, not mutually parallel affine planes. All contain the same timelike direction from P. For example, at two loops, $V_1 = \mathrm{span}\{k_2, p_1, \ldots, p_e\}$ and $V_2 = P$.

Define the matrices and positive determinants

$$\begin{aligned} G_i &= G(k_i, \ldots, k_l, p_1, \ldots, p_e), & h_i &= -\det G_i, \quad 1 \le i \le l, \\ G_{l+1} &= E, & h_{l+1} &= \Delta. \end{aligned} \tag{2.7}$$

On the generic interior $h_i > 0$. Decompose

$$k_i = k_{i,\parallel} + k_{i,\perp}, \qquad k_{i,\parallel} \in V_i, \quad k_{i,\perp} \perp V_i, \qquad \rho_i = -k_{i,\perp}^2 > 0. \tag{2.8}$$

The determinant identity proved in Appendix A.4 gives

$$\boxed{\rho_i = \frac{\det G_i}{\det G_{i+1}} = \frac{h_i}{h_{i+1}}.} \tag{2.9}$$

Both determinants in the first ratio are negative. Their ratio is the nonnegative transverse squared radius; the individual determinants need not be positive.

Start the coordinate proof in generic integer dimension $D \ge e + l$. This ensures that the full list of vectors can be independent and that $n_i \ge 1$. The formulas are derived on the interior, where the inverses and square roots exist. Boundary contributions and singular integrals are understood by a common regulator and analytic continuation, rather than by dividing by a vanishing minor.

2.3 The parallel-coordinate Jacobian

Choose the ordered basis

$$(u_1, \ldots, u_{\alpha_i}) = (k_{i+1}, \ldots, k_l, p_1, \ldots, p_e)$$

of V_i and expand

$$k_{i,\parallel} = \sum_{b=1}^{\alpha_i} (c_i)_b u_b, \qquad (\tau_i)_a = -k_i \cdot u_a. \tag{2.10}$$

Since $k_{i,\perp} \cdot u_a = 0$ and $(G_{i+1})_{ab} = -u_a \cdot u_b$,

$$(\tau_i)_a = -\sum_b (c_i)_b u_b \cdot u_a = \sum_b (G_{i+1})_{ab} (c_i)_b, \qquad \boxed{\tau_i = G_{i+1} c_i.} \tag{2.11}$$

There is no extra minus sign in this relation. It follows that

$$\mathrm{d}^{\alpha_i}\tau_i = |\det G_{i+1}|\, \mathrm{d}^{\alpha_i} c_i = h_{i+1}\, \mathrm{d}^{\alpha_i} c_i. \tag{2.12}$$

Let x_i be the components of $k_{i,\parallel}$ in a Minkowski-orthonormal basis of V_i. If the columns of J_i are the components of the u_b in that basis,

$$x_i = J_i c_i, \qquad G_{i+1} = -J_i^T \eta_\parallel J_i, \qquad \eta_\parallel = \mathrm{diag}(1, -1, \ldots, -1). \tag{2.13}$$

There is one time direction and $\alpha_i - 1$ spatial directions, so

$$\det G_{i+1} = (-1)^{\alpha_i} (\det J_i)^2 \det \eta_\parallel = -(\det J_i)^2. \tag{2.14}$$

Consequently

$$\mathrm{d}^{\alpha_i} k_{i,\parallel} = |\det J_i|\, \mathrm{d}^{\alpha_i} c_i = \sqrt{h_{i+1}}\, \mathrm{d}^{\alpha_i} c_i. \tag{2.15}$$

Combining this with (2.12),

$$\boxed{\mathrm{d}^{\alpha_i} k_{i,\parallel} = \frac{\mathrm{d}^{\alpha_i}\tau_i}{\sqrt{h_{i+1}}}.} \tag{2.16}$$

This step separates two different Jacobians: one from an orthonormal coordinate system to the generally nonorthogonal coefficients c_i, and one from c_i to the projection scalar products τ_i.

2.4 The transverse measure and one-step formula

Set $\xi_i = -k_i^2$. By (2.11),

$$-k_{i,\parallel}^2 = c_i^T G_{i+1} c_i = \tau_i^T G_{i+1}^{-1} \tau_i. \tag{2.17}$$

The orthogonal decomposition therefore gives

$$\rho_i = \xi_i - \tau_i^T G_{i+1}^{-1} \tau_i. \tag{2.18}$$

With the other loop momenta held fixed, the transformation $(\xi_i, \tau_i) \mapsto (\rho_i, \tau_i)$ has a block triangular Jacobian whose diagonal is $(1, I_{\alpha_i})$. Hence its determinant is 1. Introduce the block of scalar-product measures

$$\mathrm{d}\mu_i = \mathrm{d}\xi_i \; \mathrm{d}^{\alpha_i}\tau_i = \mathrm{d}(-k_i^2) \prod_{r=i+1}^{l} \mathrm{d}(-k_i \cdot k_r) \prod_{a=1}^{e} \mathrm{d}(-k_i \cdot p_a). \tag{2.19}$$

The transverse space has dimension n_i and is spacelike. After its angular integration,

$$\mathrm{d}^{n_i} k_{i,\perp} \;\longrightarrow\; \frac{\pi^{n_i/2}}{\Gamma(n_i/2)} \rho_i^{n_i/2-1} \, \mathrm{d}\rho_i. \tag{2.20}$$

The arrow denotes a measure reduction inside an angular-invariant integral. Define

$$\beta_i = \frac{n_i}{2} - 1 = \frac{D - l - e + i - 2}{2}. \tag{2.21}$$

Using (2.16) and (2.18),

$$\mathrm{d}^D k_i \;\longrightarrow\; \frac{\pi^{n_i/2}}{\Gamma(n_i/2)} \frac{\rho_i^{\beta_i}}{\sqrt{h_{i+1}}} \, \mathrm{d}\mu_i. \tag{2.22}$$

Substitute (2.9) and keep the original loop normalization:

$$\boxed{\frac{\mathrm{d}^D k_i}{i\pi^{D/2}} \;\longrightarrow\; \frac{\pi^{-\alpha_i/2}}{i\,\Gamma(n_i/2)} h_i^{\beta_i} h_{i+1}^{-\beta_i - 1/2} \, \mathrm{d}\mu_i.} \tag{2.23}$$

The factor $1/i$ remains because this is a real Minkowski coordinate calculation. At this stage no complex rotation of the energy variable has supplied a compensating factor. The scalar integrand is invariant under rotations of $k_{i,\perp}$ that fix V_i, since every remaining momentum lies in V_i.

2.5 Telescoping products and the final formula

The successive powers satisfy $\beta_{i+1} = \beta_i + 1/2$. For an interior factor h_j, $2 \le j \le l$, its total exponent in the product of (2.23) is

$$\beta_j - \beta_{j-1} - \tfrac{1}{2} = 0.$$

Thus

$$\begin{aligned}\prod_{i=1}^{l} h_i^{\beta_i} h_{i+1}^{-\beta_i - 1/2} &= h_1^{\beta_1} h_{l+1}^{-\beta_l - 1/2} \\ &= [-\mathcal{B}]^{(D-l-e-1)/2} \Delta^{(-D+e+1)/2}.\end{aligned} \tag{2.24}$$

All intermediate Gram determinants cancel. The remaining combinatorial factors are

$$\sum_{i=1}^{l} \alpha_i = el + \frac{l(l-1)}{2} = N_V - l, \qquad \prod_{i=1}^{l} \Gamma(n_i/2) = \prod_{j=1}^{l} \Gamma\Big(\frac{D-e+1-j}{2}\Big). \tag{2.25}$$

Each independent scalar product occurs once in (2.19), and hence

$$\prod_{i=1}^{l} \mathrm{d}\mu_i = \mathrm{d}^{N_V}\sigma = \frac{\mathrm{d}^{N_V} z}{|\det C|}. \tag{2.26}$$

Substituting these identities into (2.3) gives the direct real-Minkowski Baikov representation:

$$\boxed{\begin{aligned} I_{\mathbf{a}} = {} & \frac{i^{-l}\pi^{-(N_V-l)/2}[-det(p_1,\cdots,p_e)]^{(-D+e+1)/2}}{|\det C| \prod_{j=1}^{l} \Gamma\Big(\frac{D-e+1-j}{2}\Big)} \\ & \times \int_{\mathcal{C}_{\mathrm{M}}} \mathrm{d}^{N_V} z\, [-\mathcal{B}(z)]^{(D-l-e-1)/2} \prod_{s=1}^{N_V} (z_s - i0)^{-a_s}. \end{aligned}} \tag{2.27}$$

The polynomial convention is still $\mathcal{B} = \det G(k_1,\cdots,k_l,p_1,\cdots,p_e)$, whereas the real radial factor appearing in this Lorentzian region is $-\mathcal{B} \ge 0$. The full prefactor includes i^{-l} from the normalization of the original l loop measures.

2.6 Domain, boundaries and continuation

For a description independent of the chosen nested order, split all loop momenta relative to the external space P and write

$$G_1 = \begin{pmatrix} A & H \\ H^T & E \end{pmatrix}, \qquad A_{ij} = -k_i \cdot k_j, \quad H_{ia} = -k_i \cdot p_a, \quad T = A - HE^{-1}H^T. \tag{2.28}$$

Appendix A.5 proves

$$T_{ij} = -k_{i,\perp P} \cdot k_{j,\perp P}, \qquad T \succeq 0, \qquad \mathcal{B} = (\det E) \det T = -\Delta \det T. \tag{2.29}$$

Here $\perp P$ refers to the common external orthogonal complement, rather than to the stage-dependent $V_i^{\perp}$. Thus, in the starting integer dimension,

$$\boxed{\mathcal{C}_{\mathrm{M}} = \{z \in \mathbb{R}^{N_V} : T(z) \succeq 0\}.} \tag{2.30}$$

The rank condition $\operatorname{rank} T \leq D - e$ is automatic when $D - e \geq l$. In smaller integer dimension it becomes an additional constraint, which is why the full-dimensional coordinate proof starts with $D \geq e + l$.

For completeness, every matrix $T \succeq 0$ can be realized as the Euclidean Gram matrix of l vectors in $\mathbb{R}^{D-e}$ when $D - e \geq l$, for example by a Cholesky or spectral factorization. Their parallel components are determined by H and E. This shows that (2.30) describes precisely the scalar-product image of the real momenta in the starting dimension. The interior has $T \succ 0$; the closure includes the degenerate boundaries.

Positive semidefiniteness means $b^T T b \geq 0$ for every real b; it does *not* mean $T_{ij} \geq 0$ entry by entry. Moreover, the single determinant condition $-\mathcal{B} \geq 0$ is necessary but, at several loops, is generally not sufficient.

The real-coordinate derivation is first valid wherever a common regulator permits the angular integrations and changes of variables. The Feynman prescription is kept during this calculation. Dimensional regularization is then understood as meromorphic continuation of the regulated formulas, not as a literal Euclidean sphere in a noninteger-dimensional real vector space.

Comparing with a Wick-rotated textbook formula requires a consistent choice of contour orientation and branches of complex powers. Do not replace $(-\mathcal{B})^{\gamma}$ by $\mathcal{B}^{\gamma}$ while dropping its phase, or remove i^{-l} without a corresponding contour rotation. For purely spacelike or degenerate-null external spans, the same algebraic Gram identities may remain useful, but the real transverse spherical proof does not automatically apply.

A Gram determinants, areas and orthogonal projections

A.1 Euclidean geometry and the induced metric

For Euclidean vectors $v_1, \ldots, v_r$, define

$$H_{ab} = v_a \cdot_{\mathrm{E}} v_b. \tag{A.1}$$

The parallelotope $X(t) = \sum_a t_a v_a$, $0 \le t_a \le 1$, has

$$\mathrm{d}X \cdot_{\mathrm{E}} \mathrm{d}X = \sum_{a,b} H_{ab}\, \mathrm{d}t_a\, \mathrm{d}t_b. \tag{A.2}$$

Thus H is the metric in the coordinates supplied by the chosen edge vectors. Its Euclidean volume is $\sqrt{\det H}$. At two vectors,

$$\det H = |u|^2|v|^2 - (u \cdot_{\mathrm{E}} v)^2 = |u|^2|v|^2 \sin^2\theta = (\text{base} \times \text{height})^2. \tag{A.3}$$

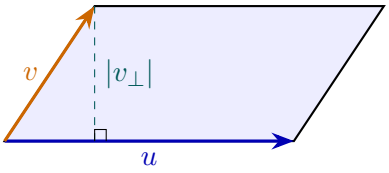


Figure 3: Euclidean illustration only: $\det H = |u|^2|v_\perp|^2$. The Minkowski sign is treated separately below.

More generally, if the edge list is changed by an invertible matrix M, then $H' = M^T H M$ and $\det H' = (\det M)^2 \det H$. This is precisely the square of the corresponding volume scaling. A shear has determinant one and preserves it.

A.2 Wedge products and how the metric acts

The exterior product $u \wedge v$ represents a directed area element. It is antisymmetric and bilinear, with $u \wedge u = 0$. It can be constructed without a metric; assigning it a squared magnitude requires the metric induced on bivectors:

$$\langle u \wedge v, a \wedge b \rangle_g = g(u,a)g(v,b) - g(u,b)g(v,a). \tag{A.4}$$

In particular,

$$\langle u \wedge v, u \wedge v \rangle_g = u^2 v^2 - (u \cdot v)^2. \tag{A.5}$$

The wedge retains the plane and its orientation; its squared magnitude does not retain that orientation. Also, $u \wedge (v + cu) = u \wedge v$: adding a parallel component shears the parallelogram without changing its directed area element.

A.3 Minkowski signs and the tensor contraction

Write the bivector as $W = \frac{1}{2} W^{\mu\nu} e_\mu \wedge e_\nu$, where

$$W^{\mu\nu} = u^\mu v^\nu - u^\nu v^\mu, \qquad W_{\mu\nu} = g_{\mu\alpha} g_{\nu\beta} W^{\alpha\beta}. \tag{A.6}$$

The metric acts on both indices. Since summing over both ordered pairs counts each coordinate plane twice,

$$\begin{aligned} \langle W, W \rangle_g &= \tfrac{1}{2} W^{\mu\nu} W_{\mu\nu} \\ &= \tfrac{1}{2} (u^\mu v^\nu - u^\nu v^\mu)(u_\mu v_\nu - u_\nu v_\mu) \\ &= (u \cdot u)(v \cdot v) - (u \cdot v)^2. \end{aligned} \tag{A.7}$$

For a general exterior degree r, the induced metric satisfies

$$\langle q_1 \wedge \cdots \wedge q_r, q_1 \wedge \cdots \wedge q_r \rangle_g = \det(g(q_a, q_b)). \tag{A.8}$$

Our Gram convention has an additional minus sign in each row, so

$$\det G(q_1, \ldots, q_r) = (-1)^r \langle q_1 \wedge \cdots \wedge q_r, q_1 \wedge \cdots \wedge q_r \rangle_g. \tag{A.9}$$

At $r = 2$ that extra factor is $+1$; at odd degree it must not be omitted. Unlike the positive Euclidean squared norm, the induced Minkowski square can be negative. For $W = p \wedge k$ in the rest frame of a timelike p,

$$W^{0a} = \sqrt{s}\, y^a, \quad W^{ab} = 0, \quad W_{0a} = -W^{0a}, \qquad \tfrac{1}{2} W^{\mu\nu} W_{\mu\nu} = -s \sum_a (y^a)^2 = -s\rho. \tag{A.10}$$

The nonnegative area magnitude in this nondegenerate two-plane is therefore $\sqrt{|\det G(k,p)|} = \sqrt{s}\sqrt{\rho}$. Its metric square is $-s\rho$. This sign is caused by the timelike-spacelike signature, not by reversing the orientation of the wedge.

In Euclidean space, zero Gram determinant is equivalent to linear dependence. In Minkowski space an independent list can span a degenerate subspace. For example, in signature $(+,-,-)$, $u = (1,1,0)$ and $v = (0,0,1)$ are independent, but $u^2 = u \cdot v = 0$, so their Gram determinant vanishes. Their wedge is nonzero but has zero induced square. For the bubble with timelike p, by contrast, $\det G(k,p) = -s\rho = 0$ forces $k_\perp = 0$ and hence $k \parallel p$.

A.4 The one-vector Gram-ratio identity

Let $V = \mathrm{span}\{u_1, \ldots, u_\alpha\}$ be nondegenerate and set $S = G(u_1, \ldots, u_\alpha)$. Define

$$\xi = -k^2, \qquad b_a = -k \cdot u_a, \qquad k_\parallel = \sum_a c_a u_a. \tag{A.11}$$

Orthogonality of $k_\perp = k - k_\parallel$ to V implies

$$b = Sc, \qquad c = S^{-1}b, \qquad -k_\parallel^2 = c^T S c = b^T S^{-1} b. \tag{A.12}$$

Thus

$$-k_\perp^2 = \xi - b^T S^{-1} b. \tag{A.13}$$

The full Gram matrix is

$$G(k, u_1, \ldots, u_\alpha) = \begin{pmatrix} \xi & b^T \\ b & S \end{pmatrix}. \tag{A.14}$$

Multiplication on the left by a determinant-one block matrix gives

$$\begin{pmatrix} 1 & -b^T S^{-1} \\ 0 & I \end{pmatrix} \begin{pmatrix} \xi & b^T \\ b & S \end{pmatrix} = \begin{pmatrix} \xi - b^T S^{-1} b & 0 \\ b & S \end{pmatrix}. \tag{A.15}$$

Taking determinants proves

$$\boxed{\frac{\det G(k, u_1, \ldots, u_\alpha)}{\det G(u_1, \ldots, u_\alpha)} = \xi - b^T S^{-1} b = -k_\perp^2.} \tag{A.16}$$

This is an algebraic identity for any nondegenerate V. The nonnegativity of its right-hand side additionally requires that $V^\perp$ be spacelike.

A.5 Several transverse momenta and the Schur complement

Use the common external decomposition

$$k_i = k_{i,\|P} + k_{i,\perp P}, \qquad k_{i,\|P} = \sum_a (c_i)_a p_a. \tag{A.17}$$

The matrices in (2.28) satisfy, in row notation,

$$H_i = c_i E, \qquad c_i = H_i E^{-1}, \qquad k_{i,\|P} \cdot k_{j,\|P} = -(H E^{-1} H^T)_{ij}. \tag{A.18}$$

Subtract the parallel scalar products from the full scalar products:

$$T_{ij} = A_{ij} - (H E^{-1} H^T)_{ij} = -k_{i,\perp P} \cdot k_{j,\perp P}. \tag{A.19}$$

For every real column $b \in \mathbb{R}^l$,

$$b^T T b = -\left(\sum_i b_i k_{i,\perp P} \right)^2 \geq 0, \tag{A.20}$$

because $P^\perp$ is spacelike. This proves $T \succeq 0$ as a matrix statement. For instance, $\left(\begin{smallmatrix} 1 & -1 \\ -1 & 1 \end{smallmatrix}\right) \succeq 0$ has negative off-diagonal entries. It is therefore incorrect to require $T_{ij} \geq 0$ entrywise.

Finally, block elimination yields

$$\begin{pmatrix} I & -HE^{-1} \\ 0 & I \end{pmatrix} \begin{pmatrix} A & H \\ H^T & E \end{pmatrix} = \begin{pmatrix} T & 0 \\ H^T & E \end{pmatrix}. \tag{A.21}$$

Taking determinants,

$$\boxed{\mathcal{B} = \det E \det T = -\Delta \det T.} \tag{A.22}$$

This factorization explains both the real-domain sign of $\mathcal{B}$ and the external Gram factor. It also separates two kinds of degeneracy: a singular external Gram matrix E prevents this projection, whereas a singular T describes a boundary of the transverse Gram domain when E remains invertible.

B Hyperspherical measures

This appendix fixes the angular convention used throughout the notes. Let n be the ambient Euclidean dimension. Its unit sphere is S^{n-1}, and $\mathrm{d}\Omega_{n-1}$ denotes the surface measure on that sphere. The subscript counts the angular dimensions, not the ambient dimensions.

Radial and angular coordinates

For $y \in \mathbb{R}^n$, write

$$y = r\,u(\theta), \qquad r = |y| \geq 0, \qquad u \cdot u = 1. \tag{B.1}$$

For integer $n \geq 2$, one possible parametrization uses $\theta_1, \ldots, \theta_{n-2} \in [0, \pi]$ and $\phi \in [0, 2\pi)$:

$$y_j = r \left(\prod_{a=1}^{j-1} \sin\theta_a \right) \cos\theta_j, \qquad 1 \leq j \leq n-2, \tag{B.2}$$

$$y_{n-1} = r \left(\prod_{a=1}^{n-2} \sin\theta_a \right) \cos\phi, \qquad y_n = r \left(\prod_{a=1}^{n-2} \sin\theta_a \right) \sin\phi. \tag{B.3}$$

Empty products equal one. In the Jacobian matrix, the radial column is $\partial_r y = u$, whereas each of the $n-1$ angular columns has the form $\partial_{\theta_a} y = r\,\partial_{\theta_a} u$. Factoring out one r from each angular column gives

$$\boxed{\mathrm{d}^n y = r^{n-1}\,\mathrm{d}r\;\mathrm{d}\Omega_{n-1}}, \qquad \mathrm{d}\Omega_{n-1} = \left[\prod_{a=1}^{n-2} (\sin\theta_a)^{n-1-a}\,\mathrm{d}\theta_a \right] \mathrm{d}\phi. \tag{B.4}$$

This is a coordinate change; it does not require rotational symmetry of the integrand. Rotational symmetry is needed only to perform the angular integral independently.

Introducing $\rho = r^2$ yields

$$\mathrm{d}r = \frac{\mathrm{d}\rho}{2\sqrt{\rho}}, \qquad \boxed{\mathrm{d}^n y = \frac{1}{2}\rho^{n/2-1}\,\mathrm{d}\rho\;\mathrm{d}\Omega_{n-1}}. \tag{B.5}$$

The radial transformation preserves orientation for $r > 0$. The origin is a coordinate singularity, rather than an additional sign choice.

Total solid angle from a Gaussian integral

Evaluate the same Gaussian integral in Cartesian and radial coordinates:

$$\int_{\mathbb{R}^n} \mathrm{d}^n y\, e^{-y^2} = \prod_{a=1}^{n} \int_{-\infty}^{\infty} \mathrm{d}y_a\, e^{-y_a^2} = \pi^{n/2}, \tag{B.6}$$

$$\int_{\mathbb{R}^n} \mathrm{d}^n y\, e^{-y^2} = \Omega_{n-1} \int_0^{\infty} \mathrm{d}r\, r^{n-1} e^{-r^2} = \frac{\Omega_{n-1}}{2} \Gamma\left(\frac{n}{2}\right). \tag{B.7}$$

Consequently,

$$\boxed{\Omega_{n-1} = \int \mathrm{d}\Omega_{n-1} = \frac{2\pi^{n/2}}{\Gamma(n/2)}}. \tag{B.8}$$

For a function depending only on y^2, equations (B.5) and (B.8) give

$$\int_{\mathbb{R}^n} \mathrm{d}^n y\, F(y^2) = \frac{\pi^{n/2}}{\Gamma(n/2)} \int_0^{\infty} \mathrm{d}\rho\, \rho^{n/2-1} F(\rho). \tag{B.9}$$

The derivation starts at integer dimension. Subject to the usual convergence and regularization conditions, its final integral formula can then be continued in the dimension. For $n = 1$, the same result uses $S^0 = \{-1, +1\}$ and $\Omega_0 = 2$.

Example: the transverse space at $D = 4$

In the rest frame of a timelike external momentum, write $k_\perp = (0, \mathbf{y})$ and $\rho = -k_\perp^2 = |\mathbf{y}|^2$. At $D = 4$, the transverse dimension is $n = 3$. Explicitly,

$$\mathbf{y} = (r\sin\theta\cos\phi,\ r\sin\theta\sin\phi,\ r\cos\theta), \qquad \left|\det \frac{\partial \mathbf{y}}{\partial(r,\theta,\phi)}\right| = r^2 \sin\theta. \tag{B.10}$$

Thus

$$\mathrm{d}^3 k_\perp = r^2\, \mathrm{d}r\ \mathrm{d}\Omega_2 = \frac{1}{2}\rho^{1/2}\, \mathrm{d}\rho\ \mathrm{d}\Omega_2, \qquad \Omega_2 = 4\pi. \tag{B.11}$$

For general D, replace n by $D - 1$:

$$\mathrm{d}^{D-1} k_\perp = \frac{1}{2}\rho^{(D-3)/2}\, \mathrm{d}\rho\ \mathrm{d}\Omega_{D-2}, \qquad \Omega_{D-2} = \frac{2\pi^{(D-1)/2}}{\Gamma((D-1)/2)}. \tag{B.12}$$