Differential equations

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Why do we study differential equations for Feynman integrals? After integration-by-parts (IBP) reduction, a family of Feynman integrals is expressed in terms of a finite set of master integrals. We use Minkowski-space denominators and the normalized measure for each loop,

$$D_j = m_j^2 - q_j^2 - \mathrm{i}0, \qquad I_{a_1\cdots a_n} = \int \prod_{r=1}^{L} \frac{\mathrm{d}^D \ell_r}{\mathrm{i}\pi^{D/2}} \prod_{j=1}^{n} D_j^{-a_j} \tag{1}$$

After the loop integrations, the result depends only on scalar kinematic quantities such as $p_i \cdot p_j$ and m_i^2. To investigate the structure of the master integrals, we differentiate with respect to these quantities and reduce the resulting integrals back to the master basis. This produces a system of differential equations with rational coefficients in the kinematic variables and the spacetime dimension D.

1 One-loop vacuum

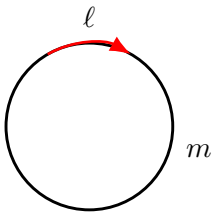

Figure 1: One-loop vacuum diagram

There is only one independent scalar product, ℓ^2, so no additional irreducible scalar product (ISP) is required. We define

$$D_1 = -\ell^2 + m^2 - \mathrm{i}0, \qquad I_a(m^2) = \int \frac{\mathrm{d}^D \ell}{\mathrm{i}\pi^{D/2}} \frac{1}{D_1^a}. \tag{2}$$

The IBP recurrence relation and master integral obtained with LiteRed are

$$(D-2a)I_a + 2am^2 I_{a+1} = 0, \qquad I_{a+1} = \frac{2a-D}{2am^2} I_a, \qquad \mathrm{MI} = \{I_1\}. \tag{3}$$

In particular,

$$I_2 = \frac{2-D}{2m^2} I_1. \tag{4}$$

The master integral I_1 depends only on m^2. The IBP coefficients are rational functions of m^2 and D, so its dependence on the mass can be determined as follows.

Method 1: differentiation with respect to the squared mass

Set $x = m^2$. Since $\partial_x D_1 = 1$, we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x} I_1(x) &= -\int \frac{\mathrm{d}^D \ell}{\mathrm{i}\pi^{D/2}} \frac{1}{(-\ell^2 + x - \mathrm{i}0)^2} = -I_2(x) \\ &= \frac{D-2}{2x} I_1(x). \end{aligned} \tag{5}$$

Separation of variables gives

$$I_1(x) = C_1 x^{D/2-1}, \qquad C_1 = I_1(1). \tag{6}$$

We now determine C_1 using the Alpha representation:

$$\frac{1}{-\ell^2 + 1 - \mathrm{i}0} = \mathrm{i} \int_0^\infty \mathrm{d}\alpha \, e^{\mathrm{i}\alpha(\ell^2 - 1 + \mathrm{i}0)}. \tag{7}$$

Therefore,

$$I_1(1) = \frac{1}{\pi^{D/2}} \int_0^\infty \mathrm{d}\alpha \, e^{-\mathrm{i}\alpha - 0^+\alpha} \int \mathrm{d}^D \ell \, e^{\mathrm{i}\alpha\ell^2}. \tag{8}$$

The Minkowski-space Fresnel integral is

$$\int \mathrm{d}^D \ell \, e^{\mathrm{i}\alpha\ell^2} = \pi^{D/2} \alpha^{-D/2} \mathrm{i}^{\,1-D/2}. \tag{9}$$

It follows that

$$\begin{aligned} I_1(1) &= \mathrm{i}^{\,1-D/2} \int_0^\infty \mathrm{d}\alpha \, \alpha^{-D/2} e^{-\mathrm{i}\alpha - 0^+\alpha} \\ &= \mathrm{i}^{\,1-D/2} \mathrm{i}^{-(1-D/2)} \Gamma\left(1 - \frac{D}{2}\right) = \Gamma\left(1 - \frac{D}{2}\right). \end{aligned} \tag{10}$$

Thus, the master integral is

$$I_1(x) = \Gamma\left(1 - \frac{D}{2}\right) x^{D/2-1}. \tag{11}$$

Method 2: dimensional analysis

The squared mass x is the only scale. We may therefore write $I_1(x) = x^{\#} C_1$. Since

$$[I_1] = [\mathrm{d}^D \ell][D_1^{-1}] = M^{D-2}, \qquad [x^{\#}] = M^{2\#}. \tag{12}$$

we find $2\# = D - 2$, or $\# = D/2 - 1$. Using the same boundary value $C_1 = I_1(1)$ reproduces the result above.

2 One-loop triangle with one internal mass

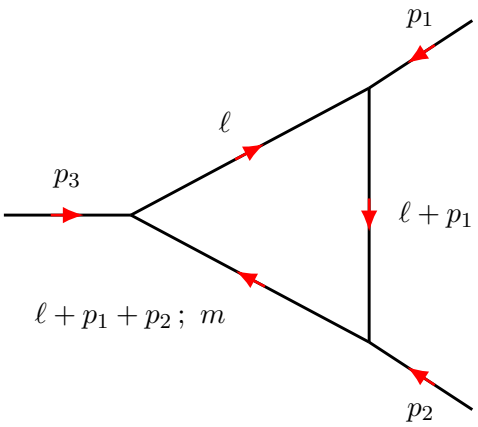

Figure 2: One-loop triangle

Take all external momenta to be incoming, so that $p_1 + p_2 + p_3 = 0$, and set

$$p_1^2 = p_2^2 = 0, \qquad s = (p_1 + p_2)^2 = 2p_1 \cdot p_2 = p_3^2 \tag{13}$$

The denominators and integral family are

$$\begin{aligned} D_1 &= -\ell^2 - \mathrm{i}0, & D_2 &= -(\ell + p_1)^2 - \mathrm{i}0, \\ D_3 &= -(\ell + p_1 + p_2)^2 + m^2 - \mathrm{i}0, & I_{a_1a_2a_3} &= \int \frac{\mathrm{d}^D \ell}{\mathrm{i}\pi^{D/2}} \frac{1}{D_1^{a_1} D_2^{a_2} D_3^{a_3}}. \end{aligned} \tag{14}$$

The three scalar products ℓ^2, $\ell \cdot p_1$ and $\ell \cdot p_2$ can all be expressed through the denominators, so no additional ISP is needed. We choose the master integrals returned by LiteRed in the order

$$\text{MIs} = \{I_{001}, I_{101}\}. \tag{15}$$

The IBP relations used in this section are

$$I_{002} = \frac{2 - D}{2m^2} I_{001}, \tag{16}$$

$$-I_{102} = \frac{D - 2}{2m^2(m^2 - s)} I_{001} + \frac{D - 3}{m^2 - s} I_{101}, \tag{17}$$

$$I_{111} = \frac{(2 - D) I_{001}/m^2 - 2(D - 3) I_{101}}{(D - 4)(m^2 - s)}. \tag{18}$$

The IBP reduction and identification of master integrals are performed with LiteRed.

Dimensional analysis and separation of the overall scale

Both I_{001} and I_{101} depend only on s and m^2. We first separate the overall scale by writing

$$I_{101}(s; m^2) = s^{\#_2} (I_{101}(x)|_{s=1})$$

Here $x = m^2/s$, and $I_{101}(x)|_{s=1}$ denotes the dimensionless integral. Since

$$[I_{101}] = M^{D-4}, \qquad [I_{001}] = M^{D-2}, \tag{19}$$

we obtain, in a region with a fixed branch for the power functions,

$$I_{101}(s;m^2) = s^{D/2-2} I_{101}(1;x), \tag{20}$$

$$I_{001}(s;m^2) = s^{D/2-1} I_{001}(1;x). \tag{21}$$

The overall powers are $D/2-2$ and $D/2-1$, respectively. After setting $s=1$, the squared mass is x.

Solving for I_{001}

A shift of the loop momentum turns I_{001} into the tadpole of the preceding section. Hence,

$$\frac{\mathrm{d}}{\mathrm{d}x} I_{001}(1;x) = -I_{002}(1;x) = \frac{D-2}{2x} I_{001}(1;x), \tag{22}$$

$$I_{001}(1;x) = \Gamma\left(1-\frac{D}{2}\right) x^{D/2-1}. \tag{23}$$

Restoring the overall scale gives

$$I_{001}(s;m^2) = \Gamma\left(1-\frac{D}{2}\right) (m^2)^{D/2-1} \tag{24}$$

Solving for I_{101}

Differentiating with respect to the squared mass in the third denominator introduces a minus sign and raises its power by one:

$$\frac{\mathrm{d}}{\mathrm{d}x} I_{101}(1;x) = -I_{102}(1;x). \tag{25}$$

Using Eq. (17) and substituting the known expression for I_{001} gives

$$\frac{\mathrm{d}}{\mathrm{d}x} I_{101}(1;x) = \frac{D-2}{2x(x-1)} I_{001}(1;x) + \frac{D-3}{x-1} I_{101}(1;x), \tag{26}$$

$$\frac{\mathrm{d}I_{101}(1;x)}{\mathrm{d}x} - \frac{D-3}{x-1} I_{101}(1;x) = -\Gamma\left(2-\frac{D}{2}\right) \frac{x^{D/2-2}}{x-1} \tag{27}$$

Here we have used

$$\left(\frac{D}{2}-1\right)\Gamma\left(1-\frac{D}{2}\right) = -\Gamma\left(2-\frac{D}{2}\right). \tag{28}$$

For $x>1$, an integrating factor with a positive real base is

$$\exp\left[-\int \frac{D-3}{x-1}\mathrm{d}x\right] = (x-1)^{3-D}. \tag{29}$$

Consequently,

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[(x-1)^{3-D} I_{101}(1;x)\right] = -\Gamma\left(2-\frac{D}{2}\right) x^{D/2-2}(x-1)^{2-D}. \tag{30}$$

The homogeneous solution is $C(x-1)^{D-3}$. To apply variation of constants, set

$$I_{101}(1;x) = (x-1)^{D-3} C(x), \tag{31}$$

Then

$$C'(x) = -\Gamma\left(2 - \frac{D}{2}\right) x^{D/2-2}(x-1)^{2-D}, \tag{32}$$

$$C(x) = C(x_0) - \Gamma\left(2 - \frac{D}{2}\right) \int_{x_0}^{x} \mathrm{d}t\, t^{D/2-2}(t-1)^{2-D}. \tag{33}$$

The general solution for I_{101} is therefore

$$I_{101}(1;x) = (x-1)^{D-3}\left[C(x_0) - \Gamma\left(2 - \frac{D}{2}\right) \int_{x_0}^{x} \mathrm{d}t\, t^{D/2-2}(t-1)^{2-D}\right] \tag{34}$$

The remaining integration constant must be fixed by a boundary condition.

3 Massless one-loop box

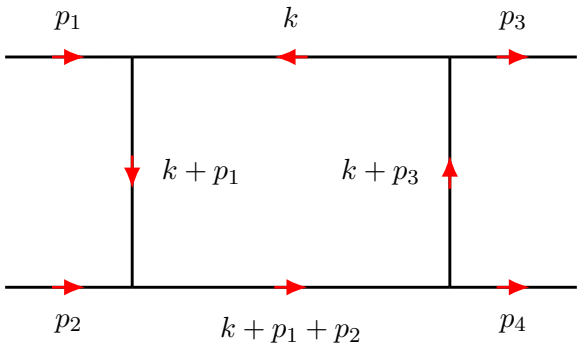

Figure 3: Massless one-loop box

We use two incoming and two outgoing external momenta:

$$p_1+p_2=p_3+p_4, \quad p_i^2=0, \quad s=(p_1+p_2)^2, \quad t=(p_1-p_3)^2, \quad u=-s-t. \tag{35}$$

It follows that

$$2p_1\cdot p_2=s, \quad 2p_1\cdot p_3=-t, \quad 2p_2\cdot p_3=s+t. \tag{36}$$

The denominators and integral family are defined by

$$\begin{aligned} D_1&=-k^2-\mathrm{i}0, & D_2&=-(k+p_1)^2-\mathrm{i}0, \\ D_3&=-(k+p_1+p_2)^2-\mathrm{i}0, & D_4&=-(k+p_3)^2-\mathrm{i}0, \end{aligned} \tag{37}$$

$$I_{a_1a_2a_3a_4}=\int\frac{\mathrm{d}^Dk}{\mathrm{i}\pi^{D/2}}\frac{1}{D_1^{a_1}D_2^{a_2}D_3^{a_3}D_4^{a_4}}. \tag{38}$$

The master integrals obtained with LiteRed are

$$\text{MIs}=\{I_{0101},I_{1010},I_{1111}\}. \tag{39}$$

For clarity, we first collect the LiteRed reduction relations needed in the differentiation below. For I_{0101} and I_{1010}, we use

$$I_{0,0,0,2}=I_{0,0,2,0}=0, \tag{40}$$

$$I_{-1,1,0,2}=I_{0,1,-1,2}=\frac{D-2}{2}I_{0101}, \tag{41}$$

$$I_{0102}=\frac{D-3}{t}I_{0101}, \tag{42}$$

$$I_{1,-1,2,0}=I_{1,0,2,-1}=\frac{D-2}{2}I_{1010}, \tag{43}$$

$$I_{1020}=\frac{D-3}{s}I_{1010} \tag{44}$$

For the derivatives of I_{1111}, we need

$$I_{0112} = I_{1102} = \frac{2(D-3)}{t^2} I_{0101}, \tag{45}$$

$$I_{1012} = \frac{4(D-3)}{(D-6)s^2} I_{1010}, \tag{46}$$

$$I_{1112} = -\frac{4(D-5)(D-3)}{(D-6)s^2 t} I_{1010} + \frac{D-5}{t} I_{1111}, \tag{47}$$

$$I_{0121} = \frac{4(D-3)}{(D-6)t^2} I_{0101}, \tag{48}$$

$$I_{1021} = I_{1120} = \frac{2(D-3)}{s^2} I_{1010}, \tag{49}$$

$$I_{1121} = -\frac{4(D-5)(D-3)}{(D-6)st^2} I_{0101} + \frac{D-5}{s} I_{1111}. \tag{50}$$

These are algebraic reduction relations for generic D. The intermediate denominators $D-6$ cancel in the final differential equations.

The chain rule

For a set of kinematic invariants s_r, the chain rule gives

$$p_i \cdot \frac{\partial}{\partial p_j} = \sum_r \left(p_i \cdot \frac{\partial s_r}{\partial p_j} \right) \frac{\partial}{\partial s_r}. \tag{51}$$

Constructing ∂_s

Keep p_1 and p_3 fixed, and write $\partial_s|_t = v_s \cdot \partial_{p_2}$. Preserving p_2^2 and p_4^2, while requiring $\partial_s s = 1$, gives

$$2v_s \cdot p_2 = 0, \quad 2v_s \cdot p_4 = 0, \quad 2v_s \cdot (p_1 + p_2) = 1. \tag{52}$$

With the ansatz $v_s = ap_1 + bp_2 + cp_3$, these constraints become

$$\begin{cases} bs - ct = 1, \\ as + c(s+t) = 0, \\ a(s+t) - bt + cs = 0. \end{cases} \tag{53}$$

The first two equations give $b = (1+ct)/s$ and $a = -c(s+t)/s$. Substitution into the third yields $-t[1 + 2c(s+t)] = 0$. Thus,

$$a = \frac{1}{2s}, \quad b = \frac{2s+t}{2s(s+t)}, \quad c = -\frac{1}{2(s+t)}. \tag{54}$$

We therefore obtain

$$v_s = \frac{p_1 + p_2}{2s} + \frac{p_2 - p_3}{2(s+t)}, \qquad \partial_s|_t = v_s \cdot \partial_{p_2}. \tag{55}$$

Constructing ∂_t

Keep p_1 and p_2 fixed, and write $\partial_t|_s = v_t \cdot \partial_{p_3}$. The corresponding constraints are

$$2v_t \cdot p_3 = 0, \quad -2v_t \cdot p_4 = 0, \quad 2v_t \cdot (p_3 - p_1) = 1. \tag{56}$$

Set $v_t = ap_1 + bp_2 + cp_3$. Using the first constraint, we can arrange the system as

$$\begin{cases} -at + b(s+t) = 0, \\ a(s+t) - bt + cs = 0, \\ -bs + ct = 1. \end{cases} \tag{57}$$

The first equation gives $b = at/(s+t)$, and the second gives $c = -a(s+2t)/(s+t)$. Substitution into the third yields $-2at = 1$, so

$$a = -\frac{1}{2t}, \quad b = -\frac{1}{2(s+t)}, \quad c = \frac{s+2t}{2t(s+t)}. \tag{58}$$

Hence,

$$v_t = -\frac{p_1 - p_3}{2t} - \frac{p_2 - p_3}{2(s+t)}, \qquad \partial_t|_s = v_t \cdot \partial_{p_3}. \tag{59}$$

These operators apply at generic kinematic points with $st(s+t) \neq 0$.

Differentiation with respect to t and reduction to master integrals

Only D_4 depends on p_3. Therefore,

$$v_t \cdot \partial_{p_3} D_4 = -2v_t \cdot (k + p_3), \tag{60}$$

$$\partial_t I_{a_1a_2a_3a_4} = 2a_4 \int \frac{\mathrm{d}^D k}{\mathrm{i}\pi^{D/2}} \frac{v_t \cdot (k+p_3)}{D_1^{a_1} D_2^{a_2} D_3^{a_3} D_4^{a_4+1}}. \tag{61}$$

Using

$$2k \cdot p_1 = D_1 - D_2, \quad 2k \cdot p_2 = D_2 - D_3 - s, \quad 2k \cdot p_3 = D_1 - D_4 \tag{62}$$

together with $v_t \cdot p_3 = 0$, we find

$$2v_t \cdot (k + p_3) = \frac{tD_1 + sD_2 + tD_3 - (s+2t)D_4 + st}{2t(s+t)}. \tag{63}$$

For the three master integrals, this gives

$$\begin{aligned} \partial_t I_{0101} &= \frac{tI_{-1,1,0,2} + sI_{0,0,0,2} + tI_{0,1,-1,2} - (s+2t)I_{0101} + stI_{0102}}{2t(s+t)} \\ &= \frac{t(D-2) - (s+2t) + s(D-3)}{2t(s+t)} I_{0101} = \frac{D-4}{2t} I_{0101}, \end{aligned} \tag{64}$$

$$\partial_t I_{1010} = 0, \tag{65}$$

$$\partial_t I_{1111} = \frac{tI_{0112} + sI_{1012} + tI_{1102} - (s+2t)I_{1111} + stI_{1112}}{2t(s+t)}. \tag{66}$$

Substituting the LiteRed relations listed above into the last equation and collecting terms gives the numerator

$$\frac{4(D-3)}{t}I_{0101} - \frac{4(D-3)}{s}I_{1010} + [(D-6)s - 2t]I_{1111}. \tag{67}$$

Thus,

$$\partial_t I_{1111} = \frac{2(D-3)}{t^2(s+t)}I_{0101} - \frac{2(D-3)}{st(s+t)}I_{1010} + \frac{(D-6)s-2t}{2t(s+t)}I_{1111}. \tag{68}$$

Differentiation with respect to s and reduction to master integrals

Only D_3 depends on p_2. Thus,

$$v_s \cdot \partial_{p_2} D_3 = -2v_s \cdot (k + p_1 + p_2), \tag{69}$$

$$\partial_s I_{a_1a_2a_3a_4} = 2a_3 \int \frac{\mathrm{d}^D k}{\mathrm{i}\pi^{D/2}} \frac{v_s \cdot (k+p_1+p_2)}{D_1^{a_1} D_2^{a_2} D_3^{a_3+1} D_4^{a_4}}. \tag{70}$$

Since $2v_s \cdot (p_1 + p_2) = 1$, the numerator can be written as

$$\begin{aligned} 2v_s \cdot (k + p_1 + p_2) &= 1 + 2v_s \cdot k \\ &= 1 + \frac{D_1 - D_3 - s}{2s} + \frac{D_2 - D_3 - s - D_1 + D_4}{2(s+t)} \\ &= \frac{tD_1 + sD_2 - (2s+t)D_3 + sD_4 + st}{2s(s+t)}. \end{aligned} \tag{71}$$

Therefore,

$$\partial_s I_{0101} = 0, \tag{72}$$

$$\begin{aligned} \partial_s I_{1010} &= \frac{tI_{0020} + sI_{1,-1,2,0} - (2s+t)I_{1010} + sI_{1,0,2,-1} + stI_{1020}}{2s(s+t)} \\ &= \frac{s(D-2) - (2s+t) + t(D-3)}{2s(s+t)} I_{1010} = \frac{D-4}{2s} I_{1010}, \end{aligned} \tag{73}$$

$$\partial_s I_{1111} = \frac{tI_{0121} + sI_{1021} - (2s+t)I_{1111} + sI_{1120} + stI_{1121}}{2s(s+t)}. \tag{74}$$

After applying the LiteRed reduction, the numerator of the last equation becomes

$$-\frac{4(D-3)}{t}I_{0101} + \frac{4(D-3)}{s}I_{1010} + [(D-6)t - 2s]I_{1111}. \tag{75}$$

Hence,

$$\partial_s I_{1111} = -\frac{2(D-3)}{st(s+t)}I_{0101} + \frac{2(D-3)}{s^2(s+t)}I_{1010} + \frac{(D-6)t-2s}{2s(s+t)}I_{1111} \tag{76}$$

Matrix form of the differential equations

Define $\boldsymbol{I} = (I_{0101}, I_{1010}, I_{1111})^{\mathsf{T}}$. Then

$$\partial_s \boldsymbol{I} = A_s \boldsymbol{I}, \qquad \partial_t \boldsymbol{I} = A_t \boldsymbol{I} \tag{77}$$

where

$$A_s = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \dfrac{D-4}{2s} & 0 \\ -\dfrac{2(D-3)}{st(s+t)} & \dfrac{2(D-3)}{s^2(s+t)} & \dfrac{(D-6)t-2s}{2s(s+t)} \end{pmatrix}, \tag{78}$$

$$A_t = \begin{pmatrix} \dfrac{D-4}{2t} & 0 & 0 \\ 0 & 0 & 0 \\ \dfrac{2(D-3)}{t^2(s+t)} & -\dfrac{2(D-3)}{st(s+t)} & \dfrac{(D-6)s-2t}{2t(s+t)} \end{pmatrix} \tag{79}$$

Direct verification of the integrability condition

Differentiating the matrix entries gives

$$\partial_s A_t - \partial_t A_s = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -\dfrac{4(D-3)}{st^2(s+t)} & \dfrac{4(D-3)}{s^2t(s+t)} & 0 \end{pmatrix}. \tag{80}$$

The difference of the matrix products is

$$A_t A_s - A_s A_t = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \dfrac{4(D-3)}{st^2(s+t)} & -\dfrac{4(D-3)}{s^2t(s+t)} & 0 \end{pmatrix}. \tag{81}$$

The entries cancel pairwise, and hence

$$\partial_s A_t - \partial_t A_s + A_t A_s - A_s A_t = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{82}$$

The two differential equations therefore satisfy the integrability condition.

Solutions for the master integrals

First consider $I_{0101}(t)$. Its differential equation is

$$\frac{\mathrm{d}}{\mathrm{d}t} I_{0101}(t) = \frac{D-4}{2t} I_{0101}(t). \tag{83}$$

Integrating on a fixed branch for $t > 0$ gives

$$I_{0101}(t) = t^{D/2-2} C_1, \qquad C_1 = I_{0101}(1). \tag{84}$$

The exponent # can also be determined directly by dimensional analysis:

$$[I_{0101}] = [\mathrm{d}^D k]\,[D_2]^{-1}[D_4]^{-1} = M^{D-4}, \qquad [t^{\#}] = M^{2\#}, \qquad \# = \frac{D}{2} - 2. \tag{85}$$

To determine the normalization, start from the integral definition,

$$I_{0101}(t) = \int \frac{\mathrm{d}^D k}{\mathrm{i}\pi^{D/2}} \frac{1}{[-(k+p_1)^2 - \mathrm{i}0][-(k+p_3)^2 - \mathrm{i}0]}, \qquad t = (p_1 - p_3)^2 = -2p_1 \cdot p_3. \tag{86}$$

Set $\ell = k + p_1$ and $Q = p_3 - p_1$, so that $Q^2 = t$. Then

$$I_{0101}(t) = \int \frac{\mathrm{d}^D \ell}{\mathrm{i}\pi^{D/2}} \frac{1}{(-\ell^2 - \mathrm{i}0)[-(\ell+Q)^2 - \mathrm{i}0]}. \tag{87}$$

We use the massless bubble result derived in the companion notes *Parameter representations*:

$$\begin{aligned} &\int \frac{\mathrm{d}^D \ell}{\mathrm{i}\pi^{D/2}} \frac{1}{(-\ell^2 - \mathrm{i}0)^{a_1}[-(\ell+Q)^2 - \mathrm{i}0]^{a_2}} \\ &\quad = (-Q^2 - \mathrm{i}0)^{D/2-a_1-a_2} \frac{\Gamma(a_1+a_2-D/2)\Gamma(D/2-a_1)\Gamma(D/2-a_2)}{\Gamma(a_1)\Gamma(a_2)\Gamma(D-a_1-a_2)}. \end{aligned} \tag{88}$$

Setting $a_1 = a_2 = 1$ and $Q^2 = t$ gives

$$I_{0101}(t) = (-t - \mathrm{i}0)^{D/2-2} \frac{\Gamma(2-D/2)\Gamma^2(D/2-1)}{\Gamma(D-2)}. \tag{89}$$

Similarly, for the other bubble master integral, take $Q = p_1 + p_2$, with $Q^2 = s$:

$$I_{1010}(s) = \int \frac{\mathrm{d}^D k}{\mathrm{i}\pi^{D/2}} \frac{1}{(-k^2 - \mathrm{i}0)[-(k+p_1+p_2)^2 - \mathrm{i}0]}. \tag{90}$$

Its differential equation, $\frac{\mathrm{d}I_{1010}}{\mathrm{d}s} = \frac{D-4}{2s} I_{1010}$, together with Eq. (88), yields

$$I_{1010}(s) = (-s - \mathrm{i}0)^{D/2-2} \frac{\Gamma(2-D/2)\Gamma^2(D/2-1)}{\Gamma(D-2)} \tag{91}$$

The normalization constants of both bubble master integrals are thus fixed by their values at $s = 1$ and $t = 1$ on the chosen branch. These expressions can be substituted into the right-hand sides of the differential equations for $I_{1111}(s,t)$.

4 Two-loop sunrise with two distinct masses

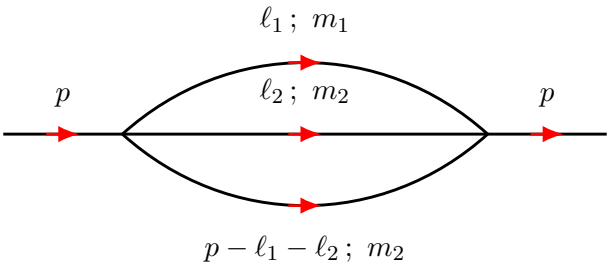

Figure 4: Two-loop sunrise

Define $p^2=-s$. With two loop momenta and one external momentum, there are five independent scalar products. We supplement the three physical propagators by D_4 and D_5 to obtain a complete basis:

$$\begin{aligned} D_1 &= -\ell_1^2+m_1^2-\mathrm{i}0, & D_2 &= -\ell_2^2+m_2^2-\mathrm{i}0,\\ D_3 &= -(p-\ell_1-\ell_2)^2+m_2^2-\mathrm{i}0, & D_4 &= p\cdot\ell_1, \qquad D_5 = p\cdot\ell_2 \end{aligned} \tag{92}$$

The integral family for this diagram is

$$I_{a_1a_2a_3a_4a_5}(s;m_1^2,m_2^2)=\int\frac{\mathrm{d}^D\ell_1}{\mathrm{i}\pi^{D/2}}\frac{\mathrm{d}^D\ell_2}{\mathrm{i}\pi^{D/2}}\prod_{j=1}^{5}D_j^{-a_j} \tag{93}$$

The top sector is $(1,1,1,0,0)$. The master integrals obtained with LiteRed are

$$\text{MIs}=\{I_{01100},I_{10100},I_{11100},I_{21100},I_{12100}\}. \tag{94}$$

As before, we first separate the overall scale. Introduce

$$x_1=\frac{m_1^2}{s}, \qquad x_2=\frac{m_2^2}{s}. \tag{95}$$

The two-loop integral has mass dimension $M^{2D-2\sum_j a_j}$, so

$$I_{\boldsymbol{a}}(s;m_1^2,m_2^2)=s^{D-\sum_j a_j}I_{\boldsymbol{a}}(1;x_1,x_2). \tag{96}$$

The corresponding powers of the overall scale for the five master integrals are

$$D-2, \quad D-2, \quad D-3, \quad D-4, \quad D-4. \tag{97}$$

In the following, all dimensionless integrals are evaluated at $s=1$, so that $p^2=-1$. To shorten the formulas, we suppress the common arguments $(1;x_1,x_2)$ without changing the integral indices or definitions.

LiteRed reduction relations

We first list the equal-mass symmetry relations and the reductions of subtopologies:

$$I_{02100} = I_{01200} = \frac{2-D}{2x_2} I_{01100}, \tag{98}$$

$$I_{20100} = \frac{2-D}{2x_1} I_{10100}, \qquad I_{10200} = \frac{2-D}{2x_2} I_{10100}, \tag{99}$$

$$I_{11200} = I_{12100}, \qquad I_{21200} = I_{22100}. \tag{100}$$

The higher-sector reductions are longer. Define the common coefficient polynomial

$$\Delta = x_1^2 + 2x_1(1-4x_2) + (1+4x_2)^2. \tag{101}$$

The three combinations with raised propagator powers needed below have the following reductions:

Reduction of $-2I_{31100}$

$$\begin{aligned} -2I_{31100} = & -\frac{(D-2)^2(-1+3x_1-4x_2)}{2x_1(1+x_1)\Delta} I_{01100} + \frac{(D-2)^2}{x_1\Delta} I_{10100} \\ & + \frac{(D-3)(3D-8)[(1+x_1)^2 - 2(x_1-3)x_2 + 8x_2^2]}{2x_1(1+x_1)\Delta} I_{11100} \\ & + \frac{4(D-3)x_2(1+x_2)(1+x_1+4x_2)}{x_1(1+x_1)\Delta} I_{12100} \\ & + \frac{I_{21100}}{2x_1(1+x_1)\Delta}\Big\{(1+x_1)^2[D-4+(5D-16)x_1] \\ & \qquad + 8(1-x_1)[D-4+(3D-10)x_1]x_2 \\ & \qquad + 16[D-4+(3D-10)x_1]x_2^2\Big\}. \end{aligned} \tag{102}$$

Reduction of $-I_{22100}$

$$\begin{aligned} -I_{22100} = & \frac{(D-2)^2}{2x_2\Delta} I_{01100} - \frac{(D-2)^2(1+x_1+4x_2)}{4x_1x_2\Delta} I_{10100} \\ & - \frac{(D-3)(3D-8)}{\Delta} I_{11100} - \frac{(D-3)(-1+3x_1-4x_2)}{\Delta} I_{21100} \\ & + \frac{(D-3)(1+x_1-8x_2)}{\Delta} I_{12100}. \end{aligned} \tag{103}$$

Reduction of $-2I_{13100} - I_{12200}$

$$\begin{aligned}-2I_{13100} - I_{12200} = &-\frac{(D-2)^2(1+x_1+4x_2)}{4x_2^2\Delta} I_{01100} + \frac{2(D-2)^2}{x_2\Delta} I_{10100} \\ &+ \frac{(D-3)(3D-8)(1+x_1+4x_2)}{2x_2\Delta} I_{11100} \\ &+ \frac{2(D-3)x_1(1+x_1)}{x_2\Delta} I_{21100} \\ &+ \frac{I_{12100}}{2x_2\Delta}\Big\{(D-4)(1+x_1)^2 + 4[(5D-17) \\ &\qquad + (11-3D)x_1]x_2 + 16(4D-13)x_2^2\Big\} \end{aligned} \tag{104}$$

These relations use the conventions $p^2 = -s$ and $m_2 = m_3$ for the integral family defined above.

Differentiation with respect to the mass ratios

At fixed s,

$$\partial_{x_1} = \frac{s}{2m_1}\partial_{m_1}, \qquad \partial_{x_2} = \frac{s}{2m_2}\partial_{m_2}. \tag{105}$$

The second mass occurs in both D_2 and D_3. Hence,

$$\partial_{x_1} I_{\boldsymbol{a}} = -s a_1 I_{a_1+1,a_2,a_3,a_4,a_5}, \tag{106}$$

$$\partial_{x_2} I_{\boldsymbol{a}} = -s a_2 I_{a_1,a_2+1,a_3,a_4,a_5} - s a_3 I_{a_1,a_2,a_3+1,a_4,a_5}. \tag{107}$$

Setting $s = 1$, the derivatives of the five master integrals are

$$\partial_{x_1} I_{01100} = 0, \qquad \partial_{x_2} I_{01100} = -I_{02100} - I_{01200} = \frac{D-2}{x_2} I_{01100}, \tag{108}$$

$$\partial_{x_1} I_{10100} = -I_{20100} = \frac{D-2}{2x_1} I_{10100}, \qquad \partial_{x_2} I_{10100} = -I_{10200} = \frac{D-2}{2x_2} I_{10100}, \tag{109}$$

$$\partial_{x_1} I_{11100} = -I_{21100}, \qquad \partial_{x_2} I_{11100} = -I_{12100} - I_{11200} = -2I_{12100}, \tag{110}$$

$$\partial_{x_1} I_{21100} = -2I_{31100}, \qquad \partial_{x_2} I_{21100} = -I_{22100} - I_{21200} = -2I_{22100}, \tag{111}$$

$$\partial_{x_1} I_{12100} = -I_{22100}, \qquad \partial_{x_2} I_{12100} = -2I_{13100} - I_{12200}. \tag{112}$$

Substituting Eqs. (102)–(104) gives a closed system involving only the five master integrals. In particular, $\partial_{x_2} I_{21100} = 2\partial_{x_1} I_{12100}$. The full matrix system follows directly from these reduction relations.

Solving the master integrals

For the first master integral,

$$\partial_{x_2} I_{01100} = \frac{D-2}{x_2} I_{01100} \quad \Longrightarrow \quad I_{01100} = C_1 x_2^{D-2} \tag{113}$$

The linear change of variables $q = p - \ell_1 - \ell_2$ factorizes the two loop integrations:

$$\begin{aligned} I_{01100}(1; x_1, x_2) &= \left[\int \frac{\mathrm{d}^D \ell_2}{\mathrm{i}\pi^{D/2}} \frac{1}{-\ell_2^2 + x_2 - \mathrm{i}0}\right]\left[\int \frac{\mathrm{d}^D q}{\mathrm{i}\pi^{D/2}} \frac{1}{-q^2 + x_2 - \mathrm{i}0}\right] \\ &= \Gamma^2\left(1 - \frac{D}{2}\right) x_2^{D-2}. \end{aligned} \tag{114}$$

Similarly,

$$I_{10100}(1; x_1, x_2) = \Gamma^2\left(1 - \frac{D}{2}\right)(x_1 x_2)^{D/2-1}. \tag{115}$$

Neither of these integrals depends on the original external momentum. Substituting them into the remaining equations gives a 3×3 inhomogeneous coupled system for I_{11100}, I_{21100} and I_{12100}.

5 Integrability conditions

5.1 Two independent variables

Let $\boldsymbol{I} = (I_1, \ldots, I_n)^\mathsf{T}$ be a complete vector of independent master integrals. For two independent variables x_1 and x_2, the differential equations take the form

$$\partial_{x_1} \boldsymbol{I} = A_1 \boldsymbol{I}, \qquad \partial_{x_2} \boldsymbol{I} = A_2 \boldsymbol{I} \tag{116}$$

Mixed partial derivatives must commute. Therefore,

$$\begin{aligned} 0 &= \partial_{x_1} \partial_{x_2} \boldsymbol{I} - \partial_{x_2} \partial_{x_1} \boldsymbol{I} \\ &= \partial_{x_1} (A_2 \boldsymbol{I}) - \partial_{x_2} (A_1 \boldsymbol{I}) \\ &= (\partial_{x_1} A_2 + A_2 A_1 - \partial_{x_2} A_1 - A_1 A_2)\, \boldsymbol{I} \end{aligned} \tag{117}$$

For a complete independent master-integral system, this implies the integrability condition

$$\partial_{x_1} A_2 - \partial_{x_2} A_1 + A_2 A_1 - A_1 A_2 = 0 \tag{118}$$

5.2 Three independent variables

If there is a third equation, $\partial_{x_3} \boldsymbol{I} = A_3 \boldsymbol{I}$, we must check the three pairs of variables (x_1, x_2), (x_1, x_3) and (x_2, x_3). The corresponding integrability conditions are

$$\begin{aligned} &\partial_{x_1} A_2 - \partial_{x_2} A_1 + A_2 A_1 - A_1 A_2 = 0, \\ &\partial_{x_1} A_3 - \partial_{x_3} A_1 + A_3 A_1 - A_1 A_3 = 0, \\ &\partial_{x_2} A_3 - \partial_{x_3} A_2 + A_3 A_2 - A_2 A_3 = 0 \end{aligned} \tag{119}$$

The number of conditions is determined by the number of independent variables, rather than the number of master integrals.