On the Linear Regime of the Characteristic formulation of General relativity in the Minkowski and Schwarzschild's Backgrounds

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Bishop, for a Schwarzschild's background, and Mädler, for a Minkowski's background, were able to show that it is possible to derive a fourth order ordinary differential equation, called master equation, for the J metric variable of the Bondi-Sachs metric. Once β, another Bondi-Sachs potential, is obtained from the field equations, and J is obtained from the master equation, the other metric variables are solved integrating directly the rest of the field equations. In the past, the master equation was solved for the first multipolar terms, for both the Minkowski's and Schwarzschild's backgrounds. Also, Mädler recently reported a generalisation of the exact solutions to the linearised field equations when a Minkowski's background is considered, expressing the master equation family of solutions for the vacuum in terms of Bessel's functions of the first and the second kind. Here, we report new solutions to the master equation for any multipolar moment l, with and without matter sources in terms only of the first kind Bessel's functions for the Minkowski, and in terms of the Confluent Heun's functions (Generalised Hypergeometric) for radiative (nonradiative) case in the Schwarzschild's background. We particularize our families of solutions for the known cases for $l = 2$ reported previously in the literature and find complete agreement, showing the robustness of our results. Finally, we apply this formalism to a binary point particle system, obtaining the Peters and Mathews gravitational radiation formula.



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1. Eth formalism

- Originally based on the stereographic projection.[1]
- Regularises the differential angular operators, e.g. the Legendrian.
- It is a variant of Newman-Penrose formalism.
- Here, spin-weighted scalars are constructed by projection on tangent vectors to the unit sphere.
- Here differential operators that inherits the properties of the covariant derivatives are constructed.
- These operators are capable to rise or lower the spin-weight of the spin-weighted functions.

[1] E. T. Newman and R. Penrose. *Journal of Mathematical Physics.* **7** 5 863–870 (1966) ; J. N. Goldberg et al. *Journal of Mathematical Physics.* **8** 11 2155–2161 (1967) .

1.1. Non-conformal projections on the sphere

- The finite coverage of the sphere removes the coordinate singularities in the angular differential operators.
- They are non-conformal transformation $\mathbb{R}^2 \to \mathbb{R}^2$
- They help to improve numerical calculations providing quasi-regular grids.
- Ideally they must be light, i.e., with an adequate geometry in order to minimise interpolation schemes.
- Discretisation schemes used originally in weather studies.[2]
- There are infinite discretisation schemes, however two of them are the most used in general relativity. The stereographic projection in two charts and the gnomonic projection in six charts.

[2] C. Ronchi, R. Iacono, and P. Paolucci. *Journal of Computational Physics.* **124** 1 93 –114 (1996) ; R. Sadourny, A. Arakawa, and Y. Mintz. *Monthly Weather Review.* **96** 6 351 –356 (1968) ; R. Sadourny. *Monthly Weather Review.* **100** 2 136 –144 (1972) .

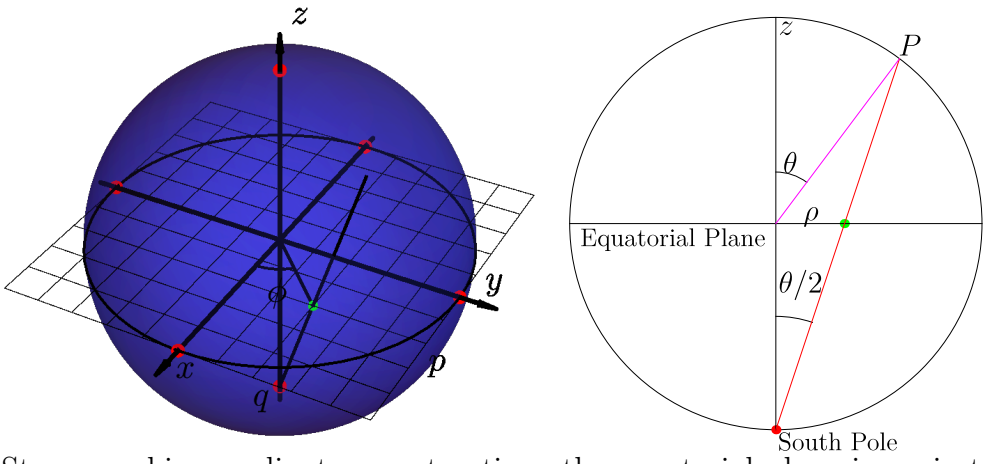


Figure 1: Stereographic coordinates construction: the equatorial plane is projected from the south pole to the surface of the unit sphere. The interior points to the equator are projected to the north hemisphere, whereas the exterior points are projected to the south.

- Stereographic coordinates defined from

$$\rho = \tan\left(\frac{\theta}{2}\right), \quad \zeta = \rho e^{i\phi}, \quad q = \Re(\zeta), \quad p = \Im(\zeta), \tag{1}$$

- The relationship between north and south charts is given by

$$\zeta_N = \zeta_S^{-1} \tag{2}$$

- Unit sphere metric $d\Omega^2 = q_{AB}dx^a dx^B$ in diagonal form

$$d\theta^2 + \sin^2\theta d\phi^2 = \frac{4}{\left(1+\zeta\overline{\zeta}\right)^2}\left(dq^2 + dp^2\right). \tag{3}$$

- and it supposed that the metric can e decomposed as

$$q_{AB} = q_{(A}\overline{q}_{B)}, \quad \text{with} \quad \overline{q}_B q^B = 2, \qquad \overline{q}_B\overline{q}^B = 0 \tag{4}$$

- It is found that

$$q_A = \frac{2}{\left(1+\zeta\overline{\zeta}\right)}\left(\delta^3{}_A + i\delta^4{}_A\right) \qquad \overline{q}_A = \frac{2}{\left(1+\zeta\overline{\zeta}\right)}\left(\delta^3{}_A - i\delta^4{}_A\right)$$

$$q_A = \delta_A{}^3 + i\sin\theta\delta_A{}^4 \qquad \overline{q}_A = \delta_A{}^3 - i\sin\theta\delta_A{}^4$$

- The transformation rules for the dyads result as.[3]

$$q^A_N = e^{i\alpha}q^A_S, \qquad e^{i\alpha} = -\frac{\overline{\zeta}_S}{\zeta_S}, \tag{5}$$

[3]Newman and Penrose, see n. 1; Goldberg et al., see n. 1; J. M. Stewart. Cambridge Monographs on Mathematical Physics. Cambridge University Press, 1993. ISBN: 9780521449465.

1.2. Spin-weighted Functions and Spin-weighted Differential Operators

- The spin-weighted functions ${}_s\Psi$ with spin-weight s can be constructed from a tensor $\Psi^{a_1\cdots a_n}{}_{b_1\cdots b_m}$ in the tangent space to the unit sphere, as

$$ {}_s\Psi = \tilde{\Lambda}_{\tilde{B}_{1m}} \tilde{\Lambda}^{\tilde{A}_{1n}} \Psi^{\tilde{B}_{1m}}{}_{\tilde{A}_{1n}}. \tag{6} $$

  where

$$ \tilde{\Lambda}_{\tilde{A}_{ab}} = \prod_{i=a}^{b} \Lambda_{A_i}, \quad \tilde{\Lambda}^{\tilde{A}_{ab}} = \prod_{i=a}^{b} \Lambda^{A_i}\ , \quad \Psi^{\tilde{B}_{1m}}{}_{\tilde{A}_{1n}} = \Psi^{B_1\cdots B_m}{}_{A_1\cdots A_n}\ . \tag{7} $$

  and Λ_{a_i} and Λ^{b_j} can take the values q_{a_i} or $\overline{q}_{a_i}$ and q^{b_j} or $\overline{q}^{b_j}$ respectively.

- The spin-weight s of the functions ${}_s\Psi$ depends on the number of q_{a_i}, $\overline{q}_{a_i}$, q^{a_i} or $\overline{q}^{a_i}$ used to construct the functions.

- If $m, \tilde{s}, n, r$ are the number of vectors $q^{a_i}, \overline{q}^{a_i}, q_{a_i}, \overline{q}_{a_i}$ respectively, used to construct the spin-weighted scalar ${}_s\Psi$, its spin-weight is given by

$$ s = 2(m+n) - (r+\tilde{s}). \tag{8} $$

- $\eth$ and $\overline{\eth}$ operators are defined from

$$\eth\ {}_s\Psi = q^D \tilde{\Lambda}_{\tilde{B}_{1m}} \tilde{\Lambda}^{\tilde{A}_{1n}} \Psi^{\tilde{B}_{1m}}{}_{\tilde{A}_{1n}|D}, \qquad \overline{\eth}\ {}_s\Psi = \overline{q}^D \tilde{\Lambda}_{\tilde{B}_{1m}} \tilde{\Lambda}^{\tilde{A}_{1n}} \Psi^{\tilde{B}_{1m}}{}_{\tilde{A}_{1n}|D}, \tag{9}$$

- They result, after some algebra, in

$$\eth\ {}_s\Psi = q^D\ {}_s\Psi_{,D} + s\,\Omega\ {}_s\Psi, \qquad \overline{\eth}\ {}_s\Psi = \overline{q}^D\ {}_s\Psi_{,D} - s\overline{\Omega}\ {}_s\Psi. \tag{10}$$

  where

$$\Omega = \frac{1}{2} q^A \overline{q}^B q_{AB}. \tag{11}$$

- In stereographic coordinates they take the form

$$\eth = \frac{1+|\zeta|^2}{2}\left(\partial_q + i\partial_p\right) + s\zeta\ , \quad \overline{\eth} = \frac{1+|\zeta|^2}{2}\left(\partial_q - i\partial_p\right) - s\overline{\zeta}. \tag{12}$$

- In spherical coordinates they take the form

$$\eth = \partial_\theta + i\csc\theta\partial_\phi - s\cot\theta \quad \text{and} \quad \overline{\eth} = \partial_\theta - i\csc\theta\partial_\phi + s\cot\theta. \tag{13}$$

- Eigenvalue equation for the commutator

$$\left[\overline{\eth}, \eth\right] \ {}_s\Psi = 2s \ {}_s\Psi \tag{14}$$

- Spin-weighted spherical harmonics

$$\eth \ {}_sY_{lm} = \left(l(l+1) - s(s+1)\right)^{1/2} \ {}_{s+1}Y_{lm}, \tag{15}$$

$$\overline{\eth} \ {}_sY_{lm} = -\left[l(l+1) - s(s-1)\right]^{1/2} \ {}_{s-1}Y_{lm}. \tag{16}$$

- $$ {}_sY_{lm} = \begin{cases} \left(\dfrac{(l-s)!}{(l+s)!}\right)^{1/2} \eth^s \ {}_0Y_{lm} & \text{for} \quad 0 \le s \le l \\ (-1)^s \left(\dfrac{(l+s)!}{(l-s)!}\right)^{1/2} \overline{\eth}^{-s} \ {}_0Y_{lm} & \text{for} \quad -l \le s \le 0 \end{cases}, \tag{17}$$

- $$ {}_sZ_{lm} = \begin{cases} \dfrac{i}{\sqrt{2}} \left((-1)^m \ {}_sY_{lm} + \ {}_sY_{l\ -m}\right) & \text{for} \quad m < 0 \\ {}_sY_{lm} & \text{for} \quad m = 0 \\ \dfrac{1}{\sqrt{2}} \left({}_sY_{lm} + (-1)^m \ {}_sY_{l\ -m}\right) & \text{for} \quad m > 0 \end{cases}, \tag{18}$$

Ver[4]

[4] Y. Zlochower et al. *Physical Review D.* **68** 8 084014 (2003) .

2. Linear Regime

2.1. Bondi-Sachs metric

- The Bondi-Sachs metric,[5][6] reads as

$$ds^2 = -\left(\frac{Ve^{2\beta}}{r}\right)du^2 - 2e^{2\beta}dudr + r^2 h_{AB}\left(U^A du - dx^A\right)\left(U^B du - dx^B\right), \tag{19}$$

where

$$2h_{AB}dx^A dx^B = \left(e^{2\gamma} + e^{2\delta}\right)d\theta^2 + 4\sin\theta\sinh(\gamma-\delta)d\theta d\phi + \sin^2\theta\left(e^{-2\gamma} + e^{-2\delta}\right)d\phi^2. \tag{20}$$

and $\|h_{AB}\| = \sin\theta$

- $(V/r)e^{2\beta}$ corresponds to α^2, U^μ corresponds with the shift vector in ADM formulation, β is the redshift, h_{AB} is the metric of the angular 2-manifold.

[5] H. Bondi, M. G. J. van der Burg, and A. W. K. Metzner. *Proceedings of the Royal Society of London A: Mathematical, Physical and Engineering Sciences.* **269** 1336 21–52 (1962) .

[6] R. K. Sachs. *Proceedings of the Royal Society of London A: Mathematical, Physical and Engineering Sciences.* **270** 1340 103–126 (1962) .

- V is re-defined as

$$V = r + w \tag{21}$$

- and it is introduced the spin-weighted quantities

$$J = h_{AB} q^A q^B, \quad \overline{J} = h_{AB} \overline{q}^A \overline{q}^B, \quad U = U^A q_A, \quad \overline{U} = U^A \overline{q}_A \tag{22}$$

- Bondi's gauge

$$1 = K^2 - J\overline{J} \tag{23}$$

- Bondi-Sachs Metric in terms of the spin-weighted functions $J, \overline{J}, K, U, \overline{U}$

$$\begin{aligned} ds^2 = & -\left(e^{2\beta}\left(1+\frac{w}{r}\right) - r^2(J\overline{U}^2 + U^2\overline{J} + 2KU\overline{U})\right) du^2 - 2e^{2\beta} du dr \\ & - \frac{2r^2\left((K+\overline{J})U + (J+K)\overline{U}\right)}{1+|\zeta|^2} dq du \\ & - \frac{2ir^2\left((K-\overline{J})U + (J-K)\bar{U}\right)}{1+|\zeta|^2} dp du + \frac{2r^2\left(J+2K+\overline{J}\right)}{(1+|\zeta|^2)^2} dq^2 \\ & - \frac{4ir^2\left(J-\overline{J}\right)}{(1+|\zeta|^2)^2} dq dp - \frac{2r^2\left(J-2K+\overline{J}\right)}{(1+|\zeta|^2)^2} dp^2. \end{aligned} \tag{24}$$

2.2. Linear Regime for the BS metric

- $$ds^2 = -\left(1+\frac{w}{r}+2\beta\right)du^2 - 2(1+2\beta)dudr - 2r^2\frac{(U+\overline{U})}{1+|\zeta|^2}dqdu$$
$$- 2r^2\frac{i(U-\overline{U})}{1+|\zeta|^2}dpdu + 2r^2\frac{\left(2+J+\overline{J}\right)}{\left(1+|\zeta|^2\right)^2}dq^2$$
$$- 4ir^2\frac{(J-\overline{J})}{(1+|\zeta|^2)^2}dqdp - 2r^2\frac{\left(-2+J+\overline{J}\right)}{\left(1+|\zeta|^2\right)^2}dp^2, \qquad (25)$$

- or for Minkowski's background

$$ds^2 = -du^2 - 2dudr + \frac{4r^2}{\left(1+|\zeta|^2\right)^2}\left(dq^2+dp^2\right) - \left(\frac{w}{r}+2\beta\right)du^2$$
$$-4\beta dudr - \frac{2r^2}{1+|\zeta|^2}du\left((U+\overline{U})dq - i(U-\overline{U})dp\right)$$
$$-4ir^2\frac{(J-\overline{J})}{(1+|\zeta|^2)^2}dqdp + \frac{2r^2\left(J+\overline{J}\right)}{\left(1+|\zeta|^2\right)^2}\left(dq^2-dp^2\right), \qquad (26)$$

- Making

$$V = r + w - 2M \tag{27}$$

we obtain perturbations to Schwarzschild's background from the Bondi-Sachs metric in the linear regime

$$\begin{aligned} ds^2 = & -\left(1 - \frac{2M}{r}\right) du^2 - 2dudr + \frac{4r^2}{\left(1+|\zeta|^2\right)^2}\left(dq^2 + dp^2\right) \\ & -\left(\frac{w}{r} + 2\beta\right) du^2 - 4\beta dudr - 4ir^2 \frac{(J - \overline{J})}{(1+|\zeta|^2)^2} dqdp \\ & -\frac{2r^2}{1+|\zeta|^2} du\left((U + \overline{U})dq - i(U - \overline{U})dp\right) \\ & +\frac{2r^2\left(J + \overline{J}\right)}{\left(1+|\zeta|^2\right)^2}\left(dq^2 - dp^2\right), \end{aligned} \tag{28}$$

2.3. Field Equations

The Einstein field equations reads

$$E_{\mu\nu} = R_{\mu\nu} - 8\pi \left(T_{\mu\nu} - \frac{g_{\mu\nu}}{2} T \right) = 0$$

In this formalism, we decompose this system of equations as

$$\begin{aligned} E_{22} = 0, \quad E_{2A} q^A = 0, \quad E_{AB} h^{AB} = 0 \quad & \text{Hypersurface} \\ E_{AB} q^A q^B = 0 \quad & \text{Evolution} \\ E_{11} = 0, \quad E_{12} = 0, \quad E_{1A} q^A = 0 \quad & \text{Constrain} \end{aligned}$$

2.4. Field Equations

$$8\pi T_{22} = \frac{4\beta_{,r}}{r}, \tag{29a}$$

$$8\pi T_{2A} q^A = \frac{\overline{\eth} J_{,r}}{2} - \eth \beta_{,r} + \frac{2\eth\beta}{r} + \frac{\left(r^4 U_{,r}\right)_{,r}}{2r^2}, \tag{29b}$$

$$8\pi \left(h^{AB} T_{AB} - r^2 T\right) = -2\eth\overline{\eth}\beta + \frac{\eth^2 \overline{J} + \overline{\eth}^2 J}{2} + \frac{\left(r^4 \left(\overline{\eth} U + \eth \overline{U}\right)\right)_{,r}}{2r^2} + 4\beta - 2w_{,r}, \tag{29c}$$

$$8\pi T_{AB} q^A q^B = -2\eth^2\beta + \left(r^2 \eth U\right)_{,r} - \left(r^2 J_{,r}\right)_{,r} + 2r\left(rJ\right)_{,ur}, \tag{29d}$$

$$8\pi \left(\frac{T}{2} + T_{11}\right) = \frac{\eth\overline{\eth} w}{2r^3} + \frac{\eth\overline{\eth}\beta}{r^2} - \frac{\left(\eth\overline{U} + \overline{\eth} U\right)_{,u}}{2} + \frac{w_{,u}}{r^2} + \frac{w_{,rr}}{2r} - \frac{2\beta_{,u}}{r}, \tag{29e}$$

$$8\pi \left(\frac{T}{2} + T_{12}\right) = \frac{\eth\overline{\eth}\beta}{r^2} - \frac{\left(r^2 \left(\eth\overline{U} + \overline{\eth} U\right)\right)_{,r}}{4r^2} + \frac{w_{,rr}}{2r}, \tag{29f}$$

$$8\pi T_{1A}q^A = \frac{\overline{\eth} J_{,u}}{2} - \frac{\eth^2 \overline{U}}{4} + \frac{\eth\overline{\eth} U}{4} + \frac{1}{2}\left(\frac{\eth w}{r}\right)_{,r} - \eth\beta_{,u} + \frac{\left(r^4 U\right)_{,r}}{2r^2} - \frac{r^2 U_{,ur}}{2} + U. \tag{29g}$$

$$ {}_s f = \sum_{l=0}^{\infty}\sum_{m=l}^{l} \Re\left(f_{lm} e^{i|m|\tilde{\phi}}\right)\ \eth^s\ Z_{lm}, \tag{30}$$

where ${}_s f = \{\beta, w, J, \overline{J}, U, \overline{U}\}$, $Z_{lm} = {}_0 Z_{lm}$, $\tilde{\phi} := \tilde{\phi}(u)$, $m \in \mathbb{Z}$, $m \in [-l, l]$ and $l \geq 0$. In previous works similar expansions were performed, where $\phi = \nu u$[7]

$$\beta_{lm,r} = 2\pi \int_\Omega d\Omega\ \overline{Z}_{lm} \int_0^{2\pi} d\tilde{\phi}\ e^{-i|m|\tilde{\phi}} r T_{22}, \tag{31a}$$

$$-\frac{(l+2)(l-1)J_{lm,r}}{2} - \beta_{lm,r} + \frac{2\beta_{lm}}{r} + \frac{\left(r^4 U_{lm,r}\right)_{,r}}{2r^2} = \frac{8\pi}{\sqrt{l(l+1)}} \int_\Omega d\Omega\ \overline{Z}_{lm} \int_0^{2\pi} d\tilde{\phi}\ e^{-i|m|\tilde{\phi}} T_{2A} q^A, \tag{31b}$$

[7] N. T. Bishop. *Classical and Quantum Gravity.* **22** 12 2393 (2005) ; M. C. Babiuc et al. *Physical Review D.* **79** 8 084011 (2009) ; N. Bishop, D. Pollney, and C. Reisswig. *Classical and Quantum Gravity.* **28** 15 155019 (2011) ; C. E. Cedeño and J. C. N. Araujo. *Classical and Quantum Gravity.* **33** 105010 (2016) .

$$2l(l+1)\beta_{lm} + (l-1)l(l+1)(l+2)J_{lm} + \frac{l(l+1)\left(r^4\left(U_{lm}\right)\right)_{,r}}{r^2}$$
$$+ 4\beta_{lm} - 2w_{lm,r} = 8\pi \int_\Omega d\Omega\ \overline{Z}_{lm} \int_0^{2\pi} d\tilde{\phi}\ e^{-i|m|\tilde{\phi}} \left(h^{AB}T_{AB} - r^2 T\right), \tag{31c}$$

$$- 2\beta_{lm} + \left(r^2 U_{lm}\right)_{,r} - \left(r^2 J_{lm,r}\right)_{,r} + 2i|m|r\dot{\tilde{\phi}}\left(r J_{lm}\right)_{,r}$$
$$= \frac{8\pi}{\sqrt{(l-1)l(l+1)(l+2)}} \int_\Omega d\Omega\ \overline{Z}_{lm} \int_0^{2\pi} d\tilde{\phi}\ e^{-i|m|\tilde{\phi}} T_{AB} q^A q^B, \tag{31d}$$

$$
\begin{aligned}
&- \frac{l(l+1)w_{lm}}{2r^3} - \frac{l(l+1)\beta_{lm}}{r^2} + i|m|l(l+1)\dot{\tilde{\phi}}U_{lm} + \frac{i|m|\dot{\tilde{\phi}}w_{lm}}{r^2} \\
&+ \frac{w_{lm,rr}}{2r} - \frac{2i|m|\dot{\tilde{\phi}}\beta_{lm}}{r} + \frac{2\beta_{lm,r}}{r} + \beta_{lm,rr} - 2\dot{\tilde{\phi}}\beta_{lm,r} \\
&= 8\pi \int_\Omega d\Omega\ \overline{Z}_{lm} \int_0^{2\pi} d\tilde{\phi}\ e^{-i|m|\tilde{\phi}} \left(\frac{T}{2} + T_{11} \right), \qquad (31e) \\
&- \frac{l(l+1)\beta_{lm}}{r^2} + \frac{l(l+1)\left(r^2 U_{lm}\right)_{,r}}{2r^2} + \frac{w_{lm,rr}}{2r} \\
&= 8\pi \int_\Omega d\Omega\ \overline{Z}_{lm} \int_0^{2\pi} d\tilde{\phi}\ e^{-i|m|\tilde{\phi}} \left(\frac{T}{2} + T_{12} \right), \qquad (31f)
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{i|m|(l+2)(l-1)J_{lm}\dot{\tilde{\phi}}}{2} + \frac{1}{2}\left(\frac{w_{lm}}{r}\right)_{,r} - i|m|\dot{\tilde{\phi}}\beta_{lm} + \frac{\left(r^4 U_{lm,r}\right)_{,r}}{2r^2} \\
&- \frac{i|m|r^2\dot{\tilde{\phi}}}{2} U_{lm,r} + U_{lm} = \frac{8\pi}{\sqrt{l(l+1)}} \int_\Omega d\Omega\ \overline{Z}_{lm} \int_0^{2\pi} d\tilde{\phi}\ e^{-i|m|\tilde{\phi}} T_{1A} q^A, \qquad (31g)
\end{aligned}
$$

3. Master Equation and its solutions

$$x = \frac{1}{r} \tag{32}$$

$$A_{lm} = 2\pi \int_\Omega d\Omega\ \overline{Z}_{lm} \int_0^{2\pi} d\tilde{\phi}\ e^{-i|m|\tilde{\phi}} x T_{22}, \tag{33a}$$

$$B_{lm} = \frac{16\pi}{\sqrt{l(l+1)}} \int_\Omega d\Omega\ \overline{Z}_{lm} \int_0^{2\pi} d\tilde{\phi}\ e^{-i|m|\tilde{\phi}} x T_{2A} q^A, \tag{33b}$$

$$D_{lm} = \frac{8\pi}{\sqrt{(l-1)l(l+1)(l+2)}} \int_\Omega d\Omega\ \overline{Z}_{lm} \int_0^{2\pi} d\tilde{\phi}\ e^{-i|m|\tilde{\phi}} T_{AB} q^A q^B. \tag{33c}$$

$$\begin{aligned} &-2x^4 J_{lm,xxxx} - 4x^2 \left(2x + i|m|\dot{\tilde{\phi}}\right) J_{lm,xxx} \\ &+ 2x \left(2i|m|\dot{\tilde{\phi}} + x(l+2)(l-1)\right) J_{lm,xx} = H_{lm}(x), \end{aligned} \tag{34}$$

where the source terms is given by

$$H_{lm}(x) = 2x B_{lm,x} + 2x^2 D_{lm,xx} - 8x\beta_{lm,x} - 2B_{lm} - 4D_{lm} \tag{35}$$

If $\tilde{J}_{lm} = J_{lm,xx}$

$$
\begin{aligned}
&- 2x^4 \tilde{J}_{lm,xx} - 4x^2 \left(2x + i|m|\dot{\tilde{\phi}}\right) \tilde{J}_{lm,x} \\
&+ 2x \left(2i|m|\dot{\tilde{\phi}} + x(l+2)(l-1)\right) \tilde{J}_{lm} = H_{lm}.
\end{aligned} \tag{36}
$$

known as *master equation* for the Minkowski background.[8][9][10]
For a Schwarzschild's background[6]:

$$
\begin{aligned}
&J_{lm,xxxx} x^4 (2Mx - 1) + J_{lm,xxx} \left(2x^3(7Mx - 2) - 2ix^2 \dot{\tilde{\phi}}\,|m|\right) \\
&+ J_{lm,xx} \left(2ix\dot{\tilde{\phi}}\,|m| + (l-1)(l+2)x^2 + 16Mx^3\right) = G_{lm}(x),
\end{aligned} \tag{37}
$$

where

$$
G_{lm}(x) = \frac{H_{lm}(x)}{2}. \tag{38}
$$

[8] Bishop, see n. 7.
[9] T. Mädler. *Physical Review D.* **87** 10 104016 (2013) .
[10] C. E. Cedeño and J. C. N. Araujo. *Physical Review D.* **92** 12 124015 (2015) .

3.1. Minkowski's case, non-radiative solutions

- $m = 0,\ H(x) = 0$

$$(k - l + 1)(k + l + 2) = 0, \quad \Rightarrow \quad \tilde{J}_{l0}(x) = C_1 x^{l-1} + C_2 x^{-(l+2)}.$$

then

$$J_{l0}(x) = C_1 x^{l+1} + C_2 x^{-l} + C_3 x + C_4. \tag{39}$$

- $m = 0,\ H(x) \neq 0$

$$\begin{aligned} J_{l0}(x) =& C_1 x^{l+1} + C_2 x^{-l} + C_3 x + C_4 + \int_a^x dv \int_b^v dw\, w^{-(l+2)} \int_c^w dy\, \frac{H(y) y^{l-1}}{4l+2} \\ & - \int_d^x dv \int_e^v dw\, w^{l-1} \int_f^w dy\, \frac{H(y) y^{-(l+2)}}{4l+2}, \end{aligned} \tag{40}$$

where $a, b, \cdots \in \mathbb{R}$.

3.2. Minkowski's case, radiative solutions

- $m \neq 0,\ |m| \leq l$ for $l > 0,\ H(x) = 0$
  This solutions can be expressed as linear combinations of the first and second kind spherical Bessel's functions.[11]

$$\tilde{J}_{lm} = \frac{C_1 2^{\frac{1}{2}-2l} z^{3/2} e^{\frac{1}{2}i(\pi l+2z)} \Gamma\left(\frac{1}{2}-l\right)\left(K J_{-l-\frac{1}{2}} + L J_{\frac{1}{2}-l}\right)}{(l-1)l} + \frac{i C_2 2^{2l+\frac{5}{2}} z^{3/2} e^{iz-\frac{i\pi l}{2}} \Gamma\left(l+\frac{3}{2}\right)\left(K J_{l+\frac{1}{2}} + L J_{l-\frac{1}{2}}\right)}{(l+1)(l+2)}, \tag{41}$$

where $J_n := J_n(z)$,

$$z = \frac{|m|\dot{\tilde{\phi}}}{x}, \tag{42}$$

$$K = -i(l(l-1)+2iz) - 2z(l-iz), \tag{43a}$$
$$L = -2z(z-i), \tag{43b}$$
$$S = l(l-1) + 2iz. \tag{43c}$$

[11] Mädler, see n. 9.

$$J_{lm} = -\frac{iC_1 2^{\frac{1}{2}-2l}\dot{\tilde{\phi}}^2 |m|^2 z^{-1/2} e^{\frac{1}{2}i(\pi l+2z)}\Gamma\left(\frac{1}{2}-l\right)\left(-2zJ_{\frac{1}{2}-l}+\overline{S}J_{-l-\frac{1}{2}}\right)}{l^2\left(l^2-1\right)}$$
$$-\frac{C_2 2^{2l+\frac{5}{2}}\dot{\tilde{\phi}}^2 |m|^2 z^{-1/2} e^{-\frac{1}{2}i(\pi l-2z)}\Gamma\left(l+\frac{3}{2}\right)\left(2zJ_{l-\frac{1}{2}}+\overline{S}J_{l+\frac{1}{2}}\right)}{l(l+1)^2(l+2)}$$
$$+C_3+C_4\frac{\dot{\tilde{\phi}}|m|}{z}. \tag{44}$$

- $m \neq 0,\ |m| \leq l$ for $l > 0,\ H(x) \neq 0$

$$
\begin{aligned}
J_{lm} = &- \frac{iC_1 2^{\frac{1}{2}-2l} \dot{\tilde{\phi}}^2 \left|m\right|^2 z^{-1/2} e^{\frac{1}{2}i(\pi l+2z)} \Gamma\left(\frac{1}{2}-l\right)\left(-2zJ_{\frac{1}{2}-l} + \overline{S}J_{-l-\frac{1}{2}}\right)}{l^2\left(l^2-1\right)} \\
&- \frac{C_2 2^{2l+\frac{5}{2}} \dot{\tilde{\phi}}^2 \left|m\right|^2 z^{-1/2} e^{-\frac{1}{2}i(\pi l-2z)} \Gamma\left(l+\frac{3}{2}\right)\left(2zJ_{l-\frac{1}{2}} + \overline{S}J_{l+\frac{1}{2}}\right)}{l(l+1)^2(l+2)} \\
&+ \int_b^z dy \int_a^y d\tilde{z} \left(\frac{2^{\frac{1}{2}-2l} \tilde{z}^{3/2} D_1 e^{\frac{i\pi l}{2}+i\tilde{z}} \Gamma\left(\frac{1}{2}-l\right)\left(KJ_{-l-\frac{1}{2}} + LJ_{\frac{1}{2}-l}\right)}{(l-1)l} \right. \\
&\left. + \frac{i2^{2l+\frac{5}{2}} \tilde{z}^{3/2} D_2 e^{i\tilde{z}-\frac{i\pi l}{2}} \Gamma\left(l+\frac{3}{2}\right)\left(KJ_{l+\frac{1}{2}} + LJ_{l-\frac{1}{2}}\right)}{(l+1)(l+2)} \right) \\
&+ C_3 + C_4 \frac{\dot{\tilde{\phi}}|m|}{z}.
\end{aligned}
\tag{45}
$$

where

$$D_1 = -\int_{|m|\dot{\tilde{\phi}}}^{|m|\dot{\tilde{\phi}}/z} d\tilde{z}\, \frac{2^{2l-\frac{5}{2}}\tilde{z}^{-1/2}e^{-\frac{1}{2}i(\pi l+2\tilde{z})}\Gamma\left(l+\frac{1}{2}\right)\left(KJ_{l+\frac{1}{2}}-LJ_{l-\frac{1}{2}}\right)}{(l+1)(l+2)\dot{\tilde{\phi}}^2\,|m|^2}H, \tag{46a}$$

$$D_2 = -\,i\int_{|m|\dot{\tilde{\phi}}}^{|m|\dot{\tilde{\phi}}/z} d\tilde{z}\, \frac{2^{-2l-\frac{9}{2}}\tilde{z}^{-1/2}e^{\frac{1}{2}i(\pi l-2\tilde{z})}\Gamma\left(-l-\frac{1}{2}\right)\left(KJ_{-l-\frac{1}{2}}+LJ_{\frac{1}{2}-l}\right)}{(l-1)l\dot{\tilde{\phi}}^2\,|m|^2}H, \tag{46b}$$

and

$$H := H\left(\frac{\dot{\tilde{\phi}}\,|m|}{\tilde{z}}\right) \tag{46c}$$

3.3. Schwarzschild's case, non-radiative solutions

- $m = 0$, $G(x) = 0$

$$\begin{aligned}\tilde{J}_{lm} =&(-2)^{-l-2}C_1 M^{-l-2}x^{-l-2}\,{}_2F_1(2-l,-l;-2l;2Mx)\\ &+(-2)^{l-1}C_2 M^{l-1}x^{l-1}\,{}_2F_1(l+1,l+3;2l+2;2Mx).\end{aligned} \tag{47}$$

then

$$\begin{aligned}J_{lm} =&\frac{C_1(-1)^{-l}2^{-l-2}(Mx)^{-l}\,{}_3F_2(-l-1,2-l,-l;1-l,-2l;2Mx)}{l(l+1)M^2}\\ &+\frac{C_2(-1)^{l+1}2^{l-1}x(Mx)^{l}\,{}_3F_2(l,l+1,l+3;l+2,2l+2;2Mx)}{l(l+1)M}\\ &+C_3x+C_4,\end{aligned} \tag{48}$$

where ${}_pF_q(a_1,\cdots a_p;b_1,\cdots,b_q;z)$ are the generalised hyper-geometric functions.

- $m = 0$, $G(x) \neq 0$

$$\begin{aligned}\tilde{J}_{lm} =&(-1)^{1-l}2^{-l-2}M^{-l-2}x^{-l-2}\left(A_2(-1)^{2l}2^{2l+1}M^{2l+1}x^{2l+1}\right.\\ &{}_2F_1(l+1,l+3;2l+2;2Mx) - A_1\,{}_2F_1(2-l,-l;-2l;2Mx)\\ &+ C_1(-2)^{-l-2}M^{-l-2}x^{-l-2}\,{}_2F_1(2-l,-l;-2l;2Mx)\\ &+ C_2(-2)^{l-1}M^{l-1}x^{l-1}\,{}_2F_1(l+1,l+3;2l+2;2Mx),\end{aligned} \tag{49}$$

where A_1 and A_2 are given by

$$A_1 = -\int_a^x dy\,\frac{(-2)^{l+2}H(y)M^{l+2}y^l\,{}_2F_1(l+1,l+3;2(l+1);2My)}{B_1+B_2}, \tag{50a}$$

$$A_2 = \int_b^x dy\,\frac{(-2)^{1-l}H(y)M^{1-l}y^{-l-1}\,{}_2F_1(2-l,-l;-2l;2My)}{B_1+B_2}, \tag{50b}$$

and the functions B_1 and B_2 are

$$\begin{aligned}B_1 =&(2My-1)(l-2)\,{}_2F_1(3-l,-l;-2l;2My)\\ &{}_2F_1(l+1,l+3;2(l+1);2My),\end{aligned} \tag{51a}$$

$$\begin{aligned}B_2 =&\,{}_2F_1(2-l,-l;-2l;2My)(2\,{}_2F_1(l+1,l+3;2(l+1);2My)\\ &+(l+1)\,{}_2F_1(l+2,l+3;2(l+1);2My))).\end{aligned} \tag{51b}$$

3.4. Schwarzschild's case, radiative solutions

- $m \neq 0$, $G(x) = 0$

$$\tilde{J}_{lm} = C_1 L e^{\frac{2\alpha}{xM}} x^{-4} + C_2 K \left(2Mx - 1\right)^{4\alpha-2} x^{-2-4\alpha} e^{\frac{2\alpha}{xM}}, \tag{52}$$

with

$$L = H_C\left(-4\alpha, \beta; \gamma, \delta, \epsilon, \eta\right) \quad \text{and} \quad K = H_C\left(-4\alpha, -\beta; \gamma, \delta, \epsilon, \eta\right), \tag{53}$$

where $H_C(\alpha, \beta; \gamma, \delta, \epsilon, \eta)$ are the confluent Heun's functions and

$$\alpha = i\dot{\tilde{\phi}} m M, \qquad \beta = 2 - 4\alpha, \tag{54a}$$

$$\gamma = 2, \qquad \delta = 8\alpha(\alpha - 1), \tag{54b}$$

$$\epsilon = -(l+2)(l-1) - 8\alpha(\alpha - 1), \qquad \eta = \frac{2Mx - 1}{2Mx}. \tag{54c}$$

- $m \neq 0$, $G(x) \neq 0$

$$\begin{aligned} J_{lm} = & -8Me^{\frac{2a}{Mx}}\left(-LMx + M^2x^2L + L/4\right)A_1x^{-4}\left(2Mx-1\right)^{-2} \\ & + 2Me^{\frac{2a}{Mx}}x^{2-4a}\left(2Mx-1\right)^{4a}A_2Kx^{-4}\left(2Mx-1\right)^{-2} \\ & + C_1Le^{\frac{2a}{Mx}}x^{-4} + C_2Ke^{\frac{2a}{Mx}}x^{-2-4a}\left(2Mx-1\right)^{-2+4a}, \end{aligned} \tag{55}$$

where A_1 and A_2 are the integrals

$$A_1 = \int_a^x d\tilde{x}\, \frac{\tilde{x}^2 H(\tilde{x}) e^{-\frac{2a}{M\tilde{x}}} K}{-4\,LKM\tilde{x} + 8\,LKaM\tilde{x} - LS + 2\,LM\tilde{x}S + KR - 2\,KM\tilde{x}R} \tag{56a}$$

$$A_2 = \int_b^x d\tilde{x}\, \frac{4\tilde{x}^{4a} e^{-\frac{2a}{M\tilde{x}}} H(\tilde{x}) \left(M\tilde{x} - 1/2\right)^2 \left(2M\tilde{x}-1\right)^{-4a} L}{-4\,LKM\tilde{x} + 8\,LKaM\tilde{x} - LS + 2\,LM\tilde{x}S + KR - 2\,KM\tilde{x}R}, \tag{56b}$$

where $S = K'(x)$ and $R = L'(x)$ and all functional dependence were suppressed.[12]

[12] Cedeño and Araujo, see n. 10.

3.5. Comparison

- Minkowski's case, $l = 2$, $H(x) = 0$ the family of solutions for the master equation is given by

$$\tilde{J}_{2m} = E_1 x + \frac{E_2 e^{\frac{2i\dot{\tilde{\phi}}|m|}{x}} \left(6x^3\dot{\tilde{\phi}}\,|m| - 6ix^2\dot{\tilde{\phi}}^2\,|m|^2 - 4x\dot{\tilde{\phi}}^3\,|m|^3 + 2i\dot{\tilde{\phi}}^4\,|m|^4 + 3ix^4\right)}{4x^3\dot{\tilde{\phi}}^5\,|m|^5}. \tag{57}$$

Now, substituting $l = 2$ in the family of solutions (41), one obtains

$$\begin{aligned}\tilde{J}_{2m} =& \frac{iC_1\dot{\tilde{\phi}}^3\,|m|^3\,e^{\frac{2i\dot{\tilde{\phi}}|m|}{x}}}{6x^3} - \frac{40iC_2\dot{\tilde{\phi}}^3\,|m|^3\,e^{\frac{2i\dot{\tilde{\phi}}|m|}{x}}}{x^3} - \frac{C_1\dot{\tilde{\phi}}^2\,|m|^2\,e^{\frac{2i\dot{\tilde{\phi}}|m|}{x}}}{3x^2} \\ &+ \frac{80C_2\dot{\tilde{\phi}}^2\,|m|^2\,e^{\frac{2i\dot{\tilde{\phi}}|m|}{x}}}{x^2} - \frac{iC_1\dot{\tilde{\phi}}\,|m|\,e^{\frac{2i\dot{\tilde{\phi}}|m|}{x}}}{2x} + \frac{120iC_2\dot{\tilde{\phi}}\,|m|\,e^{\frac{2i\dot{\tilde{\phi}}|m|}{x}}}{x} + \frac{1}{2}C_1 e^{\frac{2i\dot{\tilde{\phi}}|m|}{x}} \\ &- 120C_2 e^{\frac{2i\dot{\tilde{\phi}}|m|}{x}} + \frac{iC_1 x e^{\frac{2i\dot{\tilde{\phi}}|m|}{x}}}{4\dot{\tilde{\phi}}\,|m|} + \frac{iC_1 x}{4\dot{\tilde{\phi}}\,|m|} - \frac{60iC_2 x e^{\frac{2i\dot{\tilde{\phi}}|m|}{x}}}{\dot{\tilde{\phi}}\,|m|} + \frac{60ix}{\dot{\tilde{\phi}}\,|m|}. \end{aligned} \tag{58}$$

Both families of solutions, (57) and (58), are completely equivalent. Note that, the transformation between (57) and (58) is given by

$$E_1 = \frac{i\,(C_1 + 240C_2)}{4\dot{\tilde{\phi}}\,|m|}, \qquad E_2 = \frac{1}{3}\,(C_1 - 240C_2)\,\dot{\tilde{\phi}}^4\,|m|^4\,. \tag{59}$$

- For the Schwarzschild case, $l = 2$, $H(x) = 0$, the master equation is reduced to

$$x^2(2Mx-1)\tilde{J}_{2m,xx} + 2x(7Mx-2)\tilde{J}_{2m,x} + (16Mx+4)\tilde{J}_{2m} = 0, \tag{60}$$

whose families of solutions are

$$\begin{aligned}\tilde{J}_{2m} =& \frac{C_1}{x^4} - \frac{C_2\left(16M^4x^4 + 32M^3x^3 - 44M^2x^2 - 4Mx + 7\right)}{64M^5x^4(1-2Mx)^2} \\ &+ \frac{12C_2(1-2Mx)^2\log(1-2Mx)}{64M^5x^4(1-2Mx)^2}.\end{aligned} \tag{61}$$

Now, specialising the solutions (47) for $l = 2$, we find a totally equivalent solution, i.e.,

$$\tilde{J}_{lm} = \frac{D_1}{16M^4x^4} + \frac{5D_2\left(2Mx\left(2M^3x^3+4M^2x^2-9Mx+3\right)\right)}{8M^4x^4(1-2Mx)^2} + \frac{15D_2(1-2Mx)^2\log(1-2Mx)}{8M^4x^4(1-2Mx)^2}. \tag{62}$$

A series expansion of both solutions shows

$$D_1 = \frac{64C_1M^5 - 7C_2}{4M} \quad \text{and} \quad D_2 = -\frac{C_2}{10M}. \tag{63}$$

- For Schwarzschild's background and $l = 2$, the solution was shown by Bishop[13] were power series around $r = 2M$ were used.
- We expand the radiative family of solutions for the master equation (37) for $l = 2$. The $H_C(-4\alpha, \beta; \gamma, \delta, \epsilon, \eta)$ function is expressed as a Taylor series

[13] Bishop, see n. 7.

for the parameters (54) around $\eta = 0$, namely

$$H_C(-4\alpha, \beta; \gamma, \delta, \epsilon, \eta) \simeq 1 + \frac{\left((4a+1)^2 - 5 + (l-1)(l+2)\right)\eta}{-3+4\,a} + \frac{1}{8(a-1)(4a-3)}\left(\left(256a^4 + 192a^3 + 32a^2\left(l^2+l-5\right) + 4a\left(4l^2+4l-39\right) + l^4 + 2l^3 - 17l^2 - 18l + 72\right)\eta^2\right), \tag{64}$$

and for $H_C(-4\alpha, -\beta; \gamma, \delta, \epsilon, \eta)$ function one has

$$H_C(-4\alpha, -\beta; \gamma, \delta, \epsilon, \eta) \simeq 1 - \frac{\left(4a + l^2 + l\right)\eta}{4a-1} - \frac{\left(12a - l^4 - 2l^3 + l^2 + 2l\right)\eta^2}{8a(4a-1)}. \tag{65}$$

Then, from (64) and (65) we obtain that around $r = 2M$, (52) at first

order for $l = 2$ reads

$$\begin{aligned}\tilde{J}_{lm} =& C_1 \left(\frac{16e^{4\alpha}(4\alpha+12)\eta M^4}{4\alpha-3} + 16e^{4\alpha}M^4 \right) \\ &- \frac{2^{4\alpha+2}C_2 e^{4\alpha}\left(16\alpha^2+16\alpha+2\right)\eta^{4\alpha-1}\left(\frac{1}{M}\right)^{-4\alpha-2}}{4\alpha-1} \\ &+ \frac{2^{4\alpha-1}C_2 e^{4\alpha}\left(256\alpha^4+576\alpha^3+384\alpha^2+132\alpha+24\right)\eta^{4\alpha}\left(\frac{1}{M}\right)^{-4\alpha-2}}{\alpha(4\alpha-1)} \\ &- \frac{2^{4\alpha+1}C_2 e^{4\alpha}\left(256\alpha^5+896\alpha^4+1056\alpha^3\right)\eta^{4\alpha+1}\left(\frac{1}{M}\right)^{-4\alpha-2}}{3\alpha(4\alpha-1)} \\ &- \frac{2^{4\alpha+1}C_2 e^{4\alpha}\left(636\alpha^2+228\alpha+72\right)\eta^{4\alpha+1}\left(\frac{1}{M}\right)^{-4\alpha-2}}{3\alpha(4\alpha-1)} \\ &+ 2^{4\alpha+2}C_2 e^{4\alpha}\eta^{4\alpha-2}\left(\frac{1}{M}\right)^{-4\alpha-2}, \end{aligned} \tag{66}$$

that are just the family of solutions for the master equation obtained using power series around $r = 2M$.[14]

[14] Cedeño and Araujo, see n. 10.

4. Applications

4.1. Thin Shell

✡ Bishop considers a static thin shell,[15] with a matter distribution given by

$$\rho = \rho_0 \delta(r - r_0)\ {}_0Z_{2m}. \tag{4.1}$$

✡ We reproduce this example in details.

✡ This system can not produce gravitational radiation.

✡ Here it is supposed that the metric variables are decomposed as

$${}_sf = \Re\,(f_0)\ \eth^s\ Z_{2m}, \tag{4.2}$$

where $f := \beta, w, U, J$ and ${}_sf_{,u} = 0$.

[15] Bishop, see n. 7.

✡ The field equations are strongly simplified

$$\frac{d\beta_0}{dr} = 0, \tag{4.3a}$$

$$-4r\frac{dJ_0}{dr} + 4r^2\frac{dU_0}{dr} + r^3\frac{d^2U_0}{dr^2} + 4\beta_0 = 0, \tag{4.3b}$$

$$3r^2\frac{dU_0}{dr} + \frac{dw_0}{dr} - 12J_0 + 12rU_0 - 8\beta_0 = 0, \tag{4.3c}$$

$$-2r\frac{dJ_0}{dr} + r^2\frac{dU_0}{dr} - r^2\frac{d^2J_0}{dr^2} + 2rU_0 - 2\beta_0 = 0, \tag{4.3d}$$

$$-r^2\frac{d^2w_0}{dr^2} + 6w_0 + 12r\beta_0 = 0, \tag{4.3e}$$

$$6r^2\frac{dU_0}{dr} + r\frac{d^2w_0}{dr^2} + 12rU_0 - 12\beta_0 = 0, \tag{4.3f}$$

$$4r^3\frac{dU_0}{dr} + r\frac{dw_0}{dr} + r^4\frac{d^2U_0}{dr^2} + 2r^2U_0 - w_0 = 0. \tag{4.3g}$$

- The master equation is strongly simplified

$$x^3 \frac{d^2 J_2}{dx^2} + 4x^2 \frac{dJ_2}{dx} - 4x J_2 = 0, \tag{4.4}$$

✡ Family of solutions:

$$\begin{aligned}
J_0(r) &= C_1 + C_2 r^2 + \frac{C_3}{r} + \frac{C_4}{r^3}, && \text{(4.5a)}\\
\beta(r) &= \beta_0, && \text{(4.5b)}\\
U_0(r) &= -\frac{3C_4}{r^4} - \frac{C_5}{3r^3} + \frac{2C_3}{r^2} + 2C_2 r + C_6 + \frac{2\beta_0}{r}, && \text{(4.5c)}\\
w_0(r) &= -6C_2 r^3 - 6C_6 r^2 - \frac{6C_4}{r^2} + 12C_1 r - \frac{C_5}{r} + C_7 - 10 r\beta_0. && \text{(4.5d)}
\end{aligned}$$

✡ From the constraints

$$6C_6 r^2 + \frac{C_5}{r} = 0, \tag{4.6a}$$

$$12C_6 r^2 - 36C_1 r + \frac{2C_5}{r} - 3C_7 + 24 r\beta_0 = 0, \tag{4.6b}$$

$$-4C_6 r^2 + \frac{4C_5}{3r} - C_7 = 0, \tag{4.6c}$$

(4.3f) is satisfied identically.

✡ Solving C_5 in (4.6a) and replacing it in (4.6b) and (4.6c)

$$\begin{aligned} C_5 &= -6C_6 r^3, && \text{(4.7a)} \\ C_7 + 4(3C_1 - 2\beta_0)r &= 0, && \text{(4.7b)} \\ -12C_6 r^2 - C_7 &= 0. && \text{(4.7c)} \end{aligned}$$

✡ Substituting C_5 into equations (4.5) we obtain

$$U_0(r) = -\frac{3C_4}{r^4} + 3C_6 + \frac{2C_3}{r^2} + 2C_2 r + \frac{2\beta_0}{r}, \qquad \text{(4.8a)}$$

$$w_0(r) = -6C_2 r^3 - \frac{6C_4}{r^2} + 12C_1 r + C_7 - 10r\beta_0. \qquad \text{(4.8b)}$$

✡ We consider two separate regions of the space-time formed by the world tube which binds the matter distribution i.e., $r < r_0$ and $r > r_0$.

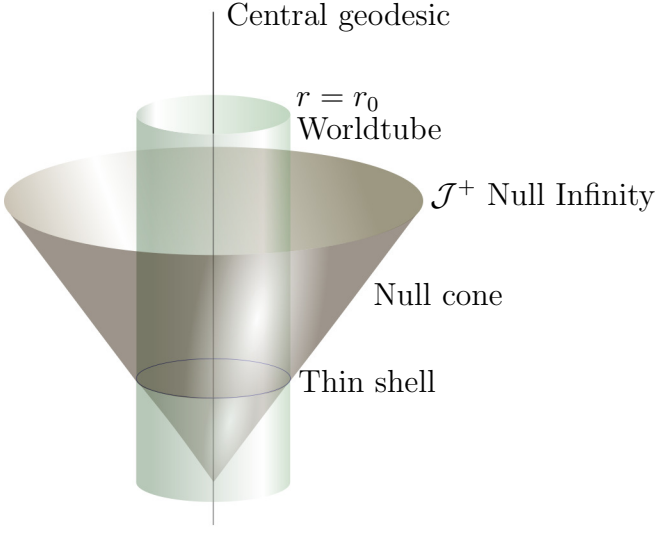


Figure 2: Sketch of the world tube generated by the thin shell. Here we note the two regions ($r < r_0$ and $r > r_0$) in which the space-time is divided.

✡ The family of solutions for the interior region can be written as

$$\beta_{0-}(r) = \beta_{0-}, \tag{4.9a}$$

$$J_{0-}(r) = C_{1-} + r^2 C_{2-} + \frac{C_{3-}}{r} + \frac{C_{4-}}{r^3}, \tag{4.9b}$$

$$U_{0-}(r) = -\frac{3C_{4-}}{r^4} + 3C_{6-} + \frac{2C_{3-}}{r^2} + 2rC_{2-} + \frac{2\beta_{0-}}{r}, \tag{4.9c}$$

$$w_{0-}(r) = -6r^3 C_{2-} - \frac{6C_{4-}}{r^2} + 12rC_{1-} + C_{7-} - 10r\beta_{0-}. \tag{4.9d}$$

✡ It is imposed convergence of the metric functions given in (30) at the vertices of the null cones.

✡ We expand the metric functions in power series of r around the vertex of the null cones and impose convergence at this limit.

✡ Substituting (4.9d) into (26) one obtains

$$g_{11-} = 6r^2 C_{2-} + \frac{6C_{4-}}{r^3} - 12C_{1-} - \frac{C_{7-}}{r} + 12\beta_{0-} + 1, \tag{4.10a}$$

$$g_{12-} = -1 - 2\beta_{0-}, \tag{4.10b}$$

$$g_{33-} = \frac{2}{(1+\zeta\bar{\zeta})^2}\left[\Re\left(r^2C_{1-} + r^4C_{2-} + rC_{3-} + \frac{C_{4-}}{r}\right)\times \left(\eth^2 + \bar{\eth}^2\right)\ {}_0Z_{2m} + 2r^2\right], \tag{4.10c}$$

$$g_{34-} = -\frac{2i}{(1+\zeta\bar{\zeta})^2}\Re\left(r^2C_{1-} + r^4C_{2-} + rC_{3-} + \frac{C_{4-}}{r}\right)\times \left(\eth^2 - \bar{\eth}^2\right)\ {}_0Z_{2m}, \tag{4.10d}$$

$$g_{44-} = -\frac{2}{(1+\zeta\bar{\zeta})^2}\left[\Re\left(r^2C_{1-} + r^4C_{2-} + rC_{3-} + \frac{C_{4-}}{r}\right)\times \left(\eth^2 + \bar{\eth}^2\right)\ {}_0Z_{2m} - 2r^2\right]. \tag{4.10e}$$

✡ When $r \to 0$, (4.10a) implies that

$$C_{4-} = 0, \tag{4.11}$$

$$C_{7-} = 0, \tag{4.12}$$

✡ Thus,

$$\lim_{r\to 0} g_{11-} = \lim_{r\to 0} \left(6r^2 C_{2-} - 12(C_{1-} + \beta_{0-}) + 1\right), \\ = -12(C_{1-} - \beta_{0-}) + 1. \tag{4.13}$$

✡ Then, if we expect a flat space-time at the null cone vertices, we must have

$$C_{1-} = \beta_{0-}. \tag{4.14}$$

✡ The convergence of J_{0-} is required at the vertex of the null cones. Thus, from (4.9b), (4.9c) and (4.14) one has

$$C_{3-} = 0, \tag{4.15}$$
$$\beta_{0-} = 0, \tag{4.16}$$
$$C_{1-} = 0. \tag{4.17}$$

✡ It implies that (4.7b) is satisfied identically, whereas from (4.7c) one obtains

$$C_{6-} = 0. \tag{4.18}$$

✡ Substituting these constants in the families of solutions (4.9) we obtain for the interior region that

$$\beta_{0-}(r) = 0, \tag{4.19a}$$
$$J_{0-}(r) = r^2 C_{2-}, \tag{4.19b}$$
$$U_{0-}(r) = 2rC_{2-}, \tag{4.19c}$$
$$w_{0-}(r) = -6r^3 C_{2-}, \tag{4.19d}$$

✡ For the exterior region

$$\beta_{0+}(r) = \beta_{0+}, \tag{4.20a}$$
$$J_{0+}(r) = C_{1+} + r^2 C_{2+} + \frac{C_{3+}}{r} + \frac{C_{4+}}{r^3}, \tag{4.20b}$$
$$U_{0+}(r) = -\frac{3C_{4+}}{r^4} + 3C_{6+} + \frac{2C_{3+}}{r^2} + 2rC_{2+} + \frac{2\beta_{0+}}{r}, \tag{4.20c}$$
$$w_{0+}(r) = -6r^3 C_{2+} - \frac{6C_{4+}}{r^2} + 12rC_{1+} + C_{7+} - 10r\beta_{0+}. \tag{4.20d}$$

✡ For convergent solutions at the null infinity $\mathcal{J}_+$, we see from (4.20b) that

$$C_{2+} = 0. \tag{4.21}$$

✡ Thus,

$$\lim_{r\to\infty} J_{0+}(r) = C_{1+}. \tag{4.22}$$

✡ We rename this constant as

$$C_{1+} = J_{0\infty}, \tag{4.23}$$

✡ Now, it is required that

$$\lim_{r\to\infty} U_{0+}(r) = 0, \qquad \Rightarrow \qquad C_{6+} = 0. \tag{4.24}$$

✡ Thus, $U_{0+}(r)$ takes the form

$$U_{0+}(r) = -\frac{3C_{4+}}{r^4} + \frac{2C_{3+}}{r^2} + \frac{2\beta_{0+}}{r}. \tag{4.25}$$

✡ The constraint (4.7c) fixes C_{7+}, i.e.,

$$C_{7+} = 0, \tag{4.26}$$

✡ When (4.26) is used on the constraint (4.7b) we find the explicit value for β_{0+}, namely

$$\beta_{0+} = \frac{3}{2} J_{0\infty}. \tag{4.27}$$

✡ Then

$$J_{0+}(r) = J_{0\infty} + \frac{C_{3+}}{r} + \frac{C_{4+}}{r^3}, \tag{4.28a}$$

$$U_{0+}(r) = -\frac{3C_{4+}}{r^4} + \frac{2C_{3+}}{r^2} + \frac{3J_{0\infty}}{r}, \tag{4.28b}$$

$$w_{0+}(r) = -\frac{6C_{4+}}{r^2} - 3rJ_{0\infty}. \tag{4.28c}$$

✡ The family of solutions for the exterior region depends only on two constants.

✡ For $l > 2$ the exterior solutions will depend only on two constants.

✡ In order to fix C_{2-}, C_{3+} and C_{4+}, we impose that at $r = r_0$:

$$\beta_0\big|_{r_{0-}}^{r_{0+}} = 2\pi r_0\rho_0, \quad J_0\big|_{r_{0-}}^{r_{0+}} = 0, \quad U_0\big|_{r_{0-}}^{r_{0+}} = 0, \quad w_0\big|_{r_{0-}}^{r_{0+}} = -2r_0\beta_{0+}, \tag{4.29}$$

where $\beta_{0+} = \beta_0(r_{0+})$, and

$$\left.\frac{dU_0}{dr}\right|_{r_{0-}}^{r_{0+}} = \frac{2\beta_{0+}}{r_0^2}, \qquad \left.\frac{dJ_0}{dr}\right|_{r_{0-}}^{r_{0+}} = 0. \tag{4.30}$$

✡ From (4.29), (4.19a) and (4.20a) the function $\beta(r)$ results in

$$\beta(r) = \beta_{0+}\Theta(r - r_0), \tag{4.31}$$

where $\Theta(r)$ is the Heaviside's function, namely

$$\Theta(r) = \begin{cases} 0 & \text{for} \quad r \leq 0 \\ 1 & \text{for} \quad r > 0 \end{cases}. \tag{4.32}$$

✡ Evaluating the continuity conditions (4.29) and (4.30) one obtains

$$C_{3+} = -r_0 J_{0\infty}, \tag{4.33a}$$

$$C_{4+} = \frac{1}{5} r_0^3 J_{0\infty}, \tag{4.33b}$$

$$C_{2-} = \frac{1}{5r_0^2} J_{0\infty}. \tag{4.33c}$$

✡ Finally, the solution for the field equations reads

$$\beta(r) = \frac{3}{2} J_{0\infty} \Theta(r - r_0), \tag{4.34a}$$

$$J(r) = \frac{r^2}{5r_0^2} J_{0\infty} \left(1 - \Theta(r - r_0)\right) + J_{0\infty} \left(1 - \frac{r_0}{r} + \frac{r_0^3}{5r^3}\right) \Theta(r - r_0), \tag{4.34b}$$

$$U(r) = \frac{2r}{5r_0^2} J_{0\infty} \left(1 - \Theta(r - r_0)\right) + J_{0\infty} \left(-\frac{3r_0^3}{5r^4} - \frac{2r_0}{r^2} + \frac{3}{r}\right) \Theta(r - r_0), \tag{4.34c}$$

$$w(r) = -\frac{6r^3}{5r_0^2} J_{0\infty} \left(1 - \Theta(r - r_0)\right) - J_{0\infty} \left(\frac{6r_0^3}{5r^2} + 3r\right) \Theta(r - r_0). \tag{4.34d}$$

✡ From (4.27) and (4.29) one obtains

$$J_{0\infty} = \frac{4}{3} \pi r_0 \rho_0. \tag{4.35}$$

✡ We plot the solutions (4.34) in Figure 3, in terms of a compactified coordinate s,

$$s = \frac{r}{r + R_0} \tag{4.36}$$

where R_0 is called *a compactification parameter*.

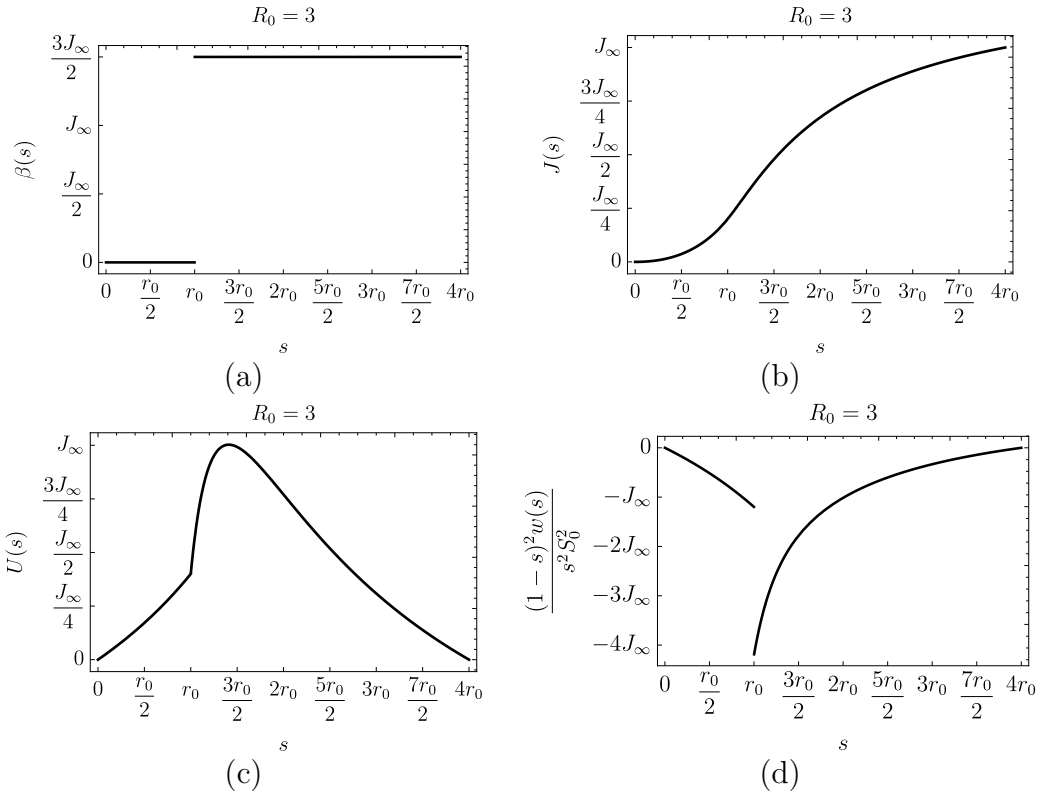


Figure 3: Metric variables as a function of the luminosity distance r for a thin shell of $r = r_0$, centred at the origin. (a) $\beta_0 := \beta_0(r)$, (b) $J_0 := J_0(r)$, (c) $U_0 := U_0(r)$, (d) $w_0 := w_0(r)$

4.2. Point Particle Binary System with Different Masses

- Bishop et. al. considered particles with equal masses.[16]
- We show that the Peters and Mathews result for the power radiated in gravitational waves[17] can be obtained by using the characteristic formulation and the News function.
- We consider that these two particles are in a Minkowski's background, in exactly the same way Peters and Mathews did in their paper of 1963 and Bishop et. al. did in their paper
- Such a system allows one to explore in full detail the boundary conditions across the hypersurfaces generated by their orbits (see Figure 4).
- The density that describes the binary system is given by

$$\rho = \frac{\delta(\theta - \pi/2)}{r^2}\left(M_1\delta(r - r_1)\delta(\phi - \nu u) + M_2\delta(r - r_2)\delta(\phi - \nu u - \pi)\right), \tag{4.37}$$

where, r_i (M_i) are the orbital radius (mass) of each particle and $r_1 < r_2$.

[16] Bishop, Pollney, and Reisswig, see n. 7.

[17] P. C. Peters and J. Mathews. *Physical Review.* **131** 1 435–440 (1963) .

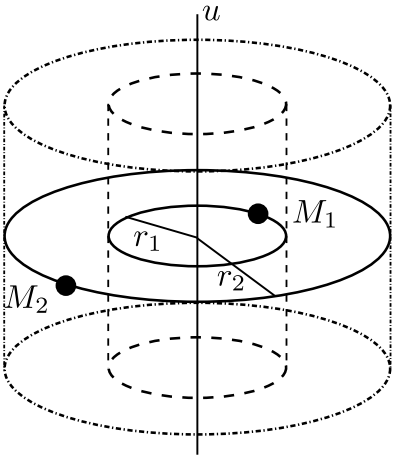


Figure 4: Binary system with the world tubes of each orbit extended along the direction of the retarded time, separating the space-time into three regions.

- The orbit of each mass generates world tubes, which are extended along the retarded time, allowing the separation of the space-time into three empty regions: inside, between and outside the matter distribution.

- The metric variables are expanded as in (30), taking $\tilde{\phi} = \nu u$, namely

$$ {}_sf = \sum_{l,m} \Re(f_{lm} e^{i|m|\nu u}) \eth^s \, {}_0Z_{lm} $$

- The families of solutions, for $l = 2$, satisfying this system of equations for the vacuum read

$$ \beta_{2m}(r) = D_{1\beta 2m}, \tag{4.38a} $$

$$J_{2m}(r) = \frac{2iD_{1\beta 2m}}{\nu r\,|m|} - \frac{D_{1J2m}(\nu r\,|m|-1)(\nu r\,|m|+1)}{6r^3} - \frac{iD_{2J2m}e^{2i\nu r|m|}(\nu r\,|m|+i)^2}{8\nu^5 r^3\,|m|^5} + \frac{D_{3J2m}(\nu r\,|m|-3i)}{\nu r\,|m|}, \tag{4.38b}$$

$$U_{2m}(r) = \frac{2D_{1\beta 2m}(\nu r\,|m|+2i)}{\nu r^2\,|m|} - \frac{D_{1J2m}\left(2\nu^2 r^2\,|m|^2+4i\nu r\,|m|+3\right)}{6r^4} - \frac{D_{2J2m}e^{2i\nu r|m|}(2\nu r\,|m|+3i)}{8\nu^5 r^4\,|m|^5} - \frac{iD_{3J2m}\left(\nu^2 r^2\,|m|^2+6\right)}{\nu r^2\,|m|}, \tag{4.38c}$$

$$w_{2m}(r) = -10rD_{1\beta 2m} + 6rD_{3J2m}(2+i\nu r\,|m|) - \frac{3iD_{2J2m}e^{2i\nu r|m|}}{4\nu^5 r^2\,|m|^5} - \frac{iD_{1J2m}((1+i)\nu r\,|m|-i)(1+(1+i)\nu r\,|m|)}{r^2}, \tag{4.38d}$$

- For the interior region and $l = 2$

$$\beta_{2m-}(r) = 0, \tag{4.39a}$$

$$J_{2m-}(r) = \frac{D_{2J2m-}}{24\nu^5 r^3 |m|^5}\Big(2\nu^3 r^3 |m|^3 - 3i\nu^2 r^2 |m|^2 e^{2i\nu r|m|} - 3i\nu^2 r^2 |m|^2 + 6\nu r |m| e^{2i\nu r|m|} + 3ie^{2i\nu r|m|} - 3i\Big), \tag{4.39b}$$

$$U_{2m-}(r) = -\frac{iD_{2J2m-}}{24\nu^5 r^4 |m|^5}\Big(2\nu^4 r^4 |m|^4 + 6\nu^2 r^2 |m|^2 - 6i\nu r |m| e^{2i\nu r|m|} - 12i\nu r |m| + 9e^{2i\nu r|m|} - 9\Big), \tag{4.39c}$$

$$w_{2m-}(r) = \frac{D_{2J2m-}}{4\nu^5 r^2 |m|^5}\Big(2i\nu^4 r^4 |m|^4 + 4\nu^3 r^3 |m|^3 - 6i\nu^2 r^2 |m|^2 - 6\nu r |m| - 3ie^{2i\nu r|m|} + 3i\Big). \tag{4.39d}$$

- For the intermediate region, the same structure of the general solutions is maintained.
- There is no reason to discard any particular term, or to establish any relationship between the constants as occurs for the interior region.

- For the exterior region, since that regularity is required at the null infinity, the coefficient of the exponential factor ($\exp(2i\nu r|m|)$) must be null.
- All constants of the form D_{2Jlm+}, with $l = 2, 3, \cdots$, must be zero. Therefore, the number of degrees of freedom for the exterior family of solutions is reduced in one parameter.
- A family of solutions for the field equations (29a)-(29g), with eight parameters to be determined, for describing the whole space-time is obtained.
- To fix these eight constants, it is necessary to impose additional boundary conditions in particular across the time-like world tubes generated by their orbits.
- These boundary conditions across the world tubes, i.e. when $r = r_i$, $i = 1, 2$, come from imposing discontinuities on the metric coefficients, i.e.,

$$\begin{aligned} &[g_{11}]_{r_i} = 0, \ [g_{12}]_{r_i} = \Delta g_{12}|_{r_i}, \ [g_{1A}]_{r_i} = 0, \ [g_{22}]_{r_i} = 0, \\ &[g_{2A}]_{r_i} = 0, \ [g_{3\mu}]_{r_i} = 0, \ [g_{4\mu}]_{r_i} = 0, \end{aligned} \tag{4.40}$$

and on their first derivatives,

$$\left[g'_{\mu\nu}\right]_{r_i} = \Delta g'_{\mu\nu}, \;\; \mu,\nu = 1,\cdots 4, \tag{4.41}$$

where the brackets mean $[f(r)]_{r_i} = f(r)|_{r_i+\epsilon} - f(r)|_{r_i-\epsilon}$.

- From (25), (4.40) and (4.41), the coefficients β_{lm}, J_{lm}, U_{lm} and w_{lm} satisfy

$$\begin{aligned} [w_{lm}(r_j)] &= \Delta w_{jlm}, \quad & [\beta_{lm}(r_j)] &= \Delta\beta_{jlm}, \\ [J_{lm}(r_j)] &= 0, & [U_{lm}(r_j)] &= 0, \end{aligned} \tag{4.42}$$

- and their first derivatives satisfy

$$\begin{aligned} [w'_{lm}(r_j)] &= \Delta w'_{jlm}, \quad & [\beta'_{lm}(r_j)] &= \Delta\beta'_{jlm}, \\ [J'_{lm}(r_j)] &= \Delta J'_{jlm}, & [U'_{lm}(r_j)] &= \Delta U'_{jlm}, \end{aligned} \tag{4.43}$$

where $j = 1, 2$, and Δw_{jlm}, $\Delta\beta_{jlm}$, $\Delta w'_{jlm}$, $\Delta\beta'_{jlm}$, $\Delta J'_{jlm}$ and $\Delta U'_{jlm}$ are functions to be determined.

- Equations (4.42) and (4.43) were solved simultaneously for both world tubes

$$\Delta\beta_{jlm} = b_{jlm}, \quad \Delta w_{jlm} = -2r_j b_{jlm}, \tag{4.44a}$$

where b_{jlm} are constants. Note that, this last fact implies that $\Delta\beta'_{jlm} = 0$.

- We obtained the jumps for the first derivative of the J_{lm} and U_{lm} functions

$$\Delta J'_{jlm} = \frac{8\nu^2 r_j b_{jlm} |m|^2}{(l-1)l(l+1)(l+2)}, \tag{4.44b}$$

$$\Delta U'_{jlm} = 2b_{ilm}\left(\frac{1}{r_i^2} - \frac{4i\nu|m|}{l(l+1)r_i}\right). \tag{4.44c}$$

- Then, it is possible to write ${}_sf_{lm}$ as

$$\begin{aligned} {}_sf_{lm}(r) = {}_sf_{1lm}(r)\left(1-\Theta(r-r_1)\right) + {}_sf_{2lm}(r)\left(\Theta(r-r_1)-\Theta(r-r_2)\right) \\ + {}_sf_{3lm}(r)\Theta(r-r_2), \end{aligned} \tag{4.45}$$

  where ${}_sf_{lm} = \{\beta_{lm},\ J_{lm},\ U_{lm},\ w_{lm}\}$, with the first subscript on the right hand side terms indicating the interior (1), the middle (2) and the exterior (3) solutions.

- These solutions depend explicitly on two specific parameters, b_{jlm}, with $j = 1, 2$.

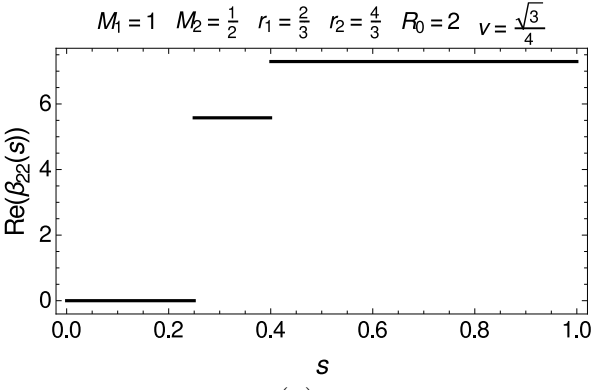
- The specific form of these relationships is obtained by just integrating (29a) across each world tube

$$b_{jlm} = 2\pi r_j \rho_{jlm}\left(1+v_j^2\right), \tag{4.46}$$

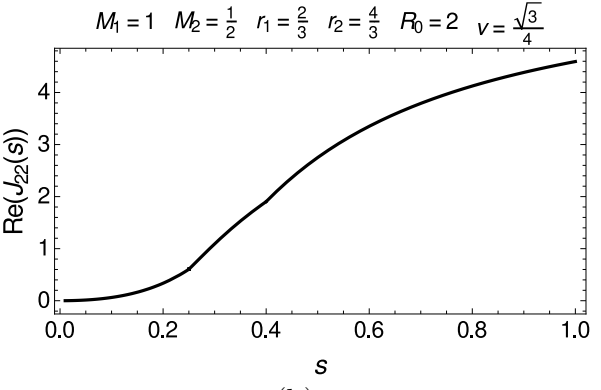
where, ρ_{jlm} are given by

$$\rho_{jlm} = \frac{1}{\pi} \int_S d(\nu u) \int_\Omega d\Omega \int_{I_j} dr \; {}_0\overline{Z}_{lm} e^{-i|m|\nu u} \rho, \tag{4.47}$$

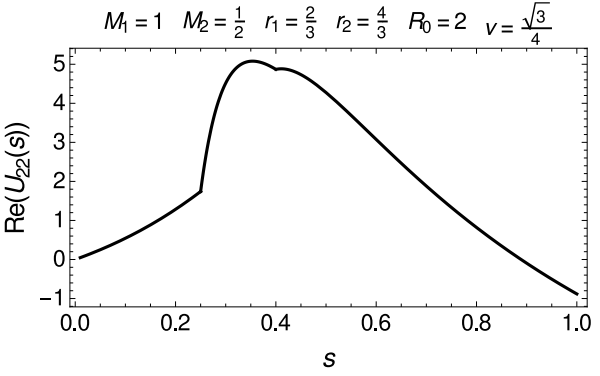
in which $S = [0, 2\pi)$, v_j is the physical velocity of the particle j in the space, and I_j is an interval ϵ around r_j that is given by $I_j = (r_j - \epsilon/2, r_j + \epsilon/2)$, with $\epsilon > 0$.



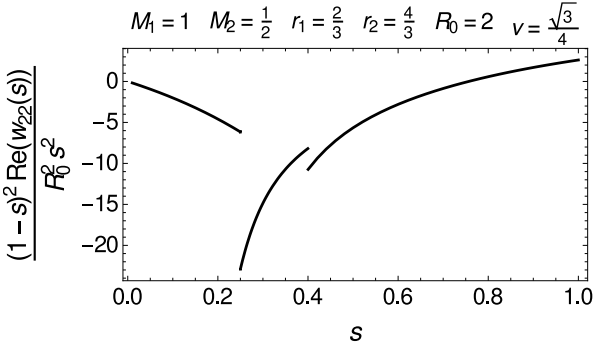
(a)



(b)



(c)



(d)

Figure 5: Real part of some components of the metric functions ($l = m = 2$) versus the compactified coordinate s (see the text) for a binary system with $M_1 = 1/2$, $M_2 = 1$. The angular velocity is computed by means of Kepler's third law. Here r_1 and r_2 are referred to the center of mass of the system.

- $M_1 = 1/2$, $M_2 = 1$, $R_0 = 2$, and r_1 and r_2 are referred to the centre of mass of the system

$$r_j = \frac{\mu}{M_j} d_0, \;\; j = 1, 2, \tag{4.48}$$

where μ is the reduced mass of the system and d_0 is the distance between the masses.

- ν is computed by means of Kepler's third law, i.e.,

$$\nu = \sqrt{\frac{M_1 + M_2}{d_0^3}}. \tag{4.49}$$

- Figure 6 illustrates the behaviour of β, J, U and w. We present them as functions of s and ϕ for a particular value of the retarded time u.

(a) (b)

(c) (d)

Figure 6: Snapshots of the metric variables as seen from the equatorial plane ($\theta = \pi/2$), as a function of s and ϕ for $u = \pi/2$. Here $M_1 = 16$, $M_2 = 4/3$, $r_1 = 1/13$, $r_2 = 12/13$, $R_0 = 1/2$ and $\nu = 2\sqrt{13/3}$. (a) $\beta(s,\phi)$, (b) $J(s,\phi)$, (c) $U(s,\phi)$ and (d) $W(s,\phi) = w(s,\phi)(1-s^2)/(s^2R_0^2)$.

– These functions are constructed by using Equations (30) , and the solutions for the coefficients for each l and m. In this case, we use $l \leq 8$.

– Since the first field equation for the vacuum $\beta_{,r} = 0$ implies that β_{lm} are constants along r, as sketched in Figures 5, and that β is a gauge term for the gravitational potential.

– Then, Φ can be redefined as $\Phi = w/(2r)$. These facts make the choice of the angular velocity ν as obeying Kepler's third law, completely consistent and natural.

4.2.1. Gravitational Radiation from the Binary System

– The Bondi's News function in the weak field approximation[18]

$$\mathcal{N} = \lim_{r\to\infty}\left(-\frac{r^2 J_{,ur}}{2} + \frac{\eth^2\omega}{2} + \eth^2\beta\right). \tag{4.50}$$

– Explicitly for $l \geq 2$ and $-l \leq m \leq l$, this takes the form

$$\mathcal{N} = \lim_{r\to\infty}\sum_{l,m}\Re\left(\left(-\frac{i|m|\nu r^2\left(J_{lm}\right)_{,r}}{2} + \beta_{lm} + \frac{l(l+1)J_{lm}}{4}\right)e^{i|m|\nu u}\right)\eth^2\,{}_0Z_{lm}. \tag{4.51}$$

– Substituting the coefficients of the metric variables for the exterior

[18] Bishop, see n. 7.

region, one obtains

$$
\begin{aligned}
\mathcal{N} = \quad & -\frac{i\nu^3\ {}_2S_{21}}{\sqrt{6}} - 4i\sqrt{\frac{2}{3}}\nu^3\ {}_2S_{22} - \frac{i\nu^4\ {}_2S_{31}}{\sqrt{30}(\nu-3i)} - \frac{8i\sqrt{\frac{2}{15}}\nu^4\ {}_2S_{32}}{2\nu-3i} \\
& - \frac{9i\sqrt{\frac{3}{10}}\nu^4\ {}_2S_{33}}{\nu-i} - \frac{i\nu^5\ {}_2S_{41}}{3\sqrt{10}\,(\nu^2-7i\nu-14)} - \frac{8i\sqrt{\frac{2}{5}}\nu^5\ {}_2S_{42}}{3\,(2\nu^2-7i\nu-7)} \\
& - \frac{81i\nu^5\ {}_2S_{43}}{\sqrt{10}\,(9\nu^2-21i\nu-14)} - \frac{256i\sqrt{\frac{2}{5}}\nu^5\ {}_2S_{44}}{3\,(8\nu^2-14i\nu-7)} \\
& + \frac{\nu^6\ {}_2S_{51}}{\sqrt{210}\,(i\nu^3+12\nu^2-54i\nu-90)} + \frac{16\sqrt{\frac{2}{105}}\nu^6\ {}_2S_{52}}{4i\nu^3+24\nu^2-54i\nu-45} \\
& + \frac{27\sqrt{\frac{3}{70}}\nu^6\ {}_2S_{53}}{3i\nu^3+12\nu^2-18i\nu-10} + \frac{1024\sqrt{\frac{2}{105}}\nu^6\ {}_2S_{54}}{32i\nu^3+96\nu^2-108i\nu-45} \\
& + \frac{625\sqrt{\frac{5}{42}}\nu^6\ {}_2S_{55}}{25i\nu^3+60\nu^2-54i\nu-18} + \cdots\,, \qquad (4.52)
\end{aligned}
$$

where we define the spin 2 quantity ${}_2S_{lm}$ as

$$ {}_2S_{lm} = \frac{\left(\Re(D_{1Jlm+}e^{i|m|\nu u})\ \eth^2\ {}_0Z_{lm} + \Re(D_{1Jl-m+}e^{i|m|\nu u})\ \eth^2\ {}_0Z_{l\ -m}\right)}{\sqrt{(l-1)l(l+1)(l+2)}}. \tag{4.53} $$

– Since the binary system is confined to a plane, then a natural choice to simplify the problem of expressing the News function, without loss of generality, is to put the masses to move on the equatorial plane $\theta = \pi/2$.

– Then, symmetry of reflection for the density of matter and, consequently, for the space-time.

– This choice restricts the components of the density, obtained from (4.47), to have the following form

$$ \rho_{lm} = \begin{cases} \tilde{\rho}_{lm}\dfrac{M_2 r_1^2\delta\left(r-r_2\right) + M_1 r_2^2\delta\left(r-r_1\right)}{r_1^2 r_2^2} & \text{if } l,m \text{ even} \\ \tilde{\rho}_{lm}\dfrac{M_1 r_2^2\delta\left(r-r_1\right) - M_2 r_1^2\delta\left(r-r_2\right)}{r_1^2 r_2^2} & \text{if } l,m \text{ odd,} \end{cases} \tag{4.54} $$

where $\tilde{\rho}_{lm}$ are constants.

– Therefore, for binaries of different masses, the News function (4.52) is simplified to

$$\begin{aligned}\mathcal{N} = \quad & -4i\sqrt{\frac{2}{3}}\nu^3\ {}_2S_{22} - \frac{i\nu^4\ {}_2S_{31}}{\sqrt{30}(\nu-3i)} - \frac{9i\sqrt{\frac{3}{10}}\nu^4\ {}_2S_{33}}{\nu-i} - \frac{8i\sqrt{\frac{2}{5}}\nu^5\ {}_2S_{42}}{3\left(2\nu^2-7i\nu-7\right)} \\ & - \frac{256i\sqrt{\frac{2}{5}}\nu^5\ {}_2S_{44}}{3\left(8\nu^2-14i\nu-7\right)} + \frac{\nu^6\ {}_2S_{51}}{\sqrt{210}\left(i\nu^3+12\nu^2-54i\nu-90\right)} \\ & + \frac{27\sqrt{\frac{3}{70}}\nu^6\ {}_2S_{53}}{3i\nu^3+12\nu^2-18i\nu-10} + \frac{625\sqrt{\frac{5}{42}}\nu^6\ {}_2S_{55}}{25i\nu^3+60\nu^2-54i\nu-18} + \cdots \end{aligned} \tag{4.55}$$

– When the explicit solutions are used,

$$\begin{aligned}
\mathcal{N} &= 8\sqrt{\frac{2\pi}{5}}\ {}_2L_{22}\left(\mathcal{M}_{21}+\mathcal{M}_{22}\right)\nu^3+\frac{1}{3}i\sqrt{\frac{\pi}{35}}\ {}_2L_{31}\left(\mathcal{M}_{31}-\mathcal{M}_{32}\right)\nu^4\\
&- 9i\sqrt{\frac{3\pi}{7}}\ {}_2L_{33}\left(\mathcal{M}_{31}-\mathcal{M}_{32}\right)\nu^4+\frac{8}{63}\sqrt{2\pi}\ {}_2L_{42}\left(\mathcal{M}_{41}+\mathcal{M}_{42}\right)\nu^5\\
&- \frac{128}{9}\sqrt{\frac{2\pi}{7}}\ {}_2L_{44}\left(\mathcal{M}_{41}+\mathcal{M}_{42}\right)\nu^5\frac{1}{180}i\sqrt{\frac{\pi}{154}}\ {}_2L_{51}\left(\mathcal{M}_{51}-\mathcal{M}_{52}\right)\nu^6\\
&- \frac{27}{40}i\sqrt{\frac{3\pi}{11}}\ {}_2L_{53}\left(\mathcal{M}_{51}-\mathcal{M}_{52}\right)\nu^6+\frac{625}{24}i\sqrt{\frac{5\pi}{33}}\ {}_2L_{55}\left(\mathcal{M}_{51}-\mathcal{M}_{52}\right)\nu^6\\
&+ \cdots, \qquad (4.56)
\end{aligned}$$

where,

$$\mathcal{M}_{lj}=M_j r_j^l(v_j^2+1), \qquad (4.57)$$

and ${}_2L_{lm}$ are defined as

$${}_2L_{lm}=\left(\ {}_2Z_{l\ -m}\Re(e^{i|m|\nu u})-\Re(ie^{i|m|\nu u})\ {}_2Z_{lm}\right). \qquad (4.58)$$

– As consequence of (4.54), for $M_1 = M_2 = M_0$ the terms with l odd disappear from the News function (4.56). Thus, as expected, one obtains immediately

$$\begin{aligned}\mathcal{N} \;=\; & 16\sqrt{\frac{2\pi}{5}}\nu^3 M_0 r_0^2 \left(V_0^2+1\right)\ {}_2L_{22} + \frac{16}{63}\sqrt{2\pi}\nu^5 M_0 r_0^4 \left(V_0^2+1\right)\ {}_2L_{42} \\ & -\frac{256}{9}\sqrt{\frac{2\pi}{7}}\nu^5 M_0 r_0^4 \left(V_0^2+1\right)\ {}_2L_{44} + \frac{32\sqrt{\frac{2\pi}{13}}}{1485}\nu^7 M_0 r_0^6 \left(V_0^2+1\right)\ {}_2L_{62} \\ & -\frac{8192}{495}\sqrt{\frac{\pi}{195}}\nu^7 M_0 r_0^6 \left(V_0^2+1\right)\ {}_2L_{64} \\ & +\frac{2592}{5}\sqrt{\frac{2\pi}{715}}\nu^7 M_0 r_0^6 \left(V_0^2+1\right)\ {}_2L_{66} \\ & +\cdots . \end{aligned} \tag{4.59}$$

where V_0 is the physical velocity of the masses, which is obviously tangent to the circular orbit.

– The energy lost by the system dE/du is related to the News function,[19]

[19] Bishop, see n. 7.

via

$$\frac{dE}{du} = \frac{1}{4\pi}\int_{\Omega} d\Omega\, \mathcal{N}\overline{\mathcal{N}}, \tag{4.60}$$

* which results for $M_1 \neq M_2$ in

$$\begin{aligned}\frac{dE}{du} =& \frac{32}{5}\nu^6 \left(\mathcal{M}_{21} + \mathcal{M}_{22}\right)^2 + \frac{2734}{315}\nu^8 \left(\mathcal{M}_{31} - \mathcal{M}_{32}\right)^2 \\ &+ \frac{57376}{3969}\nu^{10} \left(\mathcal{M}_{41} + \mathcal{M}_{42}\right)^2 + \frac{4010276}{155925}\nu^{12} \left(\mathcal{M}_{51} - \mathcal{M}_{52}\right)^2 \\ &+ \cdots .\end{aligned} \tag{4.61}$$

* Notice that the first term on the right side of the above equation is nothing but the power lost obtained by Peters and Mathews[20] for circular orbits.
* The other terms stand for the octupole, hexadecapole, etc contributions.

[20] Peters and Mathews, see n. 17.

4.3. Eccentric Point Particle Binary System

* Here it is introduced the eccentricity in the binary systems in the characteristic formulation, generalising the study of the previous section.
* From the density of energy and from an angular velocity that is variable in time, we deduce boundary conditions at the orbits, generalising those boundary conditions found for circular orbits.
* Also, we found the expression for the power emitted by the binary in gravitational radiation from the characteristic formulation, in agreement with the Peter and Mathews expression.[21]
* We consider in the News, those terms related to the angular velocity, disregarded in the circular case.[22]
* In this case, the density that describes the point particle binary is given by

$$\rho = \frac{\delta(\theta - \pi/2)}{r^2}\left(M_1\delta(r - r_1)\delta(\phi - \tilde{\phi}) + M_2\delta(r - r_2)\delta(\phi - \tilde{\phi} - \pi)\right), \tag{4.62}$$

[21] Peters and Mathews, see n. 17.

[22] C. E. Cedeño and J. C. N. Araujo. *General Relativity and Gravitation.* **48** 45 (2016) .

where, r_i (M_i) are the orbital radius (mass) of each particle, $r_1 < r_2$ and $\tilde{\phi} := \tilde{\phi}(u)$ is the angular position as indicated in Figure 7.

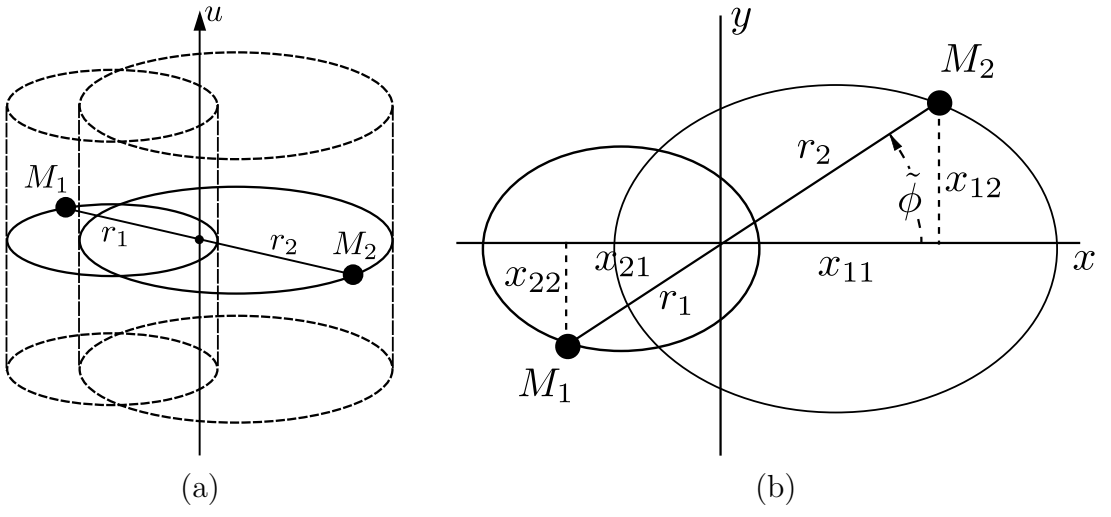


Figure 7: (a) Eccentric binary system with the world tubes of their orbits extended along the central time-like geodesic. (b) Top view of the point particle binary system, where the angular position $\tilde{\phi}$ is indicated.

* The instantaneous radius of the particle's orbits r_j and the separation between the masses d read

$$r_j = \frac{\mu d}{M_j}, \quad \mu = \frac{M_1 M_2}{M_1 + M_2}, \quad j = 1, 2, \quad d = \frac{a(1-\epsilon^2)}{1 + \epsilon \cos\tilde{\phi}}, \tag{4.63}$$

in which ϵ represents the eccentricity, and a is the semi-major axis.

* For Keplerian orbits, the angular velocity reads

$$\dot{\tilde{\phi}} = \frac{\sqrt{a(1-\epsilon^2)(M_1+M_2)}}{d^2}, \tag{4.64}$$

* We obtain the following boundary conditions:

$$\Delta\beta_{jlm} = b_{jlm}, \tag{4.65a}$$

$$\Delta w_{jlm} = -2 r_j b_{jlm}, \tag{4.65b}$$

$$\Delta J'_{jlm} = \frac{8\dot{\tilde{\phi}}^2 r_j b_{jlm} |m|^2}{(l-1)l(l+1)(l+2)}, \tag{4.65c}$$

$$\Delta U'_{jlm} = 2 b_{ilm} \left(\frac{1}{r_i^2} - \frac{4i\dot{\tilde{\phi}}|m|}{l(l+1) r_i} \right), \tag{4.65d}$$

where $\tilde{\phi} := \tilde{\phi}(u)$, $r_j := r_j(u)$, b_{jlm} are constants. $\Delta\beta'_{jlm} = 0$.

$*$ The constants D_{nFlm} depend on two parameters, namely b_{1lm} and b_{2lm}.
As an example, we show D_{1J2m+} for $|m| \neq 0$, i.e.

$$
\begin{aligned}
D_{1J2m+} =& \frac{ir_1^2 b_{12m} e^{-2ir_1\dot{\tilde{\phi}}|m|}}{\dot{\tilde{\phi}}\,|m|} - \frac{ir_1^2 b_{12m}}{\dot{\tilde{\phi}}\,|m|} + \frac{2r_1 b_{12m} e^{-2ir_1\dot{\tilde{\phi}}|m|}}{\dot{\tilde{\phi}}^2\,|m|^2} + \frac{2r_1 b_{12\mathrm{m}}}{\dot{\tilde{\phi}}^2\,|m|^2} \\
& - \frac{3ib_{12m} e^{-2ir_1\dot{\tilde{\phi}}|m|}}{\dot{\tilde{\phi}}^3\,|m|^3} - \frac{3b_{12m} e^{-2ir_1\dot{\tilde{\phi}}|m|}}{r_1\dot{\tilde{\phi}}^4\,|m|^4} - \frac{3b_{12m}}{r_1\dot{\tilde{\phi}}^4\,|m|^4} - \frac{3ib_{12m}}{r_1^2\dot{\tilde{\phi}}^5\,|m|^5} \\
& + \frac{3ib_{12m} e^{-2ir_1\dot{\tilde{\phi}}|m|}}{r_1^2\dot{\tilde{\phi}}^5\,|m|^5} + \frac{3ib_{12m}}{\dot{\tilde{\phi}}^3\,|m|^3} + \frac{ir_2^2 b_{22m} e^{-2ir_2\dot{\tilde{\phi}}|m|}}{\dot{\tilde{\phi}}\,|m|} - \frac{ir_2^2 b_{22m}}{\dot{\tilde{\phi}}\,|m|} \\
& + \frac{2r_2 b_{22\mathrm{m}} e^{-2ir_2\dot{\tilde{\phi}}|m|}}{\dot{\tilde{\phi}}^2\,|m|^2} + \frac{2r_2 b_{22m}}{\dot{\tilde{\phi}}^2\,|m|^2} - \frac{3ib_{22m} e^{-2ir_2\dot{\tilde{\phi}}|m|}}{\dot{\tilde{\phi}}^3\,|m|^3} - \frac{3b_{22m}}{r_2\dot{\tilde{\phi}}^4\,|m|^4} \\
& - \frac{3b_{22m} e^{-2ir_2\dot{\tilde{\phi}}|m|}}{r_2\dot{\tilde{\phi}}^4\,|m|^4} + \frac{3ib_{22m} e^{-2ir_2\dot{\tilde{\phi}}|m|}}{r_2^2\dot{\tilde{\phi}}^5\,|m|^5} - \frac{3ib_{22m}}{r_2^2\dot{\tilde{\phi}}^5\,|m|^5} + \frac{3ib_{22m}}{\dot{\tilde{\phi}}^3\,|m|^3}.
\end{aligned}
\tag{4.66}
$$

* The parameters b_{jlm}, $j = 1, 2$ are determined directly from (4.65a) and (31a). In particular for the binary system,

$$b_{jlm} = 2M_j \int_0^{2\pi} d\tilde{\phi}\ \frac{e^{-i|m|\tilde{\phi}}\overline{Z}_{lm}(\pi/2, \tilde{\phi} + \pi\delta_{2j})}{r_j^2}. \tag{4.67}$$

* ${}_sZ_{lm} \in \mathbb{R}$ on the equatorial plane $\theta = \pi/2$, but in general these functions are complex.
* Specifically the non-null b_{jlm}, for the firsts l and m, are given in Table 1.

Table 1: First non-null values for the constants b_{jlm}.

l	2	2	3	3
m	-2	0	-3	-1
$\frac{a\mu(\epsilon^2-1)b_{jlm}}{M_j^2}$	$\frac{i\sqrt{15\pi}}{2}$	$\sqrt{5\pi}$	$-\frac{i}{2}\sqrt{\frac{35\pi}{2}}$	$\frac{i}{2}\sqrt{\frac{21\pi}{2}}$

∗ These coefficients are written only for $m < 0$, because

$$ {}_s\overline{Z}_{lm} = (-1)^{s+m} \ {}_{-s}Z_{l(-m)}. \tag{4.68} $$

∗ For $m \neq 0$, one has

$$ b_{jlm} = i b_{jl(-m)} \quad j = 1, 2. \tag{4.69} $$

4.3.1. Gravitational Radiation Emitted by the Binary

✠ The power emitted in gravitational waves is given by

$$ \mathcal{N} = \sum_{l,m} \lim_{r\to\infty} \Re \left(\left(-\frac{i r^2 \dot{\tilde{\phi}} |m| J_{lm,r}}{2} - \frac{r^2 \dot{\tilde{\phi}} J_{lm,\tilde{\phi}r}}{2} + \frac{l(l+1) J_{lm}}{4} + \beta_{lm} \right) e^{i|m|\dot{\tilde{\phi}}} \right) \eth^2 Z_{lm}, \tag{4.70} $$

✠ The coefficients J_{lm} depend directly on the source angular position, represented by $\tilde{\phi}$. For this reason the retarded-time derivative $J_{,ur}$ is re-expressed using the chain rule.

- When the solutions to the field equations, for $r > r_2$ are substituted in (4.70), for $l = 2$, one finds,

$$\begin{aligned}\mathcal{N} =& 2i\sqrt{\frac{2}{3}}\dot{\tilde{\phi}}\left(\Re(e^{2i\tilde{\phi}}D_{2J22+})\ {}_2Z_2\ {}_2 + \Re(e^{2i\tilde{\phi}}D_{2J2-2+})\ {}_2Z_2\ {}_{-2}\right)\\ &+\frac{1}{2}\sqrt{\frac{3}{2}}\dot{\tilde{\phi}}\left(\Re(e^{2i\tilde{\phi}}D'_{2J22+})\ {}_2Z_2\ {}_2 + \Re(e^{2i\tilde{\phi}}D'_{2J2-2+})\ {}_2Z_2\ {}_{-2}\right),\end{aligned} \tag{4.71}$$

  where the prime indicates derivation with respect to $\tilde{\phi}$.
- D_{2Jlm+} depends on $\dot{\tilde{\phi}}$ and given that $\dot{\tilde{\phi}} := \dot{\tilde{\phi}}(\tilde{\phi})$, then they are functions of the retarded angular position.
- The absence of terms for $|m| = 1$ in the News expression is because $b_{j21} = b_{j2-1} = 0$ as indicated in Table 1.
- Despite $b_{jl0} \neq 0$ for $l = 2, 3, \cdots$, the terms for $m = 0$ do not enter in the News, which indicates that they are non-radiative terms.
- In the limit of low velocities, $r_1\dot{\tilde{\phi}} \ll c$, $r_2\dot{\tilde{\phi}} \ll c$ and for $l = 2$, we

find that the power lost by gravitational radiation emission reads

$$\frac{dE}{du} = \frac{32 M_1^2 M_2^2 \left(M_1 + M_2\right) \left(1 + \epsilon \cos \tilde{\phi}\right)^6}{5a^5 (1-\epsilon^2)^5} + \frac{8 M_1^2 M_2^2 \left(M_1 + M_2\right) \epsilon^2 \sin^2 \tilde{\phi} (1 + \epsilon \cos \tilde{\phi})^4}{15 a^5 (1-\epsilon^2)^5}, \tag{4.72}$$

- This corresponds to the Peters and Mathews expression for the energy lost by binary systems.[23]
- The agreement between our results and those by Peters and Mathews is in fact expected, since the system under study is the same.
- This agreement shows that characteristic formalism in the linear regime has been properly applied in the present text.
- Winicour in the 1980s decade showed that the Bondi's News function in the Quasi-Newtonian regime[24] is just

$$\mathcal{N} = \dddot{Q}, \tag{4.73}$$

[23] Peters and Mathews, see n. 17.

[24] R. A. Isaacson, J. S. Welling, and J. Winicour. *Journal of Mathematical Physics.* **26** 11 2859–2868 (1985) ; J. Winicour. *General Relativity and Gravitation.* **19** 3 281–287 (1987) .

with

$$Q = q^A \overline{q}^B Q_{AB}. \tag{4.74}$$

- The first term in the power expression (4.72) represents approximately 97% of the power emitted by the source.
- Thus, for $\epsilon < 0.5$ a reasonable approximation is just given by the first term of (4.72).
- For the case of circular orbits, the two first terms in the News (4.71), lead directly to

$$\frac{dE}{du} = \frac{32 M_1^2 M_2^2 \left(M_1 + M_2\right)}{5a^5}. \tag{4.75}$$

which corresponds in fact to the first term of (4.61).

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