

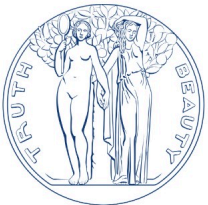
Università degli Studi di Milano · 23 Sept 2026

SubTropica

Giulio Salvatori

Based on 2406.14606 and 2604.20954 with M. Giroux & S. Mizera

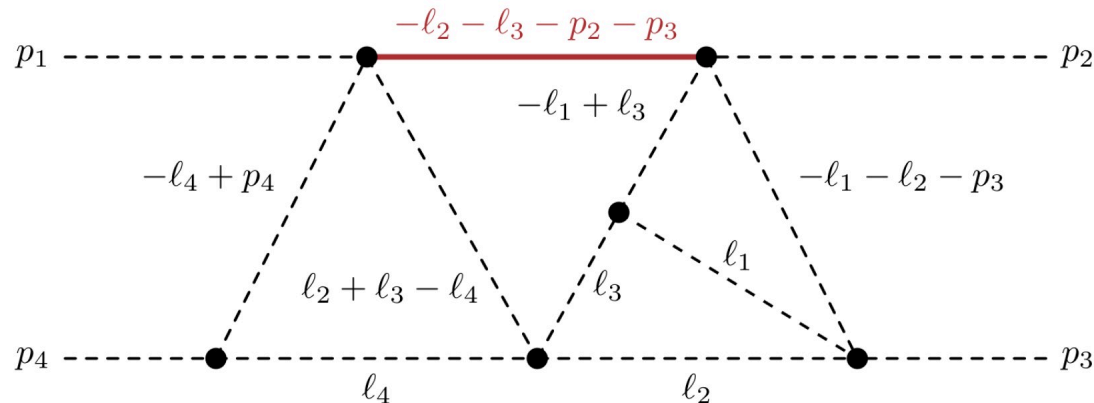
IAS Princeton



INSTITUTE FOR
ADVANCED STUDY



Feynman Integrals



Perturbative QFT speaks in **Feynman integrals**:
colliders, precision QCD & EW, gravitational
waves, cosmology...

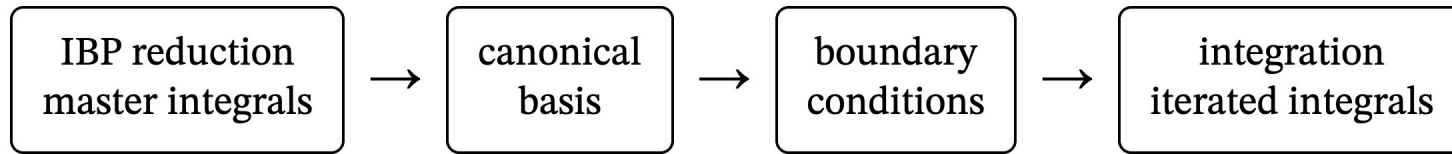
*On this side of the Atlantic Ocean I do not need to
motivate this!*

$$I_{\mathcal{V}}(s) = \frac{e^{\epsilon L \gamma_E}}{(i\pi^{D/2})^L} \int \frac{\mathcal{N} d^D \ell_1 \cdots d^D \ell_L}{(q_1^2 - m_1^2)^{\nu_1} \cdots (q_N^2 - m_N^2)^{\nu_N}}, \quad D = 4 - 2\epsilon$$

$$= \Gamma(\omega) \prod_e \frac{(-1)^{\nu_e}}{\Gamma(\nu_e)} \int \frac{d^E x}{\text{GL}(1)} \prod_e x_e^{\nu_e - 1} \tilde{\mathcal{N}} \mathcal{U}^{\omega - D/2} (-\mathcal{F})^{-\omega}, \quad \omega = \sum_e \nu_e - \frac{LD}{2}$$

Schwinger parameters: POLYNOMIALS raised to ϵ -dependent powers — the Symanzik $\mathcal{U}$ and $\mathcal{F}$

Canonical Differential Equations



$$d\vec{I}(s, \epsilon) = \epsilon dA(s) \vec{I}(s, \epsilon) \implies \vec{I} = \mathbb{P} \exp\left(\epsilon \int dA\right) \vec{I}_0, \quad A = \sum_k A_k \log W_k(s)$$

Chetyrkin, Tkachov (1981) · Laporta — hep-ph/0102033 · Henn — arXiv:1304.1806

IBP systems SWELL with loops, legs and scales

A **canonical basis** exists — but finding it is an ART

Boundary conditions: a separate computation, integral by integral

√ **are nasty:** square-root letters must be RATIONALIZED — algebraic and elliptic letters are each a story of their own

SUPERB when it works — every stage is a project of its own

Direct Integration

A COMPLEMENTARY strategy: NO reduction, NO basis, NO boundary conditions

$$I(\epsilon, s) = \int_{\mathbb{R}_{\geq 0}^n} \frac{d\alpha_1}{\alpha_1} \dots \frac{d\alpha_n}{\alpha_n} \prod_j P_j(\alpha, s)^{a_j + b_j \epsilon}$$

ONE variable at a time, in **hyperlogarithms** — integral by integral, AUTOMATICALLY

$$G(a_1, \dots, a_n; x) = \int_0^x \frac{dt}{t - a_1} G(a_2, \dots, a_n; t), \quad G(; x) = 1$$

Brown — arXiv:0804.1660 · Panzer, HyperInt — arXiv:1403.3385 · any numerator, any exponents, any Euler integral

- 1. DIVERGENCES:** poles in ϵ — one cannot expand under the integral sign
- 2. LINEAR REDUCIBILITY:** singularities must stay linear in the next variable

FIRST OBSTACLE: DIVERGENCES

Outline

1. **Polytopes In The Real World**
2. Hyper(FL)Int
3. SubTropica
4. Applications
5. Future Directions

Euler Integrals

An **Euler integral** is the Mellin transform of a product of polynomials over the positive orthant

$$I(\epsilon, s) = \int_{\mathbb{R}_{\geq 0}^n} \frac{d\alpha_1}{\alpha_1} \cdots \frac{d\alpha_n}{\alpha_n} \prod_j P_j(\alpha, s)^{a_j + b_j \epsilon}$$

$$\text{Feynman integrals: } \mathcal{I} = \mathcal{U}^{\omega - D/2} \mathcal{F}^{-\omega} \prod_e \alpha_e^{\nu_e}, \quad D = 4 - 2\epsilon, \quad \omega = \sum_e \nu_e - L D/2$$

$\mathcal{U}$ and $\mathcal{F}$ are the **Symanzik polynomials** of the graph — same class: phase space, energy correlators, cosmology, periods...

We want the Laurent expansion in the dimensional regulator

$$I(\epsilon, s) = \sum_{i=-N}^{\infty} \epsilon^i I^{(i)}(s)$$

PROBLEM: we cannot simply expand in ϵ under the integral sign

Local Finiteness

An integrand is **locally finite** if its ϵ -expansion can be performed **under the integral sign**

GOAL: expand any divergent Euler integral as

$$I(s, \epsilon) = \sum_i \varphi_i(\epsilon) I_i(s, \epsilon)$$

explicit poles in ϵ $\times$ locally finite integrals

Salvatori — The tropical geometry of subtraction schemes, arXiv:2406.14606

- **Sector decomposition:** introduces an additional scale and BREAKS linear reducibility
- **Mellin–Barnes:** trades one integral for many contour integrals
- **Forest formulae:** not LOCAL in parametric space; understood systematically only for UV

We want FEW terms, poles in LOWER dimension, linear reducibility PRESERVED

Tropical Analysis

Warm-up: $A = \int_0^\infty \int_0^\infty d\alpha_1 d\alpha_2 \alpha_1^{1+\epsilon} \alpha_2^\epsilon (\alpha_1^2 + \alpha_2 + \alpha_1 \alpha_2)^{-(2+\epsilon)}$

Rescaling $\alpha_2 \rightarrow \alpha_1^2 \alpha_2$ exposes a pole $1/\epsilon$; $\alpha_2 \rightarrow \alpha_1 \alpha_2$ exposes NONE. Which rescalings matter?

Probe a corner of the integration domain with a general rescaling

$$\alpha_i \rightarrow \alpha_i \lambda^{-\rho_i} \quad \rho = (\rho_1, \dots, \rho_n) \in \mathbb{Q}^n \quad \lambda \rightarrow 0^+$$

$$\mathcal{J}(\alpha_i \lambda^{-\rho_i}) = \lambda^{-\text{Trop } \mathcal{J}(\rho)} \mathcal{J}(\alpha, \lambda)$$

$$\text{Trop } \mathcal{J}(\rho) = a + b \epsilon$$

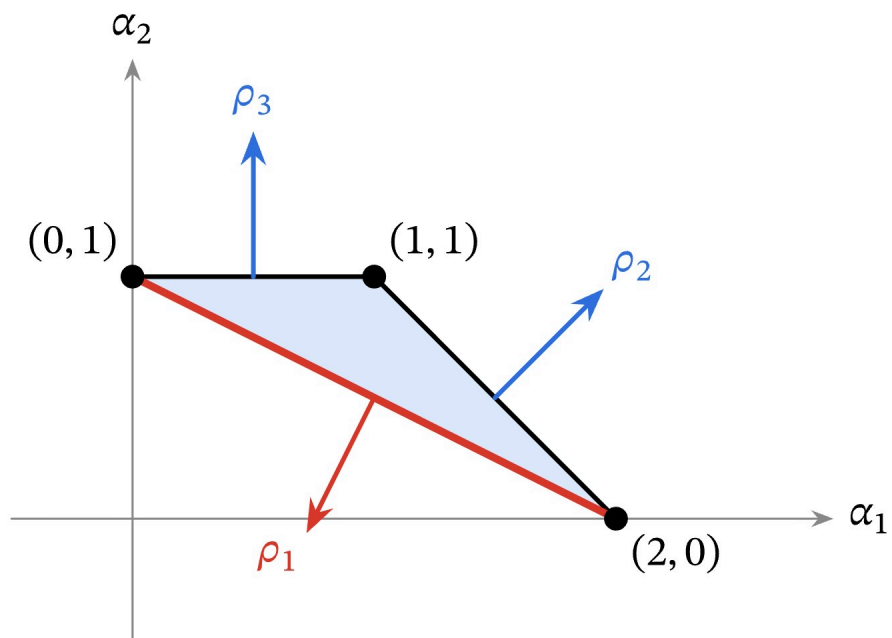
$a < 0$ convergent

$a = 0$ log divergent

$a > 0$ power divergent

No poles if $\text{Trop } \mathcal{J}(\rho)|_{\epsilon=0} < 0$ for all ρ ... infinitely many checks?

...Newton Polytopes



The Newton polytope of a polynomial is the convex hull of its exponents

$$\text{Newt } P = \text{Conv}\{m \in \mathbb{Z}^n \mid s_m \neq 0\} = \bigcap_j \{d_j - \rho_j \cdot \alpha \geq 0\}$$

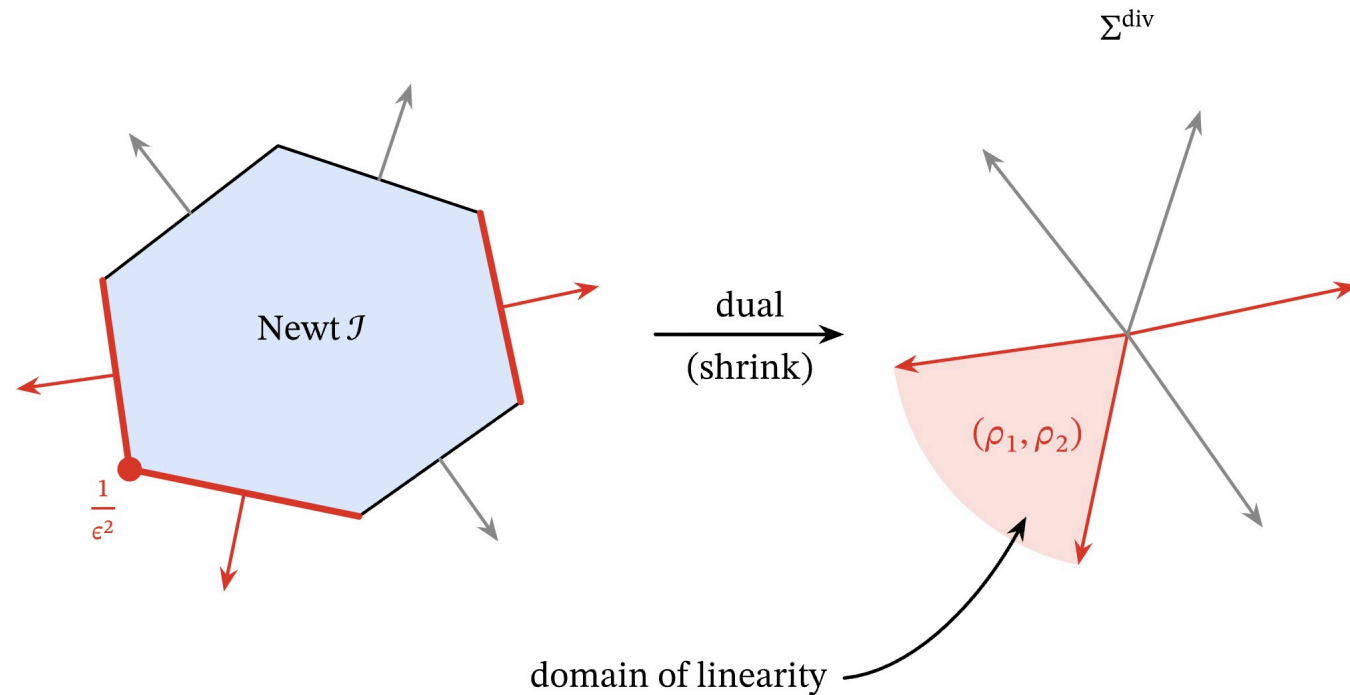
Only FINITELY many rescalings matter: the **normals to the facets**

- (i) $\{d_j - \rho_j \cdot \alpha \geq 0\}$ is a facet of $\text{Newt } \mathcal{I} \implies \text{Trop } \mathcal{I}(\rho_j) = d_j$
- (ii) elsewhere: extend by PIECEWISE LINEARITY

DIVERGENCES $\leftrightarrow$ FACETS of the Newton polytope

...Compatible Facets, Higher Poles

Divergent facets that share a vertex are **compatible**: they conspire to give higher-order poles

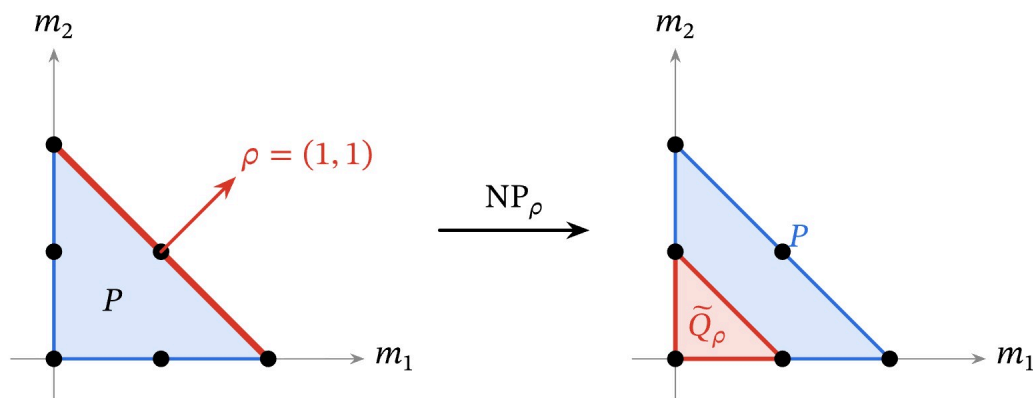


$$\Sigma^{\text{div}} = \{ \sigma = (\rho_1, \dots, \rho_m) \mid \rho_i \text{ divergent, compatible normals} \} \cup \{\emptyset\}$$

The polytope leaves an IMPRINT of itself in the pole structure of the integral

Tropical Continuation (NP)

$$\text{NP}_\rho : \quad \text{rescale } \alpha_i \rightarrow \alpha_i \lambda^{-\rho_i} \text{ and differentiate at } \lambda = 1 : \quad 0 = \int \frac{d\alpha}{\alpha} \frac{d}{d\lambda} \mathcal{I}(\alpha \lambda^{-\rho}) \Big|_{\lambda=1}$$



$$\text{Trop} \left[\frac{\text{red triangle}}{\text{blue triangle}} \right] \leq 0, \quad \text{strict on } \rho$$

$\Rightarrow$ the facet ρ is SHAVED

$$P = 1 + x_1 + x_1^2 + x_2 + x_2^2 + x_1 x_2, \quad \rho = (1, 1) : \quad \int \frac{d^2 x}{x} x_1^\epsilon x_2^\epsilon P^{-2\epsilon} = \int \frac{d^2 x}{x} x_1^\epsilon x_2^\epsilon P^{-2\epsilon} \times \frac{2 + x_1 + x_2}{P}$$

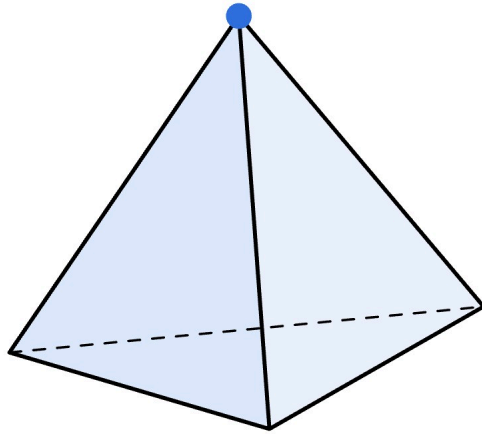
numerator: the monomials OFF the facet, weighted by their distance from it — the prefactor c/Trop equals 1 here

Price: an EXPONENTIAL proliferation of integrals, all in the SAME dimension

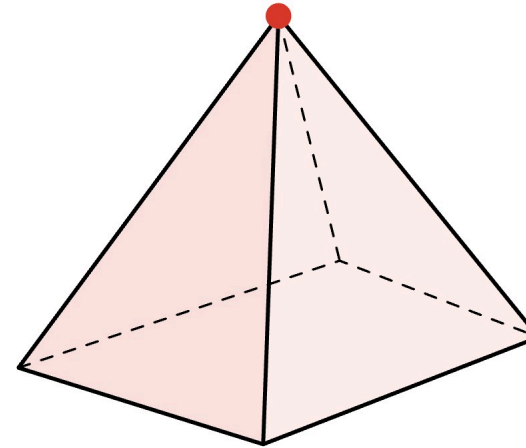
...The Geometric Property

Subtractions will need, for every divergent ρ , a dual vector w such that

$$w_\rho \cdot \rho' = -\delta_{\rho,\rho'} \quad \text{for all divergent } \rho' \text{ compatible with } \rho$$



three facets at a vertex in 3d: simplicial, w exists



four facets at a vertex in 3d: NO w

It FAILS when many divergent facets crowd together. Continue ONLY along the offending rays, then subtract:
proliferation kept to a minimum

Tropical Subtractions

$$\mathcal{J} = \left[\mathcal{J} - \mathcal{J}^{\text{ct}} \right] + \mathcal{J}^{\text{ct}}$$

Choose the counter-term so that $\mathcal{J} - \mathcal{J}^{\text{ct}}$ is **locally finite** and $\mathcal{J}^{\text{ct}}$ integrates EXACTLY, once per pole

For every $\sigma \in \Sigma^{\text{div}}$ a counter-term made of TWO factors

$$\mathcal{J}_{\sigma} = v_{\sigma} \times \mathcal{J}|_{\sigma}$$

Restriction to the face

keep only the monomials lying on the face: same singularities as the integrand near σ

$$v_{\rho} = \frac{1}{1 + \prod_i \alpha_i^{w_i}}$$

Suppression factor

kills the NEW singularities that the restriction introduces elsewhere

$$v_{(\rho_1, \dots, \rho_m)} = \prod_i v_{\rho_i}^{1 - \text{Trop } \mathcal{J}(\rho_i)}$$

...The Tropical Subtraction Formula

$$\mathcal{J}_\sigma^{\text{ren}} = \sum_{\sigma' \supseteq \sigma} (-1)^{|\sigma| - |\sigma'|} \mathcal{J}_{\sigma'} \quad \implies \quad \mathcal{J} = \sum_{\sigma \in \Sigma^{\text{div}}} \mathcal{J}_\sigma^{\text{ren}}$$

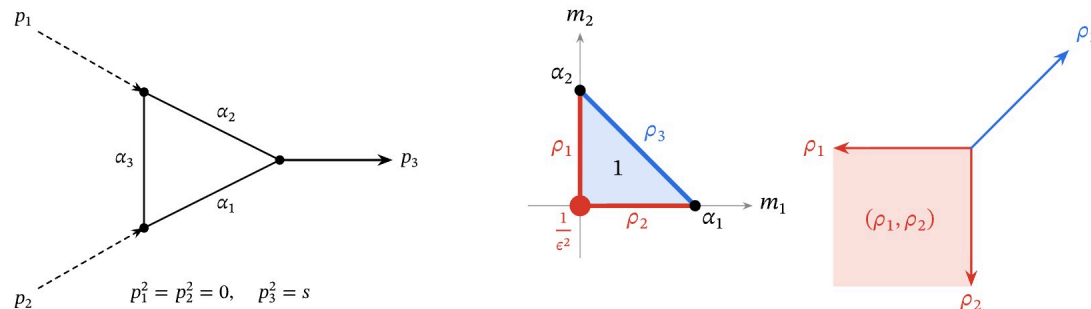
inclusion-exclusion on the faces of the divergent fan

$$\int_{\mathbb{R}_{\geq 0}^n} \frac{d\alpha}{\alpha} \mathcal{J} = \sum_{\sigma \in \Sigma^{\text{div}}} \text{Vol}(\sigma) \int_{V_\sigma} \frac{d\alpha}{\alpha} \Big|_{V_\sigma} \prod_i \alpha_i^{u_{\sigma,i}} (\mathcal{J}|_\sigma)^{\text{ren}}$$

$$\text{Vol}(\sigma) = |\det M_\sigma| \prod_{\rho \in \sigma} \frac{1}{-\text{Trop } \mathcal{J}(\rho)}$$

All integrals are **locally finite** · every pole costs one dimension LESS · as many terms as cones in the fan

...Example: The Massless Triangle



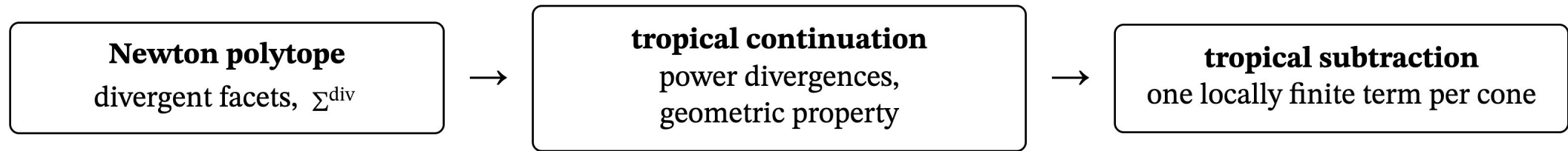
$$I(s) = -e^{\epsilon\gamma_E} \Gamma(1 + \epsilon) (-s)^{-1-\epsilon} \int_{\mathbb{R}_{\geq 0}^2} \frac{d\alpha_1}{\alpha_1} \frac{d\alpha_2}{\alpha_2} \mathcal{J}, \quad \mathcal{J} = \alpha_1^{-\epsilon} \alpha_2^{-\epsilon} (1 + \alpha_1 + \alpha_2)^{-1+2\epsilon}$$

$$\begin{aligned} \int \frac{d^2\alpha}{\alpha} \mathcal{J} &= \int \frac{d^2\alpha}{\alpha} \mathcal{J}_{\text{ren}}, & \mathcal{J}_{\text{ren}} &= \mathcal{J} - v_{\rho_1} \mathcal{J}|_{\rho_1} - v_{\rho_2} \mathcal{J}|_{\rho_2} + v_{\rho_1\rho_2} \mathcal{J}|_{\rho_1\rho_2} & \longleftrightarrow & \text{triangle} - \text{edge } \rho_1 - \text{edge } \rho_2 + \text{vertex} \\ -\frac{1}{\epsilon} \int \frac{d\alpha_2}{\alpha_2} (\mathcal{J}|_{\rho_1})^{\text{ren}}, & (\mathcal{J}|_{\rho_1})^{\text{ren}} &= \mathcal{J}|_{\rho_1} - v_{\rho_2} \mathcal{J}|_{\rho_1\rho_2} & \longleftrightarrow & \text{edge } \rho_1 - \text{vertex} \\ -\frac{1}{\epsilon} \int \frac{d\alpha_1}{\alpha_1} (\mathcal{J}|_{\rho_2})^{\text{ren}}, & (\mathcal{J}|_{\rho_2})^{\text{ren}} &= \mathcal{J}|_{\rho_2} - v_{\rho_1} \mathcal{J}|_{\rho_1\rho_2} & \longleftrightarrow & \text{edge } \rho_2 - \text{vertex} \\ +\frac{1}{\epsilon^2} (\mathcal{J}|_{\rho_1\rho_2})^{\text{ren}}, & (\mathcal{J}|_{\rho_1\rho_2})^{\text{ren}} &= \mathcal{J}|_{\rho_1\rho_2} = 1 & \longleftrightarrow & \text{vertex} \end{aligned}$$

bulk: hard, finite · edge ρ_1 : collinear to p_1 · edge ρ_2 : collinear to p_2 · vertex: soft

Bottomline: A Locally Finite Expansion

The main algorithm, in three steps



Expand each term in ϵ under the integral sign: only CONVERGENT integrals are left to do

POLYTOPES give you a LOCALLY FINITE expansion

Salvatori — arXiv:2406.14606 · Giroux, Mizera, Salvatori — arXiv:2604.20954

Outline

1. Polytopes In The Real World
2. **Hyper(FL)Int**
3. SubTropica
4. Applications
5. Future Directions

Goncharov Polylogarithms

Goncharov's **polylogarithms** $\equiv$ hyperlogarithms $H(x; a_1, \dots, a_n)$: ITERATED INTEGRALS with rational kernels

$$G(a_1, \dots, a_n; x) = \int_0^x \frac{dt}{t - a_1} G(a_2, \dots, a_n; t), \quad G(; x) = 1$$

The **word** $[\sigma_1, \dots, \sigma_w]$ labels the function, its length is the **weight** — products close into a **shuffle algebra**

$$H(z; u) H(z; v) = \sum_{w \in u \sqcup v} H(z; w) \quad \text{e.g.} \quad H(z; a) H(z; b) = H(z; a, b) + H(z; b, a)$$

$$H(z; \underbrace{0, \dots, 0}_k) = \frac{\log^k z}{k!}, \quad H(z; 1) = \log(1 - z), \quad H(z; \underbrace{0, \dots, 0}_{n-1}, 1) = -\text{Li}_n(z), \quad H(1; 0, 1) = -\zeta_2$$

$$\frac{d}{dz} H(z; \sigma_1, \sigma_2, \dots, \sigma_w) = \frac{1}{z - \sigma_1} H(z; \sigma_2, \dots, \sigma_w)$$

Differentiation PEELS OFF a letter, integration PREPENDS one

Francis Brown Enters The Arena

The massless higher-loop two-point function

F. Brown — arXiv:0804.1660

the reduction algorithms · Fubini reducible & unramified $\Rightarrow$ MZVs · planar primitive graphs to 5 loops

On the periods of some Feynman integrals

F. Brown — arXiv:0910.0114

Landau varieties of the partial integrals · **linear reducibility** · mixed Tate motives · Dodgson polynomials

Francis Brown: integrate the PARAMETRIC representation one Schwinger parameter at a time, staying inside the hyperlogarithms with RATIONAL coefficients

$$I_G^i(\alpha_{i+1}, \dots, \alpha_N) = \int_0^\infty \dots \int_0^\infty \frac{d\alpha_1 \dots d\alpha_i}{\Psi_G^2} \quad \text{singular along the Landau varieties } L_i$$

If the singularities of the partial integral are LINEAR in the next variable $\rightarrow$ CONTINUE

The Order Of Integration Matters

One variable at a time works only if, at EVERY step, the singularities of the partial integral are LINEAR in the next variable

The same integral, two orders: $f_2 = \int_0^1 \int_0^1 \frac{dx_1 dx_2}{1 - x_1 + x_1 x_2^2}$

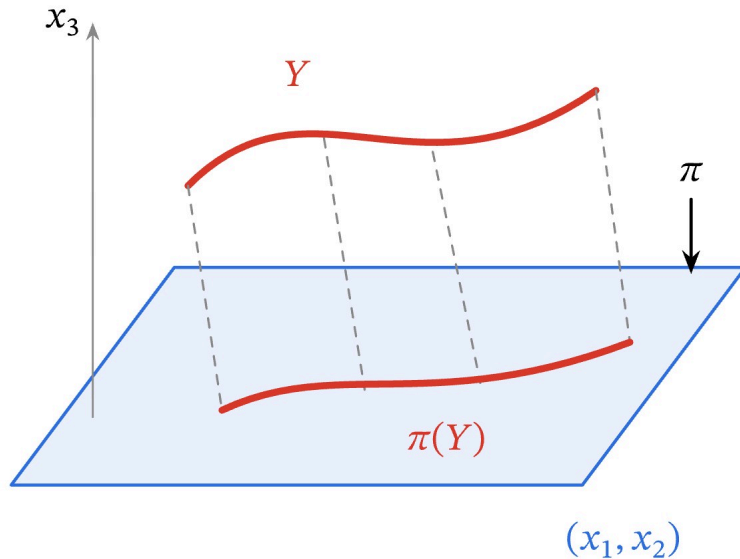
(x1, x2): linear in $x_1 \longrightarrow$ letter $x_2^2 - 1 = (x_2 - 1)(x_2 + 1) \longrightarrow f_2 = \frac{\pi^2}{4}$ ✓

(x2, x1): $\frac{1}{1 - x_1 + x_1 x_2^2} = \frac{\partial}{\partial x_2} \left[\frac{1}{\sqrt{x_1(1 - x_1)}} \arctan \frac{\sqrt{x_1} x_2}{\sqrt{1 - x_1}} \right]$ NOT a hyperlogarithm with rational letters

LINEARLY REDUCIBLE: there exists an integration order
such that ALL singularities are LINEAR with respect to the next variable

Integrate = Project = Eliminate

Integrating out a variable PROJECTS the singular variety — and ELIMINATION is algebra, not integration



Gröbner basis

$$Y = V(f_1, \dots, f_m) = V(g_1, \dots, g_k)$$

$$\pi(Y) = V(g_1, \dots, g_i) \subset V(\text{Res}(f_i, f_j))$$

Gröbner bases eliminate EXACTLY; resultants eliminate FAST, but over-approximate

(Landau, Pham, Brown) · (Panzer) · (Fevola, Mizera, Telen) · (Helmer, Papathanasiou, Tellander) · (Correia, Giroux, Mizera)

Bottleneck?

Gröbner basis → the choice of a good **HEURISTIC**

Resultants → virtually none (but it is a SUPER-approximation)

HyperInt And FLINT

Algorithms for the symbolic integration of hyperlogarithms with applications to Feynman integrals

Erik Panzer

Humboldt-Universität zu Berlin, Institut für Physik, Newton Straße 15, 12489 Berlin, Germany

Abstract

We provide algorithms for symbolic integration of hyperlogarithms multiplied by rational functions, which also include multiple polylogarithms when their arguments are rational functions. These algorithms are implemented in `Maple` and we discuss various applications. In particular, many Feynman integrals can be computed by this method.

Keywords: Feynman integrals, hyperlogarithms, polylogarithms, computer algebra, symbolic integration, ε -expansions

E. Panzer — **HyperInt** (Maple, 2014): the integration step made fully ALGORITHMIC

HyperIntica: a native Mathematica reimplementa-tion, shipped with SubTropica — good for PROTOTYPING

HyperFLINT: a C++ port · 3–10× faster · optional with the SubTropica installation

also HyperFORM (Kardos, Moch, Schnetz — arXiv:2511.19992) and HyperlogProcedures (Schnetz)

Bottleneck?

Partial Fractions (!) and picking best LR order

Bottomline: Integration = Algebra

Resultants → the integration order

$$\text{Res}_x [a x + b, c x + d] = a d - b c$$

Partial fractions → a pole becomes a letter

$$\frac{N(x)}{\prod_k (x - \sigma_k)} = \sum_k \frac{c_k}{x - \sigma_k} \quad \frac{c}{x - \sigma} \xrightarrow{f} c H(x; \sigma, \dots)$$

Shuffle-ology → move the next variable in first position

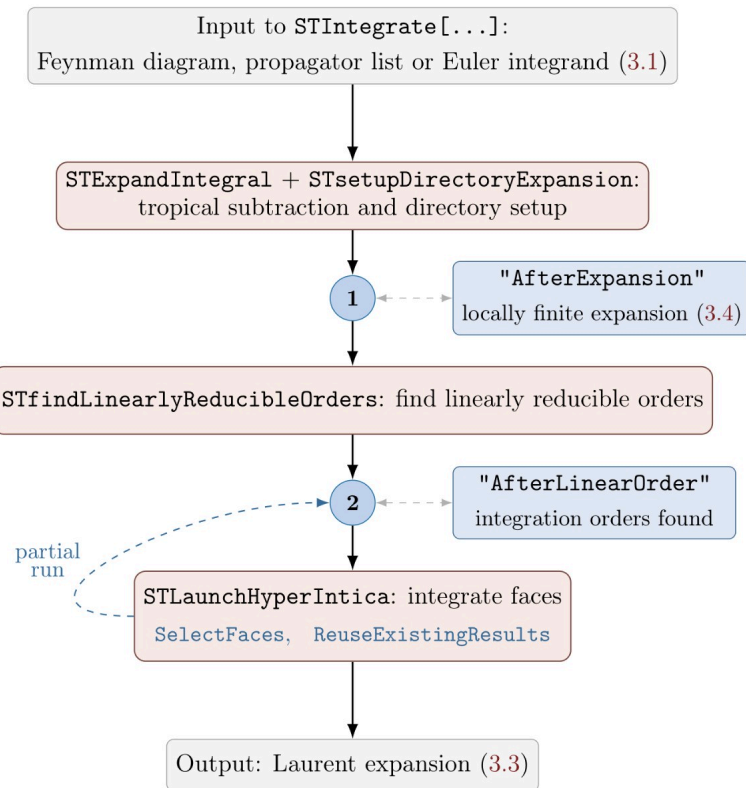
$$H(x; a) = H(a; x) + \log\left(-\frac{x}{a}\right) \quad (+ \text{ shuffles})$$

Integration is a glorified composition of //Factor, //Apart, //Permute

Outline

1. Polytopes In The Real World
2. Hyper(FL)Int
3. **SubTropica**
4. Applications
5. Future Directions

Algorithmic Flow



STIntegrate: one top-level function drives the whole pipeline

input $\rightarrow$ locally finite expansion $\rightarrow$ (best) linearly reducible order
 $\rightarrow$ hyper integration

Two CHECKPOINTS — **"StopAt"** AfterExpansion, AfterLinearOrder —
resume with **"StartAt"**

Everything stays inside Mathematica; the only required dependency is polymake

Giroux, Mizera, Salvatori — arXiv:2604.20954, Fig. 6

ONE call: from the diagram to the Laurent expansion in hyperlogarithms

Inputs

Form 0 — no arguments: the graphical interface, with a tutorial

```
STIntegrate[]
```

Form 2 — propagators + numerators, distinguished by "**Exponents**"

$$q_\mu q_\nu \int \frac{d^D \ell}{i\pi^{D/2}} \frac{\ell^\mu \ell^\nu}{[\ell^2 - m^2]^2} \quad D = 4 - 2\epsilon$$

```
STIntegrate[{l[1]^2 - mm, q[1].l[1], q[2].l[1]},  
"Exponents" -> {2, -1, -1}]  
(* Out: mm q[1].q[2]/2 (1/eps - Log[mm] + 1) + O[eps] *)
```

Form 1 — a graph {edges, nodes} with masses: the two-mass box

```
boxEdges = {{{1,2},m1}, {{2,3},0}, {{3,4},0}, {{1,4},0}};  
boxNodes = {{1,m1}, {2,0}, {3,0}, {4,M4}};  
STIntegrate[boxEdges, boxNodes]  
(* Out: (Log[MM4/s12] - Log[1 - s23/mm1]) /  
(eps (mm1 (MM4 - s12) + s12 s23)) + ... *)
```

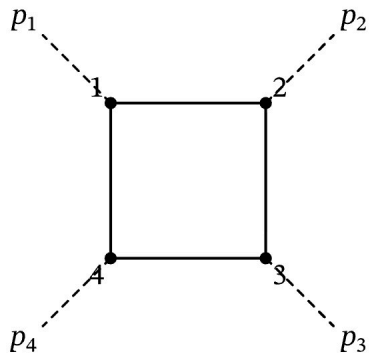
Form 3 — a raw Euler integrand with limits, like Integrate

```
STIntegrate[integrand, {x,0,1}, {y,0,∞}]  
STIntegrate[integrand, x, y] (* bare symbols: [0,∞) *)  
STIntegrate[{prefactor, integrand, vars, coeffs}]
```

Masses squared automatically ($m1 \rightarrow mm1$, $M4 \rightarrow MM4$), invariants $s12$, $s23$, ... in a cyclic basis, regulator ϵ

The same pipeline for ANY Euler integral, not only Feynman diagrams

Massless Box



```
Needs["SubTropica`"];
diag = {{{{1,2},0},{2,3},0},{3,4},0},{1,4},0}}, (* edges: {vertices, mass} *)
{{{1,0},{2,0},{3,0},{4,0}}}; (* nodes: {vertex, p^2} *)
STIntegrate[diag, "Dimension" -> 4 - 2 eps]
```

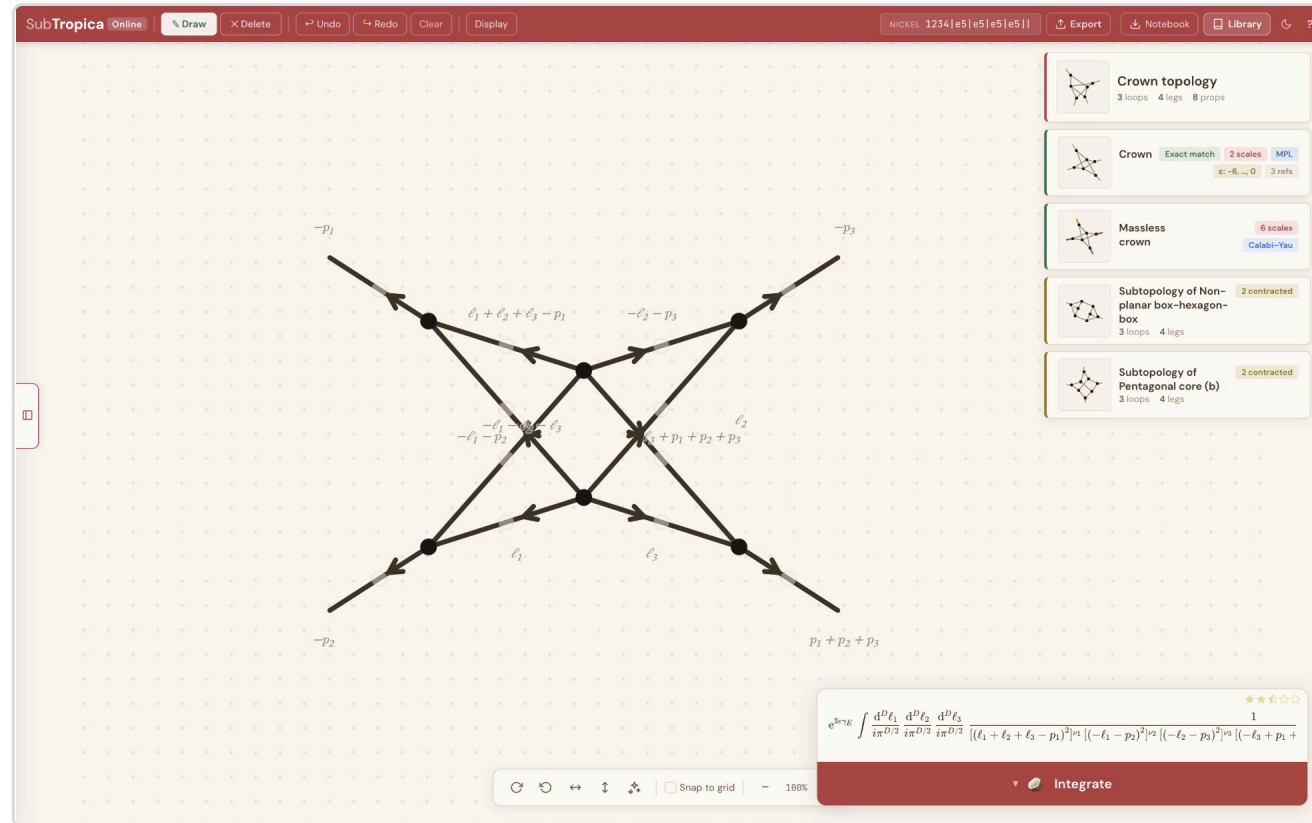
```
>> Parametric integrand: {E^(eps EulerGamma) Gamma[2+eps], (1+x1+x2+x4)^(2 eps) (-s23 x1 - s12 x2 x4)^(-2-eps), {x1,x2,x4}, {s12,s23}}
>> Time taken to find the linear orderings: 0.1s >> Time taken to integrate: 7.2s >> Overall timing: 17.6s
```

$$I_{\text{box}} = \frac{4}{s t \epsilon^2} + \frac{2 \log \frac{t}{s} - 4 \log(-t)}{s t \epsilon}$$

$$+ \frac{\pi^2 + 6 \log^2(-t) + 6 \log^2\left(-\frac{t}{s}\right) - 6 \log(-t) \log \frac{t}{s} - 12 \log\left(-\frac{t}{s}\right) \log \frac{t}{s} + 6 \log^2 \frac{t}{s} + 6 \zeta_2}{3 s t} + O(\epsilon)$$

i.e. the textbook box: $= \frac{1}{s t} \left[\frac{4}{\epsilon^2} - \frac{2 \log(-s) + 2 \log(-t)}{\epsilon} + 2 \log(-s) \log(-t) - \frac{4\pi^2}{3} \right] + O(\epsilon)$

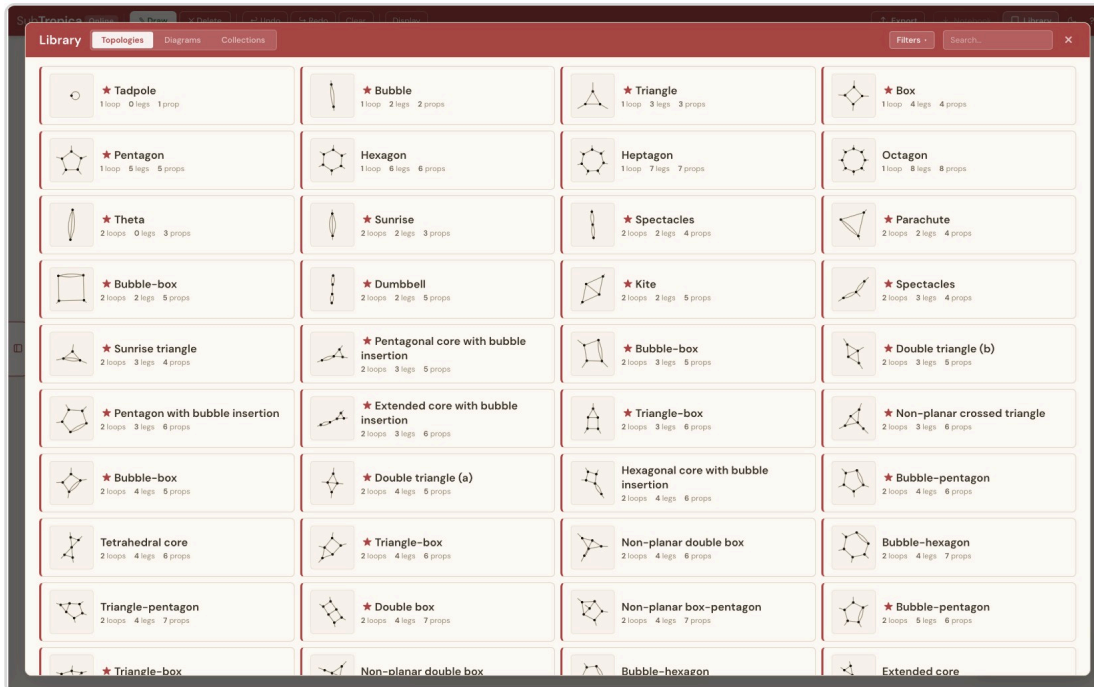
Interface



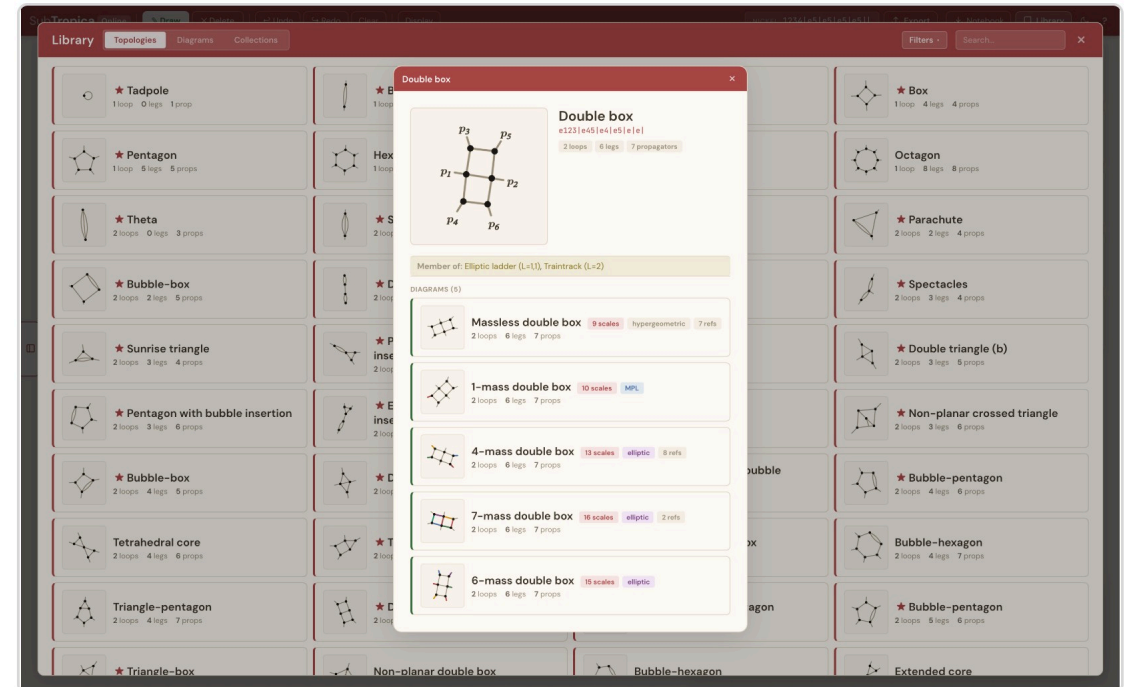
Draw the diagram in the browser at **subtropi.ca**: momentum routing, Nickel index, matching LIBRARY entries, the integrand — then **Integrate**

The same GUI opens in Mathematica with `STIntegrate[ ]`, with an interactive tutorial the first time

...The Library



Topologies · Diagrams · Collections, searchable and filterable



Every mass configuration: references, function class, scales, and the result to download

...The Papers Behind Each Diagram

The screenshot displays the Feynman Diagram Library interface. On the left, a sidebar lists various diagrams: Tadpole, Pentagon, Theta, Bubble-box, Sunrise triangle, Pentagon with bubble insertion, Bubble-box, Tetrahedral core, Triangle-pentagon, and Triangle-box. The main panel shows the 'Massless double box' diagram, which is a 2-loop, 6-legs, 7-propagators topology. It includes a diagram with external momenta p_1 through p_6 . The panel also displays the diagram's properties: 2 loops, 6 legs, 7 propagators, 9 scales, hypergeometric, $\epsilon: 0, 4$, and 7 refs. Below the diagram, a 'REFERENCES' section lists several papers:

- Elliptic Double-Box Integrals: Massless Scattering Amplitudes beyond Polylogarithms**
Jacob L. Bourjaily, Andrew J. McLeod, Marcus B. Spradlin, Matthew Von Hippel, Matthias Wilhelm
arXiv:1712.02785 (Phys.Rev.Lett. 120 (), 2018)
d=4 155 citations
- Ten-particle two-loop planar double-box integral in massless ϕ^4 theory, finite in $d=4$, expressed as elliptic integral over weight-3 polylogarithms. Dual-conformally invariant. Depends on 7 cross-ratios. First example of elliptic integrals in massless scattering amplitudes.**
- Traintracks through Calabi-Yau Manifolds: Scattering Amplitudes beyond Elliptic Polylogarithms**
Jacob L. Bourjaily, Yang-Hui He, Andrew J. McLeod, Matthew Von Hippel, Matthias Wilhelm
arXiv:1805.09326
d=4 140 citations
- 2-loop traintrack: the elliptic double-box. 2-loop double-box with 7 massless propagators, 10 massless external legs grouped into 6 momenta. Finite and dual-conformally invariant in $d=4$. Involves weight-3 hyperlogarithms integrated over an elliptic curve. Feynman parametric representation given explicitly in Eqs. (5)-(9).**
- Double box and hexagon conformal Feynman integrals**
Balasubramanian, Ananthanarayan, Sumit Banik, Samuel Friot, Shayan Ghosh
arXiv:2007.08360, Phys.Rev.D 102, 2020
 ϵ : all orders d=D 30 citations
- Off-shell massless six-point double box conformal Feynman integral with generic propagator powers a, b, c, d, e, f, ℓ in D dimensions. Written in dual position space with two integration points. Result: 44-term linear combination of generalized Horn-type multiple hypergeometric series from 9-fold MB representation.**
- Elliptic Double Box and Symbology Beyond Polylogarithms**
Alexander T. Kristensson, Matthias Wilhelm, Chi Zhang
arXiv:2106.14902 (Phys.Rev.Lett. 127 (25), 2021)

On the right, a sidebar lists other diagrams: Box, Octagon, Parachute, Spectacles, Double triangle (b), Non-planar crossed triangle, Bubble-pentagon, Bubble-hexagon, Bubble-pentagon, and Extended core.

Every diagram pulls out its literature: title, authors, arXiv, citations, abstract — with INSPIRE links and a report button

Verification

STNIntegrate: same inputs, NUMERICAL backends —
pySecDec, FIESTA, AMFlow, feynthrop — also when the integral
is not linearly reducible

```
edges = {{{1,2},m1},{1,3},m2},{1,4},m3},
        {{2,3},m4},{2,4},m5}}; (* the elliptic kite *)
nodes = {{3,M},{4,M}};
STNIntegrate[{edges, nodes}, "Substitutions" ->
{m1->67/23, m2->59/31, m3->159/31, m4->59/131,
m5->117/137, M->31 I/5}]
(* Out: -0.0708835 + O[eps] -- 1.2 s, pySecDec *)
```

STVerify: an automatic EUCLIDEAN point, sector
decomposition vs. the symbolic result evaluated with ginsh,
coefficient by coefficient

```
diag = {{{{1,2},0},{1,2},0},{2,3},0},{3,4},0},
        {{3,4},0},{1,4},0}}, {{1,0},{2,0},{3,0},{4,0}}};
result = STIntegrate[diag];
STVerify[diag, result]
(* eps^-3 ... eps^0: symbolic vs numeric *)
(* Out: "pass" -> True, rel. error ~ 1e-5 *)
```

Every analytic result is CHECKED against numerics

Database

subtropi.ca — an AI-driven library of the Feynman integrals in the literature

1,298

papers scanned

1,816

literature records

424

topologies

997

mass configurations

203

computed results

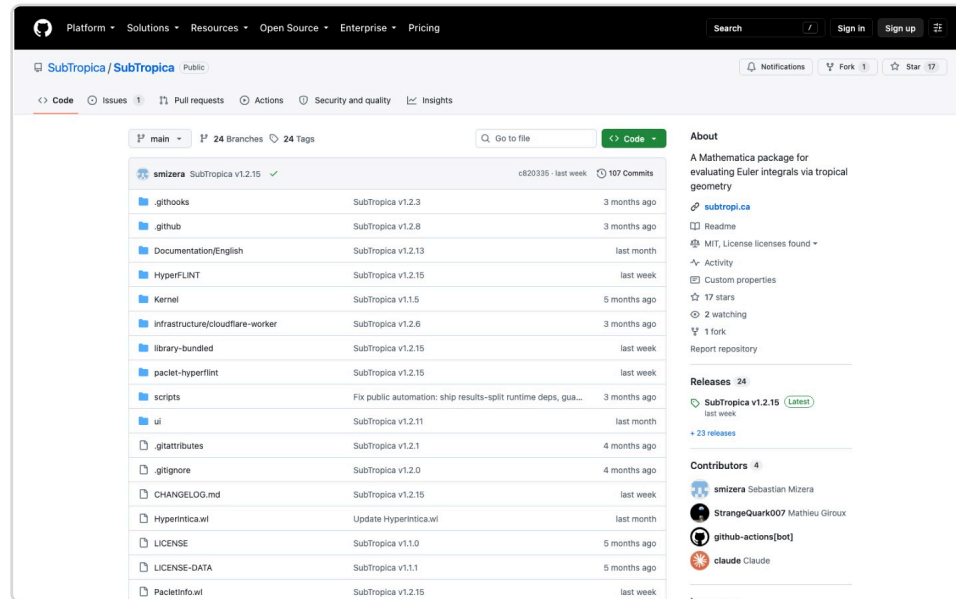
AI agents RETRIEVE and PARSE the papers, MATCH each diagram to a canonical label, VERIFY the results numerically

Draw a diagram → who computed it, which functions, the result to download

Open to submissions: the button after a successful integration, or `STSubmitResult[ ]`

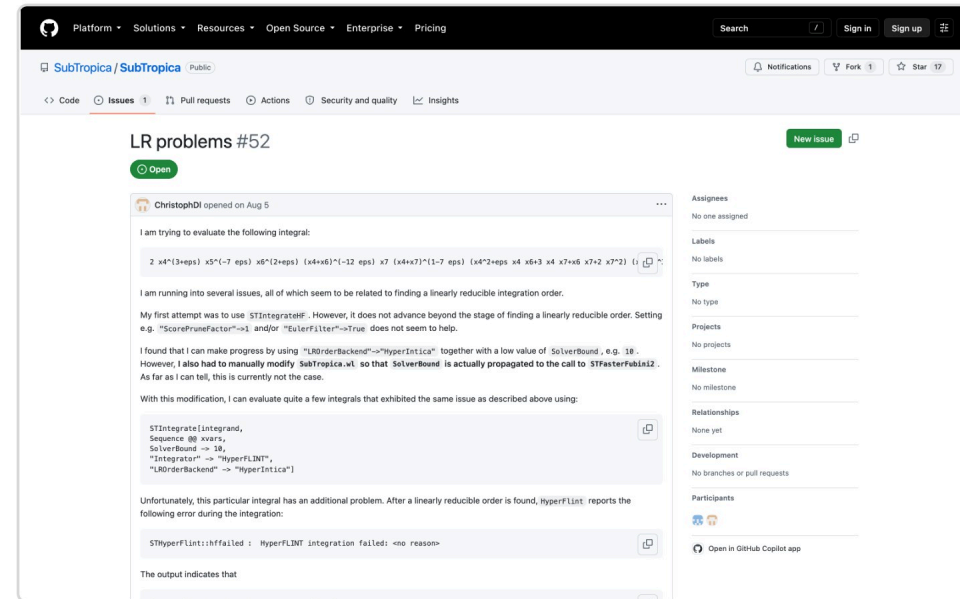
Data: CC BY-NC-SA 4.0 (no machine-learning use) · Code: MIT, github.com/SubTropica/SubTropica

...Open Source On GitHub



github.com/SubTropica/SubTropica — MIT, 24 releases, one-line install:

```
PacletInstall["https://subtropi.ca/SubTropica.paclet"]
```



Under **Issues** an automated agent assists: installation, bugs, computational tips

One thread, one month, three releases: the bugs a user hits become the next version

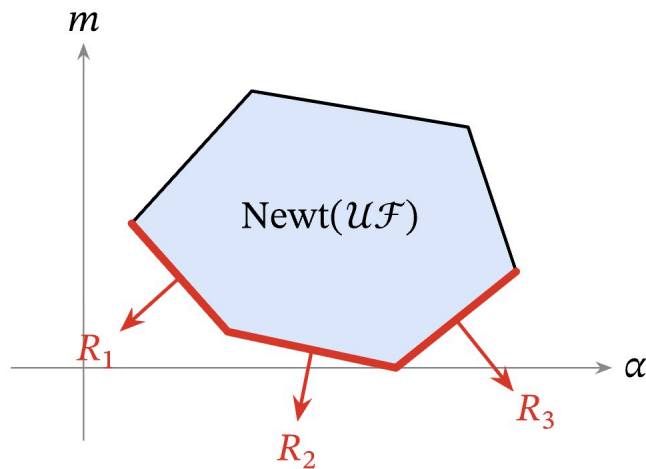
Outline

1. Polytopes In The Real World
2. Hyper(FL)Int
3. SubTropica
- 4. Applications**
5. Future Directions

Method Of Regions

A small parameter m (a mass, an off-shellness, a momentum): expand BEFORE integrating

$$\tau_m I = \sum_{\text{regions}} m^{a_R + b_R \epsilon} \times (\text{EULER integral with a SMALLER Newton polytope})$$



$$\mathcal{UF} = \sum_j c_j m^{n_{0,j}} \alpha^{n_j} \implies \text{Newt}(\mathcal{UF}) \subset \mathbb{R}^{1+n}$$

Regions $\leftrightarrow$ FACETS of the polytope facing $m \rightarrow 0$: the normal is the scaling of the Schwinger parameters

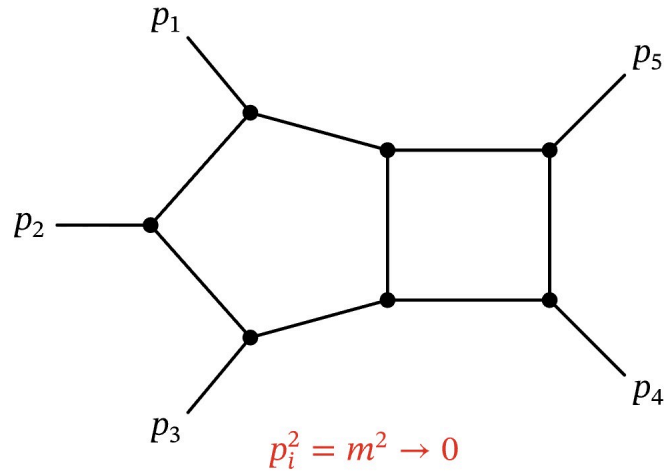
Region integrand = RESTRICTION of $\mathcal{J}$ to the facet — the same initial form as in the counter-terms

Fewer scales, but the poles in ϵ are back: exactly what tropical subtraction handles

Asymptotic expansion in terms of SIMPLER (= smaller polytopes) EULER (not Feynman!) integrals

Note: no KIRA / FIRE / ... for the IBP of the region integrals

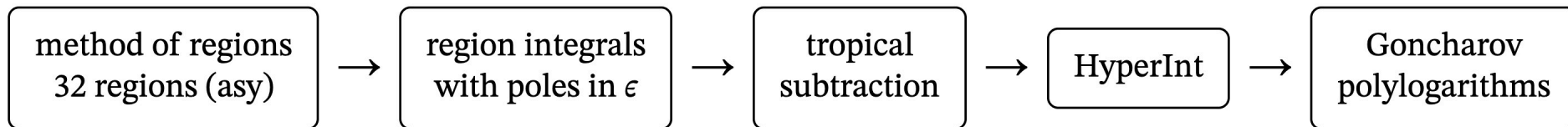
Pentabox $\rightarrow$ sYM On The Coulomb Branch



Coulomb branch of $N = 4$ sYM: the five-leg, two-loop amplitude near the mass shell — the off-shell **pentabox**

Canonical differential equations out of reach: the IBP reductions swell beyond a 2 TB cluster

“we break the barrier for the five-leg amplitude”



A. Belitsky, V. Smirnov — Tropical regions of near mass-shell pentabox, arXiv:2508.14298

Along the way: NON-SIMPLICIAL divergent fans $\rightarrow$ Nilsson–Passare · broken linear reducibility $\rightarrow$ changes of variables, algebraic roots

...And The Physics Payoff

$$s_{12}s_{23}s_{45} G_{\text{pentabox}}|_{m \rightarrow 0} = \sum_{\ell=0}^4 a_{\ell} \log^{4-\ell} m^2, \quad a_0 = 3$$

Summed over the regions all poles in ϵ cancel: the pentabox is a quartic polynomial in the infrared logarithm

Two loops: five W-boson amplitude = near-null decagon; the IR logarithms EXPONENTIATE $\rightarrow$ an all-order conjecture
(Belitsky, Bork, Lee, Onishchenko, Smirnov — 2510.16471)

Three loops: exponentiation confirmed, but each of the THREE kinematic structures gets its OWN function of the coupling
— the two-loop conjecture BREAKS (Belitsky, Bork, Lee, Smirnov — 2605.17056, 2605.17057)

Four loops are needed to close the all-loop exponentiation conjecture

Walking Sudakov: a walking anomalous dimension interpolating cusp $\leftrightarrow$ octagon (Alday, Armanini, Belitsky, Häring, Zhiboedov — 2605.16034)

Integrals that ELUDED traditional techniques, computed with polytopes

Phase Space Integration

Real-emission phase-space integrals can be brought to EULER FORM too: energies and angles become Schwinger-like parameters, the measure becomes polynomials raised to ϵ -dependent powers

$$\text{EEC}_{J_1, J_2}^{\text{real}} \propto \int_0^1 dx \frac{x^{J_1+1-2\epsilon} (1-x)^{J_2+1-2\epsilon}}{(1-zx)^{J_2+2-2\epsilon}} \mathcal{M}_{2 \rightarrow 3}(x), \quad \mathcal{M}_{2 \rightarrow 3} \propto \frac{(1-zx)^4}{x^2(1-x)^2} P(x) Q(x)$$

Already in SubTropica: the NLO energy–energy correlator in $N = 8$ supergravity (Chicherin, Korchemsky, Sokatchev, Zhiboedov — arXiv:2512.23791) and dimensionally regulated Fourier transforms of small- x physics (Brunello, Caron-Huot, Crisanti, Giroux, Smith — arXiv:2508.03794)

Herzog — arXiv:1804.07949 (geometric IR subtraction for real radiation) · Smirnov, Wunder — arXiv:2405.13120 (expansion by regions meets angular integrals) · Anastasiou, Duhr, Dulat, Mistlberger — arXiv:1302.4379

Real and virtual poles cancel only AFTER integration. Could locally finite integrands cancel them at the INTEGRAND level, real + virtual together?

OPEN QUESTION: a LOCAL subtraction for CROSS SECTIONS?

Bottomline: Put It In Euler Form

The polytope, the subtraction and the integrator never ask WHERE the integral comes from

$$I(\epsilon, s) = \int_{\mathbb{R}_{\geq 0}^n} \frac{d\alpha_1}{\alpha_1} \cdots \frac{d\alpha_n}{\alpha_n} \prod_j P_j(\alpha, s)^{a_j + b_j \epsilon}$$

Feynman integrals · phase space · Wilson-line webs · energy correlators · Fourier transforms · periods · cosmology —
all of this form

If you have an integral → put it in EULER FORM

Giroux, Mizera, Salvatori — arXiv:2604.20954, Sec. 4 and “Extending the domain of physical applications”: a challenge to the community

Outline

1. Polytopes In The Real World
2. Hyper(FL)Int
3. SubTropica
4. Applications
5. **Future Directions**

HyperFLOW

Hyperlogarithm integration = **partial fractions**, over and over

$$\frac{N(x)}{\prod_k (x - \sigma_k)^{n_k}} = \sum_k \sum_{j \leq n_k} \frac{c_{kj}}{(x - \sigma_k)^j} + \text{polynomial}$$

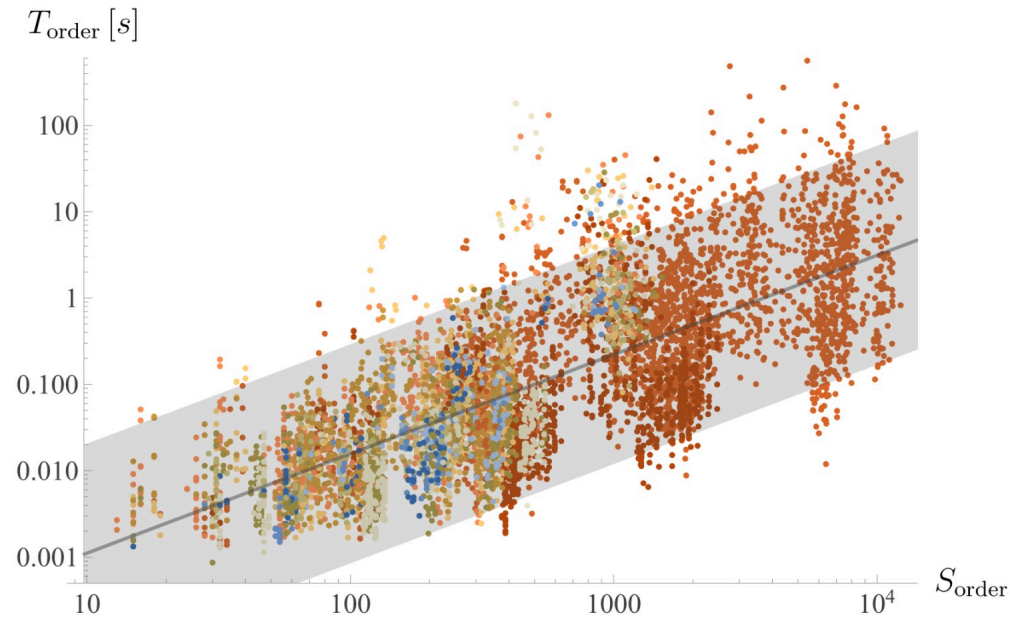
The coefficients are HUGE rational functions of the kinematics: Apart, Together, Factor dominate the run time

Do the polynomial algebra over **finite fields** with FiniteFlow, reconstruct only the FINAL rational functions

T. Peraro — arXiv:1912.03142 (FiniteFlow) · HyperFLINT already moves the arithmetic to FLINT

HyperFLOW: a finite-field & symbolic toolkit for HyperInt — work in PROGRESS

Linearly Reducible Orders



Many LR orders: which one is FASTEST?

$$S_{\text{order}} = \sum_k \text{LeafCount}(\text{polynomials met at step } k)$$

Everything is based on **HEURISTICS**

$R^2 \approx 0.55$, log-normal: three orders of magnitude of spread

Better Landau SUPER-approximation $\rightarrow$ fewer spurious polynomials $\rightarrow$ better prediction of integration complexity

Heuristics? Build an AGENTIC system and TRAIN it on lots of graphs to discover hidden patterns

Beyond Rational And GPL Functions

Linear reducibility is lost when a polynomial is QUADRATIC in every remaining variable: **square roots**

$$P = a(x_k - r_+)(x_k - r_-), \quad r_{\pm} = \frac{-b \pm \sqrt{\Delta}}{2a}, \quad \Delta = b^2 - 4ac$$

Factoring restores linearity at the price of $\sqrt{\Delta}$ — rationalize it with an **Euler transformation**

$$\sqrt{a'y^2 + b'y + c'} = \sqrt{a'}y + \sqrt{c'}t \implies y = \frac{c'(t^2 - 1)}{b' - 2\sqrt{a'}\sqrt{c'}t}, \quad \sqrt{\Delta} = \frac{\sqrt{c'}(b't - \sqrt{a'}\sqrt{c'}(t^2 + 1))}{b' - 2\sqrt{a'}\sqrt{c'}t}$$

$$\sqrt{\Delta(x_{k+1}, \dots, x_n)} \longrightarrow \sqrt{a'(x_{k+2}, \dots, x_n)}, \sqrt{c'(x_{k+2}, \dots, x_n)} \longrightarrow \dots \longrightarrow \sqrt{\text{kinematics}}$$

If a', c' are perfect squares or purely kinematic $\rightarrow$ DONE! Otherwise $\rightarrow$ ITERATE!

You stop only if you find a higher-than-quadratic discriminant... surprisingly EFFECTIVE!

...An Electroweak Example

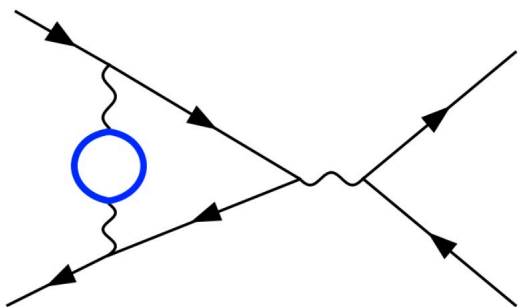


Fig. 3(e) of Armadillo, Devoto, Dradi, Vicini — arXiv:2605.28667: a two-loop Drell–Yan vertex with a fermion loop on the boson exchanged between the incoming quarks

Massless quarks · exchanged boson of mass m_2 · fermion loop of mass m_1 · vector boson of mass M ($MM = M^2$, $mm = m^2$)

SubTropica through ϵ^0 : the UV pole of the bubble + hyperlogarithms with ALGEBRAIC letters $W^\pm$

```
res = STIntegrate[graph, "Order" → 0, "Gauge" → {x4 → 1}]
```

$$\begin{aligned}
 & -\frac{\text{Log}\left[-\frac{MM}{mm2}\right]}{mm2 (MM + mm2) \text{eps}} + \left(\frac{\pi^2}{3 MM mm2 + 3 mm2^2} - \frac{2 \text{Log}\left[-\frac{MM}{mm2}\right]}{MM mm2 + mm2^2} - \frac{\text{Log}\left[-\frac{mm2}{MM}\right]}{MM mm2 + mm2^2} + \frac{\text{Log}\left[\frac{MM}{mm2}\right] \text{Log}\left[-\frac{mm2}{MM}\right]}{MM mm2 + mm2^2} + \right. \\
 & \frac{\frac{1}{2} \text{Log}\left[\frac{MM}{mm2}\right]^2 + \text{mzv}[2]}{mm2 (MM + mm2)} - \frac{1}{6 MM mm2 (MM + mm2)} \left(2 MM \pi^2 + 3 mm2 \pi^2 + 6 (MM + 3 mm2) \text{Hlog}\left[1, \left\{0, \frac{mm2}{MM + mm2}\right\}\right] + 12 MM \text{Log}\left[-\frac{MM}{mm2}\right]^2 - \right. \\
 & 18 MM \text{Log}\left[-\frac{mm2}{MM}\right] - 18 mm2 \text{Log}\left[-\frac{mm2}{MM}\right] - 6 \text{Log}\left[-\frac{MM}{mm2}\right] \left(4 MM + 3 mm2 + 2 i MM \pi + 2 MM \text{Log}[-mm2] - 3 MM \text{Log}\left[-\frac{mm2}{MM}\right] \right) + 6 MM \text{mzv}[2] \Big) - \\
 & \frac{2 \left(\text{Hlog}\left[1, \left\{0, 0, \frac{1}{1+Wm[1]}\right\}\right] - 2 \text{Hlog}\left[1, \left\{0, 0, \frac{1}{1+Wm[2]}\right\}\right] - \text{Hlog}\left[1, \left\{0, \frac{1}{1+Wm[2]}, \frac{1}{1+Wm[1]}\right\}\right] \right) (MM Wm[1] + mm1 (1 + Wm[1])^2)}{MM mm1 (MM + mm2) (1 + Wm[1])^2 (Wm[1] - Wp[1])} - \\
 & \frac{2 \left(\text{Hlog}\left[1, \left\{0, 0, \frac{1}{1+Wm[1]}\right\}\right] - 2 \text{Hlog}\left[1, \left\{0, 0, \frac{1}{1+Wp[2]}\right\}\right] - \text{Hlog}\left[1, \left\{0, \frac{1}{1+Wp[2]}, \frac{1}{1+Wm[1]}\right\}\right] \right) (MM Wm[1] + mm1 (1 + Wm[1])^2)}{MM mm1 (MM + mm2) (1 + Wm[1])^2 (Wm[1] - Wp[1])} - \\
 & \frac{\left(\text{Hlog}\left[1, \left\{0, \frac{1}{1+Wm[1]}\right\}\right] - \text{Hlog}\left[1, \left\{0, \frac{1}{1+Wm[2]}\right\}\right] - \text{Hlog}\left[1, \left\{\frac{1}{1+Wm[2]}, \frac{1}{1+Wm[1]}\right\}\right] \right) \text{Log}\left[-\frac{mm1}{MM}\right] (MM Wm[1] + mm1 (1 + Wm[1])^2)}{MM mm1 (MM + mm2) (1 + Wm[1])^2 (Wm[1] - Wp[1])} -
 \end{aligned}$$

... about 100 more lines: weight-3 hyperlogarithms in $W^\pm$, ζ_2 and ζ_3

Elliptics And Beyond

The iteration STOPS when a discriminant is quadratic in every remaining integration variable ("quadratically reducible"?)

A strong hint that you are OUTSIDE the GPLs of algebraic functions

Future: Hyper^k-Int

Closer to today: **elliptic linear reducibility** à la Hidding–Moriello (arXiv:1712.04441)

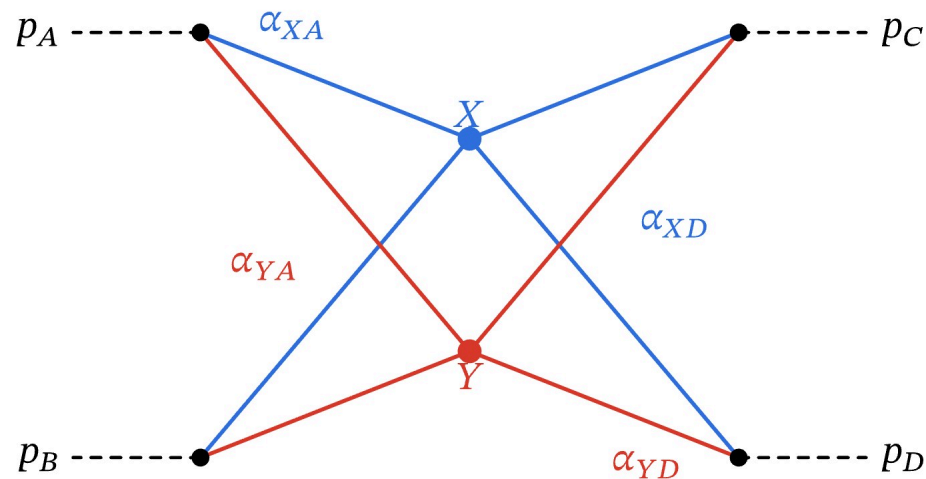
$\sqrt{\quad}$ and $\bigcirc$ are NOT a show-stopper!

Conceptual point

with DEs you rationalize $\sqrt{\text{kinematics}}$

with DI you don't! Rationalization is in parametric + kinematic space, where it is EASIER

Positivity And Glauber Gluons



The **crown**: massless, non-planar, three loops, eight propagators

Two internal vertices X, Y , each joined to all four external ones

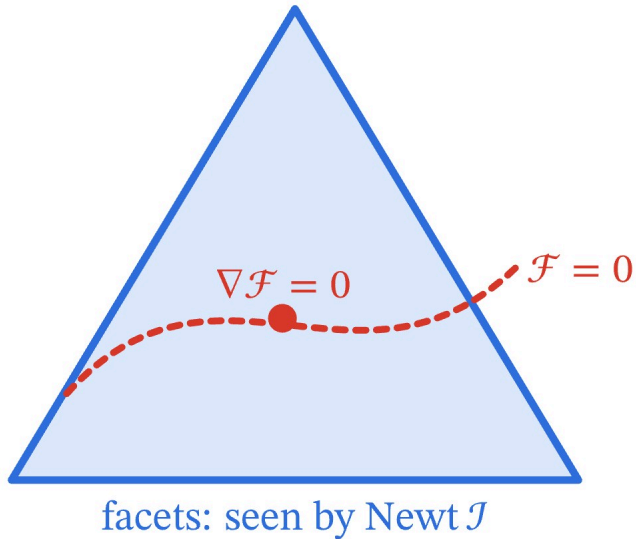
Its 2-forests organize into 2×2 **minors** of the edge variables

$$\mathcal{F} = s \Delta_{AC} \Delta_{BD} + t \Delta_{AB} \Delta_{CD}, \quad \Delta_{ij} \equiv \alpha_{Xi} \alpha_{Yj} - \alpha_{Xj} \alpha_{Yi}$$

NOT sign-definite: $\mathcal{F}$ vanishes on a hypersurface INSIDE the orthant — the Newton polytope only sees its FACETS

A $1/\epsilon$ that is neither SOFT nor COLLINEAR!

...Conjecture: A Bulk Subtraction



$$\Delta_{\text{Feynman}} = \{\alpha_i \geq 0, \sum \alpha_i = 1\}$$

Faces of the simplex $\leftrightarrow$ faces of the polytope: that is where the tropical counter-terms live

The hypersurface $\mathcal{F} = 0$ crosses the bulk; the pinch $\nabla \mathcal{F} = 0$ sits on it

$$\mathcal{J} = [\mathcal{J} - \mathcal{J}_{\mathcal{L}}] + \mathcal{J}_{\mathcal{L}}, \quad \mathcal{J}_{\mathcal{L}} = \textcolor{red}{v}_{\mathcal{L}} \times \textcolor{blue}{J}|_{\mathcal{L}}$$

CONJECTURE: subtract the integrand RESTRICTED to the locus, times a suppression factor — as for the faces

Conclusions

1. Divergences live on the FACES of the Newton polytope: tropical subtraction makes any Euler integral locally finite
2. Hyperlogarithm integration is algebra: partial fractions, shuffles, limits — and now C++ and finite fields
3. SubTropica automates the whole chain: draw a diagram, click, verify, share
4. Already used in the wild: the Coulomb-branch pentabox; phase space and correlators are next
5. Ahead: finite fields, better integration orders, elliptic letters, and positivity

Thanks for the attention!