Gluon production in Proton-Nucleus collisions from BCFW recursion

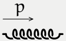
François Gelis

September 2026

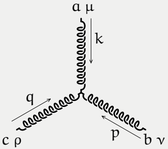
Institut de Physique Théorique
CEA/DRF Saclay

Preamble

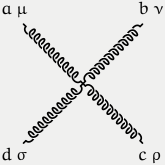
Yang-Mills theory Feynman rules



$$= \frac{-i\, g^{\mu\nu}\, \delta_{ab}}{p^2 + i0^+} + \frac{i\, \delta_{ab}}{p^2 + i0^+} \left(1 - \frac{1}{\xi}\right) \frac{p^\mu p^\nu}{p^2}$$

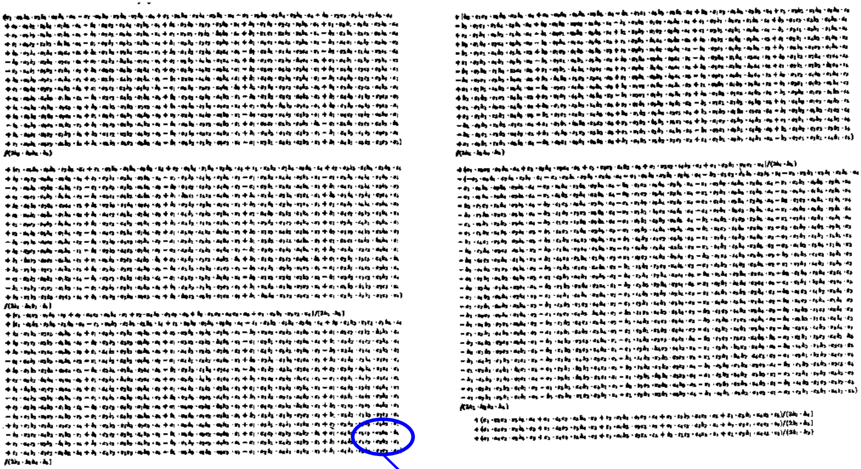


$$= -g\, f^{abc} \left\{ g^{\mu\nu}\, (k-p)^\rho + g^{\nu\rho}\, (p-q)^\mu + g^{\rho\mu}\, (q-k)^\nu \right\}$$



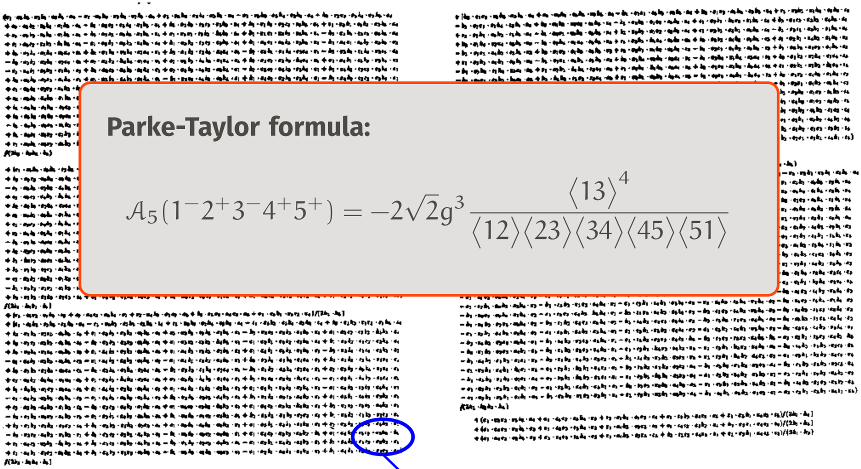
$$= -i\, g^2 \left\{ f^{abe} f^{cde} \left(g^{\mu\rho} g^{\nu\sigma} - g^{\mu\sigma} g^{\nu\rho}\right) + f^{ace} f^{bde} \left(g^{\mu\nu} g^{\rho\sigma} - g^{\mu\sigma} g^{\nu\rho}\right) + f^{ade} f^{bce} \left(g^{\mu\nu} g^{\rho\sigma} - g^{\mu\rho} g^{\nu\sigma}\right) \right\}$$

5-GLUON TREE AMPLITUDE



$$k_1 \cdot k_4\, \varepsilon_2 \cdot k_1\, \varepsilon_1 \cdot \varepsilon_3\, \varepsilon_4 \cdot \varepsilon_5$$

Slide from Zvi Bern ($\sim$ 2004)



Parke-Taylor formula:

$$\mathcal{A}_5(1^-2^+3^-4^+5^+) = -2\sqrt{2}g^3 \frac{\langle 13\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 51\rangle}$$

$$k_1 \cdot k_4 \, \varepsilon_2 \cdot k_1 \, \varepsilon_1 \cdot \varepsilon_3 \, \varepsilon_4 \cdot \varepsilon_5$$

Outline

- Color Glass Condensate
- Solutions of Yang-Mills equations
- Gluon spectrum from classical Yang-Mills
- On-shell methods
- Three-gluon amplitudes
- Scattering off an external field
- Gluon production

Color Glass Condensate

Parton evolution at small x

- The lifetime of virtual fluctuations inside hadrons increases with energy (for an observer in the lab frame)
- When this lifetime exceeds the duration of a collision, these virtual partons act as if they were on-shell
- As energy increases, more and more of these virtual excitations fulfill this criterion
- This explains the growth of the gluon distribution at small x

Gluon saturation

- At a given P_t, the transverse extent of a gluon is $\sim P_t^{-1}$
- One cannot pack an unlimited number of such gluons in the disk of a hadron, without them overlaping at some point
- When their wave-packets overlap, gluon recombinations compete with their splittings
- This leads to a slow down of the evolution with x at small-x, called *gluon saturation*
- The phase-space occupation cannot increase beyond α_s^{-1}

- In the deep saturated regime, $f(p) \sim \alpha_s^{-1}$, $A^\mu \sim g^{-1}$
- Commutators have a comparatively small effect, and the dynamics is approximately classical $\implies$ Gauge fields behave as classical solutions of Yang-Mills equations
- The YM equations are sourced by the fast color charges in the colliding hadrons
- The CGC describes them as color currents in the direction of motion of the projectiles, that do not depend on time

$$\left[D_\mu, F^{\mu\nu}\right] + J^\nu = 0, \quad J^\nu = \delta^{\nu+}\rho_1 + \delta^{\nu-}\rho_2$$

Solutions of Yang-Mills equations

Setup for solving YM equations

- YM equations are PDEs, with second order derivatives in time
- We need to prescribe A^μ and its conjugate momentum at some initial time
- In Yang-Mills theory, the color densities $\rho_{1,2}$ cannot be freely chosen in the entire spacetime. They must obey covariant current conservation

$$\left[D_\mu, J^\mu\right] = 0$$

- In practice, $\rho_{1,2}$ are chosen in the remote past (long before the collision, in a field free region), and current conservation prescribes their behavior at later times

Gauge covariance

- The YM and conservation equations are invariant under a simultaneous gauge transformation of the field A^μ and of the current J^ν,

$$A^\mu \to \Omega^\dagger A^\mu \Omega + \frac{i}{g}\Omega^\dagger \partial^\mu \Omega, \quad J^\nu \to \Omega^\dagger J^\nu \Omega$$

- Observable quantities are gauge invariant objects constructed from A^μ
- In a typical CGC calculation, the intermediate steps involve the gauge dependent A^μ, and it is only in the last stage that a gauge independent result emerges

Initial condition

- The content of the projectiles is encapsulated in the charge densities $\rho_{1,2}$. The initial field A^μ describes empty space
- The initial $A^\mu_{\rm ini}$ is a pure gauge (not necessarily zero)
- The initial $\rho_{\rm ini}$ is also gauge dependent
- If we gauge transform the initial conditions by a *space dependent* $\Omega_{\rm ini}(x)$,

$$A^\mu_{\rm ini} \to \Omega^\dagger_{\rm ini} A^\mu_{\rm ini} \Omega_{\rm ini} + \frac{i}{g}\Omega^\dagger_{\rm ini}\partial^\mu \Omega_{\rm ini}, \quad J^\nu_{\rm ini} \to \Omega^\dagger_{\rm ini} J^\nu_{\rm ini}\Omega_{\rm ini}$$

this initial gauge transformation is promoted to a global one (time independent) when solving the YM equations,

$$A^\mu \to \Omega^\dagger_{\rm ini} A^\mu \Omega_{\rm ini} + \frac{i}{g}\Omega^\dagger_{\rm ini}\partial^\mu \Omega_{\rm ini}, \quad J^\nu \to \Omega^\dagger_{\rm ini} J^\nu \Omega_{\rm ini}$$

Evolution by one timestep

- For an infinitesimal timestep dx^0, the YM equations lead to

$$\begin{aligned} F^{0i} \quad &\to \quad F^{0i} - dx^0\big([D_j, F^{ji}] + J^i\big) + \underline{i\,g\,dx^0\,[A_0, F^{0i}]} \\ A^i \quad &\to \quad A^i + dx^0 F^{0i} + \underline{dx^0\big(\partial^i A^0 - i\,g\,[A^i, A^0]\big)} \end{aligned}$$

- The underlined terms depend on A^0, for which there is no dynamical equation
- A^0 must be set by a gauge condition (this reflects the freedom to fix the gauge independently at each time)
- The underlined terms are a small gauge transformation,

$$\omega \equiv \exp(ig\theta), \quad \text{with } \theta = -dx^0 A^0$$

Overall gauge transformation

- By combining gauge transformations of the initial condition, and the freedom to choose A^0 at every timestep, we get the following overall gauge transformation

$$\Omega^\dagger(x^0,\vec{x}) = \left[\mathrm{T}\,\exp\left(ig\int_{-\infty}^{x^0} dy^0\,A^0(y^0,\vec{x})\right)\right]\Omega^\dagger_{\rm ini}(\vec{x}),$$

- No physics is contained in this transformation. But it makes the solution look very different, and the difficulty to obtain it also depends a lot on the gauge (simplest is often light-cone gauge)

Gluon spectrum from classical Yang-Mills

From A^μ to the gluon spectrum

- After we have solved the YM equations, one has to turn the gauge dependent A^μ into a gauge independent gluon spectrum
- A naive use of LSZ would suggest to define the gluon production amplitude as

$$\mathcal{M}(\mathbf{p}_a^\lambda) \equiv i \int d^3\mathbf{x}\, e^{ip\cdot x} \overleftrightarrow{\partial_0}\, A_a^\mu(x)\epsilon_{\lambda,\mu}(\mathbf{p})$$

- Given a decomposition of $A^\mu(x)$ into plane waves

$$A_a^\mu(x) = \sum_\lambda \int \frac{d^3\mathbf{k}}{(2\pi)^3 2k} \left(\alpha^*_{\mathbf{k}\lambda a}\epsilon_\lambda^{\mu*}(\mathbf{k})e^{ik\cdot x} + \alpha_{\mathbf{k}\lambda a}\epsilon_\lambda^{\mu}(\mathbf{k})e^{-ik\cdot x}\right),$$

  this is equivalent to setting

$$\mathcal{M}(\mathbf{p}_a^\lambda) = \alpha_{\mathbf{p}\lambda a}$$

CAVEATS

- A gauge transformation of A^μ produces a non-homogeneous term, $\Omega^\dagger\partial^\mu\Omega$. **Solution:** the LSZ formula is designed to capture the residue of the pole of $A^\mu(p)$ at $p^2=0$. Generically, $\Omega^\dagger\partial^\mu\Omega$ has no pole at $p^2=0$, and this term is transparent in LSZ
- The gauge transformation also produces a homogeneous color rotation of A_a^μ,

$$A_a^\mu \to \Omega^\dagger_{ab}(x)A_b^\mu(x)$$

  To remove this gauge dependence, the LSZ formula should use the solutions of

$$\left(\mathcal{D}^2 g_{\mu\nu} - \mathcal{D}_\mu\mathcal{D}_\nu\right)_{ab} a_b^\nu(x) = 0, \quad \text{with } \mathcal{D}^\mu \equiv \partial^\mu + \Omega^\dagger\partial^\mu\Omega$$

  as a basis, instead of plain plane waves

Caveats

- One more improvement is needed when the gauge transformation is time dependent at large times. By defining

$$\mathcal{M}(\mathbf{p}_a^\lambda) \equiv i \lim_{x^0 \to +\infty} \int d^3\mathbf{x} \; a_{\mathbf{p}\lambda a} \overleftrightarrow{\mathcal{D}_0} \, A_\mu ,$$

  we also eliminate the time derivatives of Ω

- The gluon spectrum can be viewed as a function of the initial value of the sources $\rho_{1,2}$, that depend on the initial gauge

  To make its expression unambiguous, we may choose the initial gauge to be the one where the initial $A^\mu_{\rm ini}$ is zero

Proton-Nucleus collisions

- The proton is considered to be a dilute hadron
  $\Longrightarrow$ The solution of YM is calculated at order one in the proton color density, and to all orders in that of the nucleus

- At this order, the YM equations can be solved analytically

  This has been done in at least three gauges: Landau ($\partial_\mu A^\mu = 0$), Fock-Schwinger ($x^+ A^- + x^- A^+ = 0$), Light-cone ($A^+ = 0$)

- The A^μ's in these gauges look extremely different (and the derivation is particularly tedious in some gauges)

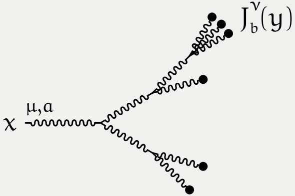
  The gluon spectrum is the same in all gauges

- It would be desirable to have a method that gives directly the gauge invariant spectrum, and bypasses all gauge dependent intermediate steps

Diagrammatic rules for A^μ

- Classical solutions can be represented as a sum of tree diagrams

$$A_a^\mu(x) = \sum_{\text{trees}}$$



- Ingredients: YM rules + color sources

- **Missing:** rules to account for $[D_\mu, J^\mu] = 0$

Current conservation

- Assume $J^\mu_a = \rho_a v^\mu$ (or a sum of two such terms)
  Current conservation reads $(v \cdot D)_{ab}\rho_b = 0$

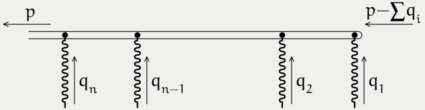
- Write $\rho_a \equiv \sum_n \rho_a^{(n)}$, with n the order in the field A^μ. Then

$$(v \cdot \partial)\rho_a^{(n)} = ig(v \cdot A)_{ab}\, \rho_b^{(n-1)}$$

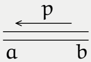
- Solution in momentum space:

$$\rho_a^{(n)}(p) = -gf^{abc} \int \frac{d^4q}{(2\pi)^4} \, \frac{v \cdot A_c(q)}{v \cdot p} \, \rho_b^{(n-1)}(p-q),$$

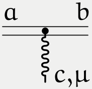
$$\rho_a(p) = \sum_{n=0}^{\infty}$$



 $= \rho^{(0)}$



$= \dfrac{\delta_{ab}}{\nu \cdot p}$



$= -g f^{abc} \nu^{\mu}$

- Initial value problem: all propagators should be retarded
- The external legs attached to the sources $\rho^{(0)}$ are not amputated. They have ν^{μ} instead of a polarization vector, and they carry a space-like momentum, that obeys $\nu \cdot p = 0$

On-shell methods

Brief history

- Yang-Mills theory known since 1953, used in high energy physics since the early 1970s. Challenging perturbative expansion:
  - Complicated vertices
  - Faster than factorial growth of the number of graphs with the number of external legs (even worse beyond tree level)

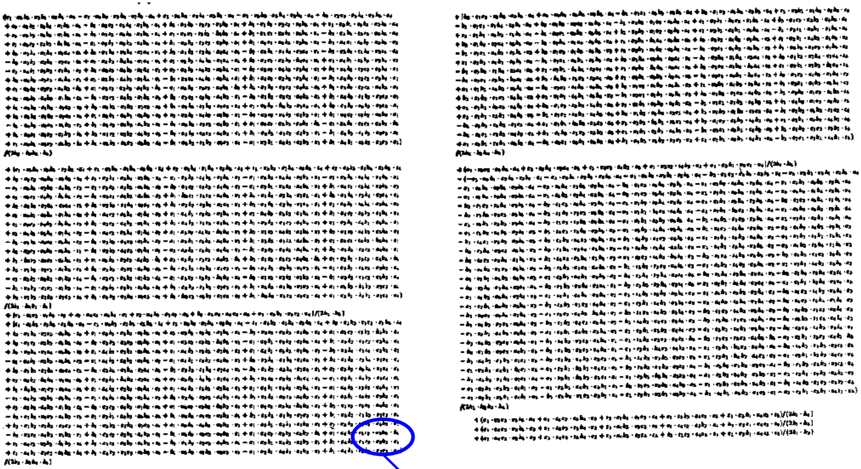
- **1986: Parke-Taylor:** one-term conjecture for any tree level MHV amplitude (2 negative helicities, all others positive)

$$\mathcal{A}_n(\cdots i^- \cdots j^- \cdots) = \# g^{n-2} \frac{\langle ij \rangle^4}{\langle 12 \rangle \langle 23 \rangle \cdots \langle n-1n \rangle \langle n1 \rangle}$$

  (for $n = 10$, there are $\sim 10^7$ graphs, $\sim 10^4$ for partial amplitudes)

- **1988: Berends-Giele:** proof by recursion on Feynman diagrams
- **2005: BCFW:** recursion among gauge invariant amplitudes

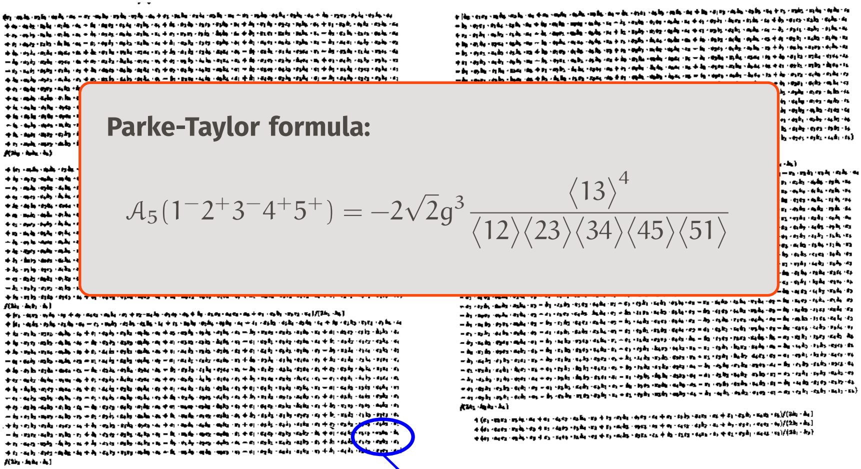
Example: 5-gluon tree amplitude



$k_1 \cdot k_4\, \varepsilon_2 \cdot k_1\, \varepsilon_1 \cdot \varepsilon_3\, \varepsilon_4 \cdot \varepsilon_5$

Slide from Zvi Bern ($\sim$ 2004)

Example: 5-gluon tree amplitude



Parke-Taylor formula:

$$\mathcal{A}_5(1^-2^+3^-4^+5^+) = -2\sqrt{2}g^3\,\frac{\langle 13\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 51\rangle}$$

$$k_1\cdot k_4\,\varepsilon_2\cdot k_1\,\varepsilon_1\cdot\varepsilon_3\,\varepsilon_4\cdot\varepsilon_5$$

- Observation: physical amplitudes (with on-shell external momenta, physical polarizations) are in general much simpler than what one may have anticipated based on their diagrammatic expansion
- Issue: perturbation theory has the same complexity whether the amplitude is physical or not. Simplifications occur only at the very end, and are hard to see except in simple cases
- Goal: design tools that give directly the simple answer for physical amplitudes, and avoid manipulating unphysical (in particular, gauge dependent) objects in the intermediate steps

Color decomposition

- $\mathfrak{su}(N)$ **Fierz identity:** any tree level amplitude of Yang-Mills theory can be decomposed as

$$\mathcal{M}_n(1_{a_1}\cdots n_{a_n}) = 2\sum_{\sigma\in\mathfrak{S}_n/\mathbb{Z}_n} \mathrm{tr}\,(t^{a_{\sigma_1}}\cdots t^{a_{\sigma_n}})\,\mathcal{A}_n(\sigma_1\cdots\sigma_n)$$

- $\mathcal{A}_n(\sigma_1\cdots\sigma_n)$ is a *color ordered* (or *partial*) amplitude
  Given by color-stripped Feynman rules; only planar diagrams

- Alternate color decomposition:

$$\mathcal{M}(1_{a_1}\ldots n_{a_n}) = -(-i)^n \sum_{\sigma\in\mathfrak{S}(3\ldots n)} f^{a_1 a_{\sigma_3} c_1} f^{c_1 a_{\sigma_4} c_2}\cdots f^{c_{n-3} a_{\sigma_n} a_2}\,\mathcal{A}_n(12\sigma_3\ldots\sigma_n)$$

  (This is just a change of color basis. The mapping between the elements of the two basis leads to the *Kleiss-Kuijf* relations)

Spinor-helicity formalism

- Define $\sigma^\mu \equiv (1, \sigma^i)$, $\overline{\sigma}^\mu \equiv (1, -\sigma^i)$

  Given p^μ, define $P \equiv p_\mu \sigma^\mu$ and $\overline{P} \equiv -p_\mu \overline{\sigma}^\mu$ (2×2 matrices)

- $p^2 = 0$ is equivalent to $\det(P) = \det(\overline{P}) = 0$, i.e., $P, \overline{P}$ are factorizable as direct products of 2-component vectors,

$$P = |\mathbf{p}]\langle\mathbf{p}|, \quad \overline{P} = |\mathbf{p}\rangle[\mathbf{p}|$$

- Useful properties:

$$\begin{aligned}
&(p+q)^2 = 2\,p\cdot q = \langle\mathbf{pq}\rangle[\mathbf{pq}],\\
&\langle\mathbf{pq}\rangle = -\langle\mathbf{qp}\rangle, \quad [\mathbf{pq}] = -[\mathbf{qp}],\\
&\langle\mathbf{pq}\rangle = [\mathbf{pq}]^* \quad \text{for real momenta}
\end{aligned}$$

Spinor-helicity formalism

- Polarization vectors for $\pm$ helicities:

$$\epsilon_-^\mu(p;q) \equiv \frac{\langle p|\overline{\sigma}^\mu|q]}{\sqrt{2}[pq]}, \quad \epsilon_+^\mu(p;q) \equiv \frac{\langle q|\overline{\sigma}^\mu|p]}{\sqrt{2}\langle qp\rangle}$$

- p^μ is the gluon momentum

  q^μ is an *auxiliary vector*. Varying q is a gauge transformation of the polarization vector,

$$\epsilon_h^\mu(p;q') = \epsilon_h^\mu(p;q) + \#\, p^\mu$$

- Auxiliary vectors can be chosen independently for each external gluon (but should be the same in all terms of an amplitude)

- The dependence on the q's always drops out in physical amplitudes (we may choose special q's to get simplifications)

Spinor-helicity formalism

- Momentum conservation:

$$\sum_i p_i^\mu = 0 \quad \Longleftrightarrow \quad \sum_i |i\rangle[i| = 0$$

- Schouten identity. Given three on-shell momenta p^μ, q^μ, r^μ:

$$|\mathbf{p}\rangle\langle\mathbf{qr}\rangle + |\mathbf{q}\rangle\langle\mathbf{rp}\rangle + |\mathbf{r}\rangle\langle\mathbf{pq}\rangle = 0$$

- Fierz identity:

$$-\tfrac{1}{2}\langle i|\overline{\sigma}^\mu|j]\,\overline{\sigma}_\mu = |i\rangle[j|$$

- **Note:** this formalism is tied to being in four dimensions
  It provides notational simplicity, but is not essential for using on-shell methods

Parameterization of off-shell momenta

- Off-shell momenta inside graphs are completely generic (but they are seldom needed)
- Off-shell momenta attached to *initial* sources $\rho^{(0)}$ must obey $v \cdot p = 0$. We consider only sources moving at the speed of light, i.e., v^μ is light-like and $v_\mu \sigma^\mu = |\boldsymbol{v}]\langle \boldsymbol{v}|$

- Given an auxiliary momentum q^μ, this momentum can be parameterized as

$$\overline{P} = \frac{p \cdot q}{v \cdot q} |\boldsymbol{v}\rangle [\boldsymbol{v}| - \kappa \frac{|\boldsymbol{v}\rangle [\mathbf{q}|}{[\boldsymbol{v}\mathbf{q}]} + \overline{\kappa} \frac{|\mathbf{q}\rangle [\boldsymbol{v}|}{\langle \mathbf{q}\boldsymbol{v}\rangle}$$

- $p_\mu p^\mu = -\kappa\overline{\kappa}$, $\overline{\kappa} = \kappa^*$ for a real momentum

  $\kappa, \overline{\kappa}$ do not depend on q^μ, but the longitudinal component does (q^μ makes the transverse plane unique, by defining it as the plane orthogonal to both p^μ and q^μ)

BCFW RECURSION

- Deformation of two external momenta in an amplitude, controlled by a complex parameter z
- **At all z:** must preserve momentum conservation, on-shell momenta must remain on-shell
- Assume $\mathcal{A}(\cdots;z) \to 0$ when $|z| \to \infty$
  Not always true, must be checked!
- Take a circle $\mathcal{C}_R$ of radius $R \to \infty$, and integrate $\mathcal{A}(...,z)/z$

$$0 = \oint_{\mathcal{C}_R} \frac{dz}{2\pi i} \frac{\mathcal{A}(\cdots,z)}{z} = \underbrace{\mathcal{A}(\cdots,z=0)}_{\substack{\text{original}\\\text{amplitude}}} + \sum_{\substack{\text{poles}\\ \neq 0}} \mathrm{Res}\left[\frac{\mathcal{A}(...,z)}{z}\right]$$

BCFW RECURSION

- Choose two external lines i, j to apply the shift to, and a *light-like* shift vector e^μ,

$$p_i \to \widehat{p}_i = p_i + z\, e, \quad p_j \to \widehat{p}_j = p_j - z\, e$$

- For any vector k, we have $\widehat{k}^2 = (k + z\, e)^2 = k^2 + 2\, z\, e \cdot k$

  Internal propagators may have a *simple pole* at a certain z_*

- At this z_*, the residue is

$$-\frac{i}{k^2} \underbrace{\sum_{h=\pm} \epsilon_h^\mu(\widehat{k}, \mathbf{q}) \epsilon_{-h}^\nu(-\widehat{k}, \mathbf{q})}_{\text{numerator of } G_0^{\mu\nu}(\widehat{k})}$$

- The sum over the non-zero poles is a sum over the internal propagators traversed by the shift

- Attach the polarization vectors to the pieces on the left and right of the propagator where the pole came from. Since $\widehat{k}^\mu$ is on-shell at the pole, this turns them into physical amplitudes (with fewer external legs)

- The recursion reads

$$\mathcal{A}(\cdots) = \sum_{\substack{\text{poles}\\ \neq 0}} \sum_{h=\pm} \mathcal{A}_{\text{L}}(\cdots \widehat{k}^h) \frac{i}{k^2} \mathcal{A}_{\text{R}}(-\widehat{k}^{-h} \cdots)$$

- **Method:**
  - Choose the pair of lines i, j and the shift vector e^μ
  - List the internal propagators that can have a pole
  - For each pole, calculate z_*, $\widehat{k}$, $\mathcal{A}_{\text{L}}$ and $\mathcal{A}_{\text{R}}$

- When applied to two on-shell momenta p_i and p_j, the shift vector should fulfill

$$e^2 = e \cdot p_i = e \cdot p_j = 0$$

- All the conditions are satisfied for

$$e^\mu = \tfrac{1}{2}\langle i|\overline{\sigma}^\mu|j] \quad \text{or} \quad e^\mu = \tfrac{1}{2}\langle j|\overline{\sigma}^\mu|i]$$

- The validity of the condition $\mathcal{A}(\cdots, z) \to 0$ when $|z| \to \infty$ depends on the helicities $h_{i,j}$. For $e^\mu = \frac{1}{2}\langle i|\overline{\sigma}^\mu|j]$,

h_i	h_j	
$-$	$+$	✓
$+$	$+$	✓
$-$	$-$	✓
$+$	$-$	✗

BCFW extension for off-shell amplitudes

- The method of shifting external momenta to split an amplitude into a pair of simpler amplitudes works also with off-shell external legs [van Hameren, 2014]
- When the line i is off-shell, the shift cannot be $\frac{1}{2}\langle i|\overline{\sigma}^\mu|j]$
  Use $\frac{1}{2}\langle v_i|\overline{\sigma}^\mu|j]$ instead, with v_i^μ the light-like direction of the source producing the gluon i
  - Poles in internal propagators: exactly as before
  - There can be a pole in the external off-shell propagator
    Gives a contribution where the off-shell gluon becomes on-shell with $+$ helicity:

$$-\frac{i}{\sqrt{2}\kappa_i}\mathcal{A}(\widehat{p}_i^+\cdots),\quad \text{with } z_* = \kappa_i/[v_i j]$$

BCFW SHIFT APPLIED TO TWO OFF-SHELL LINES

- Consider a shift applied to two off-shell external lines (i, j) produced by sources of *common* direction v^μ,

$$e^\mu = \frac{1}{2}\langle \boldsymbol{v}|\overline{\sigma}^\mu|\boldsymbol{v}] = v^\mu$$

- With this shift, the virtuality of the external lines is unchanged:

$$\widehat{p}^2 = (p + zv)^2 = p^2 + 2\,z\,\underbrace{p \cdot v}_{=\,0} + z^2\,\underbrace{v^2}_{=\,0} = p^2$$

$\Longrightarrow$ no pole in the external propagators

- Naive power counting: deformed amplitude behaves as z
- Using $v^2 = 0$, a more refined study indicates that it is z^{-1}

Three-gluon amplitudes

Fully on-shell

- With BCFW recursion, everything starts with 3-point amplitudes
- Obtained directly from Feynman rules (1 graph), or from *little group scaling* (Feynman rule only used to get the prefactor)
- Non-zero 3-gluon on-shell amplitudes:

$$\mathcal{A}_3(0^+1^+2^-) = -g\sqrt{2}\frac{[01]^3}{[12][20]}$$

$$\mathcal{A}_3(0^-1^-2^+) = g\sqrt{2}\frac{\langle 01\rangle^3}{\langle 12\rangle\langle 20\rangle}$$

(Parke-Taylor formulas for $n = 3$)

Two on-shell, One off-shell

- 3-gluon amplitudes with one off-shell: $\mathcal{A}_3(0^{h_0}1^{h_1}2^*)$
- Perturbation theory: 2 graphs

- $0^+1^+2^*$: shift $e^\mu = \frac{1}{2}\langle v_2|\overline{\sigma}^\mu|1]$ $\qquad \mathcal{A}_3(0^+1^+2^*) = 0$

- $0^+1^-2^*$: shift $e^\mu = \frac{1}{2}\langle 1|\overline{\sigma}^\mu|v_2]$

$$\mathcal{A}_3(0^+1^-2^*) = \frac{ig}{\overline{\kappa}_2}\frac{\langle 1v_2\rangle^3}{\langle v_2 0\rangle\langle 01\rangle} = \frac{ig}{\kappa_2}\frac{[v_2 0]^3}{[01][1v_2]}$$

- $0^-1^+2^*$: shift $e^\mu = \frac{1}{2}\langle v_2|\overline{\sigma}^\mu|1]$

$$\mathcal{A}_3(0^-1^+2^*) = \frac{ig}{\kappa_2}\frac{[1v_2]^3}{[v_2 0][01]} = \frac{ig}{\overline{\kappa}_2}\frac{\langle v_2 0\rangle^3}{\langle 01\rangle\langle 1v_2\rangle}$$

One on-shell, Two off-shell

- 3-gluon amplitudes with two off-shell: $\mathcal{A}_3(0^{h_0}1^*2^*)$
- Perturbation theory: 3 graphs

- $0^+1^*2^*$: shift $e^\mu = \frac{1}{2}\langle \boldsymbol{v}_2|\overline{\sigma}^\mu|\boldsymbol{v}_1]$

$$\mathcal{A}_3(0^+1^*2^*) = -\frac{g}{\sqrt{2}\,\overline{\kappa}_1\overline{\kappa}_2}\frac{\langle \boldsymbol{v}_1\boldsymbol{v}_2\rangle^3}{\langle \boldsymbol{v}_2 0\rangle\langle 0\boldsymbol{v}_1\rangle}$$

- $0^-1^*2^*$: shift $e^\mu = \frac{1}{2}\langle \boldsymbol{v}_1|\overline{\sigma}^\mu|\boldsymbol{v}_2]$

$$\mathcal{A}_3(0^-1^*2^*) = \frac{g}{\sqrt{2}\,\kappa_1\kappa_2}\frac{[\boldsymbol{v}_1\boldsymbol{v}_2]^3}{[\boldsymbol{v}_2 0]\,[0\boldsymbol{v}_1]}$$

- To square an amplitude expressed in the spinor-helicity language, we use

$$\kappa^* = \overline{\kappa}, \quad \kappa\overline{\kappa} = -k_\mu k^\mu$$
$$\big\langle \mathbf{ab}\big\rangle^* = \big[\mathbf{ab}\big], \quad \big\langle \mathbf{ab}\big\rangle\big[\mathbf{ab}\big] = 2\, a\cdot b$$

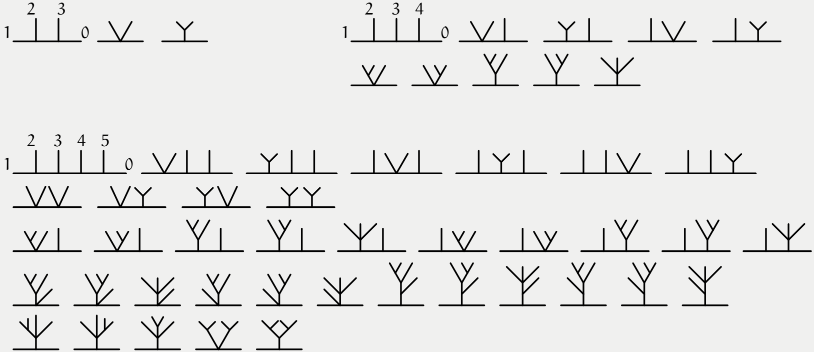
- Example:

$$\Big|\mathcal{A}_3(0^+1^*2^*)\Big|^2 = \frac{g^2}{2\,|\kappa_1|^2|\kappa_2|^2}\,\frac{\big\langle \boldsymbol{v}_1\boldsymbol{v}_2\big\rangle^3\big[\boldsymbol{v}_1\boldsymbol{v}_2\big]^3}{\big\langle \boldsymbol{v}_2 0\big\rangle\big[\boldsymbol{v}_2 0\big]\big\langle 0\boldsymbol{v}_1\big\rangle\big[0\boldsymbol{v}_1\big]}$$
$$= \frac{g^2}{p_1^2\,p_2^2}\,\frac{(v_1\cdot v_2)^3}{(p_0\cdot v_2)(p_0\cdot v_1)}$$

Scattering off an external field

Perturbation theory for partial amplitudes

- Graphs in the solution of $[D_\mu, F^{\mu\nu}] + J^\nu = 0$:



- One should also add the graphs that account for $[D_\mu, J^\mu] = 0$

- Convention: momenta are all *incoming*

  Helicities are defined with respect to incoming momenta

- We have already calculated $\mathcal{A}_3(0^+1^+2^*) = 0$
  (Scattering with helicity flip is zero)

- The only non-zero scatterings conserve helicity,

$$\mathcal{A}_3(0^+1^-2^*) = \frac{ig}{\overline{\kappa}_2} \frac{\langle 1 \boldsymbol{v}_2 \rangle^3}{\langle \boldsymbol{v}_2 0 \rangle \langle 01 \rangle}$$

Order 2 in the field

- $0^+1^+2^*3^*$: shift $e^\mu = \frac{1}{2}\big\langle \boldsymbol{v}_2 \big| \overline{\sigma}^\mu | \boldsymbol{v}_2 \big] = v_2^\mu$ on lines $(2,3)$

$$\mathcal{A}_4(0^+1^+2^*3^*) = \sum_{h=\pm} \mathcal{A}_3(0^+\widehat{1}^{+h}\widehat{3}^*)\frac{i}{(p_0+p_3)^2}\mathcal{A}_3(-\widehat{1}^{-h}\,1^+\widehat{2}^*)$$

  - $v_2 \cdot e = 0 \implies$ no poles in shifted eikonal denominators $\widehat{p}\cdot v_2$
  - h must be -1 for the amplitude on the left to be $\neq 0$
  - h must be $+1$ for the amplitude on the right to be $\neq 0$

Therefore, $\mathcal{A}_4(0^+1^+2^*3^*) = 0$

General: scattering with helicity flip is zero at all orders in the external field

Order 2 in the field

- $0^+1^-2^*3^*$: shift $e^\mu = \frac{1}{2}\langle v_2|\overline{\sigma}^\mu|v_2] = v_2^\mu$ on lines $(2,3)$

$$\mathcal{A}_4(0^+1^-2^*3^*) = \sum_{h=\pm} \mathcal{A}_3(0^+\,\widehat{1}^{+h}\,\widehat{3}^*)\frac{i}{(p_0+p_3)^2}\mathcal{A}_3(-\widehat{1}^{-h}\,1^-\widehat{2}^*)$$

- Only $h=-1$ gives a non-zero contribution

$$\mathcal{A}_3(0^+\widehat{1}^-\widehat{3}^*) = \frac{ig}{\overline{\kappa}_3}\frac{\langle\widehat{1}v_2\rangle^3}{\langle v_2 0\rangle\langle 0\widehat{1}\rangle}, \qquad \mathcal{A}_3(-\widehat{1}^+1^-\widehat{2}^*) = -\frac{ig}{\overline{\kappa}_2}\frac{\langle 1v_2\rangle^3}{\langle v_2\widehat{1}\rangle\langle\widehat{1}1\rangle}$$

$$z_* = \frac{(p_0+p_3)^2}{2p_0\cdot e}, \quad |\widehat{1}\rangle = |0\rangle + \frac{\kappa_3}{[0v_2]}|v_2\rangle$$

- Final expression:

$$\mathcal{A}_4(0^+1^-2^*3^*) = i\frac{g^2[0v_2]^2\langle 1v_2\rangle^2}{|\kappa_2|^2|\kappa_3|^2(p_0+p_3)^2}$$

Higher orders

- Use the color decomposition in terms of structure constants,

$$\mathcal{M}_{n+1}(0^+_{a_0}1^-_{a_1}2^*_{a_2}...n^*_{a_n})$$
$$= -i^n \sum_{\sigma\in\mathfrak{S}(2...n)} f^{a_0 a_{\sigma_n} d_1} f^{d_1 a_{\sigma_{n-1}} d_2} \cdots f^{d_{n-2} a_{\sigma_2} a_1} \underline{\mathcal{A}_{n+1}(0^+1^-\sigma_2^*...\sigma_n^*)}$$

- BCFW with shift $e^\mu = \frac{1}{2}\langle v_2|\overline{\sigma}^\mu|v_2]$ on lines $(n-1, n)$

$$\mathcal{A}_{n+1}(0^+1^-2^*...n^*) = \underbrace{\mathcal{A}_3(0^+ - \widehat{I}^- \widehat{n}^*)}_{\text{known}} \frac{i}{(p_0+p_n)^2} \underbrace{\mathcal{A}_n(\widehat{I}^+1^-2^*...\widehat{n-1}^*)}_{\text{previous order}}$$

$\Longrightarrow$ gives a recursion on the number of scatterings, directly at the level of scattering amplitudes (we just need to work out the values of z_*, $|\widehat{I}\rangle$ and $|\widehat{I}]$ at the pole)

Higher orders

- This leads to

$$\mathcal{A}_{n+1}(0^+1^-2^*...n^*) = i\frac{\left[0v_2\right]^2}{\left[1v_2\right]^2}\frac{\left(g\left[1v_2\right]\langle 1v_2\rangle\right)^{n-1}}{|\kappa_2|^2\cdots|\kappa_n|^2}$$
$$\times\ \frac{1}{(p_0+p_n)^2(p_0+p_n+p_{n-1})^2...(p_0+p_n+...+p_3)^2}$$

- Remarkably compact compared to what one might expect from the list of Feynman graphs
- These amplitudes are in fact the successive terms in the expansion of a Wilson line
- Alternatively, this can be obtained by solving YM equations in light-cone gauge (more involved in Landau gauge)

  Advantage of the BCFW approach: no need to fix the gauge

Gluon production

Setup, Differences

- We need the partial amplitudes $\mathcal{A}_{n+1}(0^+ 1^* \underbrace{2^* \dots n^*}_{\text{nucleus}})$
- Gluon production requires color sources with two distinct directions, v_1^μ (proton) and v_2^μ (nucleus)
- For pA collisions: one source of direction v_1^μ and many sources of direction v_2^μ
- We again use the shift $e^\mu = \frac{1}{2}\langle v_2|\overline{\sigma}^\mu|v_2] = v_2^\mu$
- Two kinds of eikonal denominators, $p \cdot v_1$ and $p \cdot v_2$
  When they are traversed by the shift:

$$p \cdot v_2 \to p \cdot v_2 - z \underbrace{e \cdot v_2}_{=0} = p \cdot v_2$$

$$p \cdot v_1 \to p \cdot v_1 - z \underbrace{e \cdot v_1}_{\neq 0}$$

$\Longrightarrow$ there are **zeroes in shifted eikonal denominators** $\widehat{p} \cdot v_1$

Order $\rho_1 \rho_2$

- At the order $\rho_1 \rho_2$,

$$\mathcal{A}_3(0^+1^*2^*) = -\frac{g}{\sqrt{2}\,\overline{\kappa}_1\overline{\kappa}_2}\,\frac{\big\langle \boldsymbol{v}_1\boldsymbol{v}_2\big\rangle^3}{\big\langle \boldsymbol{v}_2 0\big\rangle\big\langle 0\boldsymbol{v}_1\big\rangle}$$

$$\mathcal{A}_3(0^-1^*2^*) = \frac{g}{\sqrt{2}\,\kappa_1\kappa_2}\,\frac{\big[\boldsymbol{v}_1\boldsymbol{v}_2\big]^3}{\big[\boldsymbol{v}_2 0\big]\big[0\boldsymbol{v}_1\big]}$$

- The answer from PT (3 graphs) is usually written as

$$\mathcal{A}_3(0^\pm 1^*2^*) = \frac{g}{\big|\kappa_1\big|^2\big|\kappa_2\big|^2}\;L\cdot\epsilon_\pm(\mathbf{p}_0,\mathbf{q}),$$

  where L^μ is the *Lipatov vertex*,

$$L^\mu = -2p_1^\mu - \Big(2p_0^+ - \frac{\mathbf{p}_{1\perp}^2}{p_0^-}\Big)v_1^\mu - \Big(\frac{\mathbf{p}_{2\perp}^2 - \mathbf{p}_\perp^2}{p_0^+}\Big)v_2^\mu$$

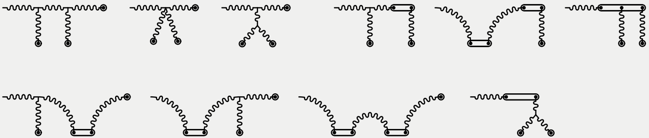
- Using the explicit form of polarization vectors, one obtains

$$L \cdot \epsilon_+(\mathbf{p}_0, \mathbf{q}) = \frac{\kappa_1 \kappa_2}{\sqrt{2}} \frac{\left\langle \boldsymbol{v}_1 \boldsymbol{v}_2 \right\rangle^3}{\left\langle 0 \boldsymbol{v}_1 \right\rangle \left\langle 0 \boldsymbol{v}_2 \right\rangle}$$

$$L \cdot \epsilon_-(\mathbf{p}_0, \mathbf{q}) = -\frac{\overline{\kappa}_1 \overline{\kappa}_2}{\sqrt{2}} \frac{\left[\boldsymbol{v}_1 \boldsymbol{v}_2 \right]^3}{\left[0 \boldsymbol{v}_1 \right] \left[0 \boldsymbol{v}_2 \right]}$$

$\Longrightarrow$ PT and BCFW agree

- Perturbation theory:



- BCFW with the shift $e^\mu = v_2^\mu$ on the lines $(2,3)$ gives:

$$\mathcal{A}_4(0^+ 1^* 2^* 3^*) = -\frac{g^2}{|\kappa_1|^2 |\kappa_2|^2 |\kappa_3|^2} \frac{\langle 0 \boldsymbol{v}_2 \rangle [0 \boldsymbol{v}_2]}{(p_0 + p_3)^2} \underbrace{\frac{\kappa_1 (\kappa_2 + \kappa_3) \langle \boldsymbol{v}_1 \boldsymbol{v}_2 \rangle^3}{\sqrt{2} \langle 0 \boldsymbol{v}_1 \rangle \langle 0 \boldsymbol{v}_2 \rangle}}_{L \cdot \epsilon_+(\boldsymbol{p}_0, \boldsymbol{q})}$$

- From the orders $\rho_1\rho_2$ and $\rho_1\rho_2^2$, we may guess

$$\begin{aligned}\mathcal{A}_{n+1}(0^+1^*...n^*) = -\frac{g^{n-1}\kappa_1(\kappa_2+\cdots+\kappa_n)}{|\kappa_1|^2|\kappa_2|^2\cdots|\kappa_n|^2}\frac{\langle v_1 v_2\rangle^3}{\sqrt{2}\langle v_2 0\rangle\langle 0 v_1\rangle}\\ \times\frac{\langle v_2 0\rangle[0 v_2]}{(p_0+p_n)^2}\cdots\frac{\langle v_2 0\rangle[0 v_2]}{(p_0+p_n+...+p_3)^2}\end{aligned}$$

- **Strategy:**
  - This ansatz is true for $n=2$ and $n=3$
  - BCFW gives a recursion from $\mathcal{A}_n$ to $\mathcal{A}_{n+1}$, that we use in order to prove the validity of the ansatz at order $n+1$, assuming it was true at order n

FROM $\rho_1\rho_2^{n-1}$ TO $\rho_1\rho_2^n$

- Assume ansatz true for $\mathcal{A}_n$
- Use a shift $e^\mu = v_2^\mu$ on the lines $(n-1, n)$
- Two poles: *gluon propagator* and *eikonal propagator* $(p\cdot v_1)^{-1}$

$$\mathcal{A}_{n+1}(0^+1^*2^*\ldots n^*) = \mathcal{A}_3(0^+ - \widehat{I}^-\widehat{n}^*)\frac{i}{K_I^2}\mathcal{A}_n(\widehat{I}^+1^*\ldots\widehat{n-1}^*) + \mathcal{A}_3(0^+ - \widehat{I}^*\widehat{n}^*)\frac{1}{K_I\cdot v_1}\mathcal{A}_n(\widehat{I}^*1^*\ldots\widehat{n-1}^*)$$

$(K_I \equiv p_0 + p_n)$

From $\rho_1\rho_2^{n-1}$ to $\rho_1\rho_2^{n}$ – gluon pole

- Pole at $z_* = (p_n^2 + 2p_0\cdot p_n)/2p_0\cdot v_2$

$$\big|\widehat{I}\big\rangle = \big|0\big\rangle + \frac{\kappa_n}{\big[0v_2\big]}\big|v_2\big\rangle, \qquad \big[\widehat{I}\big| = \big[0\big| + \frac{\overline{\kappa}_n}{\big\langle 0v_2\big\rangle}\big[v_2\big|$$

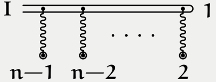
- First term produced by BCFW (using our ansatz for $\mathcal{A}_n$)

$$\begin{aligned}
&\frac{g}{\big|\kappa_n\big|^2}\frac{\big\langle v_2 0\big\rangle\big[0v_2\big]}{(p_0+p_n)^2}\,\mathcal{A}_n(\widehat{I}^+1^*\ldots\widehat{n-1}^*)\\
&= -\frac{g^{n-1}\kappa_1(\kappa_2+\ldots+\kappa_{n-1})}{\sqrt{2}\big|\kappa_1\big|^2\ldots\big|\kappa_n\big|^2}\frac{\big\langle v_1v_2\big\rangle^3\big[0v_2\big]}{\big\langle 0v_2\big\rangle\big(\big\langle v_1 0\big\rangle\big[0v_2\big]+\kappa_n\big\langle v_1v_2\big\rangle\big)}\\
&\qquad\times\frac{\big\langle v_2 0\big\rangle\big[0v_2\big]}{(p_0+p_n)^2}\cdots\frac{\big\langle v_2 0\big\rangle\big[0v_2\big]}{(p_0+p_n+\ldots+p_3)^2}.
\end{aligned}$$

- At the pole, $z_* = (p_0 + p_n)\cdot v_1$, the intermediate $\widehat{K}_I^\mu$ is off-shell, but obeys $v_1 \cdot \widehat{K}_I = 0 \implies$ valid momentum for a proton source

- The second term produced by BCFW contains

$$\mathcal{A}_n(\widehat{I}^* 1^* \ldots \widehat{n-1}^*) =$$



$$+\cdots$$

$$= \frac{(-g)^{n-2}}{|\kappa_2|^2 \ldots |\kappa_{n-1}|^2} \frac{1}{(K_I + p_{n-1})\cdot v_1} \cdots \frac{1}{(K_I + p_{n-1} + \ldots + p_3)\cdot v_1}$$

- Complete second term (written for $v_1^\mu = \delta^{\mu+}$),

$$-\frac{g^{n-1}\kappa_1\overline{\kappa}_1\kappa_n}{\sqrt{2}\big|\kappa_1\big|^2 \ldots \big|\kappa_n\big|^2} \frac{\big\langle v_1 v_2\big\rangle^3 \big[v_1 v_2\big]}{\big\langle v_2 0\big\rangle\big\langle 0 v_1\big\rangle\big(\big\langle v_1 0\big\rangle\big[0 v_2\big] + \kappa_n\big\langle v_1 v_2\big\rangle\big)}$$
$$\times \frac{-1}{p_0^- + p_n^-} \cdots \frac{-1}{p_0^- + p_n^- + \ldots + p_3^-}$$

From $\rho_1\rho_2^{n-1}$ to $\rho_1\rho_2^{n}$ – Total

- The two terms contain different denominators, e.g. $(p_0+p_n)^2$ versus $p_0^- + p_n^- \quad \Longrightarrow$ How can we combine them?

- The p_i^- dependence comes only from these denominators (the nucleus source is $\sim \delta(x^+) \quad \Longrightarrow \quad \rho_2^{(0)}(p_i)$ is independent of p_i^-)

- Anticipating the integration over the p^-'s, we can identify

$$\frac{1}{\underbrace{p_0^- + p_n^-}_{\text{pole at } p_n^- = -p_0^-}} \quad \Leftrightarrow \quad \frac{2p_0^+}{2(p_0^+ + \underbrace{p_n^+}_{=0})(p_0^- + p_n^-) - \underbrace{(\boldsymbol{p}_{0\perp}+\boldsymbol{p}_{n\perp})^2}_{\text{irrelevant shift of } p_n^- \text{ at the pole}}}$$

$$= \frac{\langle 0\boldsymbol{v}_2\rangle\,[0\boldsymbol{v}_2]}{(p_0+p_{n+1})^2}$$

- Momentum conservation:

$$\langle v_1| \Bigg[\underbrace{|0\rangle[0|}_{\overline{P}_0} + \underbrace{\frac{q_1\cdot p_1}{q_1\cdot v_1}|v_1\rangle[v_1| - \kappa_1 \frac{|v_1\rangle[q_1|}{[v_1 q_1]} + \overline{\kappa}_1 \frac{|q_1\rangle[v_1|}{\langle q_1 v_1\rangle}}_{\overline{P}_1}$$
$$+ \sum_{i\ge 2} \underbrace{\frac{q_i\cdot p_i}{q_i\cdot v_2}|v_2\rangle[v_2| - \kappa_i \frac{|v_2\rangle[q_i|}{[v_2 q_i]} + \overline{\kappa}_i \frac{|q_i\rangle[v_2|}{\langle q_i v_2\rangle}}_{\overline{P}_i} \Bigg] |v_2] = 0$$

i.e.,

$$\langle v_1 0\rangle[0 v_2] = \overline{\kappa}_1 [v_1 v_2] - (\kappa_2 + \ldots + \kappa_n)\langle v_1 v_2\rangle$$

- The sum of the two terms gives the ansatz for $\mathcal{A}_{n+1}$

Summary

- On-shell methods (spinor-helicity formalism, BCFW recursion) can be generalized to handle the amplitudes with off-shell external gluons that arise in CGC calculations

- Does not require gauge fixing

- Provides recursion relations among gauge invariant amplitudes

- Simpler than solving YM equations,
  with the added bonus of being gauge agnostic

- **Possible extensions:**
  - Other tree level processes: $g + A \to gg + A$, $g + A \to ggg + A$
  - Amplitudes with $q\bar{q}$ pairs (for massless quarks)
  - *Generalized unitarity:* break down 1-loop amplitudes into products of up to four (five in dim. reg.) tree level amplitudes

Behavior at $|z| \to \infty$

Naive power counting

- Consider the shift $e^\mu = \frac{1}{2}\langle \boldsymbol{v}|\overline{\sigma}^\mu|\boldsymbol{v}] = v^\mu$ applied to two off-shell gluons produced by sources of direction v^μ
- With this shift, the external off-shell propagators do not depend on z
- Counting rules:
    - Inner propagator $\sim z^{-1}$
    - 3-gluon vertex $\sim z$
    - 4-gluon vertex $\sim z^0$
- Worst case: 3-gluon vertices only, one more vertex that internal propagators $\Longrightarrow\ z$

Improved power counting

- The previous counting gives the graph-by-graph behavior.

  Actual behavior may be better if cancellations happen

- Write $A^\mu \equiv \mathcal{A}^\mu + a^\mu$, with a^μ a hard component that carries the momentum $\sim z$ $(\partial a \sim z)$, and a soft part $\mathcal{A}^\mu$ $(\partial\mathcal{A} \sim z^0)$,

$$\mathcal{D}^\mu = \partial^\mu - ig\mathcal{A}^\mu, \quad -ig\,\mathcal{F}^{\mu\nu} = [\mathcal{D}^\mu, \mathcal{D}^\nu]$$

$$\mathcal{L} = -\mathrm{tr}\left((\mathcal{D}^\mu a^\nu)(\mathcal{D}_\mu a_\nu)\right) + \mathrm{tr}\left((\mathcal{D}_\mu a^\mu)(\mathcal{D}_\nu a^\nu)\right) + ig\,\mathrm{tr}\left(\mathcal{F}_{\mu\nu}[a^\mu, a^\nu]\right) + \cdots$$

- Background field gauge, $\mathcal{D}_\mu a^\mu = 0$,

$$\mathcal{L}_{\text{gauge fixed}} = -\mathrm{tr}\left((\mathcal{D}^\mu a^\nu)(\mathcal{D}_\mu a_\nu)\right) + ig\,\mathrm{tr}\left(\mathcal{F}_{\mu\nu}[a^\mu, a^\nu]\right) + \cdots$$

  - First term: extended Lorentz symmetry (a^ν and $\mathcal{D}_\mu$ can be transformed independently), contains derivatives of a^ν
  - Second term: antisymmetric in the hard fields, no derivatives of the hard fields (i.e., no z)

Improved power counting

- Generic form of amputated graphs with two hard fields:

$$\mathcal{M}^{\mu\nu} = \underbrace{(c_1 z + c_0 + c_{-1} z^{-1} + \cdots)\, g^{\mu\nu}}_{\text{first term only}} + A^{\mu\nu} + R^{\mu\nu}$$

- $A^{\mu\nu}$ comes from a single insertion of $[a^\mu, a^\nu]\mathcal{F}_{\mu\nu}$ (antisymmetric in (μ,ν), order $\sim z^0$)
- $R^{\mu\nu}$ contains at least two 4-gluon vertices (order $\sim z^{-1}$)

- To obtain an amplitude, contract $\mathcal{M}^{\mu\nu}$ with the propagators of two off-shell gluons and with the directions of the sources,

$$\begin{aligned}
&(v^\alpha G_{\alpha\mu})\mathcal{M}^{\mu\nu}(G_{\nu\beta}v^\beta) \propto v_\mu \mathcal{M}^{\mu\nu} v_\nu \\
&= (c_1 z + c_0 + c_{-1} z^{-1} + \cdots)\, \underbrace{v^2}_{=0} + \underbrace{A^{\mu\nu} v_\mu v_\nu}_{=0} + \underbrace{R^{\mu\nu} v_\mu v_\nu}_{\sim z^{-1}}
\end{aligned}$$

Solution in Landau gauge

$$A_a^\mu(p) = A_{1a}^\mu(p) + \frac{ig}{p^2}\int\frac{d^2\mathbf{p}_{1\perp}}{(2\pi)^2}\Big[C^\mu(U-1)_{\mathbf{p}_{2\perp}} + \underline{D^\mu(V-1)_{\mathbf{p}_{2\perp}}}\Big]_{ab}\frac{\rho_{1b}^{(0)}(\mathbf{p}_{2\perp})}{\mathbf{p}_{2\perp}^2}$$

$$U(y^+,x^+;\mathbf{x}_\perp) = P\exp ig\int_{x^+}^{y^+} dz^+\, A_2^-(z^+,\mathbf{x}_\perp),$$

$$V(y^+,x^+;\mathbf{x}_\perp) = P\exp\frac{ig}{2}\int_{x^+}^{y^+} dz^+\, A_2^-(z^+,\mathbf{x}_\perp)$$

$$A_{1a}^\mu(p) = g\delta^{\mu+}\frac{\rho_{1a}^{(0)}(\mathbf{p}_\perp)}{\mathbf{p}_\perp^2},\quad A_{2a}^\mu(p) = g\delta^{\mu-}\frac{\rho_{2a}^{(0)}(\mathbf{p}_\perp)}{\mathbf{p}_\perp^2}$$

$$C^+ = \frac{\mathbf{p}_{1\perp}^2}{p^-},\quad C^- = \frac{\mathbf{p}_\perp^2-\mathbf{p}_{2\perp}^2}{p^+},\quad C^i = -2\mathbf{p}_{1\perp}^i$$

$$D^+ = -2p^+,\quad D^- = 2p^- - 2\frac{\mathbf{p}_\perp^2}{p^+},\quad D^i = -2\mathbf{p}_\perp^i$$

- D^μ is longitudinal when p^μ is on-shell
  $\implies$ V does not contribute to gluon production

- Quark-Antiquark production:
  - D^-/p^2 contains a term that does not go to zero when $p^- \to \infty$
    Corresponds to a singular term $\sim \delta(x^+)$ in coordinate space
  - This term leads to $Q\bar{Q}$ production inside the nucleus shockwave
    $\implies$ does quark production depend on V?
  - **NO**, but V cancels thanks to a rather weird identity,

$$\frac{ig}{2}\int_{-\infty}^{+\infty} dx^+ \; U(+\infty, x^+; \boldsymbol{x}_\perp) A_2^-(x^+, \boldsymbol{x}_\perp) \color{red}{V(x^+, -\infty; \boldsymbol{x}_\perp)}$$
$$= U(+\infty, -\infty; \boldsymbol{x}_\perp) - \color{red}{V(+\infty, -\infty; \boldsymbol{x}_\perp)}$$