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Spin bundles and Dirac equation in FRW space

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Motivation

01

$$(i\gamma^\mu \partial_\mu - m)\psi = 0$$

We already know the Dirac equation in Minkowski space, but how will spinors behave in a curved space-time?

02

$$\mathcal{L}_{KG} = \frac{1}{2} g^{\mu\nu} \nabla_\mu \nabla_\nu \phi - \frac{1}{2} m^2 \phi^2 + \frac{1}{2} \xi R \phi^2$$

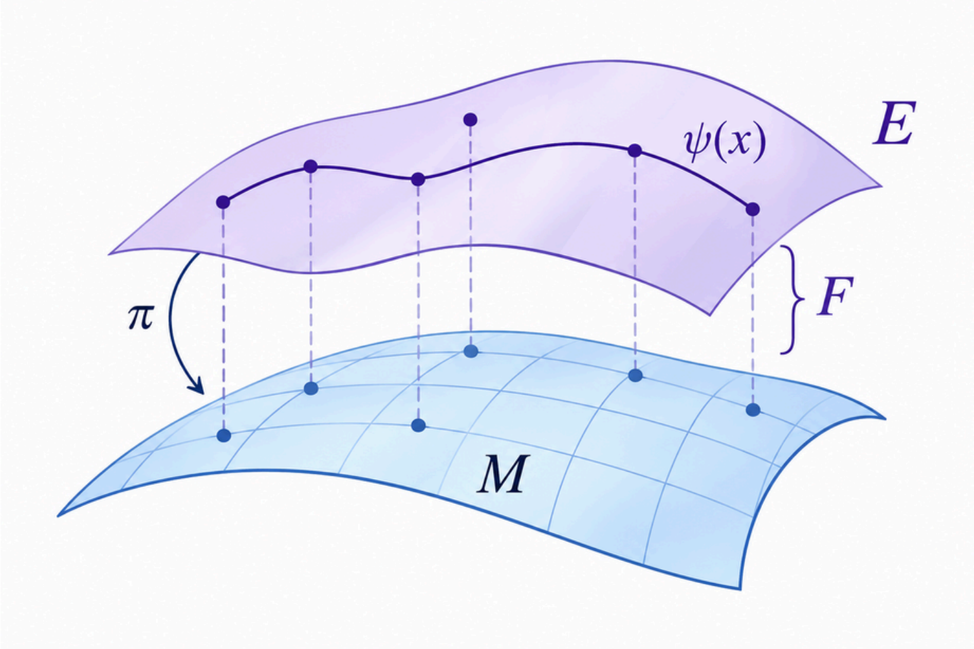
$$\mathcal{L}_{BP} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + \frac{1}{2} A_\mu A^\mu; \;\; F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu$$

We already know the equations for a Klein-Gordon and a Proca Field since we know how to calculate the covariant derivative of both scalars and tensors.

03

$$(i\gamma^\mu \nabla_\mu - m)\psi = 0?$$

Derivation and fiber bundles



The fields that we derive are not over the Manifold but over fiber bundles defined on it.

$$\nabla_\mu \phi = \partial_\mu \phi$$

$$\nabla_\mu A^\nu = \partial_\mu A^\nu + \Gamma^\nu_{\mu\lambda} A^\lambda$$

$$\nabla_\rho T^{\mu\nu} = \partial_\rho T^{\mu\nu} + \Gamma^\mu_{\rho\lambda} T^{\lambda\nu} + \Gamma^\nu_{\rho\lambda} T^{\mu\lambda}$$

$$\phi \in \mathcal{M} \times \mathbb{R} \qquad A^\mu \in T\mathcal{M} \qquad T^{\mu\nu} \in T\mathcal{M} \otimes T\mathcal{M}$$

Build any fiber bundle

A principal bundle over a manifold $\mathcal{M}$ is a $P(M, G)$ where G is the estructure group and $F \cong G$

We can build a fiber bundle E with fiber type F by considering a representation on G in F and

Let $p \in P, v \in F, g \in G$

We define an equivalence relation on $P \times F$ by

$$(p, v) \sim (pg, \rho(g^{-1})v)$$

The associated P fiber bundle to F, E, is

$$E \equiv (P \times_\rho F) / \sim$$

Spin bundles I

In our case

$$\mathcal{M} \longrightarrow \text{Lorentzian space time}$$
$$G \longrightarrow \text{Lorentz group } \mathrm{SO}^+(1,3)$$

We take $P_{\mathrm{SO}^+(1,3)}$ as our principal bundle where the fiber is defined by

$$F_p := L_p\mathcal{M} = \{(e_0, e_1, e_2, e_3) \text{ basis of } T_p\mathcal{M}\}$$

$$P_{\mathrm{SO}^+(1,3)} = \bigcup_{\mathcal{M}} L_p\mathcal{M}$$

Spin bundles II

From QFT we have

$$\phi \to D^{(0,0)}, A^{\mu} \to D^{(1/2,1/2)}, \psi \to D^{(1/2,0)} \oplus D^{(0,1/2)}$$

Spinors don't transform under $\psi \not\to \mathrm{SO}^{+}(1,3)$

We have to consider $\mathrm{Spin}(1,3) \cong \mathrm{SL}(2,\mathbb{C})$

$$\lambda : \mathrm{Spin}(1,3) \longrightarrow \mathrm{SO}^{+}(1,3),\ \ker\lambda = \{+1,-1\}$$

And naturally,

$$\Lambda : P_{\mathrm{Spin}(1,3)} \to P_{\mathrm{SO}^{+}(1,3)}$$

Spin bundles III

Take the transition functions on each group

$$g_{ij} : U_i \cap U_j \to \mathrm{SO}^+(1,3)$$
$$\tilde{g}_{ij} : U_i \cap U_j \to \mathrm{Spin}(1,3)$$

$$\lambda(\tilde{g}_{ij}\tilde{g}_{jk}\tilde{g}_{ki}) = \lambda(\tilde{g}_{ij})\lambda(\tilde{g}_{jk})\lambda(\tilde{g}_{ki}) = g_{ij}g_{jk}g_{ki} = I \quad \longrightarrow \quad \tilde{g}_{ij}\tilde{g}_{jk}\tilde{g}_{ki} = \pm I$$

Take $\tilde{g}'_{ij} = (-1)^{\epsilon_{ij}}\tilde{g}_{ij}$ s.t. $\epsilon_{ij} \in \mathbb{Z}_2$

$$\lambda(\tilde{g}'_{ij}) = \lambda(\tilde{g}_{ij}) = g_{ij}$$

"Choose" the correct signs

Stiefel Witney second class condition

Spin structure

$$\begin{array}{ccc} \mathrm{Spin}(1,3) & \xrightarrow{\lambda} & \mathrm{SO}^+(1,3) \\ \uparrow{\scriptstyle \exp_{\mathrm{Spin}}} & & \uparrow{\scriptstyle \exp_{\mathrm{SO}}} \\ \mathfrak{spin}(1,3) & \xrightarrow{d\lambda} & \mathfrak{so}^+(1,3) \end{array}$$

Spin bundles IV

Thus Λ is well defined and so is $P_{\mathrm{Spin}(1,3)}$

$$S_{\mathrm{D}} := (P_{\mathrm{Spin}(1,3)} \times_{\rho} \mathbb{C}^4)/\sim$$

Where Dirac spinors are $\psi \in \Gamma(S_{\mathrm{D}})$

And transform like $\psi(U_i) \to \psi(U_j) = \rho(\tilde{g}_{ij})\psi(U_i) = S(\Lambda)\psi = \Lambda_{1/2}\psi$

Levi-civita connection

For Levi civita connection, we define a connection 1-form

$$\omega \in \Omega^1(P_{\mathrm{GL}}, \mathfrak{gl}(4,\mathbb{R}))$$

$$\omega = \omega_\mu{}^\rho{}_\nu E_\rho{}^\nu dx^\mu$$

such that $\nabla_\mu = \partial_\mu + \omega_\mu$

defined by $\nabla_\rho g_{\mu\nu} = 0 \to \Gamma^\lambda_{\mu\nu}$ Christoffel symbols

In $\mathrm{GL}(4,\mathbb{R})$, $(E_\rho{}^\nu)^\alpha{}_\beta = \delta^\alpha_\rho \delta^\nu_\beta$

So, given a vector V

$$(E_\rho{}^\nu V)^\alpha = \delta^\alpha_\rho V^\nu$$

$$\omega_\mu V^\alpha = \Gamma^\rho_{\mu\nu}(E_\rho{}^\nu V)^\alpha = \Gamma^\alpha_{\mu\nu} V^\nu$$

Spin connection

In a local frame, we define a connection 1-form

$$\omega_L \in \Omega^1(P_{\mathrm{SO}(1,3)}, \mathfrak{so}(1,3))$$

Consider latin indices as local and greek as global

local $\to \eta_{ab}$
global $\to g_{\mu\nu}$

Because of the antisymmetry of M generators

$$\omega = \frac{1}{2}\omega_\mu{}^{ab} M_{ab} dx^\mu$$

Remember $\lambda : \mathrm{Spin}(1,3) \to \mathrm{SO}(1,3)$
$d\lambda : \mathfrak{spin}(1,3) \to \mathfrak{so}(1,3)$

The spin connection is given, under a $\mathrm{Spin}(1,3)$-rep ρ

$$\omega = \frac{1}{2}\omega_\mu{}^{ab} d\rho(M_{ab}) dx^\mu = \frac{1}{2}\omega_\mu{}^{ab} \Sigma_{ab} dx^\mu$$

$$\Sigma_{ab} = \frac{1}{4}[\gamma_a, \gamma_b]$$

From local to global

Different forms

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu$$

$$\omega^a = e^a_\mu dx^\mu$$

$$ds^2 = \eta_{ab} \omega^a \omega^b$$

$$g_{\mu\nu} = e^a_\mu e^b_\mu \eta_{ab}$$

We impose

$$\nabla_\mu e^a_\nu = 0$$

$$\nabla_\mu e^a_\nu = \partial_\mu e^a_\nu - \Gamma^\rho_{\mu\nu} e^a_\rho + \omega_\mu{}^a{}_b e^b_\nu$$

$$\omega_\mu{}^a{}_b = e^b_\nu (\Gamma^\lambda_{\mu\nu} e^a_\lambda - \partial_\mu e^a_\nu)$$

Another form for $\omega_{\mu}{}^{a}{}_{b}$

Imposing torsion zero in 1st Cartan equation

$$d\omega^{a} + \omega^{a}{}_{b} \wedge \omega^{b} = 0$$

In components

$$\partial_{\mu} e^{a}_{\nu} - \partial_{\nu} e^{a}_{\mu} + \omega_{\mu}{}^{a}{}_{b} e^{b}_{\nu} - \omega_{\nu}{}^{a}{}_{b} e^{b}_{\mu} = 0$$

Using connection antisymmetry

$$\omega_{cda} = \frac{1}{2}(C_{dac} - C_{acd} - C_{cda})$$

Lower indices, turn them into local

$$e^{\mu}_{c} e^{\nu}_{d} C_{\nu ab} + \omega_{cad} - \omega_{dac} = 0$$

$$\omega_{\mu ab} = \frac{1}{2} e^{\nu}_{a} e^{\rho}_{b} (e_{c\mu}(\partial_{\rho} e^{c}_{\nu} - \partial_{\nu} e^{c}_{\rho}) + e_{c\nu}(\partial_{\mu} e^{c}_{\rho} - \partial_{\rho} e^{c}_{\mu}) + e_{c\rho}(\partial_{\nu} e^{c}_{\mu} - \partial_{\mu} e^{c}_{\nu}))$$

Dirac equation

Gamma matrices are local, but covariant derivative has global indices

In general $A^\mu = e^\mu_a(x) A^a$

Then $\gamma^\mu(x) = e^\mu_a(x)\gamma^a$

The Clifford algebra

$$\{\gamma^\mu, \gamma^\nu\} = \{e^\mu_a \gamma^a, e^\nu_b \gamma^b\} = 2\eta^{ab} I e^\mu_a e^\nu_b$$

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} I$$

Finally,

$$(i\gamma^\mu(x)\nabla_\mu - m)\psi = 0$$

$$\left(ie^\mu_a(x)\gamma^a\left(\partial_\mu + \frac{1}{2}\omega_{\mu ab}\frac{1}{4}[\gamma^a, \gamma^b]\right) - m\right)\psi = 0$$

Why is important?

–Fermionic CGPP

–Fermionic dark matter models

–Fermionic fields evolution in early universe

–Fermions energy–momentum tensor and back reaction

FRW in conformal time

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = a(\eta)\eta_{\mu\nu}dx^\mu dx^\nu \qquad g_{\mu\nu} = a^2(\eta)\eta_{\mu\nu}$$

Thus, we can choose $e^a_\mu(\eta) = a(\eta)\delta^a_\mu,\ e^b_\nu(\eta) = \dfrac{1}{a(\eta)}\delta^b_\nu$

By using this, we get $\Gamma^\lambda_{\mu\nu} = \dfrac{1}{a}(\partial_\mu a\delta^\lambda_\nu + \partial_\nu a\delta^\lambda_\mu - \partial_\sigma a\,\eta^{\lambda\sigma}\eta_{\mu\nu})$

$$\omega_{\mu ab} = \frac{\dot{a}}{a}(\eta_{\mu a}\delta^0_b - \eta_{\mu b}\delta^0_a)$$

Dirac curved action

Now, taking the action $S = \int d^4x \sqrt{-g} \bar{\Psi} (i\gamma^\mu(x) \nabla_\mu - m) \Psi$

Consider $\gamma^\mu(x) = \gamma^\mu(\eta) = \frac{1}{a(\eta)} \delta^\mu_a \gamma^a = \frac{1}{a(\eta)} \gamma^\mu$

$g_{\mu\nu} = a^2 \eta_{\mu\nu} \longrightarrow g = -a^8 \longrightarrow \sqrt{-g} = a^4$

$$S = \int d^4x \, a^4 \bar{\Psi} \left(i \frac{1}{a} \gamma^\mu \partial_\mu + \frac{3}{2} i H \gamma^0 - m \right) \Psi$$

Weyl tranformation and Dirac eq

Consider a Weyl transformation $g_{\mu\nu} = a^2 \tilde{g}_{\mu\nu}, \Psi = a^{-3/2}\psi, e^a_\mu = \frac{1}{a}\tilde{e}^\mu_a$

$$S = \int d^4x \, \bar{\psi}(i\gamma^\mu \partial_\mu - a(\eta)m)\psi$$

By applying $\frac{\delta S}{\delta \bar{\Psi}}$, $(i\gamma^\mu \partial_\mu - a(\eta)m)\psi = 0$

field's comoving mass $\psi = a^{\frac{3}{2}}\Psi$

Instantaneous solution basis

Take a solution basis $\{U_{k,r}, V_{k,r}\}$

$$U_{kr} = \frac{e^{i\vec{k}\cdot\vec{x}}}{(2\pi)^{3/2}} \begin{pmatrix} u^A_{k\eta} \\ r u^B_{k\eta} \end{pmatrix} \otimes h_{\hat{k}r}$$

$$V_{kr} = \frac{e^{-i\vec{k}\cdot\vec{x}}}{(2\pi)^{3/2}} \begin{pmatrix} -(u^B)^*_{k\eta} \\ r(u^A)^*_{k\eta} \end{pmatrix} \otimes h_{\hat{k},-r} e^{-i\phi}$$

Using Dirac representation for Dirac matrices

$$i\partial_0 \begin{pmatrix} u^A \\ u^B \end{pmatrix} = \begin{pmatrix} am & k \\ k & -am \end{pmatrix} \begin{pmatrix} u^A \\ u^B \end{pmatrix}$$

Instantaneous solution basis

By diagonalizing $\begin{pmatrix} am & k \\ k & -am \end{pmatrix}$ we get $\lambda = \pm\omega = \pm\sqrt{k^2 + a^2m^2}$

By $Q = \begin{pmatrix} k/N_- & k/N_+ \\ (\omega - am)/N_- & -(\omega + am)/N_+ \end{pmatrix} = U^T$

$$U\left(i\partial_\eta(U^TUu) = \begin{pmatrix} am & k \\ k & -am \end{pmatrix} U^TUu\right)$$

$$i\frac{aHk_pm}{2(k_p^2 + m^2)}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}u' + i\partial_\eta u' = \begin{pmatrix} \omega & 0 \\ 0 & -\omega \end{pmatrix}u'$$

$$i\partial_\eta u' = \begin{pmatrix} \omega & i\mathfrak{X} \\ -i\mathfrak{X} & -\omega \end{pmatrix}u' \xrightarrow{a\to 0} i\partial_\eta u' = \begin{pmatrix} \omega & 0 \\ 0 & -\omega \end{pmatrix}u'$$

Conclusions

1. Constructed the spinor bundle from the Lorentz frame bundle using the spin representation
2. Derived the spin connection from the Levi–Civita connection
3. Obtained the Dirac equation in a curved spacetime.
4. Applied the formalism to a spatially flat FRW background.
5. Showed that the conformally rescaled fermion evolves with a time-dependent effective mass

Outlook

1. Fermionic gravitational particle production
2. Particle creation and cosmological relic abundance.
3. Fermionic energy–momentum tensor and backreaction.

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Thank You

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