

Approximate Methods to model the non-linear growth of cosmological large scale structure

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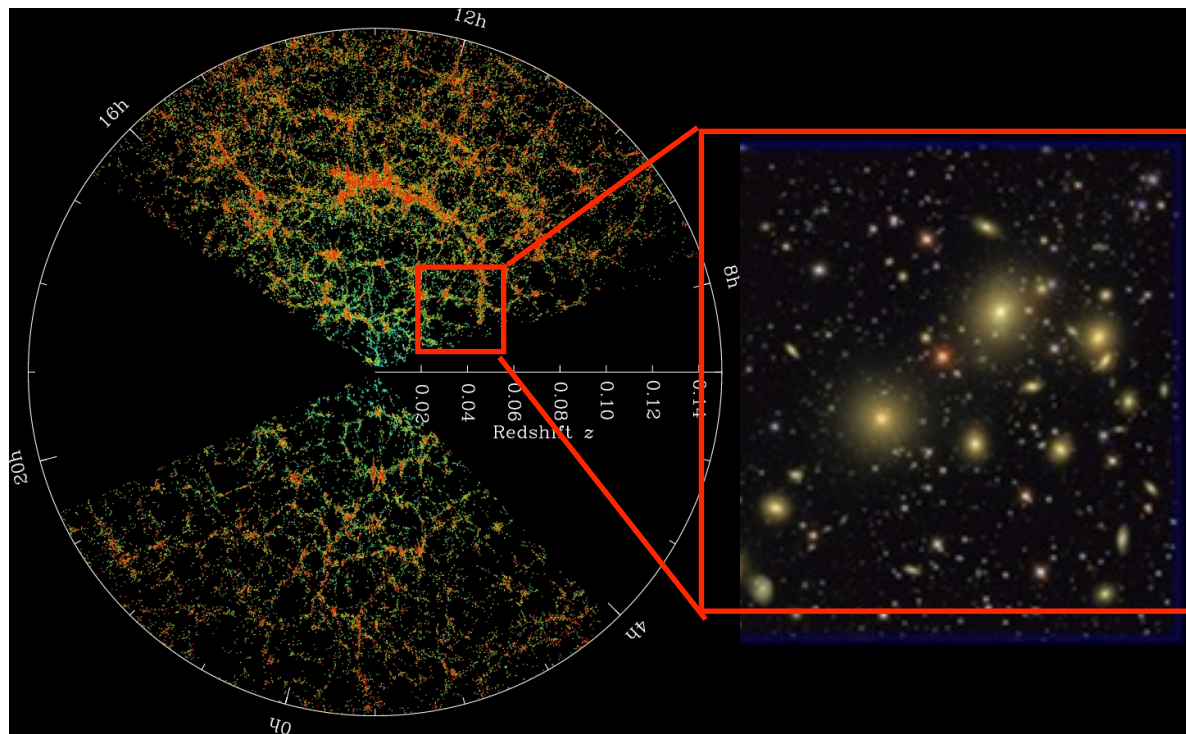


IMSc Cosmic Connections - 2nd Edition

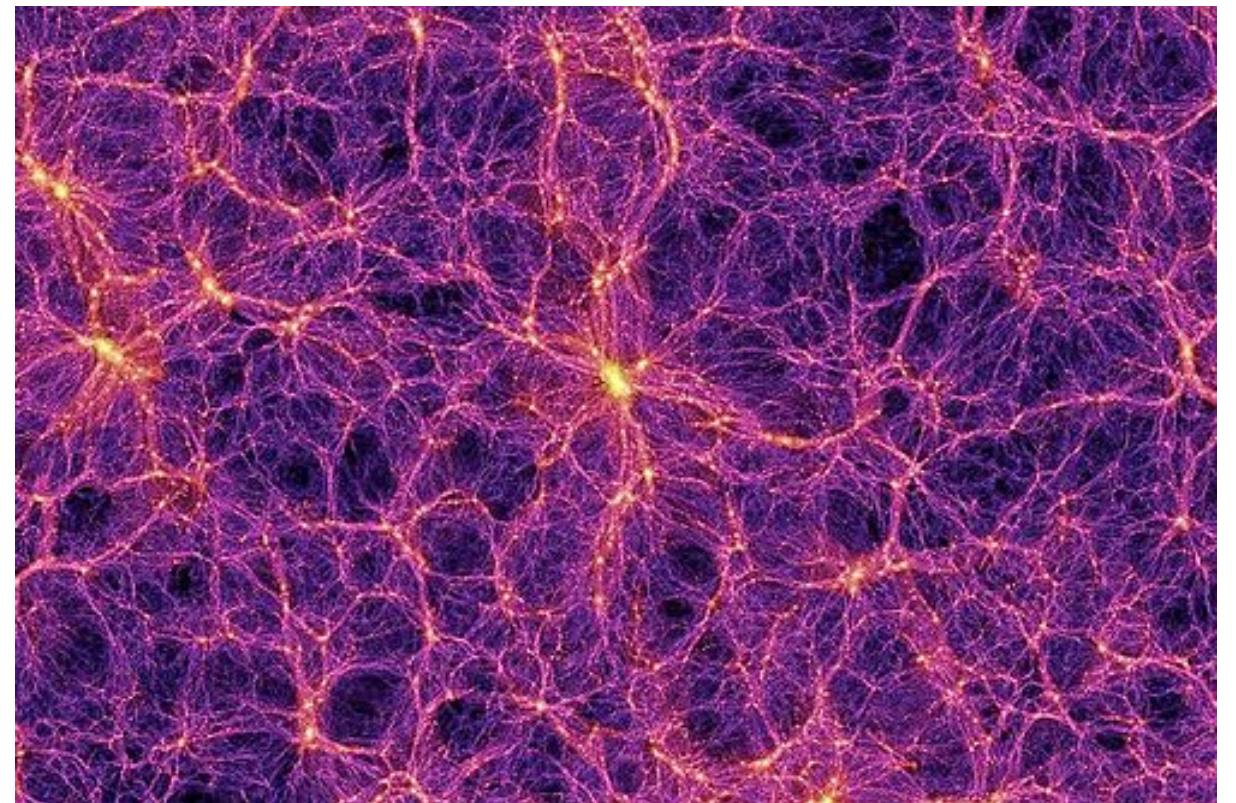
7th Sept. 2026



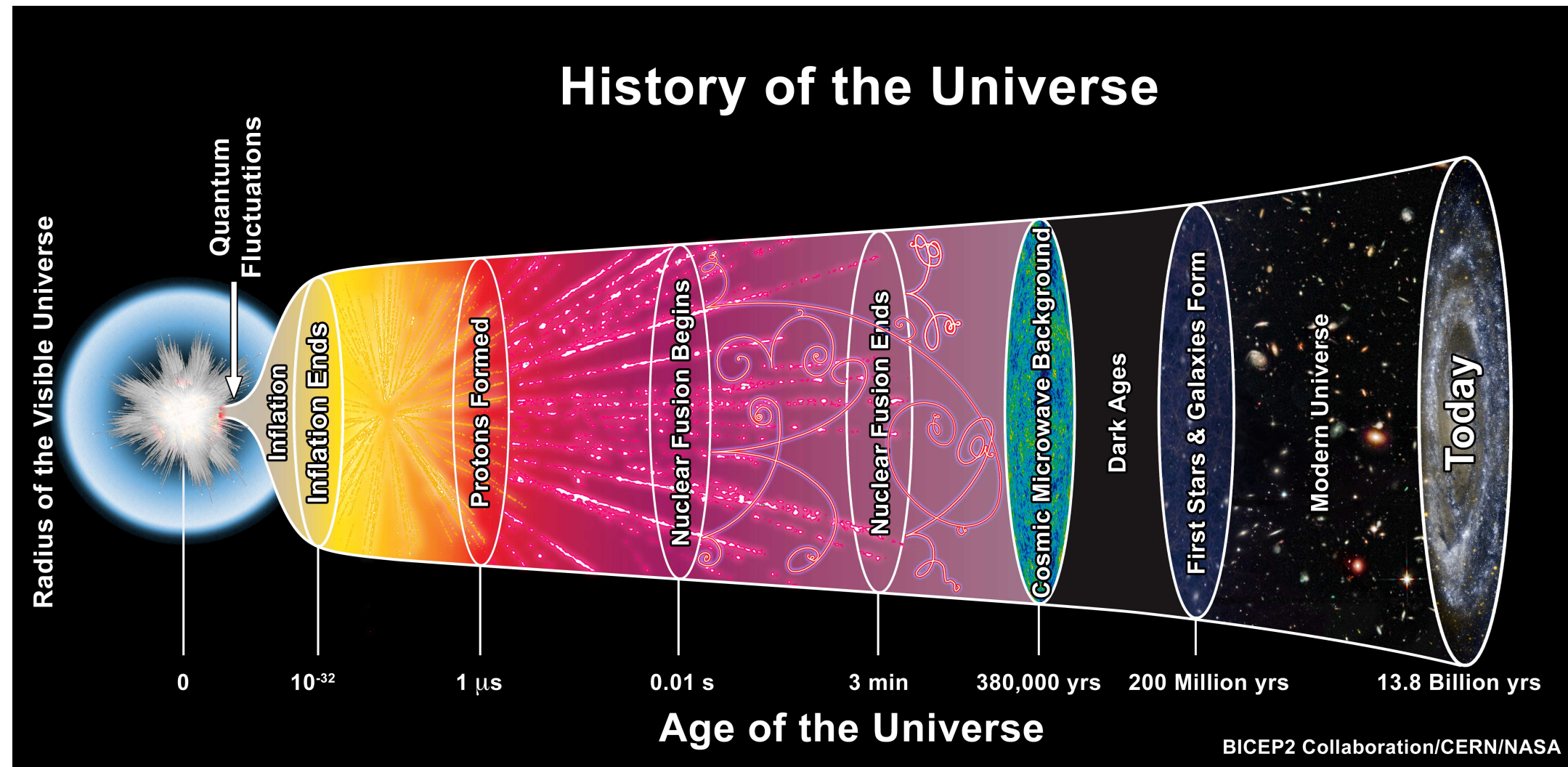
Why study Large Scale Structure



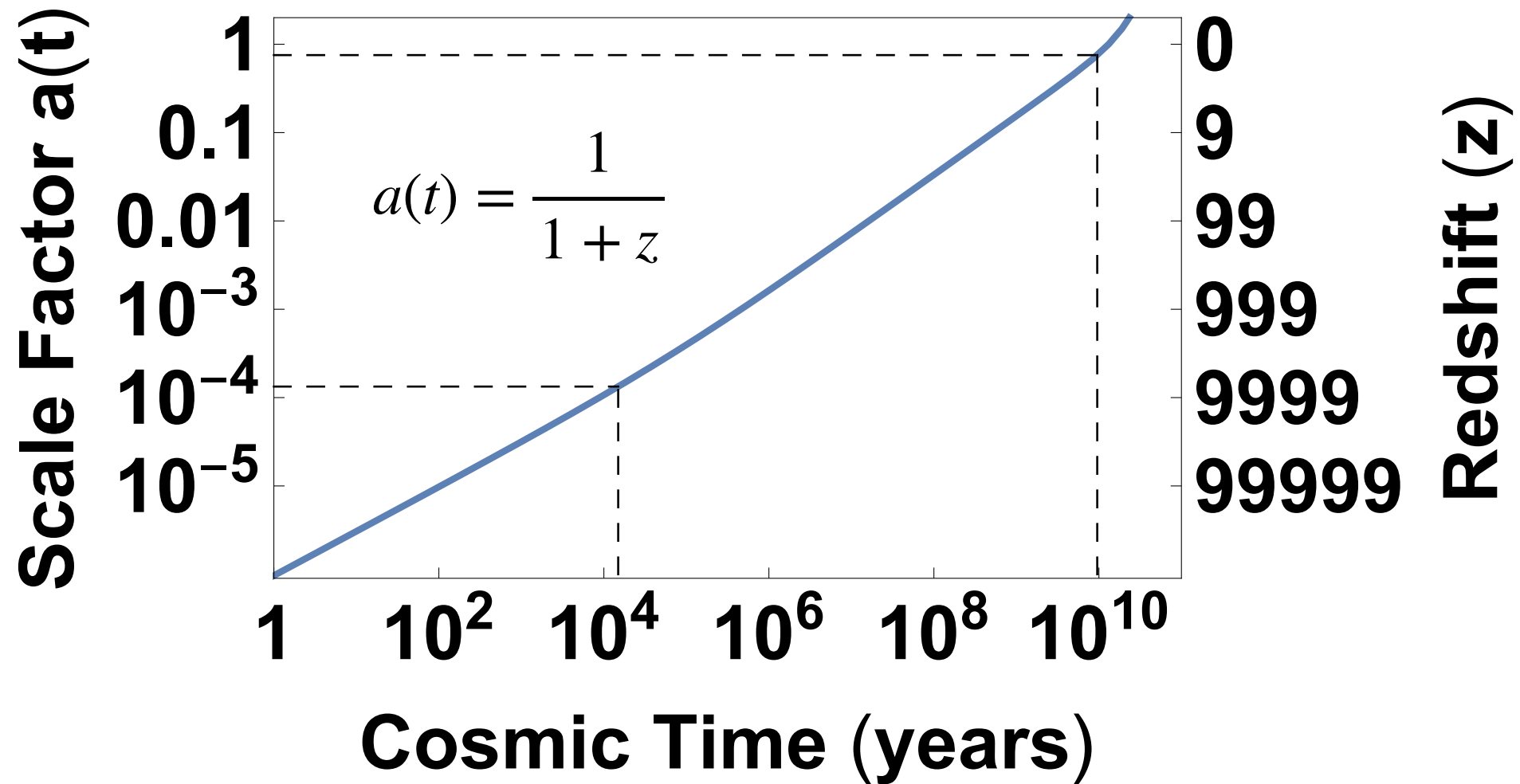
- Euclid
- DESI
- Vera Rubin Observatory (LSST)
- Nancy Grace Roman Space Telescope
- 4MOST
- SPHEREx
- SDSS-V
- SKA
- SPT
- SO
- And more



Why study Large Scale Structure



Time Units



$$H^2 = \frac{\dot{a}^2}{a^2} = \frac{8\pi G}{3} \left(\frac{\rho_{r,0} a_0^4}{a^4} + \frac{\rho_{m,0} a_0^3}{a^3} + \rho_{\Lambda,0} + \frac{K}{a^2} \right)$$

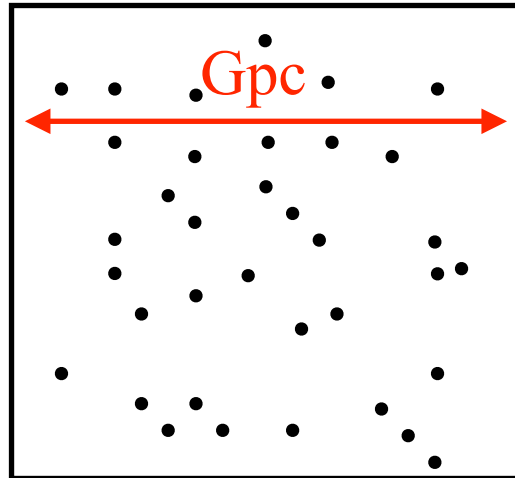
Outline of the rest of the talk

What are the equations ?

What are the tools to solve these equations ?

What can you do with these tools ?

The evolution equations



- Fluid Limit: the mean free path is much smaller than the overall scale of interest.
(intergalactic distance $\sim 1 \text{ Mpc} \ll \text{Gpc}$ Hubble scale)
- Single stream limit
- Dark matter is cold and is the only clustering component.

Newtonian hydrodynamic equations for a compressible, inviscid, self-gravitating fluid. These equations are valid on **sub-horizon scales for standard gravity models - including Λ CDM** (Sub-Horizon Approximation)

Continuity Equation

$$\left. \frac{\partial \rho_{tot}}{\partial t} \right|_r + (\mathbf{u} \cdot \nabla_r) \rho_{tot} = -\rho_{tot} (\nabla_r \cdot \mathbf{u})$$

Euler Equation

$$\left. \frac{\partial \mathbf{u}}{\partial t} \right|_r + (\mathbf{u} \cdot \nabla_r) \mathbf{u} = -\nabla_r \Psi_{tot} \quad (\mathbf{u} = \dot{\mathbf{r}})$$

**Force law:
Newton's law of
Gravity**

$$\nabla_x^2 \Psi_{tot} = 4\pi G \rho_{tot}$$

Initial conditions: initial density and velocity fields; distribution given by inflation.

Linear Theory Takeaways

When fluctuations are small, the evolution equations can be linearized in terms of the variables

$$\text{Fractional Overdensity} \quad \delta(\mathbf{r}, t) = \frac{\rho_{tot}(\mathbf{r}, t)}{\bar{\rho}(t)} - 1$$

$$\text{Peculiar velocity} \quad \mathbf{v} = \mathbf{u} - H\mathbf{r}$$

$$\text{Peculiar potential} \quad \Psi = \Psi_{tot} - \bar{\Psi}$$

$$\delta'' + \left(2 + \frac{H'}{H}\right) \delta' - \frac{3}{2} \Omega_m(a) \delta = 0. \quad \text{With initial conditions } \delta(a_i), \delta'(a_i)$$

- Standard second order equation for δ - cosmology dependence through $H(a)$
- The growth rate defined as $f = \frac{d \ln \delta}{d \ln a}$ is one of the primary quantities that observations try to constrain. (e.g. DESI measurements Qin *et al* 2025, EUCLID Collaboration 2024)
- Breakdown when $\delta \sim 1$

Linear Theory Takeaways

General Solution

$$\delta(a) = c_+ \delta_+(a) + c_- \delta_-(a)$$

For a pure matter cosmology, $\Omega_m = 1$
analytic solutions

$$\delta_+(a) = a$$

Growing mode

$$\delta_-(a) = a^{-3/2}$$

Decaying mode

*Impose: No
perturbations
at the big bang i.e.,
set $c_- = 0$*

Density and velocity perturbations are coupled.

$$\frac{\nabla_r \cdot \mathbf{v}}{H} = -f\delta = -\Omega_m^\gamma \delta$$

Continuity Eq. No decaying mode

Growth index

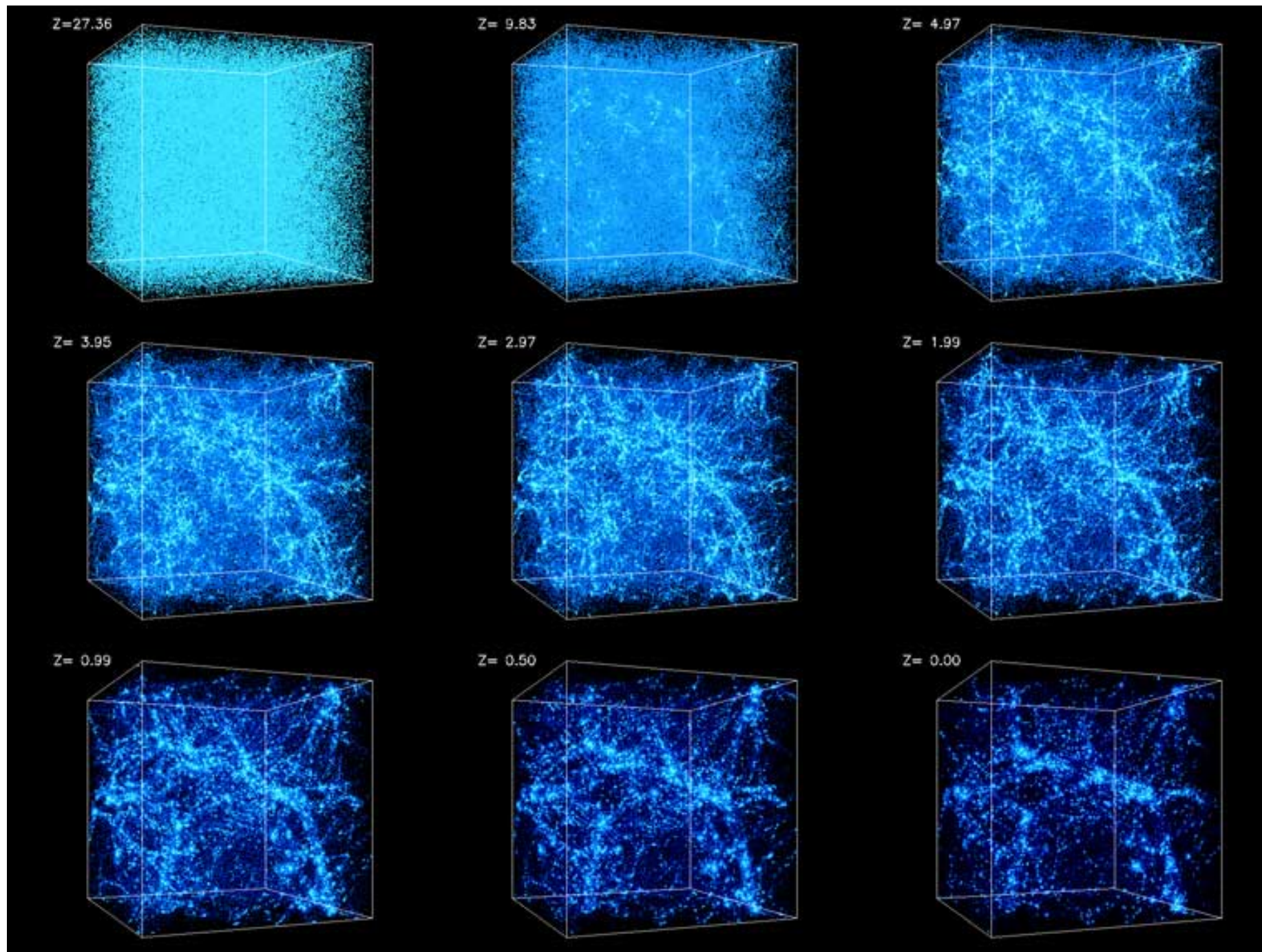
Many observational efforts to constrain it.

Guzzo *et al* 2008
Majerotto *et al* 2012
Amendola *et al.* 2018
Alam *et al* 2021

Initial Velocity field is proportional to the acceleration field i.e $\mathbf{v} \propto \nabla \Phi$
Linear Density-Velocity Divergence relation

Close-to-exact tool to solve the non-linear regime

N-body simulations



Pros:

- Almost exact

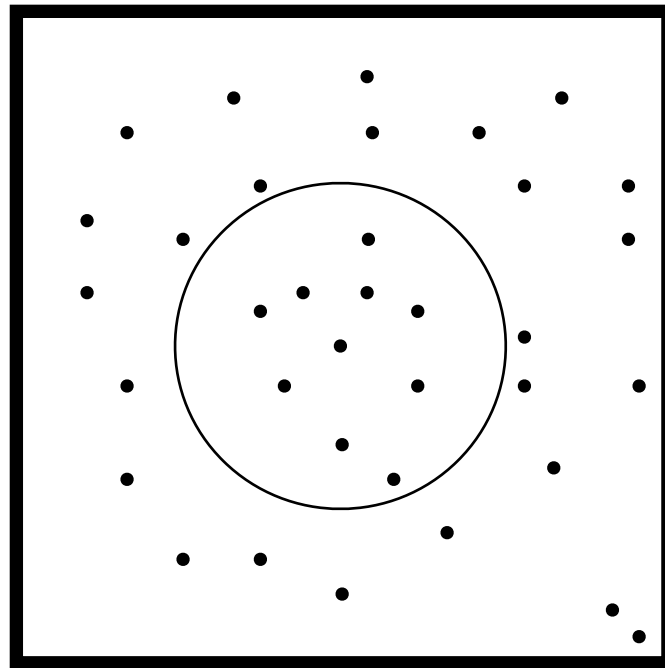
Cons:

- Discrete particle representation
- Start at relatively low redshifts

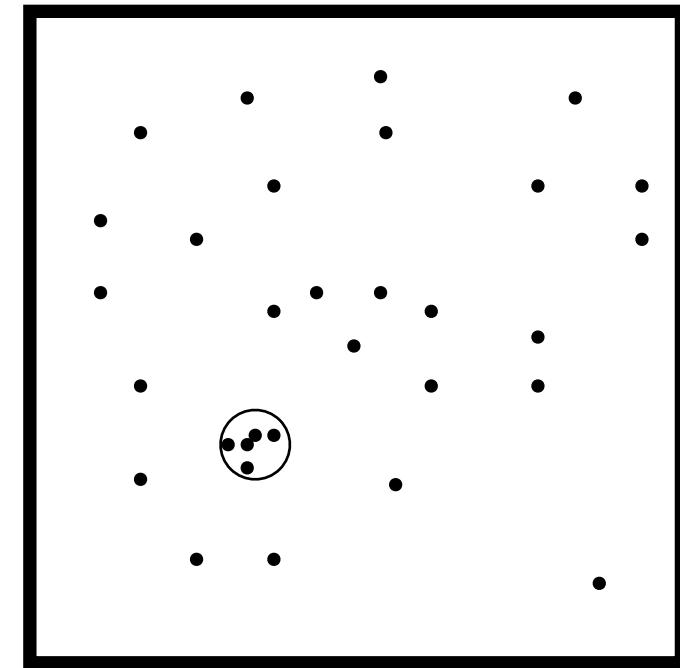
Kravtsov UChicago Box size = 43 Mpc; 1 week on 16 processors: 2 million particles
GADGET: Millennium-XXL Project 1 billion particles in 4Gpc

Analogy for shot-noise

Area of the square = 1
Actual Area of the circle = 0.5



Area of the square = 1
Actual Area of the circle = 0.05



Example: estimate the area of a circle using Monte Carlo method

$N = 100$

$N_{\text{circle}} \sim 50$

Estimated Area of circle = 0.5 (for unit square)

Error ~ 1 particle (49 or 51) in 100

Area = 0.49 or 0.51

Error in estimation is 0.5 ± 0.01

Relative Error $\sim 2\%$

$N = 100$

$N_{\text{circle}} \sim 5$

Estimated Area of circle = 0.05

Error ~ 1 particle (4 or 6)

Area = 0.04 or 0.06

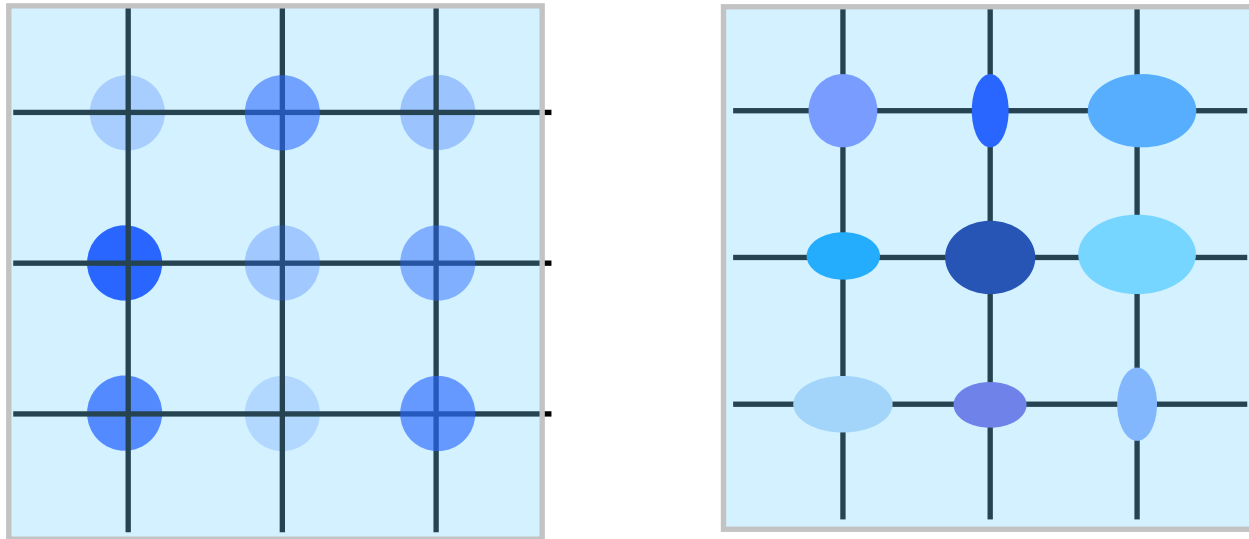
Error in estimation is 0.05 ± 0.01

Relative Error $\sim 20\%$

**At early epochs densities are small and hence the earlier the start time, the higher the number of particles required.
Usually simulations start at $z \sim 99$.**

Approximate Tools to solve the non-linear regime

Spherical or Triaxial Collapse



Patch of the universe

Pros:

- Simple ODEs

Cons:

- Non-local

1. SN-G **MNRAS** (2013) 428
2. SN-G and Singhal, **MNRAS** (2016) 457
3. SN-Gand Arya, **MNRAS** (2018) 475
4. A. Mondal and S N-G (**MNRAS** 2020) 498
5. S N-G and S. Choudhury (**MNRAS** 2022)
6. S N-G and Reddy Vaka (**JCAP** 2026)

Perturbation Theory

Eulerian
Frame

Lagrangian
Frame

Pros:

- Analytic
- Smooth description
- Non-local

Cons:

- Higher orders cumbersome
- Convergence issues

1. SN-G and Chernoff, **MNRAS** (2011) 410
2. SN-Gand Chernoff, **MNRAS** (2013) 431
3. SN-G and Bhattacharjee **Frontiers in Physics** Vol 6
4. Das and SN-G(in preparation)

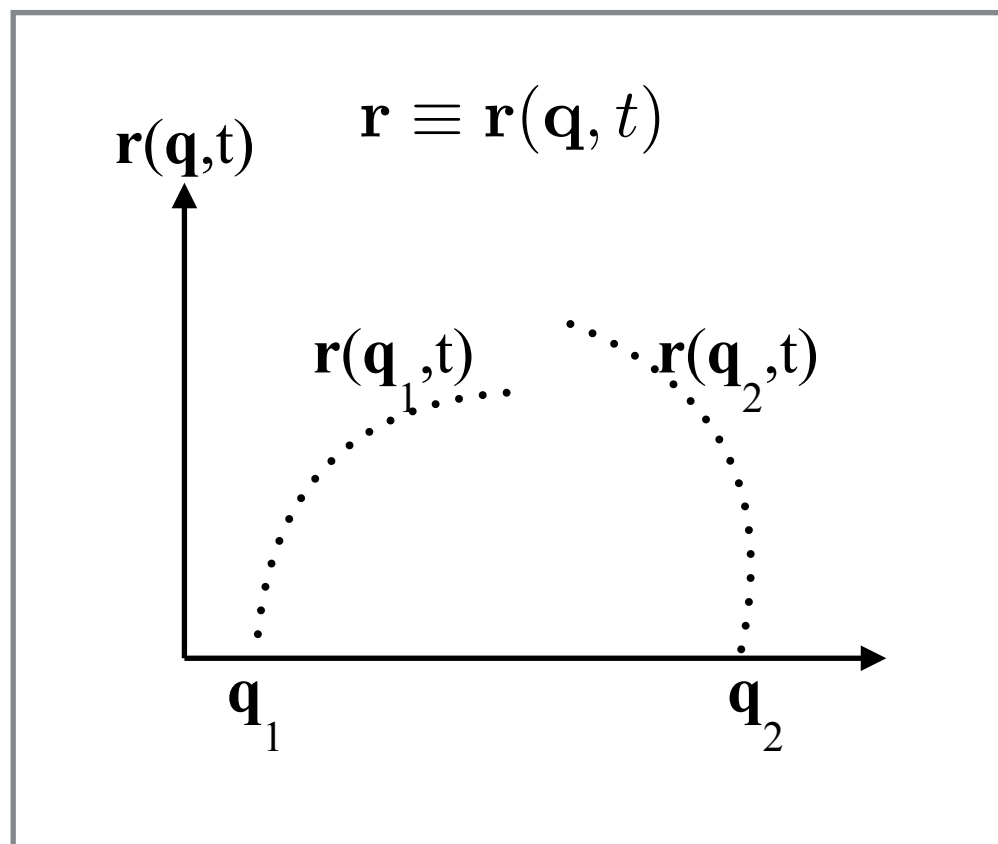
PART I - Lagrangian Perturbation Theory

Fluid description in the Lagrangian frame

Eulerian vs. Lagrangian frames of a fluid

Eulerian frame: Density and velocity are the dependent variables solved as a function of physical position r and time t .

Lagrangian Frame: displacement is the main variable, solved as a function of initial position and time.



Sub-horizon,
Newton's laws $\ddot{\mathbf{r}} = -\nabla_r \Psi$

$$\nabla_r^2 \Psi = 4\pi G \rho$$

$$\mathbf{r} = \mathbf{r}(\mathbf{q}, t)$$

$$\rho(\mathbf{q}, t) = \frac{\rho_{in} J_{in}}{J(\mathbf{q}, t)}$$

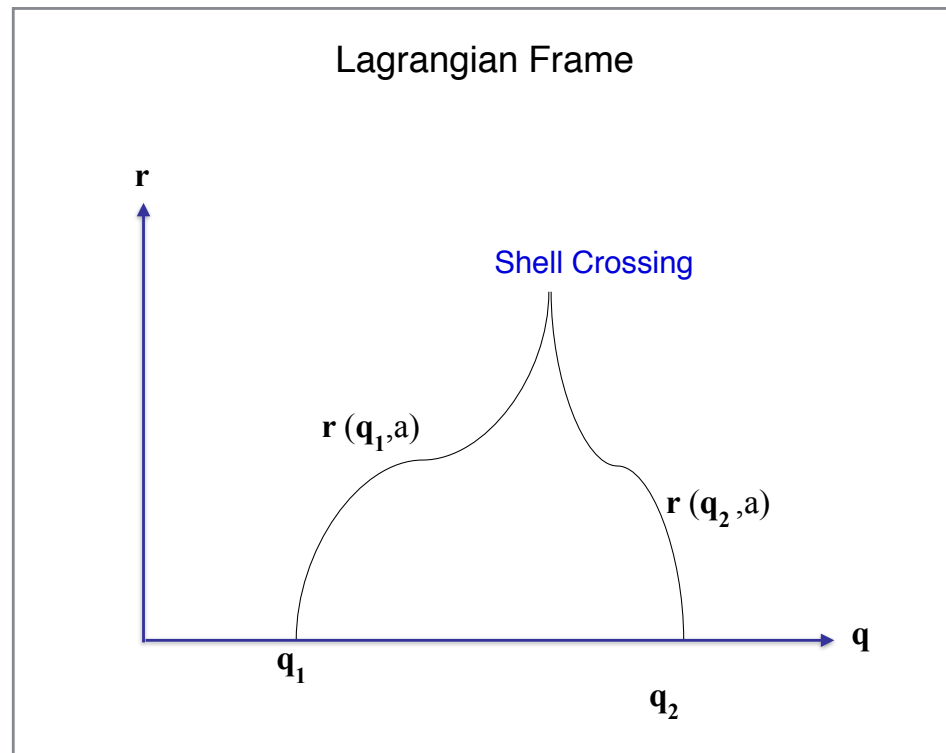
where $J(\mathbf{q}, t) = \det \left(\frac{\partial r_i}{\partial q_j} \right)$

Initial conditions: initial density and velocity fields are known.

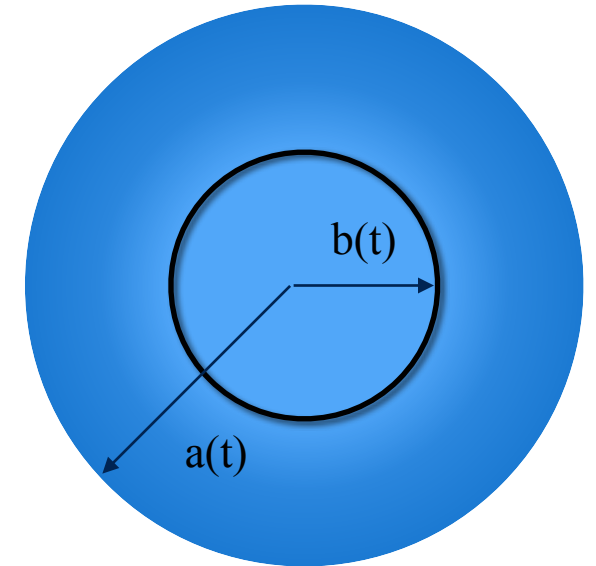
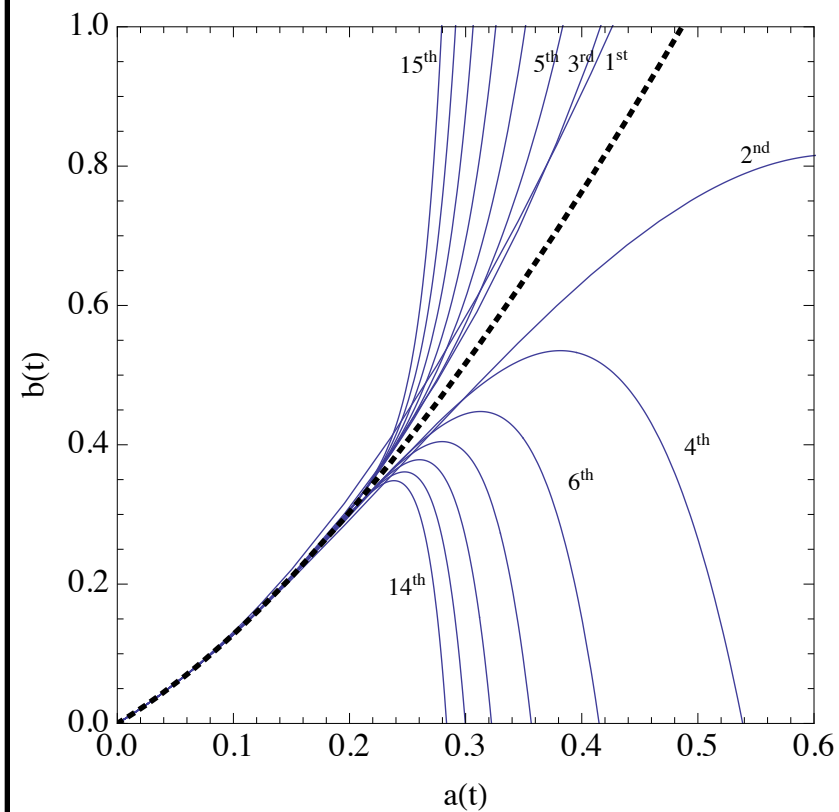
Perturbative Ansatz:

$$\mathbf{r} = a(t)\mathbf{q} + \mathbf{s}(\mathbf{q}, t) = a(t)\mathbf{q} + \sum_{i=1}^n \mathbf{s}^i(\mathbf{q}, t)\epsilon^i = a(t)\mathbf{q} + \sum_{i=1}^n \{ \mathbf{s}^{i,L}(\mathbf{q}, t) + \mathbf{s}^{i,T}(\mathbf{q}, t) \} \epsilon^i$$

Problems with LPT



1. Breaks down when trajectories cross:
shell crossing regime



$a(t)$: scale factor of background

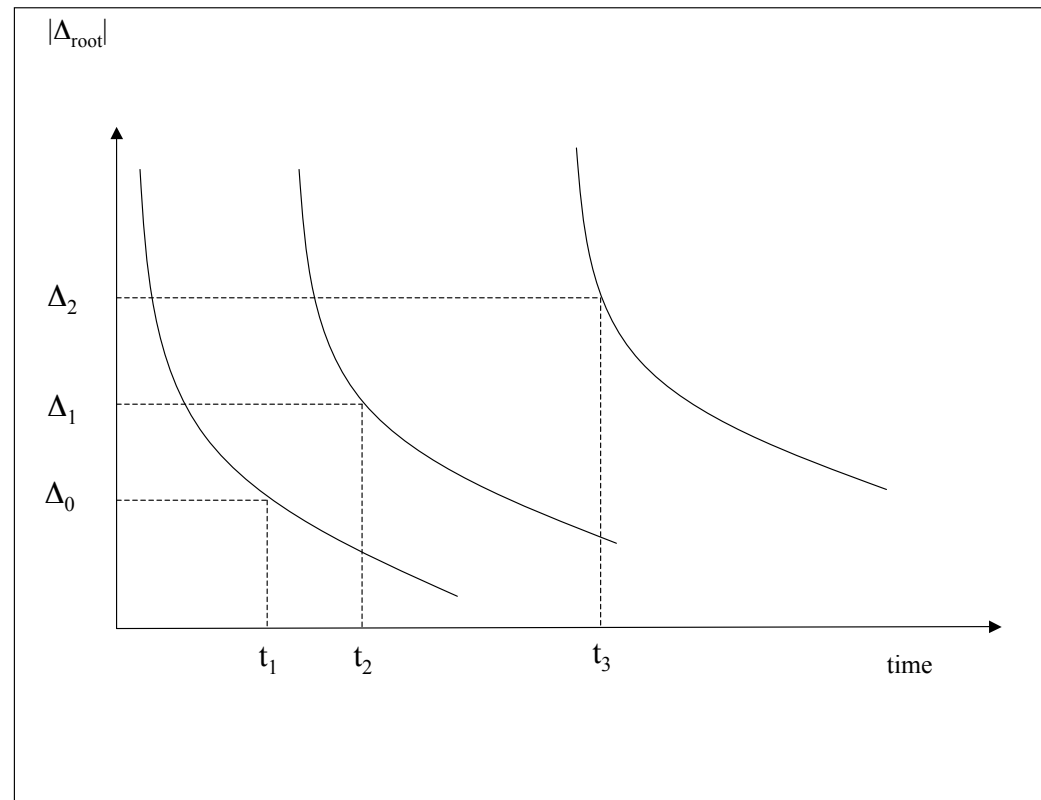
$b(t)$: scale factor of spherical void.

2. Does **not converge** in spherical **voids**.
First pointed out by Sahni and Shandarin MNRAS 1996.

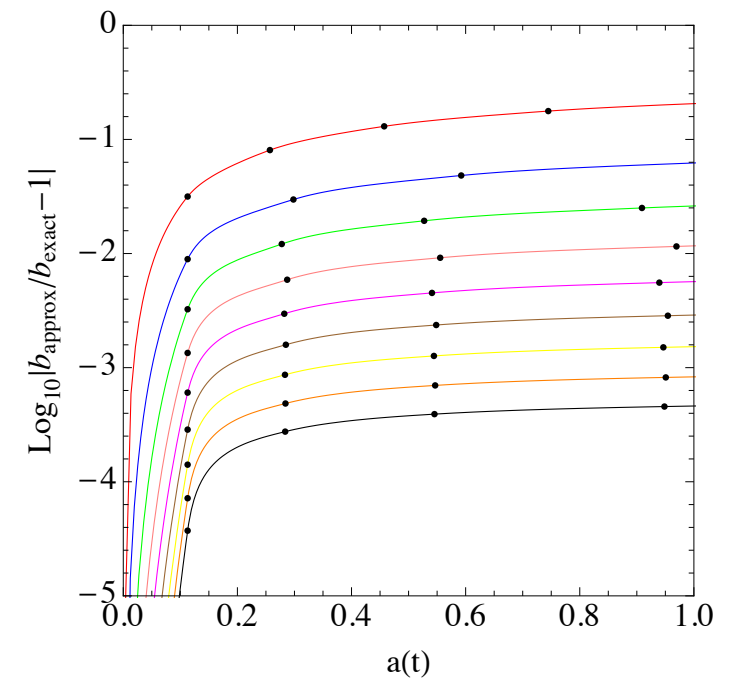
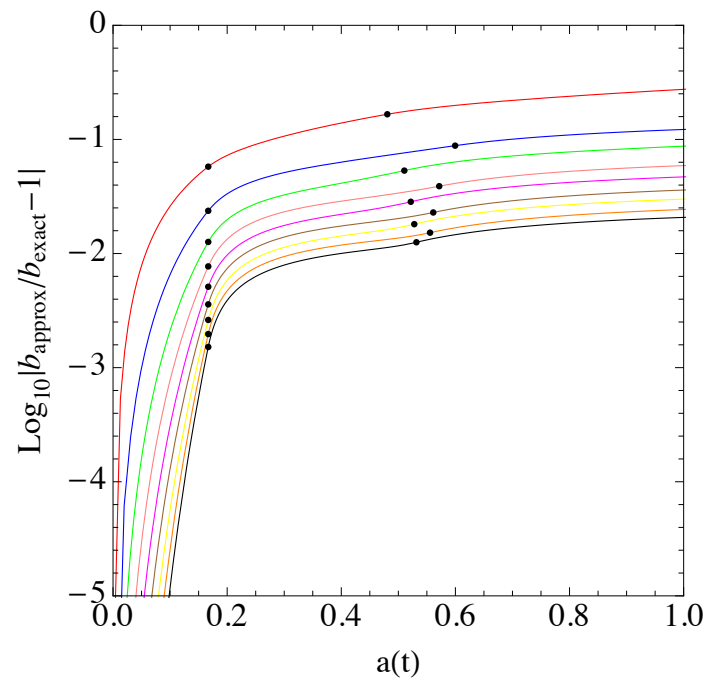
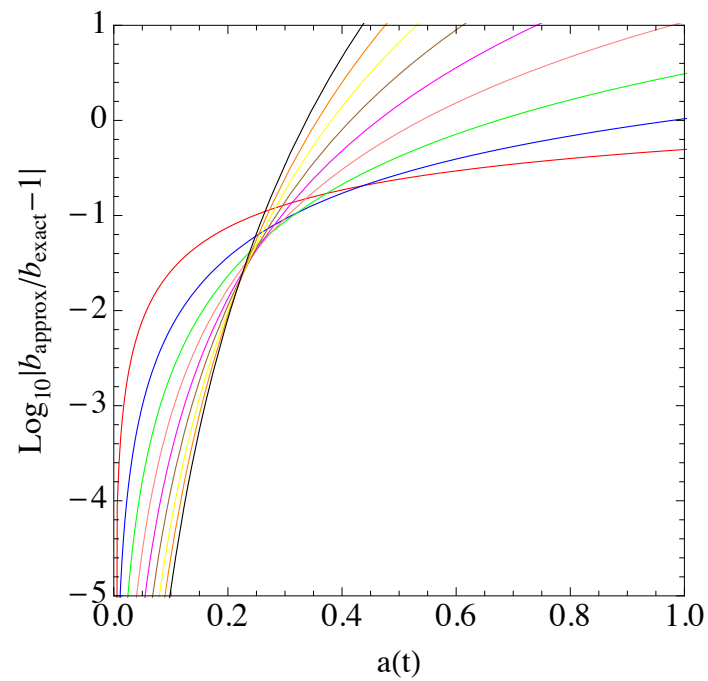
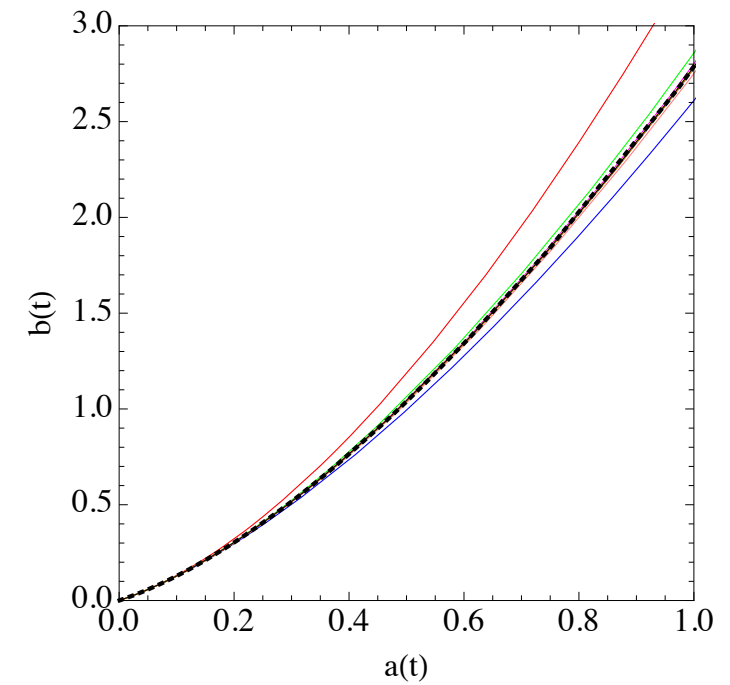
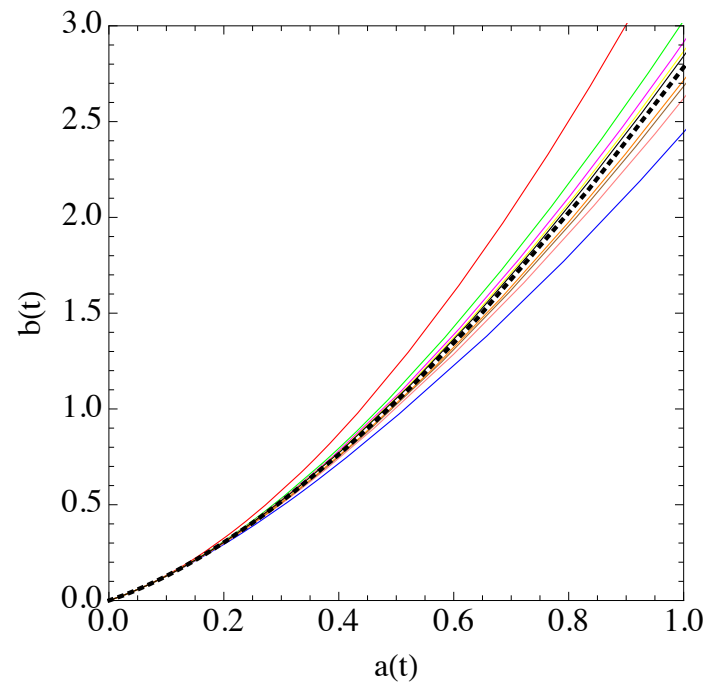
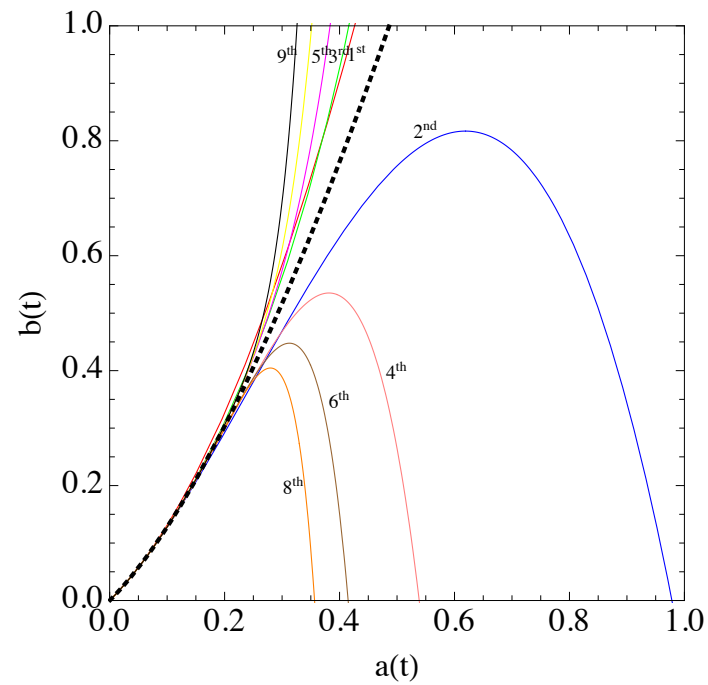
1. Nadkarni-Ghosh and Chernoff, **MNRAS** (2011) 410
2. Nadkarni-Ghosh and Chernoff, **MNRAS** (2013) 428

Why does this happen and what to do about it

- Examination of the spherical problem shows that there are closed 'mirror models' for the open model, which causes this break down.
- Proposed the method of **LPT Re-expansions** - iteratively evolve the system, each step subject to a pre-determined time of validity. And we formally proved that the method converges



Demonstration of the scheme

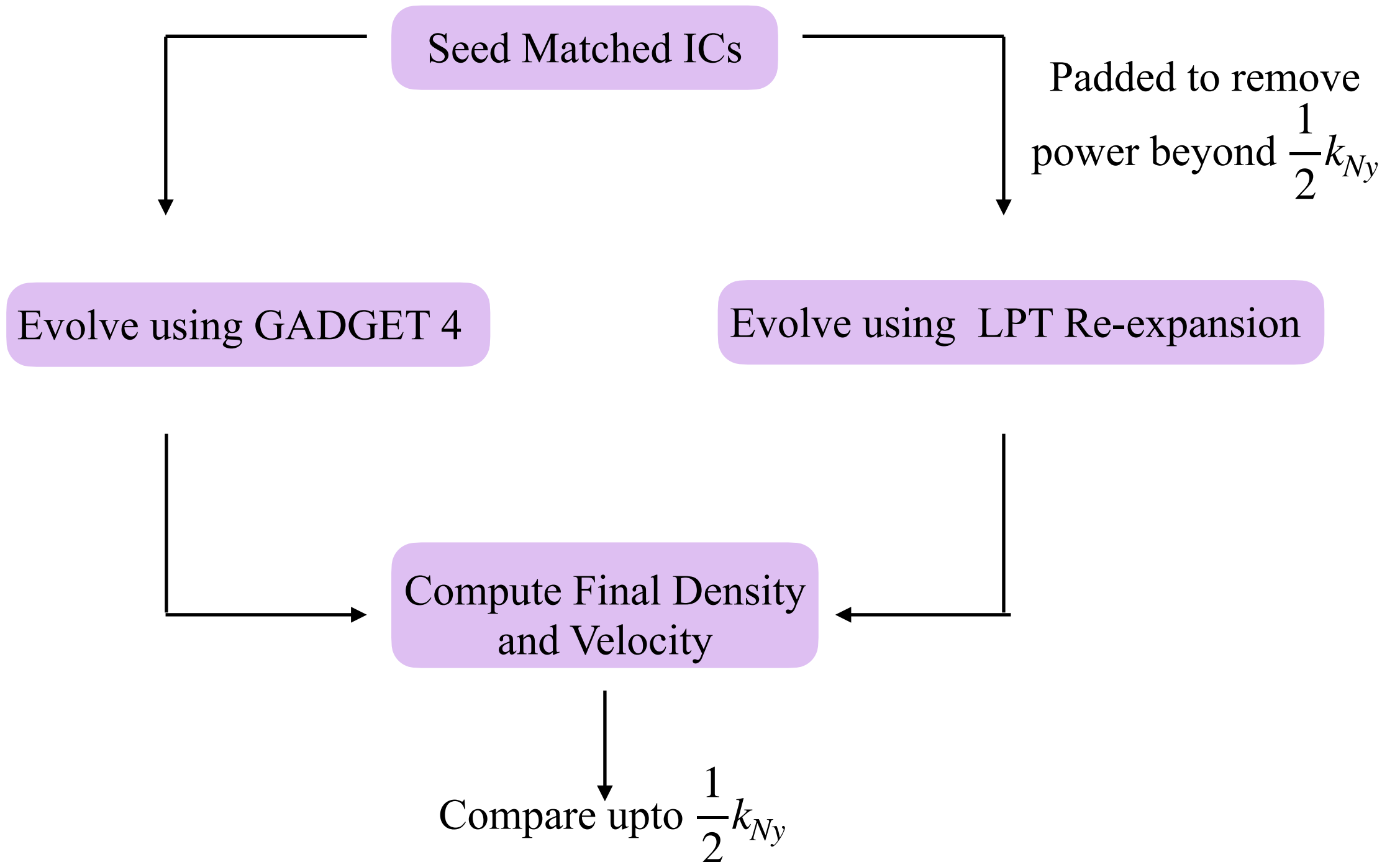


One step

Three steps

Five steps

Comparison to seed matched GADGET-4 output

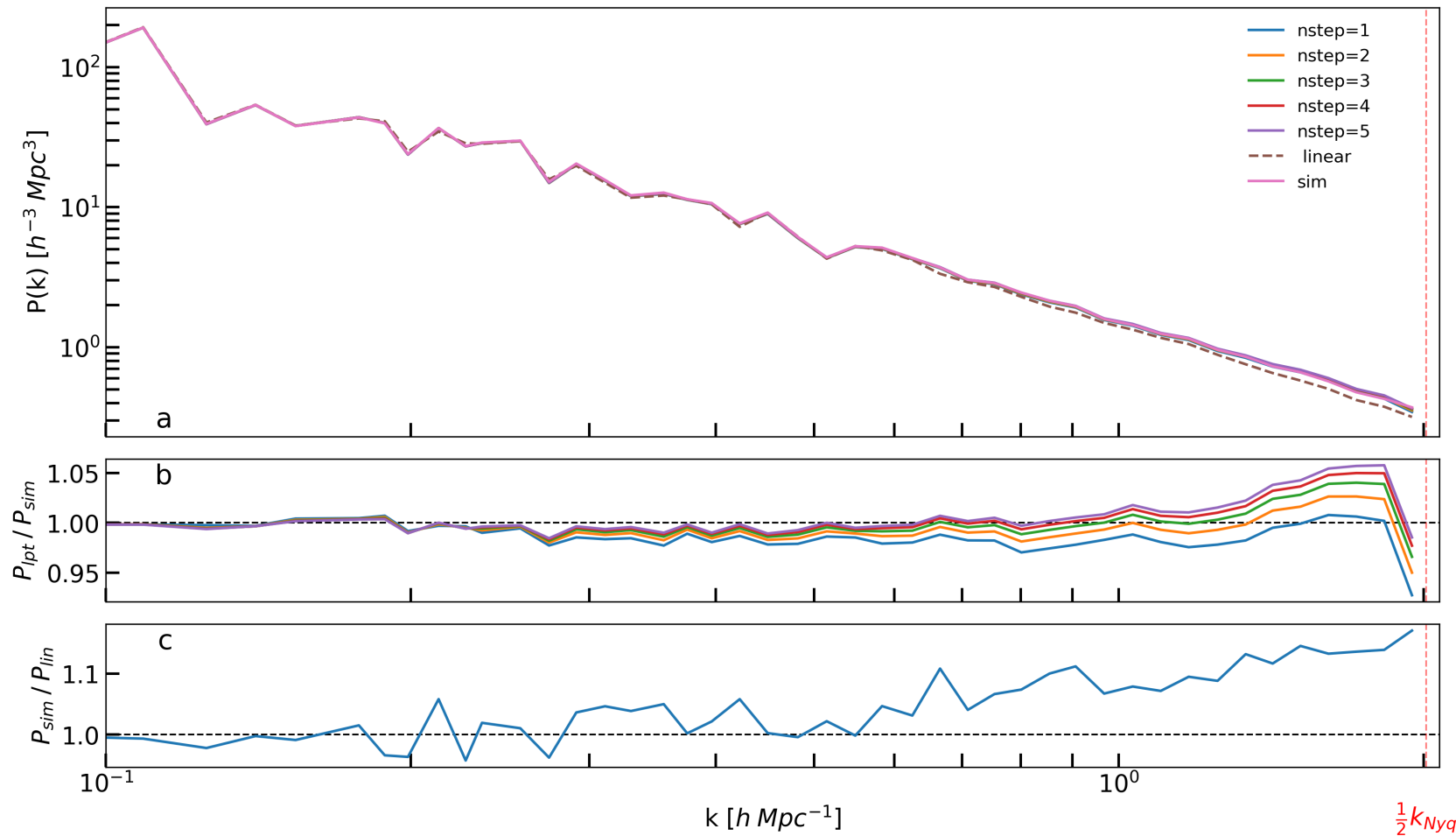


Small boxes : $L_{box} = 100h^{-1}Mpc$ $N_{grid} = 128(128^3)$

Matter power spectrum

Single Realization

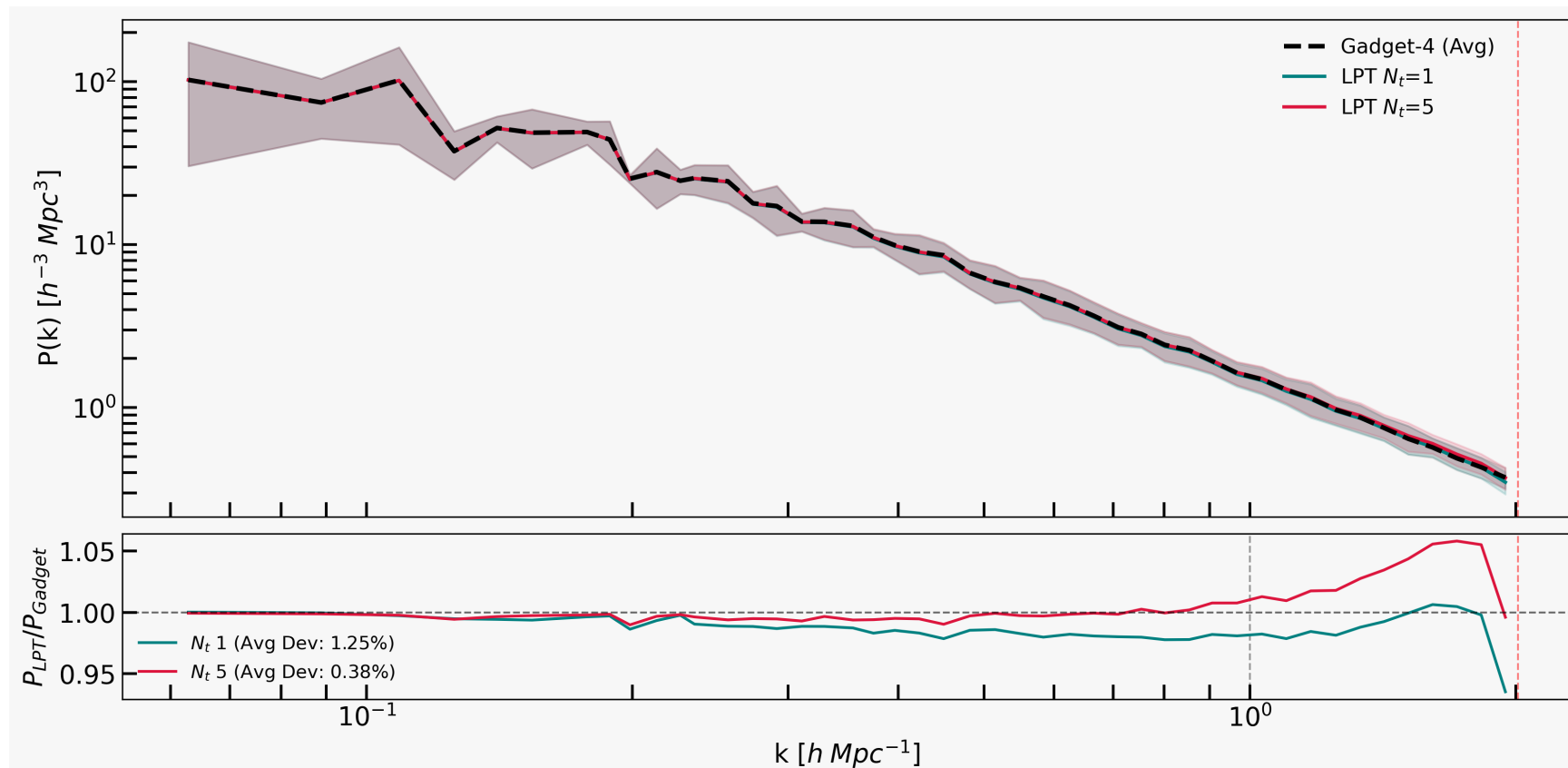
$$z_i = 99, z_f = 10$$



Checked for
Cauchy Convergence
of errors

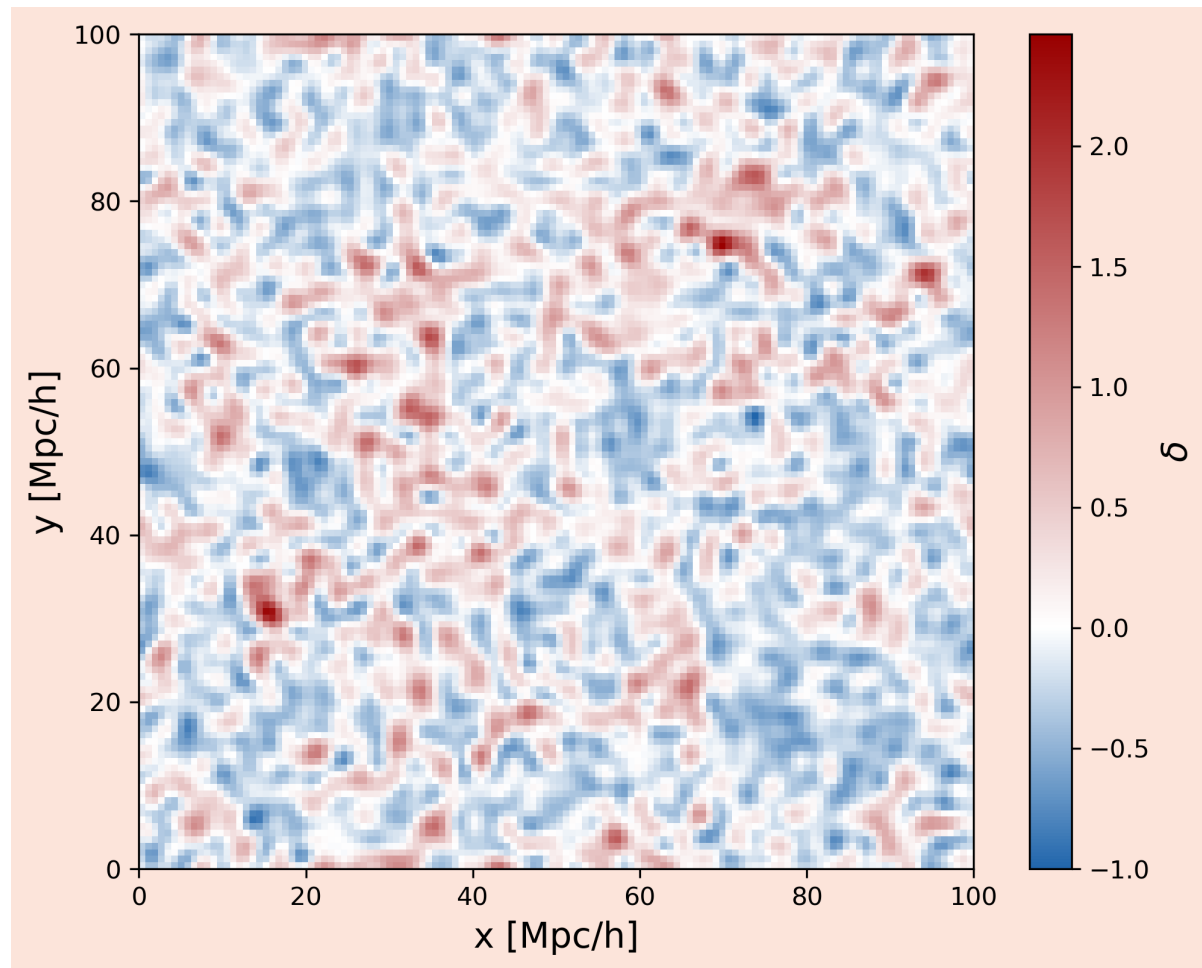
Average over
5 Realizations

$$z_i = 99, z_f = 10$$



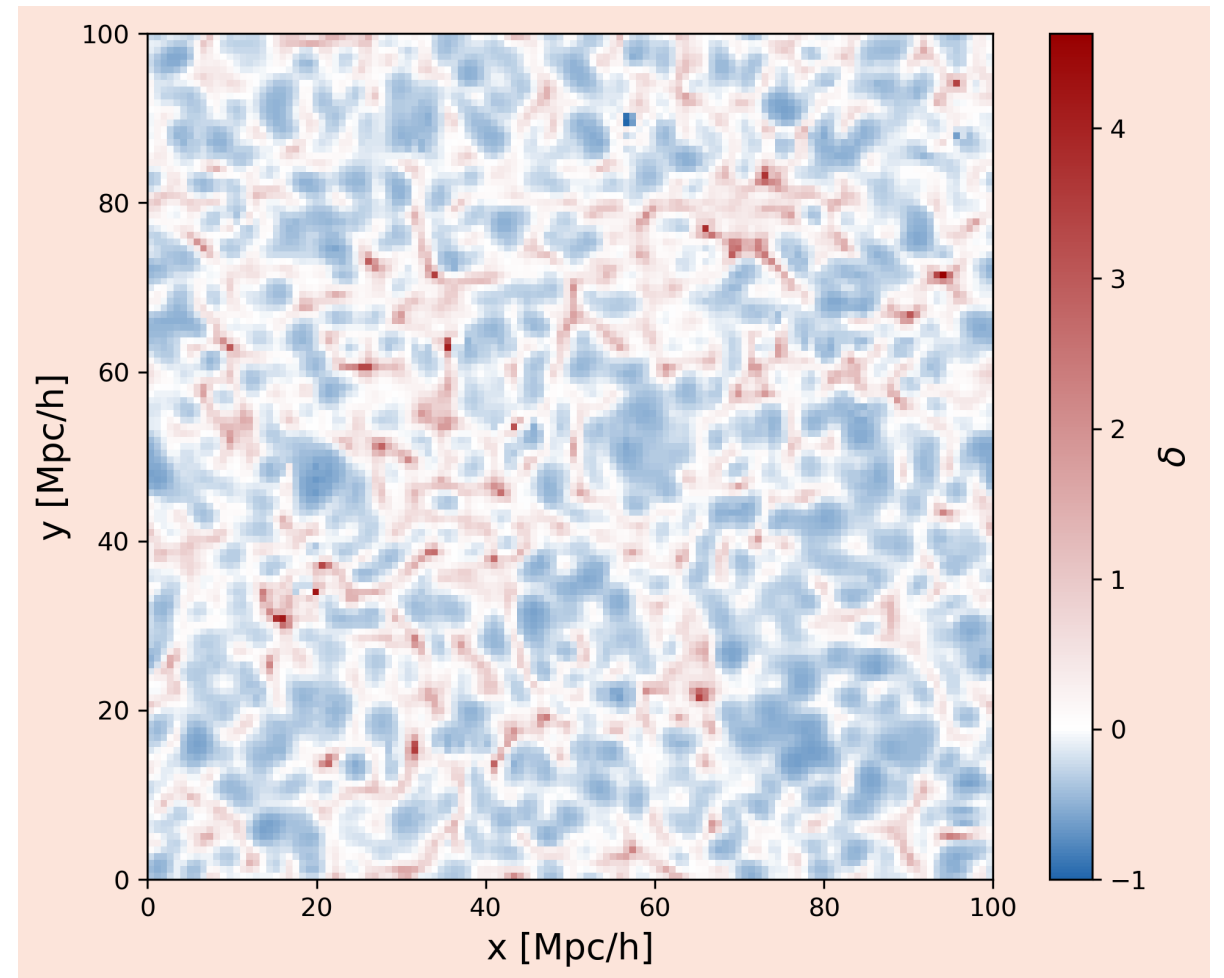
- High k - single step better
- Nyquist errors
- Simulations affected by Transients

Density Slices in real space



Filtered Gadget output at $z = 10$

$$k < \frac{1}{2}k_{Ny}$$



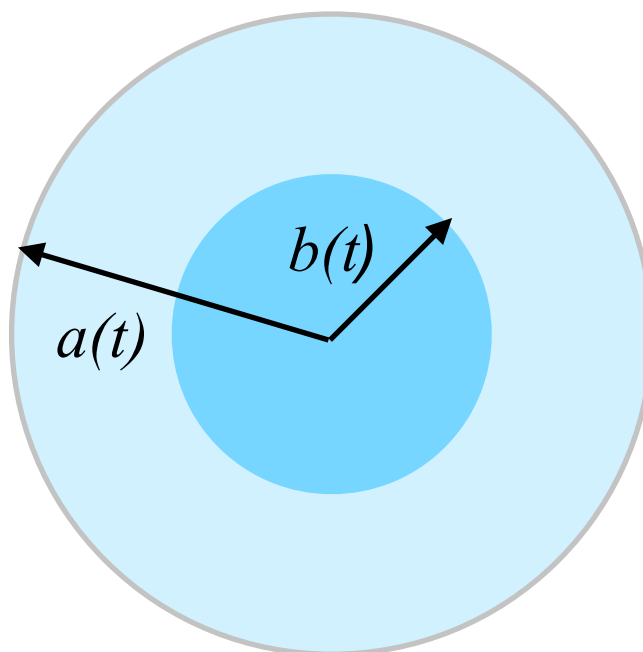
5-step LPT output at $z = 10$

- Quantitative comparison in progress
- Parallelize the codes

PART II - Geometric Models

Non-linear extension in spherical symmetry

Spherical top-hat
Constant density
perturbation



$$\delta = \frac{\rho}{\bar{\rho}} - 1$$

$$\delta_v = \frac{1}{H} \frac{\dot{b}}{b} - 1 = \frac{\Theta}{3} = \frac{\nabla_x \cdot \mathbf{v}}{3aH}$$

Method A

Equation for $b(t)$
(Lagrangian)

$$b(t) = a(t)A(t)$$

$$\frac{\ddot{b}}{b} = -\frac{H_0^2}{2} \left[\frac{\Omega_{m,0}(1 + \delta_i)}{b^3} - 2\Omega_{\Lambda,0} \right]$$

Initial conditions

$$b(a_i) = a_i, \dot{b}(a_i) = \dot{a}_i(1 + \delta_{v,i})$$

Method B

Equation for δ and δ_v
(Eulerian)

Continuity

$$\delta' = -3\delta_v(1 + \delta)$$

Euler

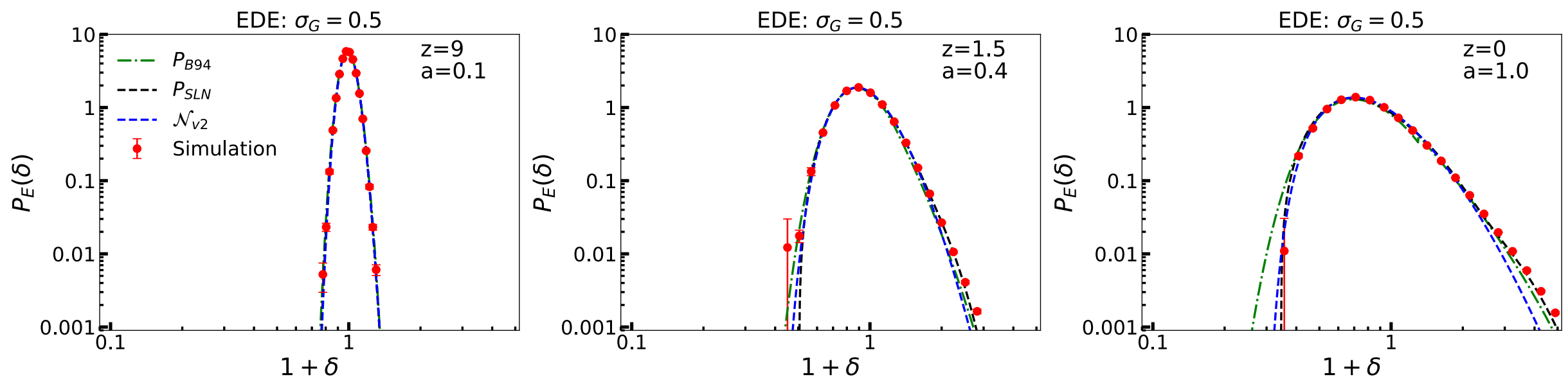
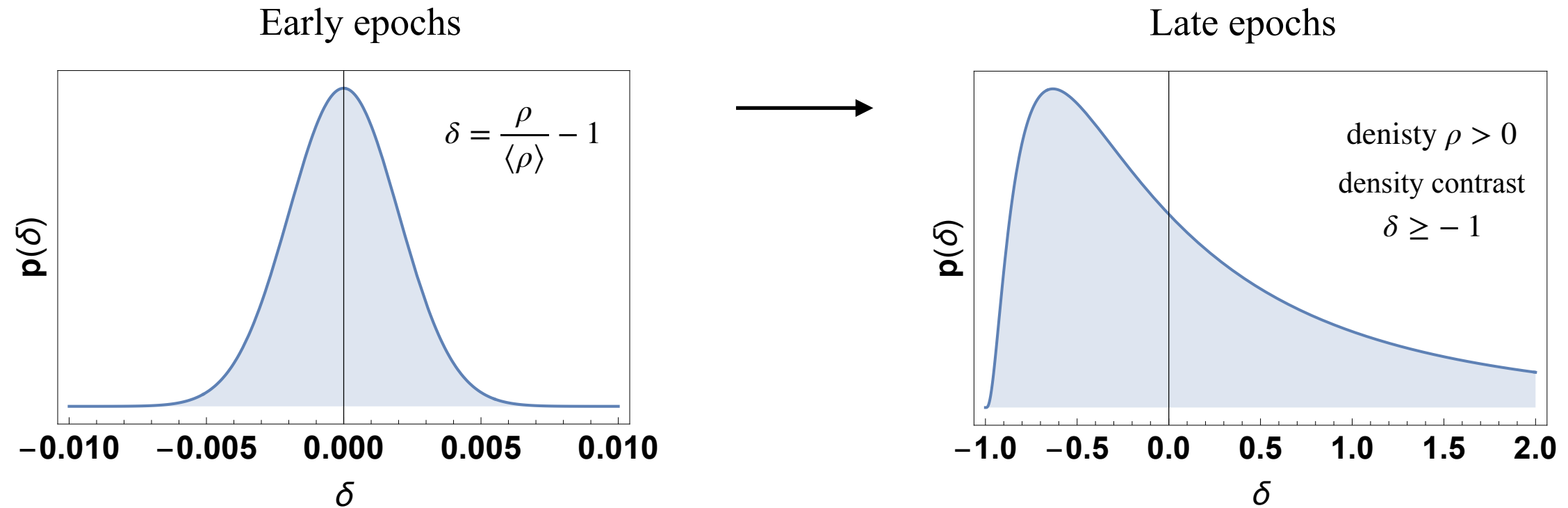
$$\delta'_v = -\frac{1}{2} \left[\Omega_m(a)\delta - \delta_v \{ \Omega_m(a) - 2\Omega_\Lambda(a) + 2\delta_v^2 \} \right]$$

Initial conditions

$$\delta_i, \delta_{v,i}$$

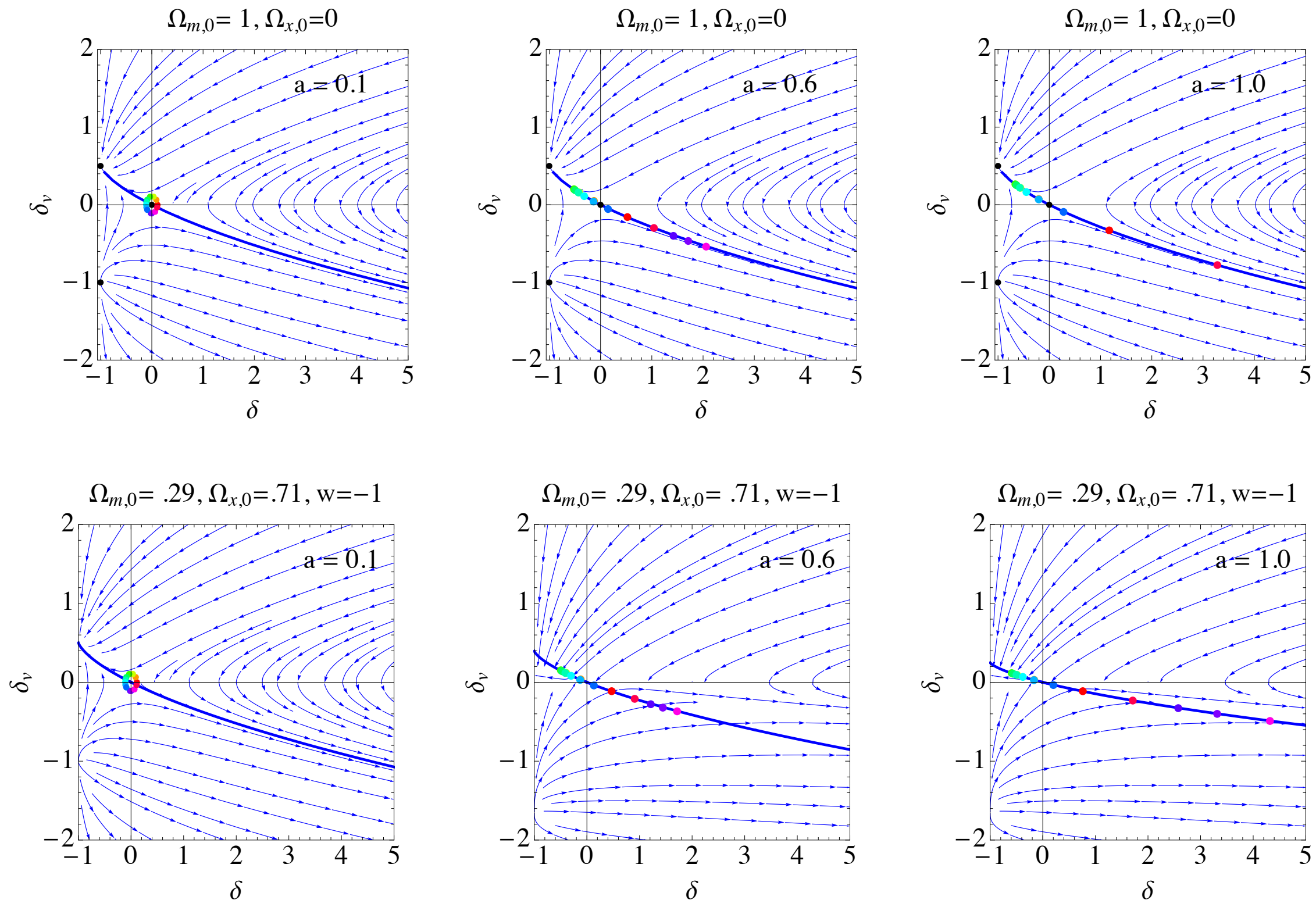
Evolution of the one-point pdf in early dark energy models

Cannot give 2-pt estimates like power spectrum but can give 1 pt PDF



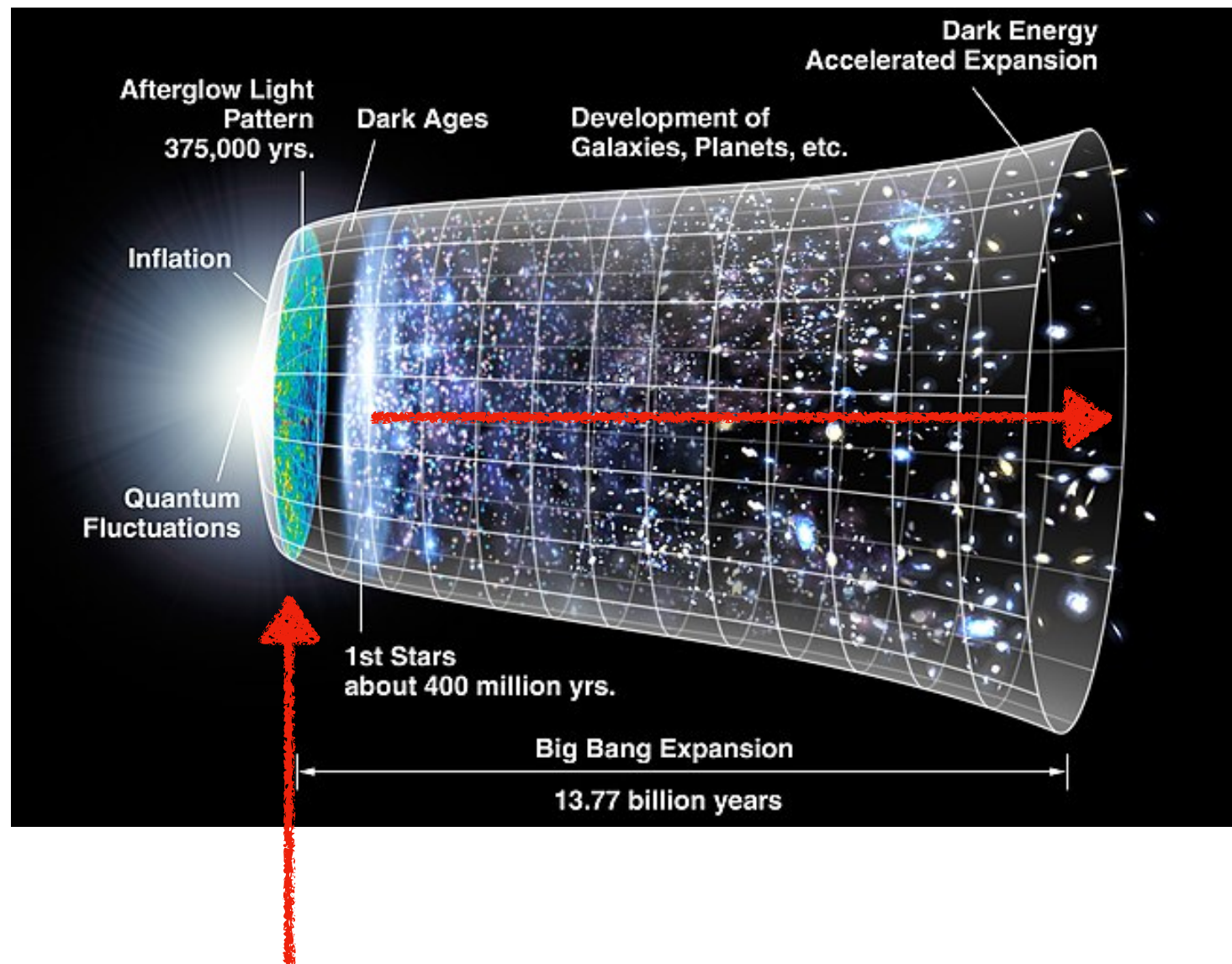
Evolved an ensembles of spherical overdensities and compared resulting pdf with N-body fits and PT estimates. Found good agreement

Non-linear density velocity-divergence relation



The non-linear DVDR is an invariant set in phase space.

Cosmological Parameter degeneracies



Those that describe evolution
 Ω_m , Ω_Λ , w etc.

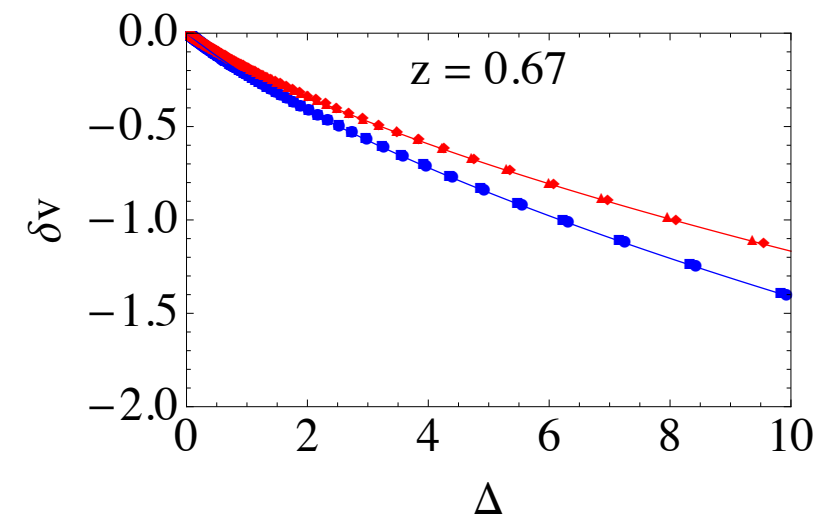
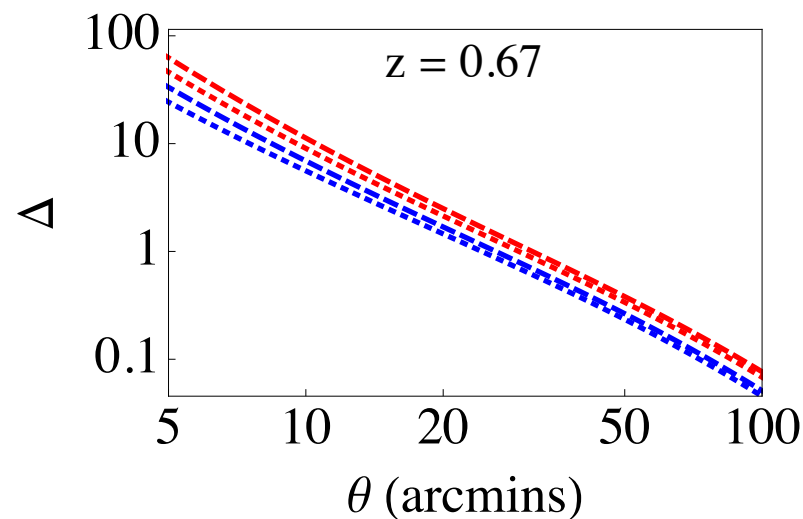
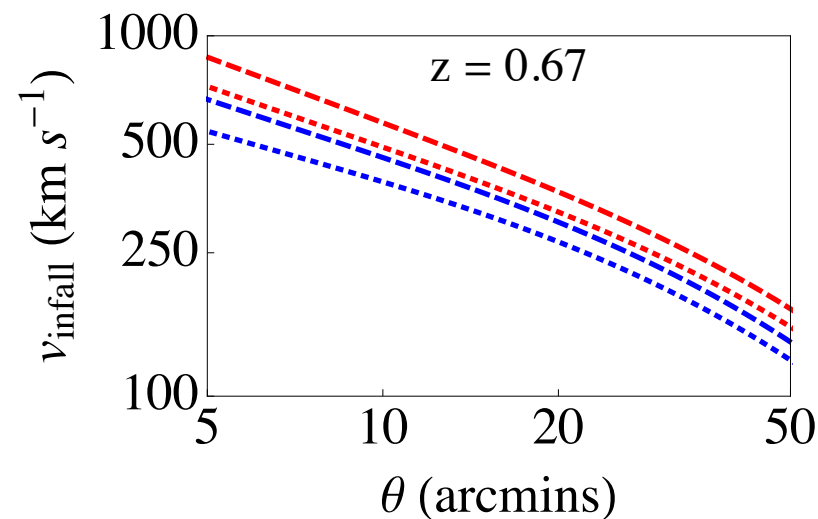
Those that describe initial conditions:
 n_s , σ_8 , etc.

Degenracies:

Higher initial amplitude can mimic higher matter content or one in which dark energy sets in later

Removal of parameter degeneracies

Initial profile is radially varying - progenitor profile for a cluster, was evolved using the equations for spherical symmetry

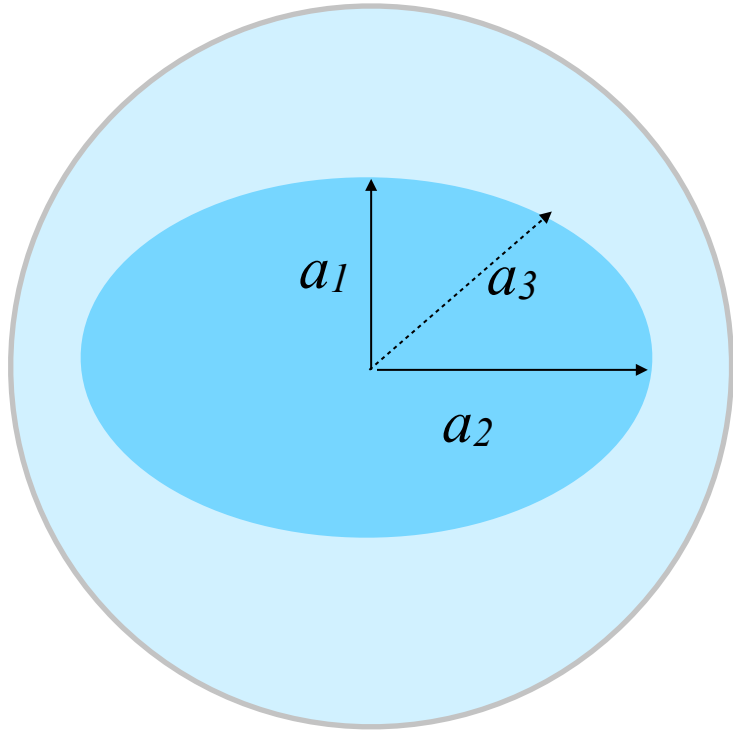


- **Red** and **Blue**: Dark energy sets in earlier in the blue so density grows slower.
- **Dashed** and **Dotted**: two different values of σ_8 (denotes the amplitude of initial fluctuations) Dotted is lower σ_8 so density grows slower.

Current work in progress: Check if this is true in GADGET outputs

With graduate student Wasim Mondal

Triaxial collapse



Traditional formulation in terms of the axis ratios involves time-consuming elliptic integrals

$$\lambda_{a,i} = 1 - \frac{a_i}{a}$$

Eigenvalues of the deformation tensor

$$\lambda_{v,i} = \frac{1}{H} \frac{\dot{a}_i}{a_i} - 1$$

Eigenvalues of the velocity derivative tensor

$$\lambda_{d,i} = \frac{\partial \phi}{\partial x_i \partial x_j}$$

Eigenvalues of the Hessian of the gravitational field

Bond and Myers 1996

S. N-G and Akshat Singhal MNRAS

$$\ddot{a}_i = -4\pi G \left[\bar{\rho}_m \left(\frac{1}{3} + \frac{\delta \alpha_i}{2} \right) + \lambda_{ext,i} \right] a_i$$

$$a_{i,init} = a_{init}(1 - \lambda_i)$$

$$\dot{a}_{i,init} = (1 + \lambda_{v,i,init}) H_{init} a_{i,init}$$

where $\alpha_i = a_1 a_2 a_3 \int_0^\infty \frac{d\tau}{(a_i^2 + \tau) \prod_{j=1}^3 (a_j^2 + \tau)^{1/2}}$

$$\sum_{i=1}^3 \alpha_i = 2, \lambda_{ext,i} = \frac{5}{4} \left(\alpha_i - \frac{2}{3} \right) \quad \text{an} \quad 1 + \delta = \frac{a^3}{a_1 a_2 a_3}$$

$$\frac{d\lambda_{a,i}}{d \ln a} = -\lambda_{v,i}(1 - \lambda_{a,i})$$

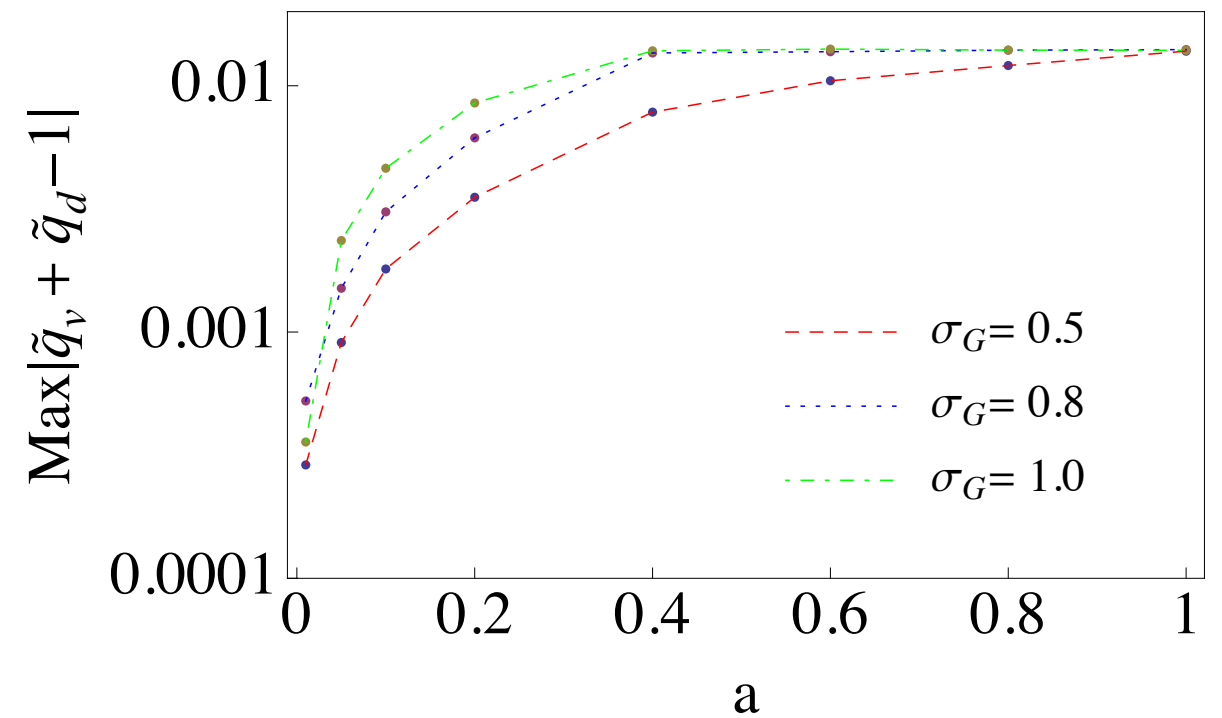
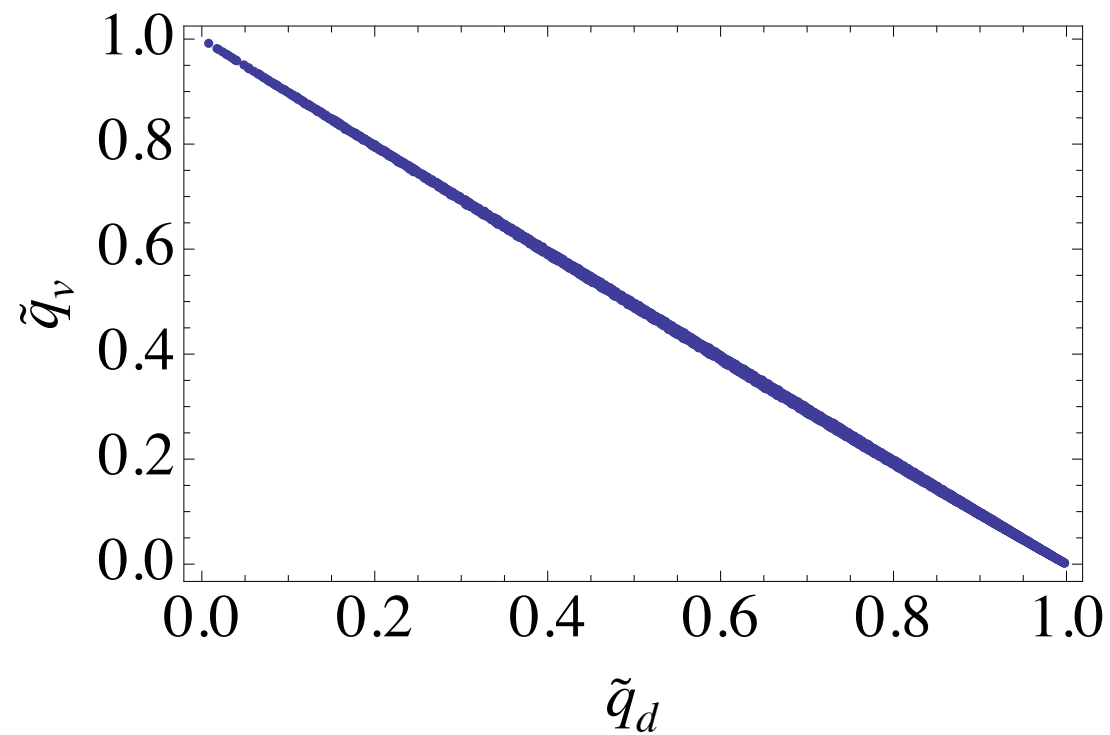
$$\frac{d\lambda_{v,i}}{d \ln a} = -\frac{1}{2} \left[3\Omega_m(a)\lambda_{d,i} - \{ \Omega_m(a) - 2\Omega_\Lambda(a) - 2 \} \lambda_{v,i} + 2\lambda_{v,i}^2 \right]$$

$$\frac{d\lambda_{d,i}}{d \ln a} = -(1 + \delta) \left(\delta + \frac{5}{2} \right)^{-1} \left(\lambda_{d,i} + \frac{5}{6} \right) \sum_{j=1}^3 \lambda_{v,j}$$

$$+ \left(\lambda_{d,i} + \frac{5}{6} \right) \sum_{i=1}^3 (1 + \lambda_{v,i}) - \left(\delta + \frac{5}{2} \right) (1 + \lambda_{v,i})$$

$$+ \sum_{j \neq i} \frac{\left\{ \lambda_{d,j} - \lambda_{d,i} \right\} \cdot \left\{ (1 - \lambda_{a,i})^2 (1 + \lambda_{v,i}) - (1 - \lambda_{a,j})^2 (1 + \lambda_{v,j}) \right\}}{(1 - \lambda_{a,i})^2 - (1 - \lambda_{a,j})^2}.$$

Phase Space Invariants: density-velocity subspace



Finding Invariants

$$\tilde{q} = \frac{\lambda_{\max} - \lambda_{\text{inter}}}{\lambda_{\max} - \lambda_{\min}}$$

$$\tilde{q}_v + \tilde{q}_d = 1$$

Summary of approximation methods

Lagrangian Perturbation Theory Re-expansions:

- Perturbatively solve for displacement field in an iterative way
- **Shot-noise free**, fine-tune to required precision by controlling order and number of time steps
- **Versatility**: The scheme is general and **allows for transverse velocities**
- **Computational Efficiency**: The developed code is comparable to Gadget in single core runs. However, in HPC, Gadget is faster. Currently not parallelized.
- Applications to modelling primordial Non-Gaussianity, warm dark matter models...

Geometric Methods

- Use the simplicity to identify **invariant relations** in dynamical variables
- Look for such invariants in N-body codes and eventually in observed data-sets
- **Future - AI-ML aided searches for more complicated models**

Thanks for listening !