

Parameter representations

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1 Propagator representation

The basic identity underlying the Alpha-parameter representation is

$$\frac{1}{(A - i0)^a} = \frac{i^a}{\Gamma(a)} \int_0^\infty d\alpha \alpha^{a-1} e^{-i\alpha(A-i0)}, \quad \text{Re } a > 0, \quad \text{Im}(A - i0) < 0 \quad (1)$$

Recall the Gamma-function identity

$$\int_0^\infty dt t^{z-1} e^{-ct} = \frac{\Gamma(z)}{c^z}, \quad \text{Re } z > 0, \quad \text{Re } c > 0, \quad (2)$$

which was derived in Eq. (212). We make the substitutions

$$t \longrightarrow \alpha, \quad z \longrightarrow a, \quad c \longrightarrow i(A - i0) \quad (3)$$

we find

$$\text{Re}[i(A - i0)] = -\text{Im}(A - i0) > 0 \quad (4)$$

Therefore, the Gamma-function identity gives

$$\int_0^\infty d\alpha \alpha^{a-1} e^{-i\alpha(A-i0)} = \frac{\Gamma(a)}{[i(A - i0)]^a} = \frac{\Gamma(a)}{i^a (A - i0)^a} \quad (5)$$

Now that the study of the preceding mathematical part is complete, we next examine the Feynman integral :

$$I_{a_1 \dots a_N}^{(L)} = \int \left(\prod_{r=1}^L \frac{d^D \ell_r}{i\pi^{D/2}} \right) \prod_{j=1}^N \frac{1}{(-q_j^2 + m_j^2 - i0)^{a_j}}$$

For the j -th propagator, we set

$$A = -q_j^2 + m_j^2 \quad (6)$$

in Eq. (1). This gives

$$\begin{aligned} \frac{1}{(-q_j^2 + m_j^2 - i0)^{a_j}} &= \frac{i^{a_j}}{\Gamma(a_j)} \int_0^\infty d\alpha_j \alpha_j^{a_j-1} e^{-i\alpha_j(-q_j^2 + m_j^2 - i0)} \\ &= \frac{i^{a_j}}{\Gamma(a_j)} \int_0^\infty d\alpha_j \alpha_j^{a_j-1} e^{i\alpha_j q_j^2} e^{-i\alpha_j(m_j^2 - i0)} \end{aligned} \quad (7)$$

Thus, every propagator is replaced by one integration over a positive Alpha parameter. Then one can see that the expression for each loop momentum appears in the exponential, and all of them are in a form that can be squared. Therefore, we can use Gaussian integrals to integrate out all the momentum variables, leaving only the α parameters, and these parameters are all greater than 0.

Next, let us look at a few examples following this line of thought.

1.1 Examples

1.1.1 One-loop vacuum

We begin with the one-loop vacuum integral shown in Fig. 1.

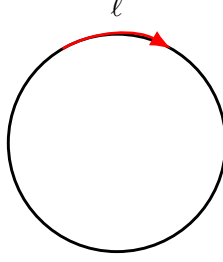


Figure 1: One-loop vacuum diagram

$$I_{a_1}(m^2) = \frac{1}{i\pi^{D/2}} \int \frac{d^D \ell}{(-\ell^2 + m^2 - i0)^{a_1}} \quad (8)$$

Here, why do we put $i\pi^{\frac{D}{2}}$ in the denominator? In principle, it would not matter what number is in the denominator, but in the calculations that follow we will find that using $i\pi^{\frac{D}{2}}$ saves a lot of repeated writing.

From the original integral to the Alpha representation

Applying Eq. (1) to the single propagator,

$$\begin{aligned} I_{a_1}(m^2) &= \frac{i^{a_1}}{i\pi^{D/2}\Gamma(a_1)} \int_0^\infty d\alpha \alpha^{a_1-1} \int d^D \ell e^{-i\alpha(-\ell^2+m^2-i0)} \\ &= \frac{i^{a_1}}{i\pi^{D/2}\Gamma(a_1)} \int_0^\infty d\alpha \alpha^{a_1-1} e^{-i\alpha(m^2-i0)} \int d^D \ell e^{i\alpha\ell^2} \\ &= \frac{i^{a_1-D/2}}{\Gamma(a_1)} \int_0^\infty d\alpha \alpha^{a_1-D/2-1} e^{-i\alpha(m^2-i0)} \end{aligned} \quad (9)$$

In the second line, the momentum dependence is separated from the mass dependence. In the third line, the Gaussian momentum integral is evaluated.

In Minkowski spacetime, the squared momentum is

$$\ell^2 = (\ell^0)^2 - \sum_{j=1}^{D-1} (\ell^j)^2, \quad \alpha > 0 \quad (10)$$

Therefore, the D -dimensional integral factorizes as

$$\int d^D \ell e^{i\alpha\ell^2} = \int_{-\infty}^\infty d\ell^0 e^{i\alpha(\ell^0)^2} \prod_{j=1}^{D-1} \int_{-\infty}^\infty d\ell^j e^{-i\alpha(\ell^j)^2} \quad (11)$$

We now use the ordinary Gaussian integral

$$\int_{-\infty}^\infty dx e^{-ax^2} = \sqrt{\frac{\pi}{a}}, \quad \text{Re } a > 0 \quad (12)$$

For the timelike component, we set $a = -i\alpha$, whereas for each spacelike component we set $a = i\alpha$. Hence,

$$\int_{-\infty}^{\infty} d\ell^0 e^{i\alpha(\ell^0)^2} = e^{i\pi/4} \sqrt{\frac{\pi}{\alpha}}, \quad \int_{-\infty}^{\infty} d\ell^j e^{-i\alpha(\ell^j)^2} = e^{-i\pi/4} \sqrt{\frac{\pi}{\alpha}}, \quad j = 1, \dots, D-1 \quad (13)$$

Substituting these one-dimensional integrals into Eq. (11), we obtain

$$\begin{aligned} \int d^D \ell e^{i\alpha \ell^2} &= \left(e^{i\pi/4} \sqrt{\frac{\pi}{\alpha}} \right) \left(e^{-i\pi/4} \sqrt{\frac{\pi}{\alpha}} \right)^{D-1} \\ &= \left(\frac{\pi}{\alpha} \right)^{D/2} \exp \left[\frac{i\pi}{4} - \frac{i\pi}{4}(D-1) \right] \\ &= \pi^{D/2} \alpha^{-D/2} \exp \left[\frac{i\pi}{2} \left(1 - \frac{D}{2} \right) \right] \end{aligned} \quad (14)$$

Using $i^\rho = \exp \left(\frac{i\pi\rho}{2} \right)$, the phase factor becomes

$$\exp \left[\frac{i\pi}{2} \left(1 - \frac{D}{2} \right) \right] = i^{1-D/2} \quad (15)$$

Consequently,

$$\int d^D \ell e^{i\alpha \ell^2} = i^{1-D/2} \pi^{D/2} \alpha^{-D/2} \quad (16)$$

With the normalized loop measure used in the Feynman integral, this gives

$$\int \frac{d^D \ell}{i\pi^{D/2}} e^{i\alpha \ell^2} = i^{-D/2} \alpha^{-D/2} \quad (17)$$

This yields the Alpha representation in Eq. (9) for the original integral in Eq. (8).

The parameter integral is

$$\int_0^\infty d\alpha \alpha^{a_1-D/2-1} e^{-i\alpha(m^2-i0)} = i^{D/2-a_1} \Gamma \left(a_1 - \frac{D}{2} \right) (m^2 - i0)^{D/2-a_1} \quad (18)$$

Here we can easily see $\text{Re}[i(m^2 - i0)] > 0$. The final result is

$$I_{a_1}(m^2) = (m^2 - i0)^{D/2-a_1} \frac{\Gamma(a_1 - D/2)}{\Gamma(a_1)} \quad (19)$$

The direct parameter integral converges initially for $\text{Re}(a_1 - D/2) > 0$. The final meromorphic expression defines the result outside this region by analytic continuation.

From Alpha parameters to Feynman parameters

To convert the Alpha representation into a Feynman-parameter representation, we insert

$$1 = \int_0^\infty d\lambda \delta(\lambda - \alpha)$$

into Eq. (9) and define

$$\alpha = \lambda x$$

The integration measure and the delta function transform as

$$d\alpha = \lambda dx, \quad \delta(\lambda - \alpha) = \delta(\lambda(1 - x)) = \frac{1}{\lambda} \delta(1 - x)$$

The two factors of λ therefore cancel:

$$d\lambda d\alpha \delta(\lambda - \alpha) = d\lambda dx \delta(1 - x)$$

Thus, the Alpha representation becomes

$$\begin{aligned} I_{a_1}(m^2) &= \frac{i^{a_1-D/2}}{\Gamma(a_1)} \int_0^\infty d\lambda \int_0^\infty d\alpha \delta(\lambda - \alpha) \alpha^{a_1-D/2-1} e^{-i\alpha(m^2-i0)} \\ &= \frac{i^{a_1-D/2}}{\Gamma(a_1)} \int_0^\infty dx \delta(1 - x) x^{a_1-D/2-1} \int_0^\infty d\lambda \lambda^{a_1-D/2-1} e^{-i\lambda x(m^2-i0)} \end{aligned} \quad (20)$$

Because $\text{Re}[i(m^2 - i0)] > 0$, we then use the Gamma-function identity

$$\int_0^\infty d\lambda \lambda^{\nu-1} e^{-i\lambda x(m^2-i0)} = i^{-\nu} \Gamma(\nu) [x(m^2 - i0)]^{-\nu}, \quad \text{Re } \nu > 0$$

Substituting this result into Eq. (20), the phase factors cancel, giving the Feynman-parameter representation

$$I_{a_1}(m^2) = \frac{\Gamma(a_1 - D/2)}{\Gamma(a_1)} \int_0^\infty dx \delta(1 - x) \frac{x^{a_1-D/2-1}}{[x(m^2 - i0)]^{a_1-D/2}} \quad (21)$$

Since there is only one propagator, the constraint fixes $x = 1$, leaving no independent Feynman parameter. Thus,

$$I_{a_1}(m^2) = \frac{\Gamma(a_1 - D/2)}{\Gamma(a_1)} (m^2 - i0)^{D/2-a_1}$$

in agreement with Eq. (19).

1.1.2 One-loop bubble

Next we consider the one-loop bubble integral shown in Fig. 2.

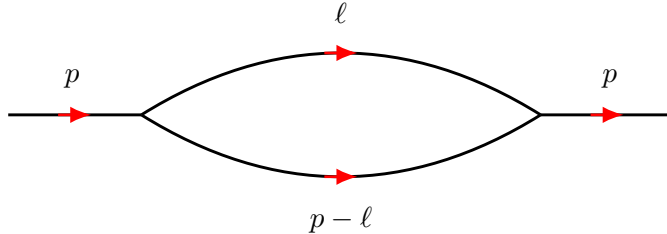


Figure 2: One-loop bubble diagram

$$I_{a_1 a_2}(p^2) = \int \frac{d^D \ell}{i\pi^{D/2}} \frac{1}{(-\ell^2 - i0)^{a_1} (-(p - \ell)^2 - i0)^{a_2}} \quad (22)$$

Here both internal propagators are massless, while the external momentum p is kept off shell in general, $s = -p^2$.

From the original integral to the Alpha representation

Applying Eq. (1) to the two propagators in Eq. (22) gives

$$I_{a_1 a_2}(p^2) = \frac{i^{a_1+a_2}}{\Gamma(a_1)\Gamma(a_2)} \int_0^\infty d\alpha_1 \int_0^\infty d\alpha_2 \alpha_1^{a_1-1} \alpha_2^{a_2-1} \int \frac{d^D \ell}{i\pi^{D/2}} \exp\{i\alpha_1 \ell^2 + i\alpha_2 (p - \ell)^2 - 0(\alpha_1 + \alpha_2)\} \quad (23)$$

Here, $-0(\alpha_1 + \alpha_2)$ is the damping factor generated by the $-i0$ prescription of the two propagators. Expanding the quadratic form gives

$$\begin{aligned} \alpha_1 \ell^2 + \alpha_2 (p - \ell)^2 &= (\alpha_1 + \alpha_2) \ell^2 - 2\alpha_2 p \cdot \ell + \alpha_2 p^2 \\ &= (\alpha_1 + \alpha_2) \left(\ell - \frac{\alpha_2}{\alpha_1 + \alpha_2} p \right)^2 + \frac{\alpha_1 \alpha_2}{\alpha_1 + \alpha_2} p^2 \end{aligned} \quad (24)$$

We therefore set

$$\ell' = \ell - \frac{\alpha_2}{\alpha_1 + \alpha_2} p, \quad d^D \ell' = d^D \ell \quad (25)$$

and use the Minkowski Gaussian integral

$$\int d^D \ell' e^{i(\alpha_1 + \alpha_2) \ell'^2} = i^{1-D/2} \pi^{D/2} (\alpha_1 + \alpha_2)^{-D/2} \quad (26)$$

Thus, the Alpha-parameter representation of the original integral is

$$\begin{aligned} I_{a_1 a_2}(p^2) &= \frac{i^{a_1+a_2-D/2}}{\Gamma(a_1)\Gamma(a_2)} \int_0^\infty d\alpha_1 \int_0^\infty d\alpha_2 \frac{\alpha_1^{a_1-1} \alpha_2^{a_2-1}}{(\alpha_1 + \alpha_2)^{D/2}} \exp\left[i \frac{\alpha_1 \alpha_2}{\alpha_1 + \alpha_2} p^2 - 0(\alpha_1 + \alpha_2)\right] \\ &= \frac{i^{a_1+a_2-D/2}}{\Gamma(a_1)\Gamma(a_2)} \int_0^\infty d\alpha_1 \int_0^\infty d\alpha_2 \frac{\alpha_1^{a_1-1} \alpha_2^{a_2-1}}{(\alpha_1 + \alpha_2)^{D/2}} \exp\left[i \frac{\alpha_1 \alpha_2}{\alpha_1 + \alpha_2} (p^2 + i0)\right] \end{aligned} \quad (27)$$

From the Alpha representation to the Feynman representation

To obtain the Feynman-parameter representation, we insert

$$1 = \int_0^\infty d\lambda \delta(\lambda - \alpha_1 - \alpha_2)$$

into Eq. (27) and, as before, define

$$\alpha_i = \lambda x_i$$

Since $\lambda = \sum_i \alpha_i$, the integration measure and the delta function transform as

$$d\alpha_1 d\alpha_2 = \lambda^2 dx_1 dx_2, \quad \delta(\lambda - \alpha_1 - \alpha_2) = \frac{1}{\lambda} \delta(1 - x_1 - x_2)$$

Therefore, one factor of λ remains:

$$d\lambda d\alpha_1 d\alpha_2 \delta(\lambda - \alpha_1 - \alpha_2) = \lambda d\lambda dx_1 dx_2 \delta(1 - x_1 - x_2).$$

Combining the remaining factor of λ from the measure with the powers in the integrand gives

$$\lambda \frac{(\lambda x_1)^{a_1-1} (\lambda x_2)^{a_2-1}}{\lambda^{D/2}} = \lambda^{a_1+a_2-D/2-1} x_1^{a_1-1} x_2^{a_2-1}$$

Thus, the Alpha representation becomes

$$\begin{aligned} I_{a_1 a_2}(p^2) &= \frac{i^{a_1+a_2-D/2}}{\Gamma(a_1)\Gamma(a_2)} \int_0^\infty d\lambda \int_0^\infty d\alpha_1 \int_0^\infty d\alpha_2 \delta(\lambda - \alpha_1 - \alpha_2) \frac{\alpha_1^{a_1-1} \alpha_2^{a_2-1}}{(\alpha_1 + \alpha_2)^{D/2}} \exp\left[i \frac{\alpha_1 \alpha_2}{\alpha_1 + \alpha_2} p^2 - 0(\alpha_1 + \alpha_2)\right] \\ &= \frac{i^{a_1+a_2-D/2}}{\Gamma(a_1)\Gamma(a_2)} \int_0^\infty dx_1 \int_0^\infty dx_2 \delta(1 - x_1 - x_2) x_1^{a_1-1} x_2^{a_2-1} \int_0^\infty d\lambda \lambda^{a_1+a_2-D/2-1} e^{-i\lambda[-x_1 x_2 p^2 - i0]} \end{aligned} \quad (28)$$

We then use the Gamma-function identity

$$\int_0^\infty d\lambda \lambda^{\nu-1} e^{-i\lambda[-x_1 x_2 p^2 - i0]} = i^{-\nu} \Gamma(\nu) [-x_1 x_2 p^2 - i0]^{-\nu}, \quad \text{Re } \nu > 0$$

Substituting this result into Eq. (28) gives the Feynman-parameter representation

$$I_{a_1 a_2}(p^2) = \frac{\Gamma(a_1 + a_2 - D/2)}{\Gamma(a_1)\Gamma(a_2)} \int_0^\infty dx_1 \int_0^\infty dx_2 \delta(1 - x_1 - x_2) \frac{x_1^{a_1-1} x_2^{a_2-1}}{[-x_1 x_2 p^2 - i0]^{a_1+a_2-D/2}} \quad (29)$$

Since there are two propagators, the constraint leaves one independent Feynman parameter. Integrating over x_2 and writing $x_1 = x$, we obtain

$$\begin{aligned} I_{a_1 a_2}(p^2) &= \frac{\Gamma(a_1 + a_2 - D/2)}{\Gamma(a_1)\Gamma(a_2)} \int_0^1 dx \frac{x^{a_1-1} (1-x)^{a_2-1}}{[-x(1-x)p^2 - i0]^{a_1+a_2-D/2}} \\ &= (-p^2)^{D/2-a_1-a_2} \frac{\Gamma(a_1 + a_2 - D/2)}{\Gamma(a_1)\Gamma(a_2)} \int_0^1 dx x^{D/2-a_2-1} (1-x)^{D/2-a_1-1} \end{aligned} \quad (30)$$

The delta-function constraint removes one Feynman-parameter integration, leaving a single independent parameter.

Using the Beta function identity

$$\int_0^1 dx x^{u-1} (1-x)^{v-1} = \frac{\Gamma(u)\Gamma(v)}{\Gamma(u+v)}, \quad \text{Re } u > 0, \quad \text{Re } v > 0$$

the final result is

$$I_{a_1 a_2}(p^2) = (-p^2 - i0)^{D/2-a_1-a_2} \frac{\Gamma(a_1 + a_2 - D/2) \Gamma(D/2 - a_1) \Gamma(D/2 - a_2)}{\Gamma(a_1)\Gamma(a_2)\Gamma(D - a_1 - a_2)} \quad (31)$$

For the displayed parameter integrals to converge literally, one may work initially in a domain where $\text{Re } a_i > 0$, $\text{Re}(a_1 + a_2 - D/2) > 0$, and $\text{Re}(D/2 - a_i) > 0$. The Gamma-function result is then continued meromorphically beyond that domain.

1.1.3 Double box

Then we study the two-loop massless double-box integral, whose topology is shown in Fig. 3.

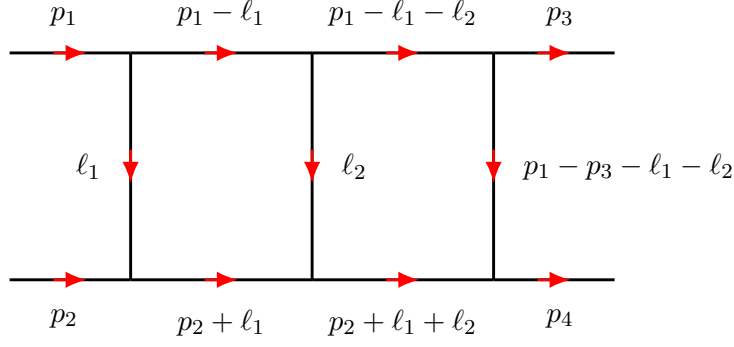


Figure 3: Double box diagram

Here we assume that all external momenta are massless and impose momentum conservation:

$$p_1 + p_2 = p_3 + p_4, \quad p_i^2 = 0, \quad i = 1, 2, 3, 4 \quad (32)$$

We choose s and t as the independent Mandelstam invariants, with $u = -s - t$:

$$s = (p_1 + p_2)^2, \quad t = (p_1 - p_3)^2, \quad u = (p_1 - p_4)^2 = -s - t \quad (33)$$

Following the momentum routing shown in Fig. 3, we define

$$\begin{aligned} q_1 &= \ell_1, & q_2 &= p_1 - \ell_1, & q_3 &= p_2 + \ell_1, & q_4 &= \ell_2 \\ q_5 &= p_1 - \ell_1 - \ell_2, & q_6 &= p_2 + \ell_1 + \ell_2, & q_7 &= p_1 - p_3 - \ell_1 - \ell_2 \end{aligned} \quad (34)$$

The corresponding two-loop integral is

$$I_{a_1 \dots a_7}(s, t) = \int \frac{d^D \ell_1}{i\pi^{D/2}} \int \frac{d^D \ell_2}{i\pi^{D/2}} \prod_{j=1}^7 \frac{1}{(-q_j^2 - i0)^{a_j}} \quad (35)$$

For later convenience, we define

$$A \equiv \sum_{j=1}^7 a_j \quad (36)$$

From the original integral to the Alpha representation

Applying Eq. (1) to all seven propagators gives

$$I_{a_1 \dots a_7}(s, t) = \frac{i^A}{\prod_{j=1}^7 \Gamma(a_j)} \left(\prod_{j=1}^7 \int_0^\infty d\alpha_j \alpha_j^{a_j-1} \right) \int \frac{d^D \ell_1}{i\pi^{D/2}} \int \frac{d^D \ell_2}{i\pi^{D/2}} \exp \left[i \sum_{j=1}^7 \alpha_j (q_j^2 + i0) \right] \quad (37)$$

Substituting the momenta defined in Eq. (34), we obtain

$$\begin{aligned} \sum_{j=1}^7 \alpha_j q_j^2 &= \alpha_1 \ell_1^2 + \alpha_2 (p_1 - \ell_1)^2 + \alpha_3 (p_2 + \ell_1)^2 + \alpha_4 \ell_2^2 + \alpha_5 (p_1 - \ell_1 - \ell_2)^2 \\ &\quad + \alpha_6 (p_2 + \ell_1 + \ell_2)^2 + \alpha_7 (p_1 - p_3 - \ell_1 - \ell_2)^2 \end{aligned} \quad (38)$$

We expand each square separately:

$$\begin{aligned}
(p_1 - \ell_1)^2 &= p_1^2 - 2p_1 \cdot \ell_1 + \ell_1^2, \\
(p_2 + \ell_1)^2 &= p_2^2 + 2p_2 \cdot \ell_1 + \ell_1^2, \\
(p_1 - \ell_1 - \ell_2)^2 &= p_1^2 - 2p_1 \cdot (\ell_1 + \ell_2) + \ell_1^2 + \ell_2^2 + 2\ell_1 \cdot \ell_2, \\
(p_2 + \ell_1 + \ell_2)^2 &= p_2^2 + 2p_2 \cdot (\ell_1 + \ell_2) + \ell_1^2 + \ell_2^2 + 2\ell_1 \cdot \ell_2, \\
(p_1 - p_3 - \ell_1 - \ell_2)^2 &= (p_1 - p_3)^2 + \ell_1^2 + \ell_2^2 + 2\ell_1 \cdot \ell_2 - 2(p_1 - p_3) \cdot (\ell_1 + \ell_2)
\end{aligned} \tag{39}$$

Using $p_1^2 = p_2^2 = 0$ and $(p_1 - p_3)^2 = t$, and then collecting equal powers of the loop momenta, Eq. (38) becomes

$$\begin{aligned}
\sum_{j=1}^7 \alpha_j q_j^2 &= (\alpha_1 + \alpha_2 + \alpha_3 + \alpha_5 + \alpha_6 + \alpha_7) \ell_1^2 \\
&\quad + (\alpha_4 + \alpha_5 + \alpha_6 + \alpha_7) \ell_2^2 + 2(\alpha_5 + \alpha_6 + \alpha_7) \ell_1 \cdot \ell_2 \\
&\quad + 2[-(\alpha_2 + \alpha_5 + \alpha_7)p_1 + (\alpha_3 + \alpha_6)p_2 + \alpha_7 p_3] \cdot \ell_1 \\
&\quad + 2[-(\alpha_5 + \alpha_7)p_1 + \alpha_6 p_2 + \alpha_7 p_3] \cdot \ell_2 + \alpha_7 t
\end{aligned} \tag{40}$$

Equation (40) allows us to identify the matrix $M^{(\alpha)}$:

$$M^{(\alpha)} = \begin{pmatrix} \alpha_1 + \alpha_2 + \alpha_3 + \alpha_5 + \alpha_6 + \alpha_7 & \alpha_5 + \alpha_6 + \alpha_7 \\ \alpha_5 + \alpha_6 + \alpha_7 & \alpha_4 + \alpha_5 + \alpha_6 + \alpha_7 \end{pmatrix} \tag{41}$$

The two linear coefficients in Eq. (40) define the vector $\mathbf{B}^{(\alpha)}$:

$$\mathbf{B}^{(\alpha)} = \begin{pmatrix} -(\alpha_2 + \alpha_5 + \alpha_7)p_1 + (\alpha_3 + \alpha_6)p_2 + \alpha_7 p_3 \\ -(\alpha_5 + \alpha_7)p_1 + \alpha_6 p_2 + \alpha_7 p_3 \end{pmatrix} \tag{42}$$

Indeed,

$$\begin{aligned}
&\begin{pmatrix} \ell_1 \\ \ell_2 \end{pmatrix}^T M^{(\alpha)} \begin{pmatrix} \ell_1 \\ \ell_2 \end{pmatrix} + 2 \begin{pmatrix} \ell_1 \\ \ell_2 \end{pmatrix}^T \mathbf{B}^{(\alpha)} \\
&= \begin{pmatrix} \ell_1 \\ \ell_2 \end{pmatrix}^T M^{(\alpha)} \begin{pmatrix} \ell_1 \\ \ell_2 \end{pmatrix} + \begin{pmatrix} \ell_1 \\ \ell_2 \end{pmatrix}^T \mathbf{B}^{(\alpha)} + \left(\mathbf{B}^{(\alpha)}\right)^T \begin{pmatrix} \ell_1 \\ \ell_2 \end{pmatrix} + \left(\mathbf{B}^{(\alpha)}\right)^T \left(M^{(\alpha)}\right)^{-1} \mathbf{B}^{(\alpha)} - \left(\mathbf{B}^{(\alpha)}\right)^T \left(M^{(\alpha)}\right)^{-1} \mathbf{B}^{(\alpha)} \\
&= \begin{pmatrix} \ell_1 \\ \ell_2 \end{pmatrix}^T M^{(\alpha)} \begin{pmatrix} \ell_1 \\ \ell_2 \end{pmatrix} + \begin{pmatrix} \ell_1 \\ \ell_2 \end{pmatrix}^T M^{(\alpha)} \left(M^{(\alpha)}\right)^{-1} \mathbf{B}^{(\alpha)} + \left(\mathbf{B}^{(\alpha)}\right)^T \left(M^{(\alpha)}\right)^{-1} M^{(\alpha)} \begin{pmatrix} \ell_1 \\ \ell_2 \end{pmatrix} \\
&\quad + \left(\mathbf{B}^{(\alpha)}\right)^T \left(M^{(\alpha)}\right)^{-1} M^{(\alpha)} \left(M^{(\alpha)}\right)^{-1} \mathbf{B}^{(\alpha)} - \left(\mathbf{B}^{(\alpha)}\right)^T \left(M^{(\alpha)}\right)^{-1} \mathbf{B}^{(\alpha)} \\
&= \left[\begin{pmatrix} \ell_1 \\ \ell_2 \end{pmatrix} + \left(M^{(\alpha)}\right)^{-1} \mathbf{B}^{(\alpha)} \right]^T M^{(\alpha)} \left[\begin{pmatrix} \ell_1 \\ \ell_2 \end{pmatrix} + \left(M^{(\alpha)}\right)^{-1} \mathbf{B}^{(\alpha)} \right] - \left(\mathbf{B}^{(\alpha)}\right)^T \left(M^{(\alpha)}\right)^{-1} \mathbf{B}^{(\alpha)}
\end{aligned} \tag{43}$$

Therefore, the completed square can be written directly as

$$\sum_{j=1}^7 \alpha_j q_j^2 = \left[\begin{pmatrix} \ell_1 \\ \ell_2 \end{pmatrix} + \left(M^{(\alpha)} \right)^{-1} \mathbf{B}^{(\alpha)} \right]^T M^{(\alpha)} \left[\begin{pmatrix} \ell_1 \\ \ell_2 \end{pmatrix} + \left(M^{(\alpha)} \right)^{-1} \mathbf{B}^{(\alpha)} \right] + \alpha_7 t - \left(\mathbf{B}^{(\alpha)} \right)^T \left(M^{(\alpha)} \right)^{-1} \mathbf{B}^{(\alpha)} \quad (44)$$

After shifting the loop momenta and relabeling the shifted integration variables as ℓ_1, ℓ_2 , the two normalized Gaussian integrals become

$$\begin{aligned} & \int \frac{d^D \ell_1}{i\pi^{D/2}} \int \frac{d^D \ell_2}{i\pi^{D/2}} \exp \left[i \begin{pmatrix} \ell_1 \\ \ell_2 \end{pmatrix}^T M^{(\alpha)} \begin{pmatrix} \ell_1 \\ \ell_2 \end{pmatrix} \right] \\ &= \int \frac{d^D \ell_1}{i\pi^{D/2}} \int \frac{d^D \ell_2}{i\pi^{D/2}} \exp \left\{ i \left[(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_5 + \alpha_6 + \alpha_7) \ell_1^2 \right. \right. \\ & \quad \left. \left. + 2(\alpha_5 + \alpha_6 + \alpha_7) \ell_1 \cdot \ell_2 + (\alpha_4 + \alpha_5 + \alpha_6 + \alpha_7) \ell_2^2 \right] \right\} \end{aligned} \quad (45)$$

We first complete the square in ℓ_1 . The quadratic form in the exponent can be written as

$$\begin{aligned} & (\alpha_1 + \alpha_2 + \alpha_3 + \alpha_5 + \alpha_6 + \alpha_7) \ell_1^2 + 2(\alpha_5 + \alpha_6 + \alpha_7) \ell_1 \cdot \ell_2 + (\alpha_4 + \alpha_5 + \alpha_6 + \alpha_7) \ell_2^2 \\ &= (\alpha_1 + \alpha_2 + \alpha_3 + \alpha_5 + \alpha_6 + \alpha_7) \left[\ell_1 + \frac{\alpha_5 + \alpha_6 + \alpha_7}{\alpha_1 + \alpha_2 + \alpha_3 + \alpha_5 + \alpha_6 + \alpha_7} \ell_2 \right]^2 \\ & \quad + \left[\alpha_4 + \alpha_5 + \alpha_6 + \alpha_7 - \frac{(\alpha_5 + \alpha_6 + \alpha_7)^2}{\alpha_1 + \alpha_2 + \alpha_3 + \alpha_5 + \alpha_6 + \alpha_7} \right] \ell_2^2 \end{aligned} \quad (46)$$

We now perform the shift

$$\ell'_1 = \ell_1 + \frac{\alpha_5 + \alpha_6 + \alpha_7}{\alpha_1 + \alpha_2 + \alpha_3 + \alpha_5 + \alpha_6 + \alpha_7} \ell_2 \quad (47)$$

For fixed ℓ_2 , this is a translation of ℓ_1 , and its Jacobian is therefore 1:

$$d^D \ell_1 = d^D \ell'_1$$

Eq. (45) therefore becomes

$$\begin{aligned} & \int \frac{d^D \ell'_1}{i\pi^{D/2}} \exp \left[i(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_5 + \alpha_6 + \alpha_7) \ell'^2_1 \right] \\ & \times \int \frac{d^D \ell_2}{i\pi^{D/2}} \exp \left\{ i \left[\alpha_4 + \alpha_5 + \alpha_6 + \alpha_7 - \frac{(\alpha_5 + \alpha_6 + \alpha_7)^2}{\alpha_1 + \alpha_2 + \alpha_3 + \alpha_5 + \alpha_6 + \alpha_7} \right] \ell_2^2 \right\} \end{aligned} \quad (48)$$

Using the normalized Minkowski Gaussian integral, the first integral is

$$\int \frac{d^D \ell'_1}{i\pi^{D/2}} \exp \left[i(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_5 + \alpha_6 + \alpha_7) \ell'^2_1 \right] = i^{-D/2} (\alpha_1 + \alpha_2 + \alpha_3 + \alpha_5 + \alpha_6 + \alpha_7)^{-D/2} \quad (49)$$

and the second integral is

$$\begin{aligned} & \int \frac{d^D \ell_2}{i\pi^{D/2}} \exp \left\{ i \left[\alpha_4 + \alpha_5 + \alpha_6 + \alpha_7 - \frac{(\alpha_5 + \alpha_6 + \alpha_7)^2}{\alpha_1 + \alpha_2 + \alpha_3 + \alpha_5 + \alpha_6 + \alpha_7} \right] \ell_2^2 \right\} \\ &= i^{-D/2} \left[\alpha_4 + \alpha_5 + \alpha_6 + \alpha_7 - \frac{(\alpha_5 + \alpha_6 + \alpha_7)^2}{\alpha_1 + \alpha_2 + \alpha_3 + \alpha_5 + \alpha_6 + \alpha_7} \right]^{-D/2} \end{aligned} \quad (50)$$

Multiplying Eqs. (49) and (50), we obtain

$$\begin{aligned} & i^{-D} (\alpha_1 + \alpha_2 + \alpha_3 + \alpha_5 + \alpha_6 + \alpha_7)^{-D/2} \left[\alpha_4 + \alpha_5 + \alpha_6 + \alpha_7 - \frac{(\alpha_5 + \alpha_6 + \alpha_7)^2}{\alpha_1 + \alpha_2 + \alpha_3 + \alpha_5 + \alpha_6 + \alpha_7} \right]^{-D/2} \\ &= i^{-D} \left[(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_5 + \alpha_6 + \alpha_7)(\alpha_4 + \alpha_5 + \alpha_6 + \alpha_7) - (\alpha_5 + \alpha_6 + \alpha_7)^2 \right]^{-D/2} \end{aligned} \quad (51)$$

On the other hand, the determinant of the explicit matrix is

$$\begin{aligned} \det M^{(\alpha)} &= \det \begin{pmatrix} \alpha_1 + \alpha_2 + \alpha_3 + \alpha_5 + \alpha_6 + \alpha_7 & \alpha_5 + \alpha_6 + \alpha_7 \\ \alpha_5 + \alpha_6 + \alpha_7 & \alpha_4 + \alpha_5 + \alpha_6 + \alpha_7 \end{pmatrix} \\ &= (\alpha_1 + \alpha_2 + \alpha_3 + \alpha_5 + \alpha_6 + \alpha_7)(\alpha_4 + \alpha_5 + \alpha_6 + \alpha_7) - (\alpha_5 + \alpha_6 + \alpha_7)^2 \end{aligned} \quad (52)$$

Hence,

$$\int \frac{d^D \ell_1}{i\pi^{D/2}} \int \frac{d^D \ell_2}{i\pi^{D/2}} \exp \left[i \begin{pmatrix} \ell_1 \\ \ell_2 \end{pmatrix}^T M^{(\alpha)} \begin{pmatrix} \ell_1 \\ \ell_2 \end{pmatrix} \right] = i^{-D} \left(\det M^{(\alpha)} \right)^{-D/2} \quad (53)$$

We therefore define $\mathcal{U}(\alpha) = \det M^{(\alpha)}$, so that

$$\begin{aligned} \mathcal{U}(\alpha) &= (\alpha_1 + \alpha_2 + \alpha_3 + \alpha_5 + \alpha_6 + \alpha_7)(\alpha_4 + \alpha_5 + \alpha_6 + \alpha_7) - (\alpha_5 + \alpha_6 + \alpha_7)^2 \\ &= (\alpha_1 + \alpha_2 + \alpha_3)(\alpha_4 + \alpha_5 + \alpha_6 + \alpha_7) + \alpha_4(\alpha_5 + \alpha_6 + \alpha_7) \end{aligned} \quad (54)$$

In the present massless case, the only nonvanishing invariant r_j^2 is

$$r_7^2 = (p_1 - p_3)^2 = t$$

Therefore, we can first define $\mathcal{F}_0(\alpha)$:

$$\mathcal{F}_0(\alpha) = -\alpha_7 t \mathcal{U}(\alpha) + \mathcal{U}(\alpha) \left(\mathbf{B}^{(\alpha)} \right)^T \left(M^{(\alpha)} \right)^{-1} \mathbf{B}^{(\alpha)} \quad (55)$$

Using

$$\mathcal{U}(\alpha) \left(M^{(\alpha)} \right)^{-1} = \begin{pmatrix} \alpha_4 + \alpha_5 + \alpha_6 + \alpha_7 & -(\alpha_5 + \alpha_6 + \alpha_7) \\ -(\alpha_5 + \alpha_6 + \alpha_7) & \alpha_1 + \alpha_2 + \alpha_3 + \alpha_5 + \alpha_6 + \alpha_7 \end{pmatrix} \quad (56)$$

we obtain

$$\begin{aligned}
\mathcal{F}_0(\boldsymbol{\alpha}) = & -\alpha_7 t \mathcal{U}(\boldsymbol{\alpha}) + (\alpha_4 + \alpha_5 + \alpha_6 + \alpha_7) [-(\alpha_2 + \alpha_5 + \alpha_7)p_1 + (\alpha_3 + \alpha_6)p_2 + \alpha_7 p_3]^2 \\
& - 2(\alpha_5 + \alpha_6 + \alpha_7) [-(\alpha_2 + \alpha_5 + \alpha_7)p_1 + (\alpha_3 + \alpha_6)p_2 + \alpha_7 p_3] [-(\alpha_5 + \alpha_7)p_1 + \alpha_6 p_2 + \alpha_7 p_3] \\
& + (\alpha_1 + \alpha_2 + \alpha_3 + \alpha_5 + \alpha_6 + \alpha_7) [-(\alpha_5 + \alpha_7)p_1 + \alpha_6 p_2 + \alpha_7 p_3]^2
\end{aligned} \tag{57}$$

The first square is

$$\begin{aligned}
& [-(\alpha_2 + \alpha_5 + \alpha_7)p_1 + (\alpha_3 + \alpha_6)p_2 + \alpha_7 p_3]^2 \\
= & -2(\alpha_2 + \alpha_5 + \alpha_7)(\alpha_3 + \alpha_6)p_1 \cdot p_2 - 2\alpha_7(\alpha_2 + \alpha_5 + \alpha_7)p_1 \cdot p_3 + 2\alpha_7(\alpha_3 + \alpha_6)p_2 \cdot p_3 \\
= & -s(\alpha_2 + \alpha_5 + \alpha_7)(\alpha_3 + \alpha_6) + t\alpha_7(\alpha_2 + \alpha_5 + \alpha_7) + (s+t)\alpha_7(\alpha_3 + \alpha_6) \\
= & -s(\alpha_2 + \alpha_5)(\alpha_3 + \alpha_6) + t\alpha_7(\alpha_2 + \alpha_3 + \alpha_5 + \alpha_6 + \alpha_7)
\end{aligned} \tag{58}$$

The second square is

$$\begin{aligned}
& [-(\alpha_5 + \alpha_7)p_1 + \alpha_6 p_2 + \alpha_7 p_3]^2 \\
= & -2\alpha_6(\alpha_5 + \alpha_7)p_1 \cdot p_2 - 2\alpha_7(\alpha_5 + \alpha_7)p_1 \cdot p_3 + 2\alpha_6\alpha_7 p_2 \cdot p_3 \\
= & -s\alpha_6(\alpha_5 + \alpha_7) + t\alpha_7(\alpha_5 + \alpha_7) + (s+t)\alpha_6\alpha_7 \\
= & -s\alpha_5\alpha_6 + t\alpha_7(\alpha_5 + \alpha_6 + \alpha_7)
\end{aligned} \tag{59}$$

The mixed scalar product is

$$\begin{aligned}
& [-(\alpha_2 + \alpha_5 + \alpha_7)p_1 + (\alpha_3 + \alpha_6)p_2 + \alpha_7 p_3] \cdot [-(\alpha_5 + \alpha_7)p_1 + \alpha_6 p_2 + \alpha_7 p_3] \\
= & -\alpha_6(\alpha_2 + \alpha_5 + \alpha_7)p_1 \cdot p_2 - \alpha_7(\alpha_2 + \alpha_5 + \alpha_7)p_1 \cdot p_3 \\
& - (\alpha_3 + \alpha_6)(\alpha_5 + \alpha_7)p_1 \cdot p_2 + \alpha_7(\alpha_3 + \alpha_6)p_2 \cdot p_3 \\
& - \alpha_7(\alpha_5 + \alpha_7)p_1 \cdot p_3 + \alpha_6\alpha_7 p_2 \cdot p_3 \\
= & -\frac{s}{2}(\alpha_2\alpha_6 + \alpha_3\alpha_5 + 2\alpha_5\alpha_6) + \frac{t\alpha_7}{2}(\alpha_2 + \alpha_3 + 2\alpha_5 + 2\alpha_6 + 2\alpha_7)
\end{aligned} \tag{60}$$

The terms proportional to s are

$$\begin{aligned}
\mathcal{F}_0(\boldsymbol{\alpha})|_s = & -s \left\{ (\alpha_4 + \alpha_5 + \alpha_6 + \alpha_7)(\alpha_2 + \alpha_5)(\alpha_3 + \alpha_6) - (\alpha_5 + \alpha_6 + \alpha_7)(\alpha_2\alpha_6 + \alpha_3\alpha_5 + 2\alpha_5\alpha_6) \right. \\
& \left. + (\alpha_1 + \alpha_2 + \alpha_3 + \alpha_5 + \alpha_6 + \alpha_7)\alpha_5\alpha_6 \right\} \\
= & -s \left[\alpha_5\alpha_6(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4) + \alpha_2\alpha_3(\alpha_4 + \alpha_5 + \alpha_6 + \alpha_7) + \alpha_3\alpha_4\alpha_5 + \alpha_2\alpha_4\alpha_6 \right]
\end{aligned} \tag{61}$$

The terms proportional to t are

$$\begin{aligned}
\mathcal{F}_0(\boldsymbol{\alpha})|_t = & t\alpha_7 \left\{ -\mathcal{U}(\boldsymbol{\alpha}) + (\alpha_4 + \alpha_5 + \alpha_6 + \alpha_7)(\alpha_2 + \alpha_3 + \alpha_5 + \alpha_6 + \alpha_7) \right. \\
& \left. - (\alpha_5 + \alpha_6 + \alpha_7)(\alpha_2 + \alpha_3 + 2\alpha_5 + 2\alpha_6 + 2\alpha_7) + (\alpha_1 + \alpha_2 + \alpha_3 + \alpha_5 + \alpha_6 + \alpha_7)(\alpha_5 + \alpha_6 + \alpha_7) \right\} \\
= & -t\alpha_1\alpha_4\alpha_7
\end{aligned} \tag{62}$$

Combining these terms, we finally obtain

$$\begin{aligned}\mathcal{F}_0(\boldsymbol{\alpha}) = & -s \left[\alpha_5 \alpha_6 (\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4) + \alpha_2 \alpha_3 (\alpha_4 + \alpha_5 + \alpha_6 + \alpha_7) \right. \\ & \left. + \alpha_3 \alpha_4 \alpha_5 + \alpha_2 \alpha_4 \alpha_6 \right] - t \alpha_1 \alpha_4 \alpha_7\end{aligned}\quad (63)$$

Restoring the Feynman prescription, we define the full polynomial by

$$\mathcal{F}(\boldsymbol{\alpha}) \equiv \mathcal{F}_0(\boldsymbol{\alpha}) - i0 \mathcal{U}(\boldsymbol{\alpha}) \sum_{j=1}^7 \alpha_j. \quad (64)$$

After performing the two Gaussian integrations, the Alpha representation becomes

$$I_{a_1 \dots a_7}(s, t) = \frac{i^{A-D}}{\prod_{j=1}^7 \Gamma(a_j)} \left(\prod_{j=1}^7 \int_0^\infty d\alpha_j \alpha_j^{a_j-1} \right) \mathcal{U}(\boldsymbol{\alpha})^{-D/2} \exp \left[-i \frac{\mathcal{F}(\boldsymbol{\alpha})}{\mathcal{U}(\boldsymbol{\alpha})} \right] \quad (65)$$

From the Alpha representation to the Feynman representation

We insert

$$1 = \int_0^\infty d\lambda \delta \left(\lambda - \sum_{j=1}^7 \alpha_j \right) \quad (66)$$

and define

$$\alpha_j = \lambda x_j \quad (67)$$

Then $x_j \geq 0$ and $\sum_{j=1}^7 x_j = 1$. The measure and delta function give

$$d\lambda \left(\prod_{j=1}^7 d\alpha_j \right) \delta \left(\lambda - \sum_{j=1}^7 \alpha_j \right) = \lambda^6 d\lambda \left(\prod_{j=1}^7 dx_j \right) \delta \left(1 - \sum_{j=1}^7 x_j \right) \quad (68)$$

From Eqs. (63) and (54), the two polynomials satisfy

$$\mathcal{U}(\lambda \mathbf{x}) = \lambda^2 \mathcal{U}(\mathbf{x}), \quad \mathcal{F}(\lambda \mathbf{x}) = \lambda^3 \mathcal{F}(\mathbf{x}) \quad (69)$$

Combining the measure, the propagator powers, and the factor $\mathcal{U}^{-D/2}$, the dependence on the overall scale is

$$\lambda^6 \prod_{j=1}^7 (\lambda x_j)^{a_j-1} \mathcal{U}(\lambda \mathbf{x})^{-D/2} = \lambda^{A-D-1} \left(\prod_{j=1}^7 x_j^{a_j-1} \right) \mathcal{U}(\mathbf{x})^{-D/2} \quad (70)$$

The remaining scale integral is

$$\int_0^\infty d\lambda \lambda^{A-D-1} \exp \left[-i\lambda \frac{\mathcal{F}(\mathbf{x})}{\mathcal{U}(\mathbf{x})} \right] = i^{-(A-D)} \Gamma(A-D) \left[\frac{\mathcal{F}(\mathbf{x})}{\mathcal{U}(\mathbf{x})} \right]^{-(A-D)} \quad (71)$$

The phase in this expression cancels the phase in Eq. (65). We finally obtain

$$I_{a_1 \dots a_7}(s, t) = \frac{\Gamma(A - D)}{\prod_{j=1}^7 \Gamma(a_j)} \left(\prod_{j=1}^7 \int_0^\infty dx_j x_j^{a_j-1} \right) \delta \left(1 - \sum_{j=1}^7 x_j \right) \frac{\mathcal{U}(\mathbf{x})^{A-3D/2}}{\mathcal{F}(\mathbf{x})^{A-D}} \quad (72)$$

Here $\mathcal{U}(\mathbf{x})$ and $\mathcal{F}(\mathbf{x})$ are obtained from Eqs. (54) and (63) by replacing every α_j by x_j .

Here we can see that D appears only in the determinant and the gamma function, so we can naturally analytically continue D to non-integer values.

2 General Feynman integral

We now consider a general integral family with L independent loop momenta and N propagators:

$$I_{a_1 \dots a_N}^{(L)} = \int \left(\prod_{r=1}^L \frac{d^D \ell_r}{i\pi^{D/2}} \right) \prod_{j=1}^N \frac{1}{(-q_j^2 + m_j^2 - i0)^{a_j}} \quad (73)$$

The momentum of each internal line can be written as

$$q_j = \sum_{r=1}^L c_{jr} \ell_r + r_j \quad (74)$$

where the c_{jr} are real numbers determined by the momentum routing, typically $0, \pm 1$, and r_j is a fixed vector constructed solely from external momenta.

From the original integral to the Alpha representation

Applying the parameter-integral identity to each propagator gives

$$I_{a_1 \dots a_N}^{(L)} = \frac{i^{\sum_{j=1}^N a_j}}{\prod_{j=1}^N \Gamma(a_j)} \left(\prod_{j=1}^N \int_0^\infty d\alpha_j \alpha_j^{a_j-1} \right) \exp \left[-i \sum_{j=1}^N \alpha_j (m_j^2 - i0) \right] \int \left(\prod_{r=1}^L \frac{d^D \ell_r}{i\pi^{D/2}} \right) \exp \left(i \sum_{j=1}^N \alpha_j q_j^2 \right) \quad (75)$$

Substituting Eq. (74) into the quadratic form in the exponent gives

$$\sum_{j=1}^N \alpha_j q_j^2 = \sum_{r,s=1}^L \left(\sum_{j=1}^N \alpha_j c_{jr} c_{js} \right) \ell_r \cdot \ell_s + 2 \sum_{r=1}^L \ell_r \cdot \left(\sum_{j=1}^N \alpha_j c_{jr} r_j \right) + \sum_{j=1}^N \alpha_j r_j^2 \quad (76)$$

This is an ordinary multivariable quadratic form. To avoid repeatedly writing the same sums, we denote the coefficients of the quadratic and linear terms by

$$M_{rs}^{(\alpha)} = \sum_{j=1}^N \alpha_j c_{jr} c_{js}, \quad B_r^{(\alpha)} = \sum_{j=1}^N \alpha_j c_{jr} r_j \quad (77)$$

Here, M is an $L \times L$ real symmetric matrix with the explicit form

$$M^{(\alpha)} = \begin{pmatrix} \sum_{j=1}^N \alpha_j c_{j1}^2 & \sum_{j=1}^N \alpha_j c_{j1} c_{j2} & \cdots & \sum_{j=1}^N \alpha_j c_{j1} c_{jL} \\ \sum_{j=1}^N \alpha_j c_{j2} c_{j1} & \sum_{j=1}^N \alpha_j c_{j2}^2 & \cdots & \sum_{j=1}^N \alpha_j c_{j2} c_{jL} \\ \vdots & \vdots & \ddots & \vdots \\ \sum_{j=1}^N \alpha_j c_{jL} c_{j1} & \sum_{j=1}^N \alpha_j c_{jL} c_{j2} & \cdots & \sum_{j=1}^N \alpha_j c_{jL}^2 \end{pmatrix} \quad (78)$$

It satisfies

$$M_{sr}^{(\alpha)} = \sum_{j=1}^N \alpha_j c_{js} c_{jr} = \sum_{j=1}^N \alpha_j c_{jr} c_{js} = M_{rs}^{(\alpha)} \quad (79)$$

Its inverse is defined by

$$\sum_{s=1}^L M_{rs}(M^{-1})_{su} = \delta_{ru}, \quad r, u = 1, \dots, L \quad (80)$$

The corresponding column vector is

$$\begin{pmatrix} B_1 \\ B_2 \\ \vdots \\ B_L \end{pmatrix} = \begin{pmatrix} \sum_{j=1}^N \alpha_j c_{j1} r_j \\ \sum_{j=1}^N \alpha_j c_{j2} r_j \\ \vdots \\ \sum_{j=1}^N \alpha_j c_{jL} r_j \end{pmatrix} \quad (81)$$

We perform the shift

$$\ell'_r = \ell_r + \sum_{s=1}^L (M^{-1})_{rs} B_s \text{ and, equivalently, } \ell_r = \ell'_r - \sum_{s=1}^L (M^{-1})_{rs} B_s \quad (82)$$

Substituting Eq. (82), we first consider the quadratic term:

$$\begin{aligned} \sum_{r,s=1}^L M_{rs} \ell_r \cdot \ell_s &= \sum_{r,s=1}^L M_{rs} \left[\ell'_r - \sum_{u=1}^L (M^{-1})_{ru} B_u \right] \cdot \left[\ell'_s - \sum_{v=1}^L (M^{-1})_{sv} B_v \right] \\ &= \sum_{r,s=1}^L M_{rs} \ell'_r \cdot \ell'_s - \sum_{r,s,u=1}^L M_{rs} (M^{-1})_{su} \ell'_r \cdot B_u - \sum_{r,s,u=1}^L M_{rs} (M^{-1})_{ru} B_u \cdot \ell'_s \\ &\quad + \sum_{r,s,u,v=1}^L M_{rs} (M^{-1})_{ru} (M^{-1})_{sv} B_u \cdot B_v \end{aligned} \quad (83)$$

$$\begin{aligned} &- \sum_{r,s,u=1}^L M_{rs} (M^{-1})_{su} \ell'_r \cdot B_u = - \sum_{r,u=1}^L \delta_{ru} \ell'_r \cdot B_u = - \sum_{r=1}^L \ell'_r \cdot B_r \\ &- \sum_{r,s,u=1}^L M_{rs} (M^{-1})_{ru} B_u \cdot \ell'_s = - \sum_{s,u=1}^L \delta_{su} B_u \cdot \ell'_s = - \sum_{s=1}^L \ell'_s \cdot B_s \\ \Rightarrow &- \sum_{r,s,u=1}^L M_{rs} (M^{-1})_{su} \ell'_r \cdot B_u - \sum_{r,s,u=1}^L M_{rs} (M^{-1})_{ru} B_u \cdot \ell'_s = -2 \sum_{r=1}^L \ell'_r \cdot B_r \end{aligned}$$

Finally, consider the terms involving only B :

$$\begin{aligned} \sum_{r,s,u,v=1}^L M_{rs}(M^{-1})_{ru}(M^{-1})_{sv} B_u \cdot B_v &= \sum_{u,v=1}^L \left[\sum_{r,s=1}^L (M^{-1})_{ur} M_{rs} (M^{-1})_{sv} \right] B_u \cdot B_v \\ &= \sum_{u,v=1}^L (M^{-1})_{uv} B_u \cdot B_v \end{aligned}$$

Therefore, the complete quadratic term can be written as

$$\sum_{r,s=1}^L M_{rs} \ell_r \cdot \ell_s = \sum_{r,s=1}^L M_{rs} \ell'_r \cdot \ell'_s - 2 \sum_{r=1}^L \ell'_r \cdot B_r + \sum_{r,s=1}^L (M^{-1})_{rs} B_r \cdot B_s \quad (84)$$

On the other hand, the linear term becomes

$$\begin{aligned} 2 \sum_{r=1}^L \ell_r \cdot B_r &= 2 \sum_{r=1}^L \left[\ell'_r - \sum_{s=1}^L (M^{-1})_{rs} B_s \right] \cdot B_r \\ &= 2 \sum_{r=1}^L \ell'_r \cdot B_r - 2 \sum_{r,s=1}^L (M^{-1})_{rs} B_s \cdot B_r = 2 \sum_{r=1}^L \ell'_r \cdot B_r - 2 \sum_{r,s=1}^L (M^{-1})_{rs} B_r \cdot B_s \end{aligned} \quad (85)$$

Adding Eqs. (84) and (85), we see that the cross terms involving $\ell'_r \cdot B_r$ cancel:

$$\begin{aligned} &\sum_{r,s=1}^L M_{rs} \ell_r \cdot \ell_s + 2 \sum_{r=1}^L \ell_r \cdot B_r \\ &= \sum_{r,s=1}^L M_{rs} \ell'_r \cdot \ell'_s - 2 \sum_{r=1}^L \ell'_r \cdot B_r + \sum_{r,s=1}^L (M^{-1})_{rs} B_r \cdot B_s + 2 \sum_{r=1}^L \ell'_r \cdot B_r - 2 \sum_{r,s=1}^L (M^{-1})_{rs} B_r \cdot B_s \\ &= \sum_{r,s=1}^L M_{rs} \ell'_r \cdot \ell'_s - \sum_{r,s=1}^L (M^{-1})_{rs} B_r \cdot B_s \end{aligned}$$

Thus, we obtain

$$\sum_{j=1}^N \alpha_j q_j^2 = \sum_{r,s=1}^L M_{rs} \ell'_r \cdot \ell'_s + \sum_{j=1}^N \alpha_j r_j^2 - \sum_{r,s=1}^L (M^{-1})_{rs} B_r \cdot B_s \quad (86)$$

The general matrix Gaussian integral derived in Appendix A.1.1 contains no Lorentz-space dimension. We now apply that result separately to each Lorentz component. In the interior of the Alpha-parameter domain, $\alpha_j > 0$. For any nonzero real L -component vector $\mathbf{v}$,

$$\mathbf{v}^T M \mathbf{v} = \sum_{r,s=1}^L v_r M_{rs} v_s = \sum_{j=1}^N \alpha_j \left(\sum_{r=1}^L c_{jr} v_r \right)^2 > 0 \quad (87)$$

provided that the routing matrix c_{jr} has rank L . Thus M is positive definite, as required by the

matrix Fresnel formulas. For every μ , introduce the L -component column vector

$$\boldsymbol{\ell}'^\mu = \begin{pmatrix} \ell_1'^\mu \\ \ell_2'^\mu \\ \vdots \\ \ell_L'^\mu \end{pmatrix} \quad (88)$$

With the mostly-minus Minkowski metric,

$$\sum_{r,s=1}^L M_{rs} \ell_r' \cdot \ell_s' = (\boldsymbol{\ell}'^0)^\text{T} M \boldsymbol{\ell}'^0 - \sum_{j=1}^{D-1} (\boldsymbol{\ell}'^j)^\text{T} M \boldsymbol{\ell}'^j \quad (89)$$

The original integral therefore factorizes into one L -dimensional matrix Fresnel integral with a positive phase and $D - 1$ such integrals with a negative phase, we find

$$\begin{aligned} & \int \left(\prod_{r=1}^L d^D \ell_r' \right) \exp \left(i \sum_{r,s=1}^L M_{rs} \ell_r' \cdot \ell_s' \right) \\ &= \left[\int d^L \boldsymbol{\ell}'^0 \exp \left(i (\boldsymbol{\ell}'^0)^\text{T} M \boldsymbol{\ell}'^0 \right) \right] \prod_{j=1}^{D-1} \left[\int d^L \boldsymbol{\ell}'^j \exp \left(-i (\boldsymbol{\ell}'^j)^\text{T} M \boldsymbol{\ell}'^j \right) \right] \\ &= \left[e^{iL\pi/4} \frac{\pi^{L/2}}{\sqrt{\det M}} \right] \left[e^{-iL\pi/4} \frac{\pi^{L/2}}{\sqrt{\det M}} \right]^{D-1} \\ &= i^{L(1-D/2)} \pi^{LD/2} (\det M)^{-D/2} \end{aligned} \quad (90)$$

For dimensional regularization this result is first obtained for integer D and then analytically continued in D . Substituting Eqs. (86) and (90) back into the integral gives

$$I_{a_1 \dots a_N}^{(L)} = \frac{i^{\sum_{j=1}^N a_j - LD/2}}{\prod_{j=1}^N \Gamma(a_j)} \left(\prod_{j=1}^N \int_0^\infty d\alpha_j \alpha_j^{a_j-1} \right) (\det M)^{-\frac{D}{2}} \quad (91)$$

$$\times \exp \left\{ -i \left[\sum_{j=1}^N \alpha_j (m_j^2 - r_j^2 - i0) + \sum_{r,s=1}^L (M^{-1})_{rs} B_r \cdot B_s \right] \right\} \quad (92)$$

To make this common structure explicit, we define

$$A \equiv \sum_{j=1}^N a_j, \quad \omega \equiv A - \frac{LD}{2} \quad (93)$$

For the Alpha parameters $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_N)$, we define

$$\mathcal{U}(\boldsymbol{\alpha}) \equiv \det M^{(\boldsymbol{\alpha})}, \quad (94)$$

$$\frac{\mathcal{F}(\boldsymbol{\alpha})}{\mathcal{U}(\boldsymbol{\alpha})} \equiv \sum_{j=1}^N \alpha_j (m_j^2 - r_j^2 - i0) + \sum_{r,s=1}^L \left[\left(M^{(\boldsymbol{\alpha})} \right)^{-1} \right]_{rs} B_r^{(\boldsymbol{\alpha})} \cdot B_s^{(\boldsymbol{\alpha})} \quad (95)$$

The general Alpha representation in Eq. (92) takes the compact form

$$I_{a_1 \dots a_N}^{(L)} = \frac{i^{A-LD/2}}{\prod_{j=1}^N \Gamma(a_j)} \left(\prod_{j=1}^N \int_0^\infty d\alpha_j \alpha_j^{a_j-1} \right) \mathcal{U}(\boldsymbol{\alpha})^{-\frac{D}{2}} \exp \left[-i \frac{\mathcal{F}(\boldsymbol{\alpha})}{\mathcal{U}(\boldsymbol{\alpha})} \right] \quad (96)$$

From Alpha Parameters to Feynman Parameters

For the Feynman parameters $\mathbf{x} = (x_1, \dots, x_N)$, we define

$$\alpha_j = \alpha x_j, \quad \alpha \equiv \sum_{j=1}^N \alpha_j, \quad M_{rs}^{(x)} \equiv \sum_{j=1}^N x_j c_{jr} c_{js}, \quad B_r^{(x)} \equiv \sum_{j=1}^N x_j c_{jr} r_j \quad (97)$$

with $x_j \geq 0$ and $\sum_{j=1}^N x_j = 1$.

Since M and B are both linear in their parameters, one has

$$M^{(\alpha)}(\alpha \mathbf{x}) = \alpha M^{(x)}(\mathbf{x}), \quad B_r^{(\alpha)}(\alpha \mathbf{x}) = \alpha B_r^{(x)}(\mathbf{x}) \quad (98)$$

and consequently

$$\left(M^{(\alpha)}(\alpha \mathbf{x}) \right)^{-1} = \alpha^{-1} \left(M^{(x)}(\mathbf{x}) \right)^{-1}, \quad \det M^{(\alpha)}(\alpha \mathbf{x}) = \alpha^L \det M^{(x)}(\mathbf{x}) \quad (99)$$

The term containing the inverse matrix therefore scales as

$$\begin{aligned} & \sum_{r,s=1}^L \left[\left(M^{(\alpha)}(\alpha \mathbf{x}) \right)^{-1} \right]_{rs} B_r^{(\alpha)}(\alpha \mathbf{x}) \cdot B_s^{(\alpha)}(\alpha \mathbf{x}) \\ &= \alpha \sum_{r,s=1}^L \left[\left(M^{(x)}(\mathbf{x}) \right)^{-1} \right]_{rs} B_r^{(x)}(\mathbf{x}) \cdot B_s^{(x)}(\mathbf{x}) \end{aligned} \quad (100)$$

It follows that

$$\begin{aligned} & \sum_{j=1}^N \alpha_j (m_j^2 - r_j^2 - i0) + \sum_{r,s=1}^L \left[\left(M^{(\alpha)} \right)^{-1} \right]_{rs} B_r^{(\alpha)} \cdot B_s^{(\alpha)} \\ &= \alpha \left[\sum_{j=1}^N x_j (m_j^2 - r_j^2 - i0) + \sum_{r,s=1}^L \left[\left(M^{(x)} \right)^{-1} \right]_{rs} B_r^{(x)} \cdot B_s^{(x)} \right] \end{aligned} \quad (101)$$

Thus, the homogeneity relations are

$$\mathcal{U}(\alpha \mathbf{x}) = \alpha^L \mathcal{U}(\mathbf{x}), \quad \mathcal{F}(\alpha \mathbf{x}) = \alpha^{L+1} \mathcal{F}(\mathbf{x}) \quad (102)$$

where

$$\mathcal{U}(\mathbf{x}) = \det M^{(x)}, \quad (103)$$

$$\frac{\mathcal{F}(\mathbf{x})}{\mathcal{U}(\mathbf{x})} = \sum_{j=1}^N x_j (m_j^2 - r_j^2 - i0) + \sum_{r,s=1}^L \left[\left(M^{(x)} \right)^{-1} \right]_{rs} B_r^{(x)} \cdot B_s^{(x)} \quad (104)$$

The integration measure on the positive orthant decomposes as

$$\prod_{j=1}^N d\alpha_j = \alpha^{N-1} d\alpha \left(\prod_{j=1}^N dx_j \right) \delta \left(1 - \sum_{j=1}^N x_j \right) \quad (105)$$

Combining this Jacobian with the powers of the parameters and with the homogeneity of $\mathcal{U}$, the dependence on the overall scale becomes

$$\alpha^{N-1} \prod_{j=1}^N (\alpha x_j)^{a_j-1} \mathcal{U}(\alpha \mathbf{x})^{-D/2} = \alpha^{A-LD/2-1} \left(\prod_{j=1}^N x_j^{a_j-1} \right) \mathcal{U}(\mathbf{x})^{-D/2} \quad (106)$$

The remaining scale integral is

$$\int_0^\infty d\alpha \alpha^{\omega-1} \exp \left[-i\alpha \frac{\mathcal{F}(\mathbf{x})}{\mathcal{U}(\mathbf{x})} \right] = i^{-\omega} \Gamma(\omega) \left[\frac{\mathcal{F}(\mathbf{x})}{\mathcal{U}(\mathbf{x})} \right]^{-\omega}, \quad \omega = A - \frac{LD}{2} \quad (107)$$

Here $\text{Re}[i \frac{\mathcal{F}(\mathbf{x})}{\mathcal{U}(\mathbf{x})}] > 0$ is always right, so just need to stasify $\text{Re} \omega > 0$.

We therefore obtain the general Feynman-parameter representation

$$\begin{aligned} I_{a_1 \dots a_N}^{(L)} &= \frac{\Gamma(A-LD/2)}{\prod_{j=1}^N \Gamma(a_j)} \left(\prod_{j=1}^N \int_0^\infty dx_j x_j^{a_j-1} \right) \delta \left(1 - \sum_{j=1}^N x_j \right) \times \mathcal{U}(\mathbf{x})^{-D/2} \left[\frac{\mathcal{F}(\mathbf{x})}{\mathcal{U}(\mathbf{x})} \right]^{-(A-LD/2)} \\ &= \frac{\Gamma(A-LD/2)}{\prod_{j=1}^N \Gamma(a_j)} \left(\prod_{j=1}^N \int_0^\infty dx_j x_j^{a_j-1} \right) \delta \left(1 - \sum_{j=1}^N x_j \right) \frac{\mathcal{U}(\mathbf{x})^{A-(L+1)D/2}}{\mathcal{F}(\mathbf{x})^{A-LD/2}} \end{aligned} \quad (108)$$

Thus, once $\mathcal{U}$ and $\mathcal{F}/\mathcal{U}$ are known, the desired parameter representation follows directly from the corresponding change of variables.

Previously, if we directly defined D as a non-integer, we would easily become quite confused about D -dimensional integrals. For example, for $D = 3.8$, what is $d^{3.8}\ell$? But when we change the Feynman integral from the momentum representation to the alpha and Feynman parameter representation, the whole integral becomes integer powers of the parameters multiplied by powers of the polynomials $\mathcal{U}$ and $\mathcal{F}$, and the dimension D appears only in the Gamma functions and in the powers of the polynomials $\mathcal{U}$ and $\mathcal{F}$. Since $\mathcal{U}$ and $\mathcal{F}$ are both polynomials, their non-integer powers are also reasonable, so after using the parameter representation we can more naturally accept Feynman integrals after dimensional regularization.

3 Lee-Pomeransky representation

The Lee-Pomeransky representation is

$$I_{a_1 \dots a_N}^{(L)} = \frac{\Gamma(D/2)}{\Gamma\left(\frac{D}{2} - \sum_{j=1}^N a_j + \frac{LD}{2}\right) \prod_{j=1}^N \Gamma(a_j)} \left(\prod_{j=1}^N \int_0^\infty dx_j x_j^{a_j-1} \right) \mathcal{G}(\mathbf{x})^{-D/2} \quad (109)$$

where

$$\mathcal{G} = \mathcal{U} + \mathcal{F}$$

First, we insert a delta function, which does not affect the result:

$$I' = \left(\prod_{i=1}^N \int dx_i x_i^{a_i-1} \right) (\mathcal{U} + \mathcal{F})^{-D/2} = \int_0^\infty d\rho \delta(\rho - \sum_i x_i) \left(\prod_{i=1}^N \int dx_i x_i^{a_i-1} \right) (\mathcal{U} + \mathcal{F})^{-D/2}$$

We then factor out the overall scale by defining

$$x_j = \rho y_j, \quad \rho = \sum_{j=1}^N x_j$$

Since $x_i \geq 0$, we also have $y_i \geq 0$. Using the homogeneity of $\mathcal{U}$ and $\mathcal{F}$, we find

$$\mathcal{G}(\rho y) = \mathcal{U}(\rho y) + \mathcal{F}(\rho y) = \rho^L \mathcal{U}(y) + \rho^{L+1} \mathcal{F}(y) = \rho^L [\mathcal{U}(y) + \rho \mathcal{F}(y)]$$

It follows that

$$\mathcal{G}(\rho y)^{-\frac{D}{2}} = \rho^{-\frac{LD}{2}} [\mathcal{U}(y) + \rho \mathcal{F}(y)]^{-\frac{D}{2}}$$

Therefore, the integral becomes

$$\begin{aligned} I' &= \int_0^\infty d\rho \frac{1}{\rho} \delta(1 - \sum_i y_i) \left(\prod_{i=1}^N \int dy_i y_i^{a_i-1} \right) \rho^{N+\sum_i a_i-N} \mathcal{G}(\rho y)^{-D/2} \\ &= \int_0^\infty d\rho \rho^{\sum_i a_i-1} \left(\prod_{i=1}^N \int dy_i y_i^{a_i-1} \right) \delta(1 - \sum_i y_i) \rho^{-LD/2} [\mathcal{U}(y) + \rho \mathcal{F}(y)]^{-D/2} \\ &= \left(\prod_{i=1}^N \int_0^\infty dy_i y_i^{a_i-1} \right) \delta\left(1 - \sum_{j=1}^N y_j\right) \int_0^\infty d\rho \rho^{\sum_i a_i-LD/2-1} [\mathcal{U}(y) + \rho \mathcal{F}(y)]^{-D/2} \\ &= \left(\prod_{j=1}^N \int_0^\infty dy_j y_j^{a_j-1} \right) \delta\left(1 - \sum_{j=1}^N y_j\right) \frac{\Gamma(\sum_i a_i - \frac{LD}{2}) \Gamma(-\sum_i a_i + \frac{D(1+L)}{2})}{\Gamma(\frac{D}{2})} \left(\frac{\mathcal{F}}{\mathcal{U}}\right)^{\frac{LD}{2}-\sum_i a_i} \mathcal{U}^{-D/2} \end{aligned} \quad (110)$$

This reproduces the Feynman-parameter representation and therefore verifies the Lee-Pomeransky representation in Eq. (109).

A Gaussian Integral, Gamma Function and Beta Function

A.1 Gaussian Integral

We begin with the standard Gaussian integral

$$I = \int_{-\infty}^{\infty} e^{-x^2} dx \quad (111)$$

Since the function e^{-x^2} is even,

$$I = 2 \int_0^{\infty} e^{-x^2} dx \quad (112)$$

First, we need to ensure that this integral is well-defined, that is, that it converges, before we can proceed to integrate over them. On the interval $[0, 1]$, the function e^{-x^2} is continuous, so $\int_0^1 e^{-x^2} dx$ is finite.

For $x \geq 1$, we have $x^2 \geq x$, and hence

$$0 \leq e^{-x^2} \leq e^{-x}, \quad \int_1^{\infty} e^{-x} dx = e^{-1} < \infty \quad (113)$$

Thus, $\int_1^{\infty} e^{-x^2} dx$ converges. Therefore, I is finite and positive, and we may square it:

$$\begin{aligned} I^2 &= \left(\int_{-\infty}^{\infty} e^{-x^2} dx \right) \left(\int_{-\infty}^{\infty} e^{-y^2} dy \right) \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-x^2} e^{-y^2} dx dy \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-(x^2+y^2)} dx dy \end{aligned}$$

Because x and y both range from 0 to infinity, over the entire two-dimensional plane we can define polar coordinates well, and their ranges can also be specified well:

$$x = r \cos \theta, \quad y = r \sin \theta \quad (114)$$

with

$$0 \leq r < \infty, \quad 0 \leq \theta < 2\pi \quad (115)$$

so

$$x^2 + y^2 = r^2 \cos^2 \theta + r^2 \sin^2 \theta = r^2 \quad (116)$$

The Jacobian determinant of this transformation is

$$\frac{\partial(x, y)}{\partial(r, \theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos \theta & -r \sin \theta \\ \sin \theta & r \cos \theta \end{vmatrix} \quad (117)$$

$$= r \cos^2 \theta + r \sin^2 \theta = r \quad (118)$$

Therefore, the area element satisfies

$$dx dy = r dr d\theta \quad (119)$$

Therefore,

$$\begin{aligned} I^2 &= \int_0^{2\pi} \int_0^\infty e^{-r^2} r \, dr \, d\theta \\ &= \left(\int_0^{2\pi} d\theta \right) \left(\int_0^\infty r e^{-r^2} \, dr \right) = 2\pi \left(\int_0^\infty r e^{-r^2} \, dr \right) \end{aligned}$$

Let

$$u = r^2, \quad du = 2r \, dr \quad (120)$$

Thus

$$\int_0^\infty r e^{-r^2} \, dr = \frac{1}{2} \int_0^\infty e^{-u} \, du = \frac{1}{2} [-e^{-u}]_0^\infty = \frac{1}{2} \quad (121)$$

and hence

$$I^2 = 2\pi \cdot \frac{1}{2} = \pi, \quad \rightarrow \int_{-\infty}^\infty e^{-x^2} \, dx = \sqrt{\pi} \quad (122)$$

We now consider the more general integral and complete the square in the exponent:

$$-ax^2 + bx = -a \left[\left(x - \frac{b}{2a} \right)^2 - \frac{b^2}{4a^2} \right] = -a \left(x - \frac{b}{2a} \right)^2 + \frac{b^2}{4a} \quad (123)$$

so

$$e^{-ax^2+bx} = e^{b^2/(4a)} e^{-a(x-\frac{b}{2a})^2} \quad (124)$$

and

$$\int_{-\infty}^\infty e^{-ax^2+bx} \, dx = e^{b^2/(4a)} \int_{-\infty}^\infty e^{-a(x-\frac{b}{2a})^2} \, dx \quad (125)$$

Let

$$u = \sqrt{a} \left(x - \frac{b}{2a} \right) \quad (126)$$

For $a > 0$,

$$du = \sqrt{a} \, dx, \quad dx = \frac{du}{\sqrt{a}} \quad (127)$$

As $x \rightarrow -\infty$, $u \rightarrow -\infty$, while as $x \rightarrow +\infty$, $u \rightarrow +\infty$. Since $a \left(x - \frac{b}{2a} \right)^2 = u^2$, we obtain

$$\begin{aligned} \int_{-\infty}^\infty e^{-ax^2+bx} \, dx &= e^{b^2/(4a)} \int_{-\infty}^\infty e^{-u^2} \frac{du}{\sqrt{a}} \\ &= \frac{e^{b^2/(4a)}}{\sqrt{a}} \int_{-\infty}^\infty e^{-u^2} \, du \\ &= \sqrt{\frac{\pi}{a}} e^{b^2/(4a)} \end{aligned} \quad (128)$$

A.1.1 Gaussian integral for a general quadratic form

Let

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} \quad (129)$$

For any real matrix A , only its symmetric part contributes to the quadratic form because

$$\mathbf{x}^T A \mathbf{x} = \mathbf{x}^T \left(\frac{A + A^T}{2} \right) \mathbf{x} \quad (130)$$

We may therefore take A to be real and symmetric. For the ordinary Gaussian integral we further require A to be positive definite, so all its eigenvalues are positive and the integral converges.

We first complete the square. Since $A^T = A$, one has

$$\begin{aligned} & - \left(\mathbf{x} - \frac{1}{2} A^{-1} \mathbf{b} \right)^T A \left(\mathbf{x} - \frac{1}{2} A^{-1} \mathbf{b} \right) \\ &= -\mathbf{x}^T A \mathbf{x} + \frac{1}{2} \mathbf{x}^T \mathbf{b} + \frac{1}{2} \mathbf{b}^T \mathbf{x} - \frac{1}{4} \mathbf{b}^T A^{-1} \mathbf{b} \\ &= -\mathbf{x}^T A \mathbf{x} + \mathbf{b}^T \mathbf{x} - \frac{1}{4} \mathbf{b}^T A^{-1} \mathbf{b}. \end{aligned} \quad (131)$$

Consequently,

$$-\mathbf{x}^T A \mathbf{x} + \mathbf{b}^T \mathbf{x} = - \left(\mathbf{x} - \frac{1}{2} A^{-1} \mathbf{b} \right)^T A \left(\mathbf{x} - \frac{1}{2} A^{-1} \mathbf{b} \right) + \frac{1}{4} \mathbf{b}^T A^{-1} \mathbf{b}. \quad (132)$$

With the translation

$$\mathbf{y} = \mathbf{x} - \frac{1}{2} A^{-1} \mathbf{b}, \quad d^n x = d^n y, \quad (133)$$

Eq. (132) gives

$$\begin{aligned} & \int d^n x \exp(-\mathbf{x}^T A \mathbf{x} + \mathbf{b}^T \mathbf{x}) \\ &= \exp\left(\frac{1}{4} \mathbf{b}^T A^{-1} \mathbf{b}\right) \int d^n y \exp(-\mathbf{y}^T A \mathbf{y}). \end{aligned} \quad (134)$$

Because A is real and symmetric, it can be diagonalized by an orthogonal matrix S :

$$A = S^T \Lambda S, \quad \Lambda = \text{diag}(\lambda_1, \lambda_2, \dots, \lambda_n), \quad \lambda_a > 0, \quad S^T S = S S^T = \mathbf{I} \quad (135)$$

Set

$$\mathbf{z} = S \mathbf{y}, \quad \mathbf{y} = S^T \mathbf{z} \quad (136)$$

Since S is orthogonal, $|\det S| = 1$ and therefore

$$d^n y = |\det S^T| d^n z = d^n z \quad (137)$$

The quadratic form becomes

$$\mathbf{y}^T A \mathbf{y} = \mathbf{y}^T S^T \Lambda S \mathbf{y} = \mathbf{z}^T \Lambda \mathbf{z} = \sum_{a=1}^n \lambda_a z_a^2 \quad (138)$$

Hence the n -dimensional integral factorizes into n ordinary one-dimensional Gaussian integrals:

$$\begin{aligned} \int d^n y e^{-\mathbf{y}^T A \mathbf{y}} &= \int d^n z \exp\left(-\sum_{a=1}^n \lambda_a z_a^2\right) \\ &= \prod_{a=1}^n \int_{-\infty}^{\infty} dz_a e^{-\lambda_a z_a^2} = \prod_{a=1}^n \sqrt{\frac{\pi}{\lambda_a}} \\ &= \pi^{n/2} \left(\prod_{a=1}^n \lambda_a\right)^{-1/2} \end{aligned} \quad (139)$$

On the other hand,

$$\begin{aligned} \det A &= \det(S^T \Lambda S) = \det(S^T) \det(\Lambda) \det(S) \\ &= (\det S)^2 \prod_{a=1}^n \lambda_a = \prod_{a=1}^n \lambda_a \end{aligned} \quad (140)$$

Combining Eqs. (134), (139), and (140), we obtain the general matrix Gaussian integral

$$\int d^n x \exp(-\mathbf{x}^T A \mathbf{x} + \mathbf{b}^T \mathbf{x}) = \frac{\pi^{n/2}}{\sqrt{\det A}} \exp\left(\frac{1}{4} \mathbf{b}^T A^{-1} \mathbf{b}\right) \quad (141)$$

A.1.2 Fresnel integrals

We now evaluate the Fresnel integrals

$$\int_{-\infty}^{\infty} dx e^{i\alpha x^2}, \quad \int_{-\infty}^{\infty} dx e^{-i\alpha x^2}, \quad \alpha > 0 \quad (142)$$

Unlike the ordinary Gaussian integral, these integrals are not absolutely convergent because

$$\left| e^{\pm i\alpha x^2} \right| = 1. \quad (143)$$

Their convergence instead comes from the rapid oscillation of the complex exponential.

For $R > S > 0$, we use

$$\frac{d}{dx} e^{i\alpha x^2} = 2i\alpha x e^{i\alpha x^2} \quad (144)$$

to write

$$e^{i\alpha x^2} = \frac{1}{2i\alpha x} \frac{d}{dx} e^{i\alpha x^2} \quad (145)$$

Integration by parts therefore gives

$$\int_S^R dx e^{i\alpha x^2} = \left[\frac{e^{i\alpha x^2}}{2i\alpha x} \right]_S^R - \frac{i}{2\alpha} \int_S^R dx \frac{e^{i\alpha x^2}}{x^2} \quad (146)$$

Expanding the boundary term in Eq. (146), we have

$$\int_S^R dx e^{i\alpha x^2} = \frac{e^{i\alpha R^2}}{2i\alpha R} - \frac{e^{i\alpha S^2}}{2i\alpha S} - \frac{i}{2\alpha} \int_S^R dx \frac{e^{i\alpha x^2}}{x^2}. \quad (147)$$

Taking the absolute value and applying the triangle inequality gives

$$\left| \int_S^R dx e^{i\alpha x^2} \right| \leq \left| \frac{e^{i\alpha R^2}}{2i\alpha R} \right| + \left| \frac{e^{i\alpha S^2}}{2i\alpha S} \right| + \frac{1}{2\alpha} \left| \int_S^R dx \frac{e^{i\alpha x^2}}{x^2} \right| \quad (148)$$

Since

$$\left| e^{i\alpha x^2} \right| = 1, \quad |i| = 1, \quad (149)$$

the first two terms become

$$\left| \frac{e^{i\alpha R^2}}{2i\alpha R} \right| = \frac{1}{2\alpha R}, \quad \left| \frac{e^{i\alpha S^2}}{2i\alpha S} \right| = \frac{1}{2\alpha S}. \quad (150)$$

For the remaining integral, the integral triangle inequality gives

$$\left| \int_S^R dx \frac{e^{i\alpha x^2}}{x^2} \right| \leq \int_S^R dx \frac{|e^{i\alpha x^2}|}{x^2} = \int_S^R \frac{dx}{x^2} = \left[-\frac{1}{x} \right]_S^R = \frac{1}{S} - \frac{1}{R} \quad (151)$$

Combining these results, we obtain

$$\begin{aligned} \left| \int_S^R dx e^{i\alpha x^2} \right| &\leq \frac{1}{2\alpha R} + \frac{1}{2\alpha S} + \frac{1}{2\alpha} \left(\frac{1}{S} - \frac{1}{R} \right) = \frac{1}{2\alpha R} + \frac{1}{2\alpha S} + \frac{1}{2\alpha S} - \frac{1}{2\alpha R} \\ &= \frac{1}{\alpha S} \end{aligned} \quad (152)$$

Notice that the two terms proportional to $\frac{1}{R}$ cancel. Hence

$$\left| \int_S^R dx e^{i\alpha x^2} \right| \leq \frac{1}{\alpha S} \rightarrow 0 \quad \text{as } S \rightarrow \infty \quad (153)$$

This bound is independent of R . Therefore, the tail of the integral can be made arbitrarily small for every $R > S$, which proves the convergence of $\int_0^\infty dx e^{i\alpha x^2}$ by the Cauchy criterion. The same argument applies to $e^{-i\alpha x^2}$. We now calculate their values.

Our strategy is first to evaluate

$$\int_0^\infty dt e^{it^2}, \quad (154)$$

and then recover the general parameter $\alpha > 0$ by a simple change of variables.

Step 1: A constant combination For $x \geq 0$, consider the expression

$$\left(\int_0^x dt e^{it^2} \right)^2 + i \int_0^1 dt \frac{e^{ix^2(1+t^2)}}{1+t^2}. \quad (155)$$

We shall prove that this expression is independent of x .

The derivative of the first term is

$$\begin{aligned}\frac{d}{dx} \left(\int_0^x dt e^{it^2} \right)^2 &= 2 \left(\int_0^x dt e^{it^2} \right) \frac{d}{dx} \left(\int_0^x dt e^{it^2} \right) \\ &= 2e^{ix^2} \int_0^x dt e^{it^2}.\end{aligned}\tag{156}$$

Here we have used the fundamental theorem of calculus:

$$\frac{d}{dx} \left(\int_0^x dt e^{it^2} \right) = e^{ix^2}.\tag{157}$$

For the second term, we first note that

$$\frac{d}{dx} e^{ix^2(1+t^2)} = 2ix(1+t^2) e^{ix^2(1+t^2)}.\tag{158}$$

Therefore,

$$\begin{aligned}\frac{d}{dx} \left[i \int_0^1 dt \frac{e^{ix^2(1+t^2)}}{1+t^2} \right] &= i \int_0^1 dt \frac{2ix(1+t^2)e^{ix^2(1+t^2)}}{1+t^2} \\ &= -2x \int_0^1 dt e^{ix^2(1+t^2)} \\ &= -2xe^{ix^2} \int_0^1 dt e^{ix^2t^2}.\end{aligned}\tag{159}$$

For $x > 0$, we make the change of variables

$$u = xt, \quad du = x dt.\tag{160}$$

When $t = 0$, we have $u = 0$, and when $t = 1$, we have $u = x$. Hence,

$$x \int_0^1 dt e^{ix^2t^2} = \int_0^x du e^{iu^2}.\tag{161}$$

Substituting Eq. (161) into Eq. (159), we obtain

$$\frac{d}{dx} \left[i \int_0^1 dt \frac{e^{ix^2(1+t^2)}}{1+t^2} \right] = -2e^{ix^2} \int_0^x du e^{iu^2}\tag{162}$$

The symbols t and u are only dummy integration variables. Consequently,

$$\int_0^x dt e^{it^2} = \int_0^x du e^{iu^2}.\tag{163}$$

Adding Eqs. (156) and (162), we find

$$\begin{aligned}
& \frac{d}{dx} \left[\left(\int_0^x dt e^{it^2} \right)^2 + i \int_0^1 dt \frac{e^{ix^2(1+t^2)}}{1+t^2} \right] \\
&= 2e^{ix^2} \int_0^x dt e^{it^2} - 2e^{ix^2} \int_0^x dt e^{it^2} \\
&= 0.
\end{aligned} \tag{164}$$

Thus, the expression in Eq. (155) is independent of x .

To determine its value, we set $x = 0$. The first term vanishes, while the exponential in the second term becomes 1. Therefore,

$$\begin{aligned}
& \left(\int_0^0 dt e^{it^2} \right)^2 + i \int_0^1 dt \frac{e^{i0^2(1+t^2)}}{1+t^2} \\
&= i \int_0^1 \frac{dt}{1+t^2} = i [\arctan t]_0^1 = \frac{i\pi}{4}
\end{aligned} \tag{165}$$

Step 2: The second term vanishes as $x \rightarrow \infty$ We now prove that

$$\lim_{x \rightarrow \infty} \int_0^1 dt \frac{e^{ix^2(1+t^2)}}{1+t^2} = 0 \tag{166}$$

Since $|e^{ix^2}| = 1$, we have

$$\left| \int_0^1 dt \frac{e^{ix^2(1+t^2)}}{1+t^2} \right| = \left| \int_0^1 dt \frac{e^{ix^2 t^2}}{1+t^2} \right|. \tag{167}$$

For $x \geq 1$, we split the integration interval at $t = 1/x$:

$$\int_0^1 dt \frac{e^{ix^2 t^2}}{1+t^2} = \int_0^{1/x} dt \frac{e^{ix^2 t^2}}{1+t^2} + \int_{1/x}^1 dt \frac{e^{ix^2 t^2}}{1+t^2}. \tag{168}$$

For the first integral,

$$\left| \int_0^{1/x} dt \frac{e^{ix^2 t^2}}{1+t^2} \right| \leq \int_0^{1/x} \frac{dt}{1+t^2} \leq \int_0^{1/x} dt = \frac{1}{x}. \tag{169}$$

For the second integral, we use

$$\frac{d}{dt} e^{ix^2 t^2} = 2ix^2 t e^{ix^2 t^2}, \tag{170}$$

which implies

$$e^{ix^2 t^2} = \frac{1}{2ix^2 t} \frac{d}{dt} e^{ix^2 t^2}. \tag{171}$$

Integration by parts then gives

$$\int_{1/x}^1 dt \frac{e^{ix^2 t^2}}{1+t^2} = \left[\frac{e^{ix^2 t^2}}{2ix^2 t(1+t^2)} \right]_{1/x}^1 - \int_{1/x}^1 dt e^{ix^2 t^2} \frac{d}{dt} \left[\frac{1}{2ix^2 t(1+t^2)} \right] \tag{172}$$

At the upper endpoint $t = 1$, the boundary term is

$$\frac{e^{ix^2}}{2ix^2(1)(1+1)} = \frac{e^{ix^2}}{4ix^2} \quad (173)$$

Therefore,

$$\left| \frac{e^{ix^2}}{4ix^2} \right| = \frac{1}{4x^2} \quad (174)$$

At the lower endpoint $t = 1/x$, we have

$$\left. \frac{e^{ix^2t^2}}{2ix^2t(1+t^2)} \right|_{t=1/x} = \frac{e^{ix^2(1/x)^2}}{2ix^2(1/x)(1+1/x^2)} = \frac{e^i}{2ix(1+1/x^2)} \quad (175)$$

It follows that

$$\left| \frac{e^i}{2ix(1+1/x^2)} \right| = \frac{1}{2x(1+1/x^2)} \leq \frac{1}{2x} \quad (176)$$

where we used $1 + 1/x^2 \geq 1$.

It remains to estimate the integral in the second line of Eq. (172). First, we calculate

$$\frac{d}{dt} \left[\frac{1}{2ix^2t(1+t^2)} \right] = \frac{1}{2ix^2} \frac{d}{dt} \left[\frac{1}{t(1+t^2)} \right]. \quad (177)$$

The remaining derivative is

$$\frac{d}{dt} \left[\frac{1}{t(1+t^2)} \right] = \frac{d}{dt} (t+t^3)^{-1} = -\frac{1+3t^2}{(t+t^3)^2} = -\frac{1+3t^2}{t^2(1+t^2)^2} \quad (178)$$

Using

$$\left| e^{ix^2t^2} \right| = 1, \quad \left| \frac{1}{2ix^2} \right| = \frac{1}{2x^2}, \quad (179)$$

we find

$$\begin{aligned} & \left| \int_{1/x}^1 dt e^{ix^2t^2} \frac{d}{dt} \left[\frac{1}{2ix^2t(1+t^2)} \right] \right| \\ & \leq \frac{1}{2x^2} \int_{1/x}^1 dt \frac{1+3t^2}{t^2(1+t^2)^2}. \end{aligned} \quad (180)$$

Equation (178) implies

$$\begin{aligned} \int_{1/x}^1 dt \frac{1+3t^2}{t^2(1+t^2)^2} &= - \int_{1/x}^1 dt \frac{d}{dt} \left[\frac{1}{t(1+t^2)} \right] \\ &= \frac{1}{(1/x)(1+1/x^2)} - \frac{1}{1(1+1)} \\ &= \frac{x}{1+1/x^2} - \frac{1}{2}. \end{aligned} \quad (181)$$

Since

$$\frac{x}{1+1/x^2} - \frac{1}{2} \leq \frac{x}{1+1/x^2} \leq x, \quad (182)$$

Eq. (180) gives

$$\left| \int_{1/x}^1 dt e^{ix^2 t^2} \frac{d}{dt} \left[\frac{1}{2ix^2 t(1+t^2)} \right] \right| \leq \frac{1}{2x^2} \left(\frac{x}{1+1/x^2} - \frac{1}{2} \right) \leq \frac{1}{2x} \quad (183)$$

Combining the two boundary estimates (174) and (176) with Eq. (183), we obtain

$$\left| \int_{1/x}^1 dt \frac{e^{ix^2 t^2}}{1+t^2} \right| \leq \frac{1}{4x^2} + \frac{1}{2x} + \frac{1}{2x} = \frac{1}{x} + \frac{1}{4x^2} \quad (184)$$

Finally, combining this result with the short-interval estimate, we find

$$\begin{aligned} \left| \int_0^1 dt \frac{e^{ix^2(1+t^2)}}{1+t^2} \right| &= \left| \int_0^1 dt \frac{e^{ix^2 t^2}}{1+t^2} \right| \\ &\leq \frac{1}{x} + \frac{1}{4x^2} + \frac{1}{2x} + \frac{1}{2x} \\ &= \frac{2}{x} + \frac{1}{4x^2} \end{aligned} \quad (185)$$

Because

$$\lim_{x \rightarrow \infty} \left(\frac{2}{x} + \frac{1}{4x^2} \right) = 0, \quad (186)$$

here proves Eq. (166).

Step 3: Evaluation of the half-line integral Taking $x \rightarrow \infty$ in Eq. (165), the second term vanishes. Since the Fresnel integral converges, we obtain

$$\left(\int_0^\infty dt e^{it^2} \right)^2 = \frac{i\pi}{4} \quad (187)$$

Using

$$i = e^{i\pi/2}, \quad (188)$$

we have

$$\frac{i\pi}{4} = \frac{\pi}{4} e^{i\pi/2}. \quad (189)$$

Its square roots are

$$\pm \frac{\sqrt{\pi}}{2} e^{i\pi/4}. \quad (190)$$

The standard alternating-lobe argument for the real and imaginary parts shows that

$$\int_0^\infty dt \cos(t^2) > 0, \quad \int_0^\infty dt \sin(t^2) > 0. \quad (191)$$

We must therefore choose the root with positive real and imaginary parts. Hence,

$$\int_0^\infty dt e^{it^2} = \frac{\sqrt{\pi}}{2} e^{i\pi/4} \quad (192)$$

Step 4: Restoring the parameter α Let

$$u = \sqrt{\alpha} x, \quad dx = \frac{du}{\sqrt{\alpha}}, \quad \alpha > 0. \quad (193)$$

Then

$$\int_0^\infty dx e^{i\alpha x^2} = \frac{1}{\sqrt{\alpha}} \int_0^\infty du e^{iu^2} = \frac{1}{\sqrt{\alpha}} \frac{\sqrt{\pi}}{2} e^{i\pi/4} = \frac{1}{2} e^{i\pi/4} \sqrt{\frac{\pi}{\alpha}} \quad (194)$$

Since $e^{i\alpha x^2}$ is an even function of x ,

$$\int_{-\infty}^\infty dx e^{i\alpha x^2} = 2 \int_0^\infty dx e^{i\alpha x^2} = e^{i\pi/4} \sqrt{\frac{\pi}{\alpha}} \quad (195)$$

Finally, the integral with the opposite sign in the exponent is the complex conjugate:

$$\int_{-\infty}^\infty dx e^{-i\alpha x^2} = \left[\int_{-\infty}^\infty dx e^{i\alpha x^2} \right]^* = e^{-i\pi/4} \sqrt{\frac{\pi}{\alpha}} \quad (196)$$

Combining the two cases, we obtain

$$\int_{-\infty}^\infty dx e^{\pm i\alpha x^2} = e^{\pm i\pi/4} \sqrt{\frac{\pi}{\alpha}}, \quad \alpha > 0 \quad (197)$$

A.2 Gamma Function

The Gamma function is defined by the following integral:

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt, \quad \operatorname{Re}(z) > 0 \quad (198)$$

Using the change of variables $t = x^2$, the Gaussian integral can be written in terms of the Gamma function:

$$\begin{aligned} \int_{-\infty}^\infty e^{-x^2} dx &= 2 \int_0^\infty e^{-x^2} dx \\ &= \int_0^\infty t^{-1/2} e^{-t} dt = \Gamma\left(\frac{1}{2}\right) \end{aligned} \quad (199)$$

Therefore,

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi} \quad (200)$$

For $\operatorname{Re}(z) > 0$, integration by parts gives

$$\Gamma(z+1) = \int_0^\infty t^z e^{-t} dt = [-t^z e^{-t}]_0^\infty + z \int_0^\infty t^{z-1} e^{-t} dt \quad (201)$$

The boundary term vanishes because

$$\lim_{t \rightarrow \infty} t^z e^{-t} = 0 \quad (202)$$

and, for $\operatorname{Re}(z) > 0$,

$$\lim_{t \rightarrow 0^+} t^z = 0 \quad (203)$$

Hence, the Gamma function satisfies the recurrence relation

$$\Gamma(z+1) = z\Gamma(z) \quad (204)$$

For a positive integer n , repeated use of the recurrence relation gives

$$\begin{aligned} \Gamma(n+1) &= n\Gamma(n) \\ &= n(n-1)\Gamma(n-1) \\ &= n(n-1)(n-2)\Gamma(n-2) \\ &\vdots \\ &= n(n-1)(n-2)\cdots 2 \cdot 1 \Gamma(1) \end{aligned} \quad (205)$$

Since

$$\Gamma(1) = \int_0^\infty e^{-t} dt = 1 \quad (206)$$

we obtain

$$\Gamma(n+1) = n! \quad n = 0, 1, 2, \dots \quad (207)$$

Now let

$$z = \epsilon, \quad \epsilon \rightarrow 0$$

and therefore

$$\Gamma(1 + \epsilon) = \epsilon \Gamma(\epsilon) \tag{208}$$

Since the Gamma function is continuous at $z = 1$,

$$\lim_{\epsilon \rightarrow 0} \Gamma(1 + \epsilon) = \Gamma(1) = 1$$

Therefore,

$$\lim_{\epsilon \rightarrow 0} \epsilon \Gamma(\epsilon) = 1 \tag{209}$$

It follows that

$$\Gamma(\epsilon) \sim \frac{1}{\epsilon} \tag{210}$$

This shows that the divergent behavior of the Γ function near 0 is $\frac{1}{\epsilon}$, and that it has only one pole.

Next we consider

$$I(a, z) = \int_0^\infty t^{z-1} e^{-at} dt, \quad \operatorname{Re} a > 0$$

and set

$$u = at$$

Because $\operatorname{Re} a > 0$, and

$$t = \frac{u}{a}, \quad dt = \frac{du}{a}$$

When t varies from 0 to ∞ , u also varies from 0 to ∞ . Therefore,

$$\begin{aligned} I(a, z) &= \int_0^\infty \left(\frac{u}{a}\right)^{z-1} e^{-u} \frac{du}{a} \\ &= a^{-(z-1)} a^{-1} \int_0^\infty u^{z-1} e^{-u} du \\ &= \frac{\Gamma(z)}{a^z} \end{aligned} \tag{211}$$

Thus, we obtain the identity

$$\int_0^\infty t^{z-1} e^{-at} dt = \frac{\Gamma(z)}{a^z}, \quad \operatorname{Re} a > 0, \quad \operatorname{Re}(z) > 0 \tag{212}$$

A.3 Beta Function

The Beta function is defined by the following integral:

$$B(x, y) = \int_0^1 t^{x-1} (1-t)^{y-1} dt, \quad \operatorname{Re}(x) > 0, \operatorname{Re}(y) > 0 \quad (213)$$

Now we set

$$\operatorname{Re}(\alpha) > 0, \quad \operatorname{Re}(\beta) > 0$$

By the definition of the Gamma function (198), we write

$$\Gamma(\alpha) = \int_0^\infty u^{\alpha-1} e^{-u} du, \quad \Gamma(\beta) = \int_0^\infty v^{\beta-1} e^{-v} dv$$

Thus

$$\begin{aligned} \Gamma(\alpha)\Gamma(\beta) &= \left(\int_0^\infty u^{\alpha-1} e^{-u} du \right) \left(\int_0^\infty v^{\beta-1} e^{-v} dv \right) \\ &= \int_0^\infty \int_0^\infty u^{\alpha-1} v^{\beta-1} e^{-(u+v)} du dv \end{aligned} \quad (214)$$

We make the change of variables

$$r = u + v, \quad t = \frac{u}{u + v}$$

The inverse transformation is

$$u = rt, \quad v = r(1-t), \quad 0 < r < \infty, \quad 0 < t < 1$$

The Jacobian determinant is

$$\frac{\partial(u, v)}{\partial(r, t)} = \begin{vmatrix} t & r \\ 1-t & -r \end{vmatrix} = -r \quad (215)$$

Thus

$$du dv = r dr dt$$

and

$$\begin{aligned} u^{\alpha-1} v^{\beta-1} &= (rt)^{\alpha-1} (r(1-t))^{\beta-1} \\ &= r^{\alpha+\beta-2} t^{\alpha-1} (1-t)^{\beta-1} \end{aligned} \quad (216)$$

Therefore

$$\begin{aligned}\Gamma(\alpha)\Gamma(\beta) &= \int_0^1 \int_0^\infty r^{\alpha+\beta-2} t^{\alpha-1} (1-t)^{\beta-1} e^{-r} r \, dr \, dt \\ &= \int_0^1 \int_0^\infty r^{\alpha+\beta-1} e^{-r} t^{\alpha-1} (1-t)^{\beta-1} \, dr \, dt \\ &= \left(\int_0^\infty r^{\alpha+\beta-1} e^{-r} \, dr \right) \left(\int_0^1 t^{\alpha-1} (1-t)^{\beta-1} \, dt \right)\end{aligned}\tag{217}$$

Because

$$\int_0^\infty r^{\alpha+\beta-1} e^{-r} \, dr = \Gamma(\alpha + \beta)$$

and

$$\int_0^1 t^{\alpha-1} (1-t)^{\beta-1} \, dt = B(\alpha, \beta)$$

Therefore,

$$B(\alpha, \beta) = \frac{\Gamma(\alpha) \Gamma(\beta)}{\Gamma(\alpha + \beta)}\tag{218}$$