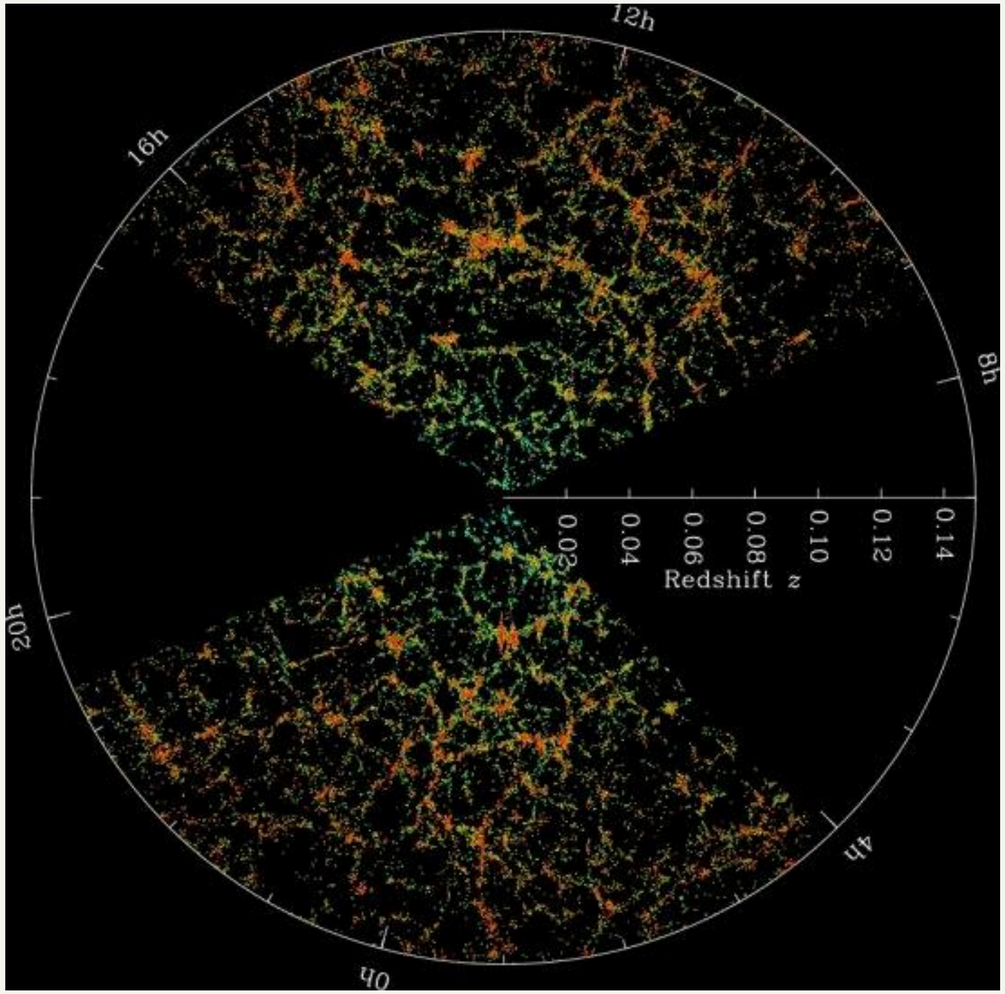
Universidad Industrial de Santander



Metric Affine Lovelock's theorem

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0.1.Lovelock's theorem (Riemmanian manifold)

Which is the most general tensor that is...

- Symmetric and second rank,
- Concomitant of the metric tensor and its first two derivatives,
- Divergenceless,

The answer depends on the dimension of the space!

D. Lovelock, J. Math. Phys., 1971.

0.1.Lovelock's theorem (Riemmanian manifold)

In the most important case in which D=4 he found

$$\mathring{A}_{ij} = \alpha_1 \mathring{G}_{ij} + \alpha_2 g_{ij},$$

which corresponds to GR!. (Having as associated action the Einstein-Hilbert one). And a boundary term:

$$\mathcal{L} = \mathring{R}^2 - 4\mathring{R}_{\mu\nu}\mathring{R}^{\mu\nu} + \mathring{R}_{\mu\nu\rho\lambda}\mathring{R}^{\mu\nu\rho\lambda}.$$

In n dimensions, the EH action is given by

$$S_G = \int \mathring{\mathbf{L}} = \int \mathring{R}\, \mathbf{vol}_g = \int \mathring{R}\, \frac{1}{n!}\sqrt{-g}\, \epsilon_{\mu_1 \ldots \mu_n} \mathrm{d}x^{\mu_1} \wedge \ldots \wedge \mathrm{d}x^{\mu_n} = \int \mathring{R}\sqrt{-g}\, \mathrm{d}^n x,$$

D. Lovelock, J. Math. Phys., 1971.

A. Mardones and J. Zanelli, Class. Quant. Grav., 1991.

0.2.Differential forms

In a general frame (or coframe), we can write

$$\alpha = \alpha_{a_1 \dots a_k} \vartheta^{a_1} \otimes \dots \otimes \vartheta^{a_k} = \frac{1}{k!} \alpha_{a_1 \dots a_k} \vartheta^{a_1} \wedge \dots \wedge \vartheta^{a_k} = \frac{1}{k!} \alpha_{a_1 \dots a_k} \vartheta^{a_1 \dots a_k},$$

where the wedge product is defined as

$$\mathbf{T} \wedge \mathbf{S} = \frac{(k+l)!}{k!l!} T_{[a_1,\dots,a_k} S_{a_{k+1},\dots,a_{k+l}]} \vartheta^{a_1} \otimes \dots \otimes \vartheta^{a_{k+l}},$$

and the dimension of the k form space in an n dimensional space is given by

$$\dim(\Omega^k(\mathcal{M})) = \frac{n!}{k!(n-k)!},$$

$$n = 2 :$$
$$\dim(\Omega^0(\mathcal{M})) = 1,$$
$$\dim(\Omega^1(\mathcal{M})) = 2,$$
$$\dim(\Omega^2(\mathcal{M})) = 1.$$

0.2.Differential forms

The Hodge duality (Hodge star) relates k-forms and (n-k)-forms by

$$\star\alpha = \frac{1}{(n-k)!k!} a^{a_1 \dots a_k} \mathcal{E}_{a_1 \dots a_k b_1 \dots b_{n-k}} \vartheta^{b_1 \dots b_{n-k}},$$

and in particular, for an n-dimensional space, we have

$$\star\vartheta^{a_1 \dots a_n} = \mathcal{E}^{a_1 \dots a_n},$$

and, in addition, the EH action written as

$$S_G = \int \mathring{R}\, \mathbf{vol}_g = \int \mathring{\mathbf{R}}^{a_1 a_2} \wedge \star\vartheta_{a_1 a_2}.$$

D. Lovelock, J. Math. Phys., 1971.

A. Mardones and J. Zanelli, Class. Quant. Grav., 1991.

B. Janssen and A. Jiménez-Cano, Phys. Lett. B, 2019.

0.3.On the integration in manifolds

Differential forms give the notion of volume element and therefore

$$Volume = \int_{\mathcal{U}} \mathbf{vol},$$

and a k-form can be “derivate” to obtain a (k+1)-form using

$$\mathrm{d}\alpha = \frac{1}{k!}\partial_{[\mu_1}\alpha_{\mu_2\ldots\mu_{k+1}]}\mathrm{d}x^{\mu_1\cdots\mu_{k+1}},$$

Having also $\mathrm{d}(\mathrm{d}\alpha) = 0$, and finally, Stoke’s theorem allows us to remove exterior derivatives

$$\int_{\mathcal{M}} \mathrm{d}\omega = \int_{\partial\mathcal{M}} \omega,$$

0.3.1.Poincaré's Lemma

Every closed k-form, when $n \geq k \geq 1$, is an exact form if it lives in an open ball in R^n .

$$\mathrm{d}\alpha = 0 \Rightarrow \mathrm{closed}. \qquad \alpha = \mathrm{d}\beta \Rightarrow \mathrm{exact}.$$

- Poincaré's lemma fails when having manifolds whose topology is non trivial, i.e., that the manifold has "holes".

J. M. Lee. Springer, 2013.

0.4.Lovelock's gravity: properties

- Propagation of only two degrees of freedom.
- The graviton is massless.
- The equations of motion for the metric are second order at most.
- Avoids Ostrogradki's instability.
- In D dimensions, the D/2-th term is a boundary term.

D. Lovelock, J. Math. Phys., 1971.

A. Mardones and J. Zanelli, Class. Quant. Grav., 1991.

B. Janssen and A. Jiménez-Cano, Phys. Lett. B, 2019.

0.5.General Manifold

Beyond the metric, we also have

$$\Gamma^{\alpha}{}_{\mu\nu} = \frac{1}{2} g^{\nu\beta}(\partial_\alpha g_{\beta\mu} + \partial_\mu g_{\alpha\beta} - \partial_\beta g_{\mu\alpha}) + \frac{1}{2}(T^{\nu}{}_{\alpha\mu} - T_{\alpha\mu}{}^{\nu} - T_{\mu\alpha}{}^{\nu}) + \frac{1}{2}(Q^{\nu}{}_{\alpha\mu} - Q_{\mu\alpha}{}^{\nu} - Q_{\alpha\mu}{}^{\nu})$$

where

$$Q_{\mu\alpha\beta} \equiv \nabla_\mu g_{\alpha\beta}, \qquad T^{\alpha}{}_{\mu\nu} \equiv \Gamma^{\alpha}{}_{\mu\nu} - \Gamma^{\alpha}{}_{\nu\mu}.$$

0.6.Metric affine gravity (MAG)

$$\mathbf{R}^{1,3} \rtimes \mathrm{O}(1,3)\,, \quad \Longrightarrow \quad \mathbf{R}^{1,3} \rtimes \mathrm{GL}(1,3)\,,$$

and non metricity modifies the inner product under parallel transport as

$$\partial_\alpha (g_{\mu\nu} V^\mu W^\nu) = \nabla_\alpha (g_{\mu\nu} V^\mu W^\nu) = Q_{\alpha\mu\nu} V^\mu W^\nu,$$

deformations of ideal crystal can be described by a metric like in GR, but local deffects, require structures like torsion and non metricity to be described.

F. W. Hehl et. al., Phys. Rept., 1995.

F. Falk, Journal of Elasticity, 1981.

0.7.On the Levi-Civita connection 1-form

In a general basis, the components of the Levi-Civita connection are given as

$$\mathring{\omega}_{cab} \equiv \frac{1}{2}(\partial_c g_{ba} + \partial_a g_{cb} - \partial_b g_{ca}) - \frac{1}{2}(\Omega_{cab} + \Omega_{bca} - \Omega_{abc}),$$

They can be used to define a 1-form

$$\mathring{\omega}_a{}^b \equiv \mathring{\omega}_{\mu a}{}^b \mathrm{d}x^\mu.$$

0.8.On the general connection

In a general basis, the components of the connection are given as

$$\begin{aligned}\omega_{cab} \equiv & \frac{1}{2}(\partial_c g_{ba} + \partial_a g_{cb} - \partial_b g_{ca}) - \frac{1}{2}(\Omega_{cab} + \Omega_{bca} - \Omega_{abc}) \\ & + \frac{1}{2}(T_{cab} + T_{bca} - T_{abc}) + \frac{1}{2}(Q_{cab} + Q_{abc} - Q_{bca}).\end{aligned}$$

Therefore we define the connection 1-form as

$$\omega_a{}^b \equiv \omega_{\mu a}{}^b \mathrm{d}x^\mu.$$

0.9.On the geometry

Defining the covariant exterior derivative as

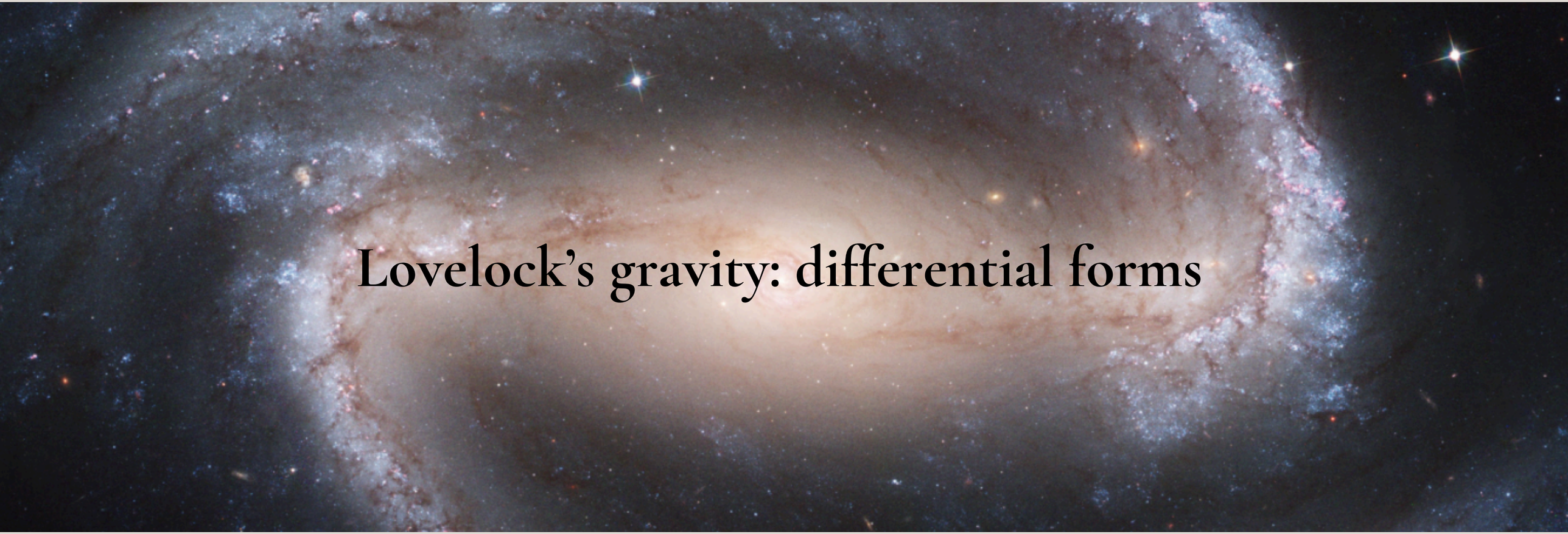
$$\mathbf{D}\alpha_{a\ldots}{}^{b\ldots} = \mathrm{d}\alpha_{a\ldots}{}^{b\ldots} + \omega_c{}^b \wedge \alpha_{a\ldots}{}^{c\ldots} + \ldots - \omega_a{}^c \wedge \alpha_{c\ldots}{}^{b\ldots} - \ldots$$

The geometry is described by

$$\mathbf{R}_a{}^b \equiv \mathrm{d}\omega_a{}^b + \omega_c{}^b \wedge \omega_a{}^c, \qquad \mathbf{DR}_a{}^b = 0,$$
$$\mathbf{T}^a \equiv \mathbf{D}\vartheta^a, \qquad \mathbf{DT}^a = \mathbf{R}_b{}^a \wedge \vartheta^b,$$
$$\mathbf{Q}_{ab} \equiv -\mathbf{D}g_{ab}. \qquad \mathbf{DQ}_{ab} = \mathbf{R}_{(ab)}.$$

A. Mardones and J. Zanelli, Class. Quant. Grav., 1991.

B. Janssen and A. Jiménez-Cano, Phys. Lett. B, 2019.



Lovelock's gravity: differential forms

A. Mardones and J. Zanelli, Class. Quant. Grav., 1991.

1.Lovelock's gravity: differential forms

1.The lagrangian is invariant under LLT, constructed out of $\vartheta^a, \omega_a{}^b$, their exterior derivatives and products thereof, without using the Hodge dual operation.

2. The connection is, by definition, torsion-free (spacetime is a Riemannian manifold).

D. Lovelock, J. Math. Phys., 1971.

A. Mardones and J. Zanelli, Class. Quant. Grav., 1991.

1.Lovelock's gravity: differential forms

Lovelock's invariants can be written as

$$\mathring{\mathbf{L}}_k^{(D)} \equiv \mathring{\mathbf{R}}^{a_1 a_2} \wedge \ldots \wedge \mathring{\mathbf{R}}^{a_{2k-1} a_{2k}} \wedge \star\vartheta_{a_1 \ldots a_{2k}},$$

For D=2:

$$\mathring{\mathbf{L}}_{\mathrm{Lov}}^{(2)} = \mathcal{E}^a{}_b \mathring{\mathbf{R}}_a{}^b = \mathrm{d}(\mathcal{E}^a{}_b \mathring{\omega}_a{}^b),$$

D. Lovelock, J. Math. Phys., 1971.

A. Mardones and J. Zanelli, Class. Quant. Grav., 1991.

B. Janssen and A. Jiménez-Cano, Phys. Lett. B, 2019.

1.Lovelock's gravity: invariants

The important case of D=4:

$$\mathring{\mathbf{L}}_1^{(4)} = \alpha_1 \mathring{\mathbf{R}}^{ab} \wedge \star\vartheta_{ab}, \qquad \mathring{\mathbf{L}}_2^{(4)} = \mathcal{E}_{abcd} \mathring{\mathbf{R}}^{ab} \wedge \mathring{\mathbf{R}}^{cd},$$

then the only contribution to the field equations is

$$\mathring{\mathbf{L}}_{\mathrm{Lov}}^{(4)} = \alpha_1 \mathring{\mathbf{R}}^{ab} \wedge \star\vartheta_{ab},$$

D. Lovelock, J. Math. Phys., 1971.

A. Mardones and J. Zanelli, Class. Quant. Grav., 1991.

B. Janssen and A. Jiménez-Cano, Phys. Lett. B, 2019.

1.Lovelock's gravity: lagrangian

In general dimensions:

$$\mathring{\mathbf{L}}^{(D)}_{\mathrm{Lov}} \equiv \sum_{i=1}^{[D/2]} \alpha_i \mathring{\mathbf{L}}^{(D)}_{i},$$

where

$$\mathring{\mathbf{L}}^{(2k)}_{k},$$

is a boundary term.

D. Lovelock, J. Math. Phys., 1971.

A. Mardones and J. Zanelli, Class. Quant. Grav., 1991.

B. Janssen and A. Jiménez-Cano, Phys. Lett. B, 2019.

Lovelock-Cartan theory of gravity (vanishing non metricity)



A. Mardones and J. Zanelli, Class. Quant. Grav., 1991.

2.Lovelock-Cartan's gravity: invariants

First family:

$$\mathbf{L}_k^{(D)} = \mathbf{R}^{a_1 a_2} \wedge \ldots \wedge \mathbf{R}^{a_{2k-1} a_{2k}} \wedge \star \vartheta_{a_1 \ldots a_{2k}},$$

for example in D=2

$$\mathbf{L}_1^{(2)} \equiv \mathbf{R}^{a_1 a_2} \wedge \star \vartheta_{a_1 a_2} = \mathcal{E}^a{}_b \mathbf{R}_a{}^b = \mathrm{d}(\mathcal{E}^a{}_b \omega_a{}^b),$$

A. Mardones and J. Zanelli, Class. Quant. Grav., 1991.

2.Lovelock-Cartan's gravity: invariants

In the important case of D=4:

$$\mathbf{L}_2^{(4)} = \mathcal{E}_{abcd}\mathbf{R}^{ab} \wedge \mathbf{R}^{cd},$$

is again a boundary term, as well as

$$\mathbf{L}_k^{(2k)} \equiv \mathbf{R}^{a_1 a_2} \wedge \ldots \wedge \mathbf{R}^{a_{2k-1} a_{2k}} \wedge \star\vartheta_{a_1 \ldots a_{2k}}.$$

A. Mardones and J. Zanelli, Class. Quant. Grav., 1991.

2.Lovelock-Cartan's gravity: torsion invariants

For the second family:

$$\mathbf{R}_A \equiv \mathbf{R}_{a_1}{}^{a_2} \wedge ... \wedge \mathbf{R}_{a_A}{}^{a_1},$$

$$\mathbf{t}_A \equiv \mathbf{T}^{a_1} \wedge \mathbf{R}_{a_1}{}^{a_2} \wedge ... \wedge \mathbf{R}_{a_A}{}^{b} \wedge \mathbf{T}_b,$$

$$\mathbf{V}_A \equiv \vartheta^{a_1} \wedge \mathbf{R}_{a_1}{}^{a_2} \wedge ... \wedge \mathbf{R}_{a_A}{}^{b} \wedge \vartheta_b,$$

$$\mathbf{K}_A \equiv \mathbf{T}^{a_1} \wedge \mathbf{R}_{a_1}{}^{a_2} \wedge ... \wedge \mathbf{R}_{a_A}{}^{b} \wedge \vartheta_b.$$

Since curvature 2-form is antisymmetric: $\mathbf{R}_A = \mathbf{t}_A = \mathbf{V}_{A+1} = 0,$ iff A is even (2,4,6,....), thanks to Q=0.

A. Mardones and J. Zanelli, Class. Quant. Grav., 1991.

2.Lovelock-Cartan's gravity: torsion invariants

Having as generalized Bianchi identities

$$\mathrm{d}\Big(\mathbf{R}_A\Big) = 0,$$

$$\mathrm{d}\Big(\mathbf{K}_A\Big) = \mathbf{V}_{A+1} + \mathbf{t}_A,$$

$$\mathrm{d}\Big(\mathbf{t}_A\Big) = -2\mathbf{K}_{A+1},$$

$$\mathrm{d}\Big(\mathbf{V}_A\Big) = -2\mathbf{K}_A,$$

A. Mardones and J. Zanelli, Class. Quant. Grav., 1991.

2.Lovelock-Cartan's gravity: torsion invariants

Therefore, the second family is given by

$$\tilde{\mathbf{L}}^{(D)}_{\mathbf{A}} \equiv \mathbf{R}_{A_1} \wedge ... \wedge \mathbf{R}_{A_r} \wedge \mathbf{t}_{B_1} \wedge ... \wedge \mathbf{t}_{B_s} \wedge \mathbf{V}_{C_1} \wedge ... \wedge \mathbf{V}_{C_u} \wedge \mathbf{K}_{D_1} \wedge ... \wedge \mathbf{K}_{D_v},$$

from which they have the most general lagrangian:

$$\mathbf{L}^{(D)}_{\mathrm{Lov}} = \sum_{i=1}^{[D/2]} \alpha_i \mathbf{L}^{(D)}_i + \sum \tilde{\alpha_{\mathbf{A}}} \tilde{\mathbf{L}}^{(D)}_{\mathbf{A}},$$

A. Mardones and J. Zanelli, Class. Quant. Grav., 1991.

2.Lovelock-Cartan's gravity: lagrangian

For D=2:

$$\mathbf{L}^{(2)}_{\mathrm{Lov}} = \mathcal{E}^{a}{}_{b}\mathbf{R}_{a}{}^{b},$$

For D=4:

$$\mathbf{L}^{(4)}_{\mathrm{Lov}} = \alpha_1 \mathbf{R}^{ab} \wedge \star\vartheta_{ab} + \alpha_2 \mathcal{E}_{abcd}\mathbf{R}^{ab} \wedge \mathbf{R}^{cd} + \tilde{\alpha}_1 \mathbf{R}_2 + \tilde{\alpha}_2 \mathbf{t}_0 + \tilde{\alpha}_3 \mathbf{V}_1$$

having as non boundary terms

$$\mathbf{L}^{(4)}_{\mathrm{Lov}} = \alpha_1 \mathbf{R}^{ab} \wedge \star\vartheta_{ab} + \tilde{\alpha}_2 \mathbf{t}_0,$$

A. Mardones and J. Zanelli, Class. Quant. Grav., 1991.

2.Lovelock-Cartan's gravity: lagrangian

D	T(D)
1	0
2	0
3	1
4	3
5	1
6	0
7	4
8	10

D	$\tilde{\mathbf{L}}_{\mathbf{A}}^{(D)}$
3	$\mathbf{K}_0^*$
4	$\mathbf{R}_2^*,\ \mathbf{V}_1,\ \mathbf{t}_0$
5	$\mathbf{K}_1^*$
6	-
7	$\mathbf{R}_2 \wedge \mathbf{K}_0^*,\ \mathbf{V}_1 \wedge \mathbf{K}_0,\ \mathbf{t}_0 \wedge \mathbf{K}_0,\ \mathbf{K}_2^*$
8	$\mathbf{R}_4^*,\ \mathbf{R}_2 \wedge \mathbf{R}_2^*,\ \mathbf{R}_2 \wedge \mathbf{V}_1,\ \mathbf{V}_3,\ \mathbf{V}_1 \wedge \mathbf{V}_1$ $\mathbf{t}_0 \wedge \mathbf{t}_0,\ \mathbf{R}_2 \wedge \mathbf{t}_0,\ \mathbf{t}_2,\ \mathbf{V}_1 \wedge \mathbf{t}_0,\ \mathbf{K}_1 \wedge \mathbf{K}_0$

A. Mardones and J. Zanelli, Class. Quant. Grav., 1991.



Metric Affine Lovelock's theorem

3.Lovelock's MAG: invariants

Again, as a first family can be used

$$\mathbf{L}_k^{(D)} = \mathbf{R}^{a_1 a_2} \wedge \ldots \wedge \mathbf{R}^{a_{2k-1} a_{2k}} \wedge \star\vartheta_{a_1 \ldots a_{2k}}.$$

For D=2,

$$\begin{aligned}
\mathcal{E}^a{}_b \mathbf{R}_a{}^b &= \mathcal{E}^a{}_b \mathrm{d}\omega_a{}^b + \mathcal{E}^a{}_b \omega_c{}^b \wedge \omega_a{}^c \\
&= \mathrm{d}(\mathcal{E}^a{}_b \omega_a{}^b) + \frac{1}{4} \mathcal{E}^a{}_b \mathbf{Q}_c{}^b \wedge \mathbf{Q}_a{}^c \\
&= \mathrm{d}(\mathcal{E}^a{}_b \omega_a{}^b) + \frac{1}{4} \mathcal{E}^a{}_b \hat{\mathbf{Q}}_c{}^b \wedge \hat{\mathbf{Q}}_a{}^c,
\end{aligned}$$

B. Janssen and A. Jiménez-Cano, Phys. Lett. B, 2019.

3.Lovelock's MAG: invariants

but can be generalized as

$$\mathbf{L}_1^{'(2)} = \mathcal{E}^{a}{}_{b}\left(\mathbf{R}_{a}{}^{b} - \frac{1}{4}\hat{\mathbf{Q}}_{c}{}^{b} \wedge \hat{\mathbf{Q}}_{a}{}^{c}\right),$$

also for D=4, in principle we have

$$\mathbf{L}_1^{(4)} = \mathbf{R}^{ab} \wedge \star\vartheta_{ab}, \qquad \mathbf{L}_2^{(4)} = \mathcal{E}_{abcd}\mathbf{R}^{ab} \wedge \mathbf{R}^{cd},$$

but can be generalized as

$$\mathbf{L}_1^{'(4)} = \left(\mathbf{R}^{a_1 a_2} - \frac{1}{4}\hat{\mathbf{Q}}_{c_1}{}^{a_2} \wedge \hat{\mathbf{Q}}^{a_1 c_1}\right) \wedge \star\vartheta_{a_1 a_2},$$

$$\mathbf{L}_2^{'(4)} = \mathcal{E}^{a_1}{}_{a_2}{}^{a_3}{}_{a_4}\left(\mathbf{R}_{a_1}{}^{a_2} - \frac{1}{4}\hat{\mathbf{Q}}_{c_1}{}^{a_2} \wedge \hat{\mathbf{Q}}_{a_1}{}^{c_1}\right) \wedge \left(\mathbf{R}_{a_3}{}^{a_4} - \frac{1}{4}\hat{\mathbf{Q}}_{c_3}{}^{a_4} \wedge \hat{\mathbf{Q}}_{a_3}{}^{c_3}\right).$$

B. Janssen and A. Jiménez-Cano, Phys. Lett. B, 2019.

J. Martínez, J. Rodríguez, Y. Rodríguez.,XXXX,XXXX.

3.Lovelock's MAG: invariants

Then, in a general number of dimensions we can writte

$$\mathbf{L}_k^{\prime(D)} = \left(\mathbf{R}^{a_1 a_2} - \frac{1}{4}\hat{\mathbf{Q}}_{c_1}{}^{a_2} \wedge \hat{\mathbf{Q}}^{a_1 c_1}\right) \wedge \ldots \wedge \left(\mathbf{R}^{a_{2k-1} a_{2k}} - \frac{1}{4}\hat{\mathbf{Q}}_{c_1}{}^{a_{2k}} \wedge \hat{\mathbf{Q}}^{a_{2k-1} c_1}\right) \wedge \star\vartheta_{a_1 \ldots a_{2k}},$$

which recovers total derivatives

$$\mathbf{L}_1^{\prime(2)} = \mathcal{E}^a{}_b \left(\mathbf{R}_a{}^b - \frac{1}{4}\hat{\mathbf{Q}}_c{}^b \wedge \hat{\mathbf{Q}}_a{}^c\right),$$

$$\mathbf{L}_2^{\prime(4)} = \mathcal{E}^{a_1}{}_{a_2}{}^{a_3}{}_{a_4} \left(\mathbf{R}_{a_1}{}^{a_2} - \frac{1}{4}\hat{\mathbf{Q}}_{c_1}{}^{a_2} \wedge \hat{\mathbf{Q}}_{a_1}{}^{c_1}\right) \wedge \left(\mathbf{R}_{a_3}{}^{a_4} - \frac{1}{4}\hat{\mathbf{Q}}_{c_3}{}^{a_4} \wedge \hat{\mathbf{Q}}_{a_3}{}^{c_3}\right).$$

B. Janssen and A. Jiménez-Cano, Phys. Lett. B, 2019.

J. Martínez, J. Rodríguez, Y. Rodríguez.,XXXX,XXXX.

3.Lovelock's MAG: TQ invariants

For the second family, we can start from

$$\mathbf{R}_A \equiv \mathbf{R}_{a_1}{}^{a_2} \wedge \ldots \wedge \mathbf{R}_{a_A}{}^{a_1},$$

$$\mathbf{t}_A \equiv \mathbf{T}^{a_1} \wedge \mathbf{R}_{a_1}{}^{a_2} \wedge \ldots \wedge \mathbf{R}_{a_A}{}^{b} \wedge \mathbf{T}_b,$$

$$\mathbf{V}_A \equiv \vartheta^{a_1} \wedge \mathbf{R}_{a_1}{}^{a_2} \wedge \ldots \wedge \mathbf{R}_{a_A}{}^{b} \wedge \vartheta_b,$$

$$\mathbf{K}_A \equiv \mathbf{T}^{a_1} \wedge \mathbf{R}_{a_1}{}^{a_2} \wedge \ldots \wedge \mathbf{R}_{a_A}{}^{b} \wedge \vartheta_b.$$

The curvature 2-form is no longer antisymmetric, i. e., A has no restrictions.

A. Mardones and J. Zanelli, Class. Quant. Grav., 1991.

3.Lovelock's MAG: TQ invariants

Sadly, the generalized Bianchi identities are no longer preserved due non metricity

$$\mathrm{d}\left(\mathbf{K}_A\right) \neq \mathbf{V}_{A+1} + \mathbf{t}_A,$$

$$\mathrm{d}\left(\mathbf{t}_A\right) \neq -2\mathbf{K}_{A+1},$$

$$\mathrm{d}\left(\mathbf{V}_A\right) \neq -2\mathbf{K}_A,$$

A. Mardones and J. Zanelli, Class. Quant. Grav., 1991.

J. Martínez, J. Rodríguez, Y. Rodríguez.,XXXX,XXXX.

3.Lovelock′s MAG: TQ invariants

Therefore, we also used

$$\mathbf{K}'_A \equiv \vartheta^{a_1} \wedge \mathbf{R}_{a_1}{}^{a_2} \wedge \ldots \wedge \mathbf{R}_{a_A}{}^{b} \wedge \mathbf{T}_b,$$

$$\mathbf{TQV}_A \equiv \mathbf{T}^{a_1} \wedge \mathbf{R}_{a_1}{}^{a_2} \wedge \ldots \wedge \mathbf{R}_{a_A}{}^{b} \wedge \mathbf{Q}_{ab} \wedge \vartheta^{a},$$

$$\mathbf{VQV}_A \equiv \vartheta^{a_1} \wedge \mathbf{R}_{a_1}{}^{a_2} \wedge \ldots \wedge \mathbf{R}_{a_A}{}^{b} \wedge \mathbf{Q}_{ab} \wedge \vartheta^{a}.$$

J. Martínez, J. Rodríguez, Y. Rodríguez.,XXXX,XXXX.

3.Lovelock's MAG: TQ invariants

finding the closed algebra

$$\mathrm{d}\mathbf{R}_A = 0,$$

$$\mathrm{d}\Big(\mathbf{K}_A\Big) = \mathbf{V}_{A+1} + \Big(\mathbf{t}_A - \mathbf{TQV}_A\Big),$$

$$\mathrm{d}\Big(\mathbf{V}_A\Big) = \mathbf{K}_A - \Big(\mathbf{K}'_A - \mathbf{VQV}_A\Big),$$

$$\mathrm{d}\Big(\mathbf{t}_A - \mathbf{TQV}_A\Big) = \Big(\mathbf{K}'_{A+1} - \mathbf{VQV}_{A+1}\Big) - \mathbf{K}_{A+1},$$

$$\mathrm{d}\Big(\mathbf{K}'_A - \mathbf{VQV}_A\Big) = \Big(\mathbf{t}_A - \mathbf{TQV}_A\Big) + \mathbf{V}_{A+1}.$$

A. Mardones and J. Zanelli, Class. Quant. Grav., 1991.

B. Janssen and A. Jiménez-Cano, Phys. Lett. B, 2019.

J. Martínez, J. Rodríguez, Y. Rodríguez.,XXXX,XXXX.

3.Lovelock's MAG: TQ invariants

Then, we can construct the second family of lagrangians as

$$\tilde{\mathbf{L}'}^{(D)}_{\mathbf{A}} \equiv \mathbf{R}_{A_1} \wedge ... \wedge \mathbf{R}_{A_r} \wedge \mathbf{TQ}_{B_1} \wedge ... \wedge \mathbf{TQ}_{B_s} \wedge \mathbf{V}_{C_1} \wedge ... \wedge \mathbf{V}_{C_u} \wedge \mathbf{K}_{D_1} \wedge ... \wedge \mathbf{K}_{D_v},$$

where we have used

$$\mathbf{TQ}_A \equiv \mathbf{t}_A - \mathbf{TQV}_A,$$

from which we have the most general lagrangian:

$$\mathbf{L}^{'(D)}_{\mathrm{Lov}} = \sum_{i=1}^{[D/2]} \alpha_i \mathbf{L}^{'(D)}_i + \sum \tilde{\alpha_{\mathbf{A}}} \tilde{\mathbf{L}'}^{(D)}_{\mathbf{A}},$$

A. Mardones and J. Zanelli, Class. Quant. Grav., 1991.

B. Janssen and A. Jiménez-Cano, Phys. Lett. B, 2019.

J. Martínez, J. Rodríguez, Y. Rodríguez.,XXXX,XXXX.

3.Lovelock's MAG: Lagrangian

Therefore, for D=2, we have

$$\mathbf{L}^{\prime(2)}_{\mathrm{Lov}} = \alpha_1 \mathcal{E}^{a}{}_{b}\left(\mathbf{R}_{a}{}^{b} - \frac{1}{4}\hat{\mathbf{Q}}_{c}{}^{b} \wedge \hat{\mathbf{Q}}_{a}{}^{c}\right) + \tilde{\alpha}_1 \mathbf{R}_1,$$

where

$$\mathbf{R}_1 = \mathrm{d}\omega_{a}{}^{a}.$$

In principle two dimensional gravity do not have a dynamic terms, but...

A. Mardones and J. Zanelli, Class. Quant. Grav., 1991.

B. Janssen and A. Jiménez-Cano, Phys. Lett. B, 2019.

J. Martínez, J. Rodríguez, Y. Rodríguez.,XXXX,XXXX.

3.Lovelock's MAG: Lagrangian

For the important case of D=4, we have found

$$\begin{aligned}\mathbf{L}^{'(4)}_{\mathrm{Lov}} =&\alpha_1\left(\mathbf{R}^{a_1a_2}-\frac{1}{4}\hat{\mathbf{Q}}_{c_1}{}^{a_2}\wedge\hat{\mathbf{Q}}^{a_1c_1}\right)\wedge\star\vartheta_{a_1a_2}\\&+\alpha_2\mathbf{L}^{'(4)}_2+\tilde{\alpha}_0\mathbf{R}_1\wedge\mathbf{R}_1+\tilde{\alpha}_1\mathbf{R}_2+\tilde{\alpha}_2\mathbf{T}\mathbf{Q}_0+\tilde{\alpha}_3\mathbf{V}_1,\end{aligned}$$

which can be reduced to

$$\begin{aligned}\mathbf{L}^{'(4)}_{\mathrm{Lov}} =&\alpha_1\left(\mathbf{R}^{a_1a_2}-\frac{1}{4}\hat{\mathbf{Q}}_{c_1}{}^{a_2}\wedge\hat{\mathbf{Q}}^{a_1c_1}\right)\wedge\star\vartheta_{a_1a_2}\\&+\tilde{\alpha}_2\mathbf{T}^b\wedge(\mathbf{T}_b-\mathbf{Q}_{ab}\wedge\vartheta^a)\end{aligned}$$

A. Mardones and J. Zanelli, Class. Quant. Grav., 1991.

B. Janssen and A. Jiménez-Cano, Phys. Lett. B, 2019.

J. Martínez, J. Rodríguez, Y. Rodríguez.,XXXX,XXXX.

3.Lovelock's MAG: Lagrangian

D	QT(D)
1	0
2	1
3	1
4	4
5	2
6	7
7	6
8	17

D	$\tilde{\mathbf{L}'}^{(D)}_{\mathbf{A}}$
2	$\mathbf{R}_1^*$
3	$\mathbf{K}_0^*$
4	$\mathbf{R}_2^*,\ \mathbf{V}_1,\ \mathbf{TQ}_0,\ \mathbf{R}_1 \wedge \mathbf{R}_1^*$
5	$\mathbf{K}_1^*,\ \mathbf{R}_1 \wedge \mathbf{K}_0^*$
6	$\mathbf{R}_3^*,\ \mathbf{R}_1 \wedge \mathbf{R}_2^*,\ \mathbf{R}_1 \wedge \mathbf{R}_1 \wedge \mathbf{R}_1^*$ $\mathbf{R}_1 \wedge \mathbf{V}_1,\ \mathbf{R}_1 \wedge \mathbf{TQ}_0,\ \mathbf{V}_2,\ \mathbf{TQ}_1$
7	$\mathbf{R}_2 \wedge \mathbf{K}_0^*,\ \mathbf{V}_1 \wedge \mathbf{K}_0,\ \mathbf{TQ}_0 \wedge \mathbf{K}_0,\ \mathbf{K}_2^*$ $\mathbf{R}_1 \wedge \mathbf{R}_1 \wedge \mathbf{K}_0^*,\ \mathbf{R}_1 \wedge \mathbf{K}_1^*$
8	$\mathbf{R}_4^*,\ \mathbf{R}_2 \wedge \mathbf{R}_2^*,\ \mathbf{R}_2 \wedge \mathbf{V}_1,\ \mathbf{V}_3,\ \mathbf{V}_1 \wedge \mathbf{V}_1$ $\mathbf{TQ}_0 \wedge \mathbf{TQ}_0,\ \mathbf{R}_2 \wedge \mathbf{TQ}_0,\ \mathbf{TQ}_2,\ \mathbf{V}_1 \wedge \mathbf{TQ}_0$ $\mathbf{K}_1 \wedge \mathbf{K}_0,\ \mathbf{R}_1 \wedge \mathbf{R}_3^*,\ \mathbf{R}_1 \wedge \mathbf{R}_1 \wedge \mathbf{R}_2^*$ $\mathbf{R}_1 \wedge \mathbf{R}_1 \wedge \mathbf{R}_1 \wedge \mathbf{R}_1^*,\ \mathbf{R}_1 \wedge \mathbf{TQ}_1$ $\mathbf{R}_1 \wedge \mathbf{R}_1 \wedge \mathbf{V}_1,\ \mathbf{R}_1 \wedge \mathbf{R}_1 \wedge \mathbf{TQ}_0,\ \mathbf{R}_1 \wedge \mathbf{V}_2,$

A. Mardones and J. Zanelli, Class. Quant. Grav., 1991.

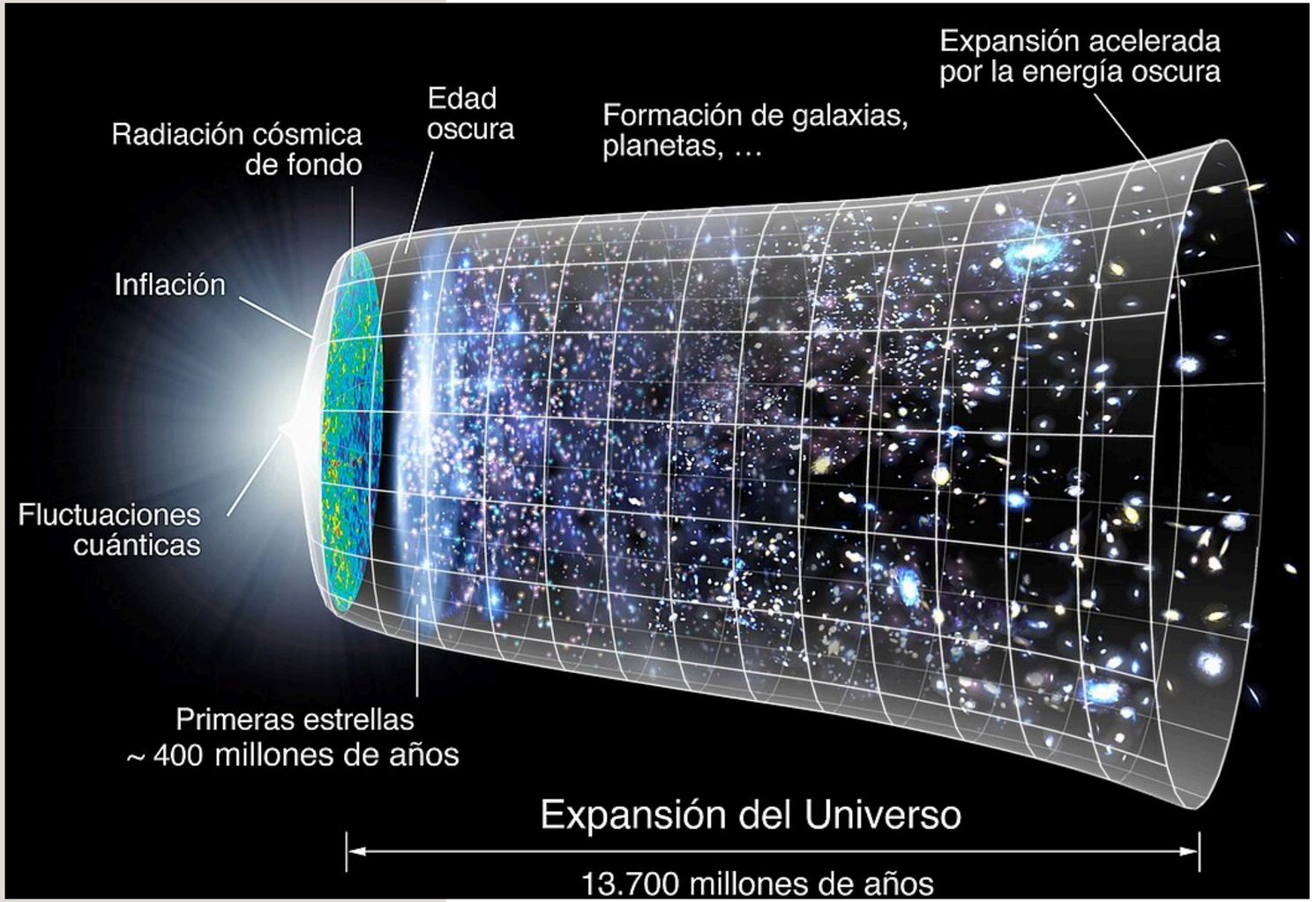
B. Janssen and A. Jiménez-Cano, Phys. Lett. B, 2019.

J. Martínez, J. Rodríguez, Y. Rodríguez.,XXXX,XXXX.

4.Conclusions, prospects...

- Even here, fourth dimensional gravity only propagates two degrees of freedom!
- Equivalence principle is naturally recovered outside material sources in a fourth dimensional manifold
- Torsion and non metricity could play a role in high density regimes: Inflation, black holes, neutron stars
- In higher than four dimensions, these new fields could propagate by the quadratic, or higher order terms, in the curvature 2-Form

A. Mardones and J. Zanelli, Class. Quant. Grav., 1991.
B. Janssen and A. Jiménez-Cano, Phys. Lett. B, 2019.
J. Martínez, J. Rodríguez, Y. Rodríguez.,XXXX,XXXX.



Gracias!

Thank you!

Grazie!