

Proving the OSV conjecture with the gravitational path integral

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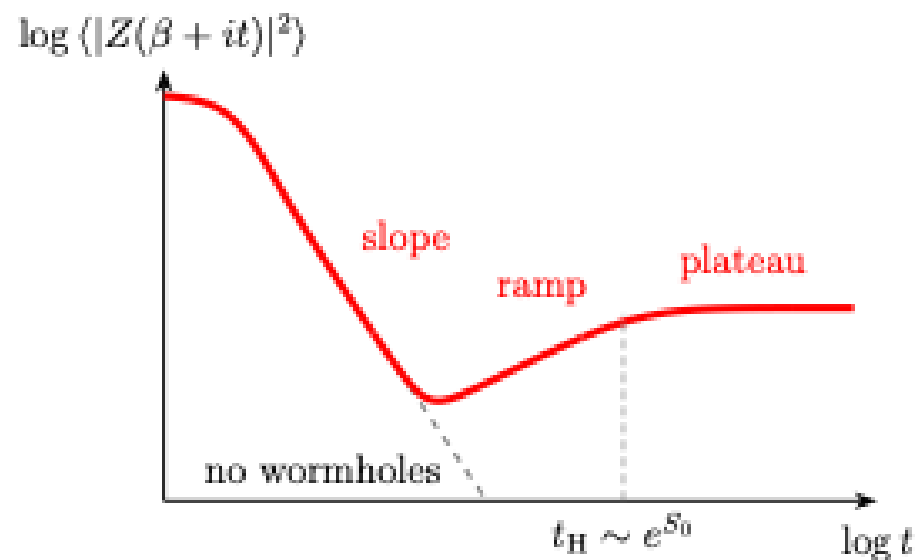
Based on work with
Nikolay Bobev, Jan Boruch, and Dawid Maskalaniec

building on earlier work with
Jan Boruch, Murat Kologlu, Sameer Murthy, and Gustavo J. Turiaci

The gravitational path integral (GPI) has become a precision tool for understanding quantum properties of black holes.

The gravitational path integral as a precision tool

Without supersymmetry: fine-grained statistics of black hole observables (e.g., the Page curve and spectral statistics)



With supersymmetry: The GPI can reproduce exact degeneracies (e.g., 1/8-BPS BHs in $\mathcal{N} = 8$ supergravity)

$$\begin{array}{c} e^{\pi\sqrt{\Delta}} \\ \text{cup} \\ c=1 \end{array} + \begin{array}{c} e^{\pi\sqrt{\Delta}/2} \\ \text{shallow cup} \\ c=2 \end{array} + \begin{array}{c} e^{\pi\sqrt{\Delta}/3} \\ \text{deep cup} \\ c=3 \end{array} + \dots = d(\Delta) \in \mathbb{Z}$$

$(\text{AdS}_2 \times S^2)/\mathbb{Z}_c$

What is still missing? The UV

In both settings, the GPI is insensitive to the interplay between **UV corrections** to the gravitational theory and the **sum over spacetimes**.

Without supersymmetry: UV effects, captured by higher-derivative corrections, are not under sufficient calculational control.

$\mathcal{N} = 8$ supergravity: the higher-derivative corrections coming from string theory truncate at tree level.

A useful middle ground:

1/2-BPS black holes in $\mathcal{N} = 2$ supergravity, obtained from a Calabi–Yau compactification of type IIA string theory where higher-derivative corrections are highly non-trivial.

An exact prediction for $\mathcal{N} = 2$ supergravity: the OSV conjecture

Ooguri, Strominger and Vafa conjectured an exact formula for the black hole partition function that should include all stringy effects:

$$Z_{\text{BH}}^{\text{OSV}}(\Gamma^I, \mu^I; \Omega_\infty) \equiv \underbrace{\text{Tr}_{\text{Fixed } \Gamma^I} (-1)^F (J_3)^2 e^{-\beta\{Q, Q^\dagger\} + \beta\mu^I \Gamma_I}}_{\text{helicity supertrace}} \stackrel{?}{=} \underbrace{|Z_{\text{top}}(t^A, g_{\text{top}})|^2}_{\text{A-model topological string}}$$

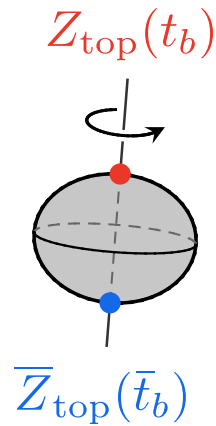
Z_{top} is the topological string partition function evaluated at the attractor values of the moduli, fixed by the charges and chemical potentials, and has a genus expansion with each order exact in α' :

$$F_{\text{top}}(t^A, g_{\text{top}}) = \log Z_{\text{top}} = \sum_{g=0}^{\infty} g_{\text{top}}^{2g-2} F_{\text{top},g}(t^A)$$

What we prove

We use the GPI to prove the OSV conjecture **to all orders in the topological string coupling** and at leading order in G_N .

New 4d supergravity saddles that compute the index to all orders in the derivative expansion: smooth supersymmetric Euclidean black holes with an $S^1_\beta \times S^2$ boundary at *finite* β .



Action localizes to the **north** and **south** poles of the horizon:

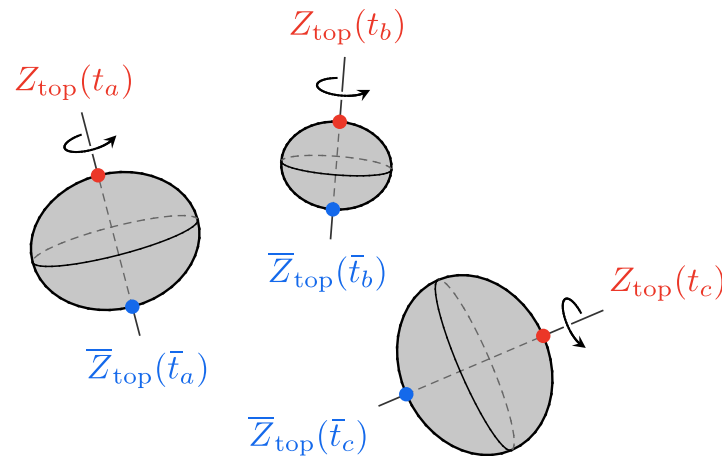
$$e^{-I_{\text{saddle}}} = \underbrace{Z_{\text{top}}(t_N, g_{\text{top}, N})}_{\text{north pole}} \underbrace{\bar{Z}_{\text{top}}(t_S, g_{\text{top}, S})}_{\text{south pole}} = |Z_{\text{top}}(t, g_{\text{top}})|^2$$

Note: Not the extremal ($\beta = \infty$) geometries, whose entropy counts bosons *plus* fermions: at finite β , the saddles directly compute their *difference*.

Why do we need a refinement?

The GPI also contains saddles whose contributions are non-perturbatively small, $\sim e^{-\#S}$, compared to the leading saddle:

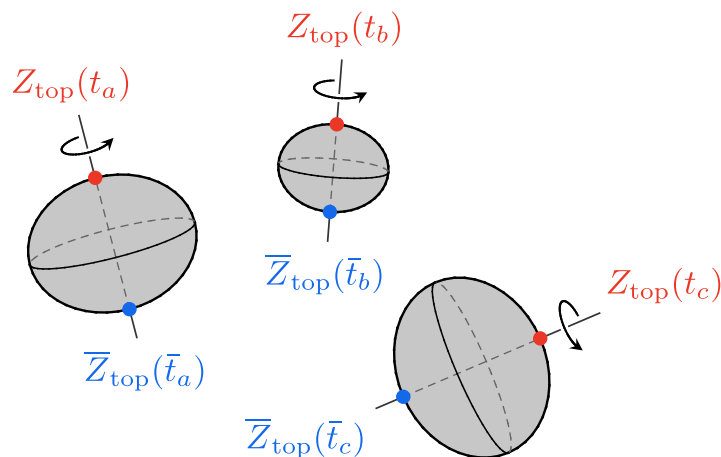
1. **Shifted saddles:** gauge-field configurations that need additional patches — responsible for the **quantization of electric charges**.
2. **Multi-center saddles:** several finite-temperature rotating black holes in equilibrium.



These are especially relevant when the single-center black hole is absent or subdominant.

A non-perturbative refinement of OSV

For all saddles we prove,



$$Z_{\text{BH}}^{\text{OSV}}(\{\Gamma_a^I\}, \mu^I, \{n_a^I\}; \Omega_\infty) \underset{\text{saddle-point approx.}}{=} \prod_{a=1}^n Z_{\text{top}}(t_{N,a}, g_{\text{top},N,a}) \bar{Z}_{\text{top}}(t_{S,a}, g_{\text{top},S,a})$$

to all orders in the topological string coupling for each non-perturbative correction in G_N .

What tools do we need?

1. Index saddles: finite-temperature, supersymmetric (complex) Euclidean black holes that directly compute the index — to all orders in the derivative expansion.

[LVI, Kologlu, Turiaci '21, Boruch, LVI, Murthy, Turiaci '23, '25, Adhikari et al. '25]

2. Equivariant localization of the gravitational action.

[Benetti Genolini et al. '19, '24, '26, Sparks et al. '26]

Key property: the Killing vector built from the Killing spinor has only isolated fixed points — the **north** and **south** poles of each horizon.

How do we compute indices directly in gravity?

$$Z_{\text{BH}}(\Gamma; \Omega_\infty) = \text{Tr}_{\text{Fixed } \Gamma} (-1)^F e^{-\beta H} = \Omega(\Gamma; \Omega_\infty) e^{-\beta M_{\text{BPS}}(\Gamma; \Omega_\infty)}$$

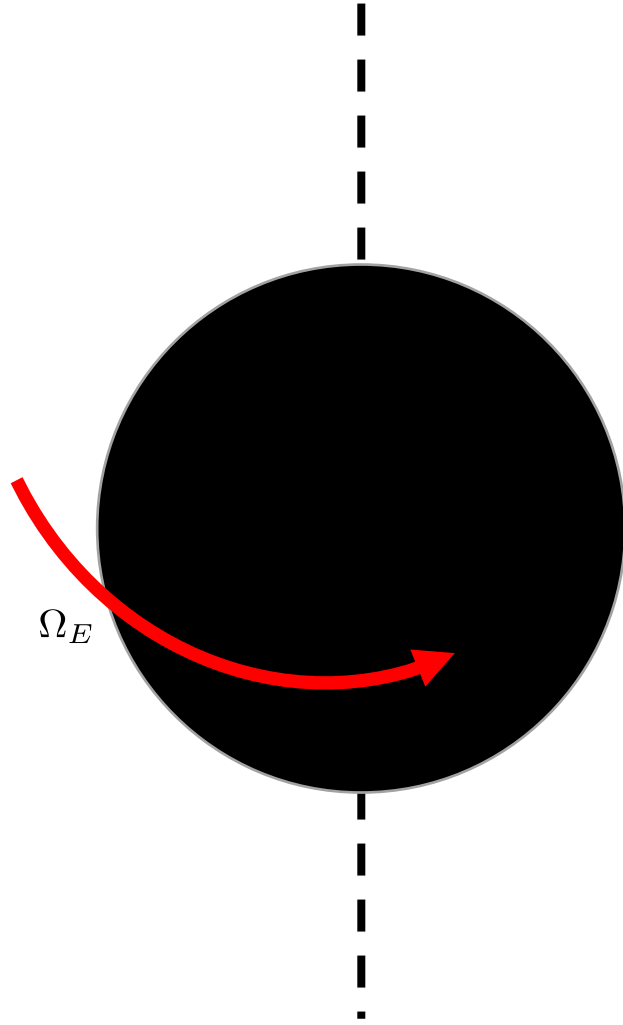
$$(-1)^F = e^{2\pi i J} \quad \rightarrow \quad \beta \Omega_E = i\beta \Omega_H = 2\pi(1 + 2n), \quad n \in \mathbb{Z}$$

Euclidean angular velocity $\Omega_E = i\Omega_H$: rotating by $2\pi(1+2n)$ around the thermal circle makes the fermions periodic.

[for the purposes of this talk there is no need to distinguish the index and the helicity supertrace]

Simplest example:
Pure (super)gravity

A black hole solution that we all hate (too complicated):
the Kerr–Newman solution



$$t \rightarrow i\tau, \quad \Omega_E = i\Omega_H$$



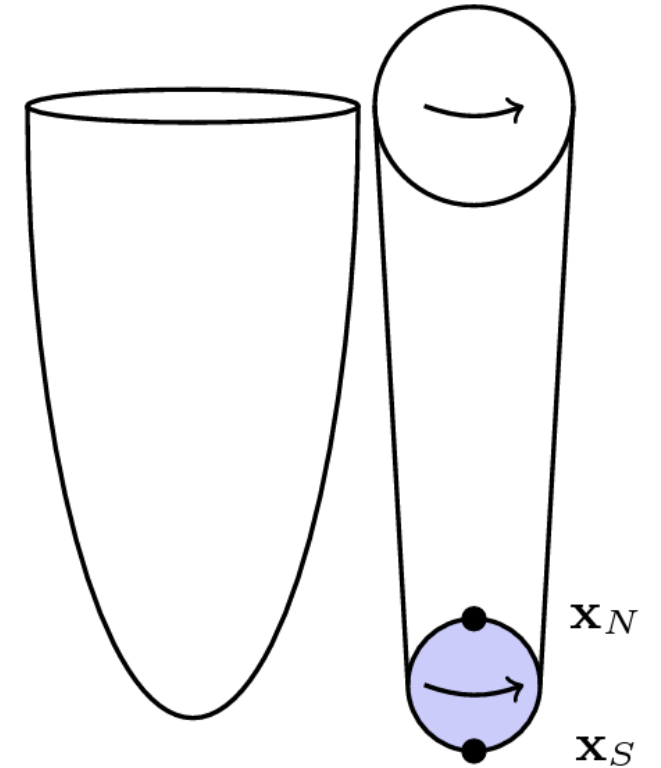
$$(\tau, \phi) \sim (\tau + \beta, \phi + \beta \Omega_E)$$



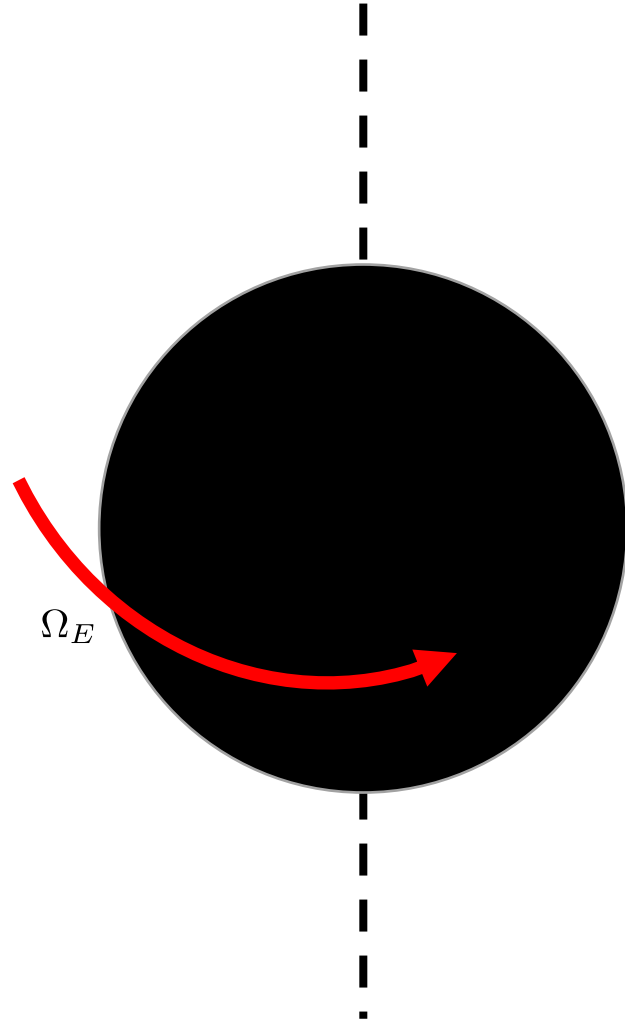
$$\psi(\tau, \phi) = -\psi(\tau + \beta, \phi + 2\pi)$$



$$\psi(\tau, \phi) = \psi(\tau + \beta, \phi)$$



A black hole solution that we all hate (too complicated):
the Kerr–Newman solution



$$t \rightarrow i\tau, \quad \Omega_E = i\Omega_H$$

$$-I_E(\beta, \Omega_E, Q) = -\beta Q + \pi Q^2 + \frac{Q^3(4\pi^2 - (\beta\Omega_E)^2)}{2\beta} + \dots$$

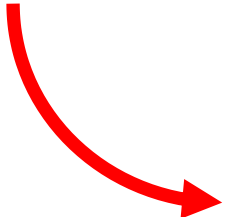
$$-I_E(\beta, \Omega_E, Q) \Big|_{\beta\Omega_E = \pm 2\pi} = \underbrace{-\beta Q + \pi Q^2}_{\text{form of an index}} \quad \text{(exact calculation)}$$

While the geometry is not spherically symmetric,
it has Killing spinors (i.e. it is supersymmetric).

Are there other supersymmetric saddles in pure (super)gravity?

General solution with Killing spinors: The Israel–Wilson solution

$$ds^2 = \frac{1}{U\tilde{U}}(d\tau + \omega_i dx^i)^2 + U\tilde{U}d\vec{x}^2, \quad \nabla^2 U = \nabla^2 \tilde{U} = 0$$

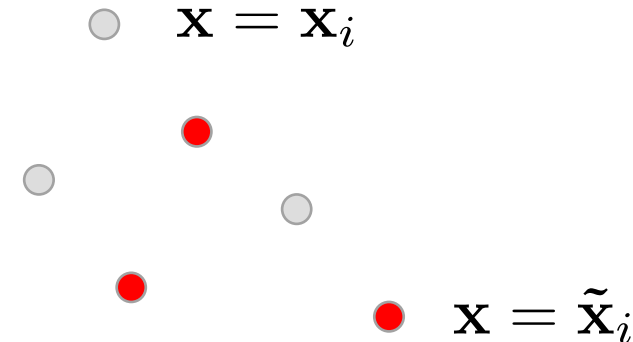


$$U(\mathbf{x}) = 1 + \sum_i \frac{a_i}{|\mathbf{x} - \mathbf{x}_i|}, \quad \tilde{U}(\mathbf{x}) = 1 + \sum_i \frac{\tilde{a}_i}{|\mathbf{x} - \tilde{\mathbf{x}}_i|},$$

Asymptotically $\mathbb{R}^3 \times S^1$ or $\mathbb{R}^{3,1}$

=

$$\# \bullet = \# \circ = N$$



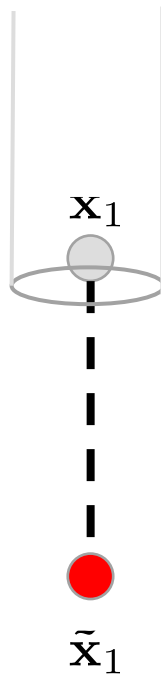
[Hartle, Hawking `72, Israel, Wilson `72, Whitt `84, Todd `85, Yuille `87, Maldacena, Michelson, Strominger `98, Dunajski, Hartnoll `06, ...]

Imposing smoothness

$$\text{If, } \mathbf{x}_i \neq \tilde{\mathbf{x}}_i \quad \epsilon_{ijk} \partial_j \omega_k = \tilde{U} \partial_i U - U \partial_i \tilde{U}$$

$$\int_{\partial S_i} \omega = 4\pi a_i \tilde{U}(\mathbf{x}_i)$$

$$\int_{\partial \tilde{S}_i} \omega = -4\pi \tilde{a}_i U(\tilde{\mathbf{x}}_i)$$



Singularity?



Imposing smoothness

$$\text{If, } \mathbf{x}_i \neq \tilde{\mathbf{x}}_i \quad \epsilon_{ijk} \partial_j \omega_k = \tilde{U} \partial_i U - U \partial_i \tilde{U}$$

$$\int_{\partial S_i} \omega = 4\pi a_i \tilde{U}(\mathbf{x}_i)$$

$$\int_{\partial \tilde{S}_i} \omega = -4\pi \tilde{a}_i U(\tilde{\mathbf{x}}_i)$$

We can perform a coordinate transformation

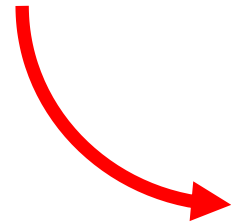
$$t \rightarrow t + \lambda(x^i) \quad \omega_i \rightarrow \omega_i + \partial_i \lambda(x) \quad t \rightarrow \begin{cases} t - 2U(\tilde{\mathbf{x}}_j) \tilde{a}_j \phi, & \text{at } \mathbf{x}_j \\ t + 2\tilde{U}(\mathbf{x}_j) a_j \phi, & \text{at } \tilde{\mathbf{x}}_j \end{cases}$$

Imposing smoothness

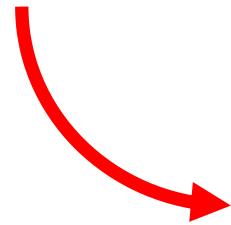
Assuming poles are different, we can perform a coordinate transformation

$$t \rightarrow t + \lambda(x^i) \qquad \omega_i \rightarrow \omega_i + \partial_i \lambda(x) \qquad t \rightarrow \begin{cases} t - 2U(\tilde{\mathbf{x}}_j) \tilde{a}_j \phi, & \text{at } \mathbf{x}_j \\ t + 2\tilde{U}(\mathbf{x}_j) a_j \phi, & \text{at } \tilde{\mathbf{x}}_j \end{cases}$$

Lorentzian: For $\omega \neq 0$ singular. $U(\mathbf{x})$ and $\tilde{U}(\mathbf{x})$ need to have the same poles



$$\bullet \quad \mathbf{x} = \mathbf{x}_i = \tilde{\mathbf{x}}_i \quad \& \quad \omega = 0$$



$$U(\mathbf{x}) = \tilde{U}(\mathbf{x}) = 1 + \sum_i \frac{Q_i}{|\mathbf{x} - \mathbf{x}_i|}$$

Majumdar–Papapetrou
extremal solution
(AdS₂ fragmentation)

Imposing smoothness

Assuming poles are different, we can perform a coordinate transformation

$$t \rightarrow t + \lambda(x^i) \qquad \omega_i \rightarrow \omega_i + \partial_i \lambda(x) \qquad t \rightarrow \begin{cases} t - 2U(\tilde{\mathbf{x}}_j) \tilde{a}_j \phi, & \text{at } \mathbf{x}_j \\ t + 2\tilde{U}(\mathbf{x}_j) a_j \phi, & \text{at } \tilde{\mathbf{x}}_j \end{cases}$$

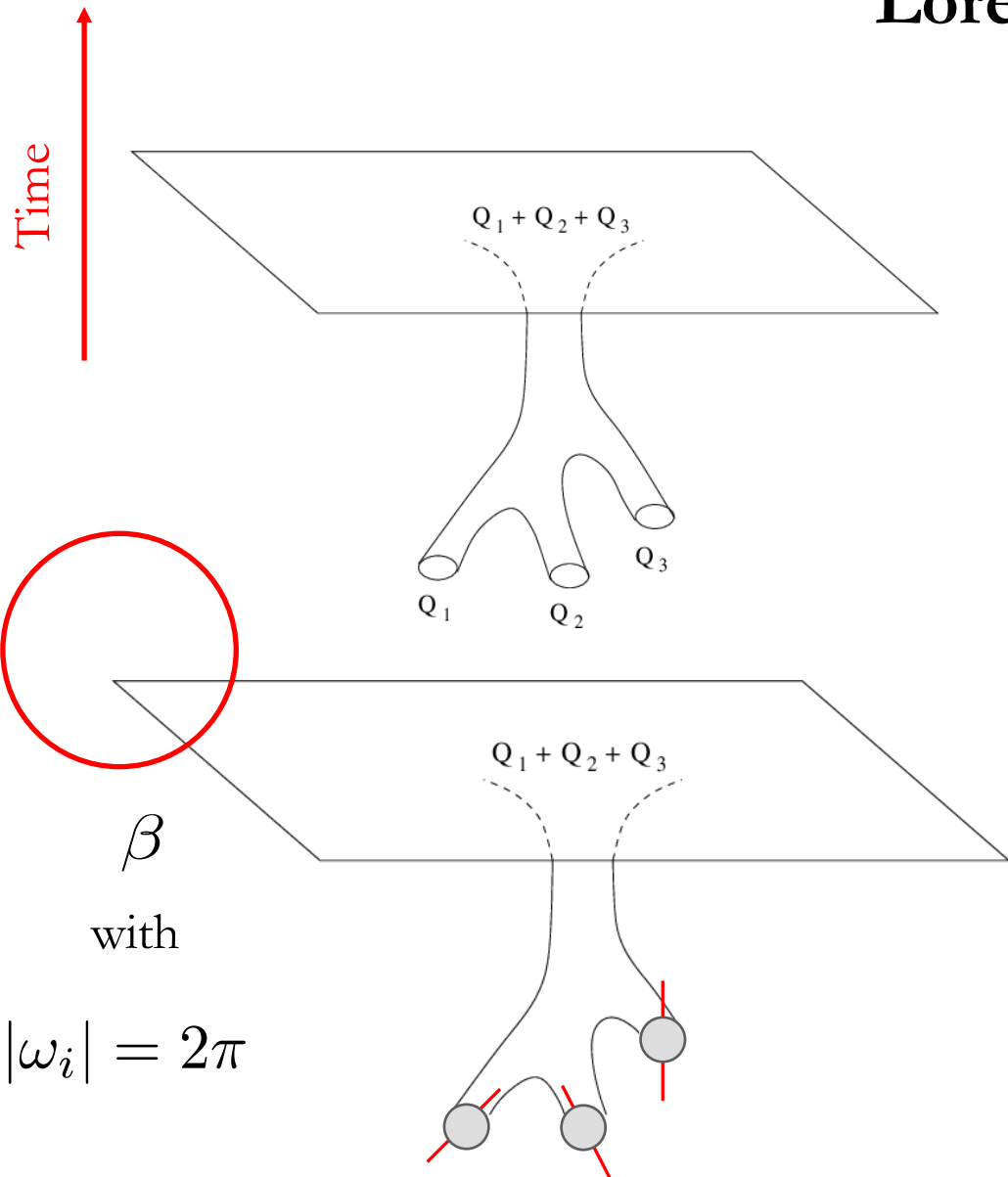
Euclidean: $t \sim t + \beta,$ $\tilde{U}(\mathbf{x}_j) a_j = \frac{\beta}{4\pi},$ $U(\tilde{\mathbf{x}}_j) \tilde{a}_j = \frac{\beta}{4\pi},$ $\forall j$

$$= \qquad =$$

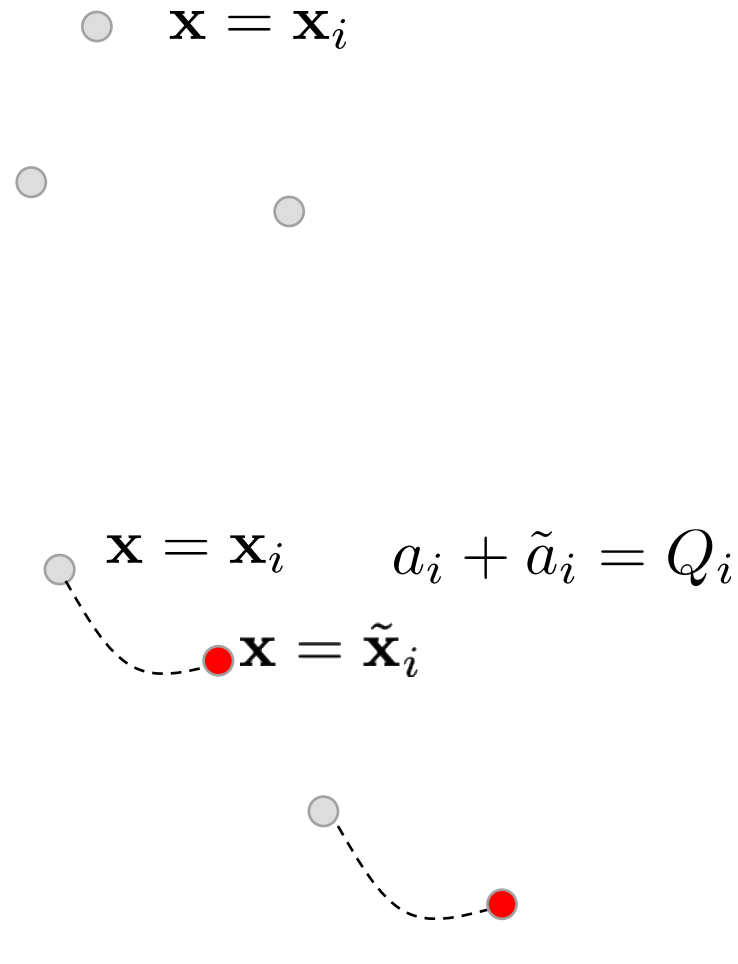
At each horizon: $\frac{1}{2\Omega} \qquad \frac{1}{2\Omega} \xrightarrow{\text{red arrow}} \text{Once again, } \beta\Omega = 2\pi$

One smoothness condition for each pole.

Lorentzian solution



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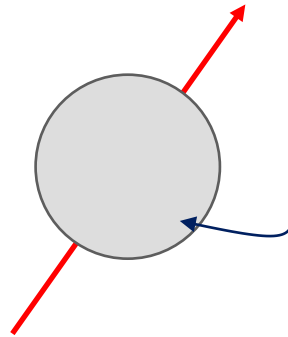
with $2N - 1$ constraints

Euclidean solution

The most basic example:
A north pole and a south pole

Back to the basic example:

If there are only two poles,
we find that this is just the finite- β Kerr–Newman solution with
Killing spinors that we discussed earlier.



The index for the black hole ensemble

For more than two poles, the geometry has **no** Killing horizon and we cannot apply the quantum statistical relation to compute the free energy [Wald].

$$-I_E = \underbrace{\pi \sum_i Q_i^2}_{\log(d_b - d_f)} - \beta \underbrace{\sum_i Q_i}_{=M_{\text{BPS}}}$$

This takes precisely the form of the free energy associated with an index.

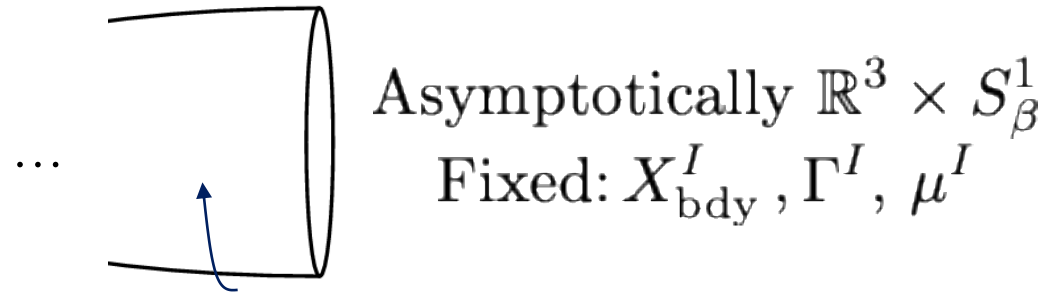
Once again, each individual area depends very non-trivially on β , but overall the Euclidean on-shell action has trivial β dependence.

How about black holes in string theory on $\mathbb{R}^4 \times CY$?

Black holes in string theory

Two complications:

- 1) Black holes in $\mathcal{N} = 2$ supergravity with vector multiplets (with **scalar moduli** X^I): generalizations of the *attractor solutions* [Ferrara, Kallosh, Strominger '95].



$$\psi(\tau, \phi) = \psi(\tau + \beta, \phi)$$

Black holes in string theory

2) UV (higher-derivative) corrections of the Calabi–Yau compactification are non-trivial and are encoded in the prepotential:

$\mathcal{F}^{(g)} \longleftrightarrow F_{\text{top},g} :$ genus- g topological string free energy

$$F_{\text{top}} = \sum_{g=0}^{\infty} g_{\text{top}}^{2g-2} F_{\text{top},g}(t) \iff \mathcal{F}(X, A) = \sum_{g=0}^{\infty} \underbrace{\mathcal{F}^{(g)}(X) A^g}_{A = \text{determined by the Weyl multiplet}}, \quad \mathcal{F}^{(g)}(\lambda X) = \lambda^{2-2g} \mathcal{F}^{(g)}(X)$$

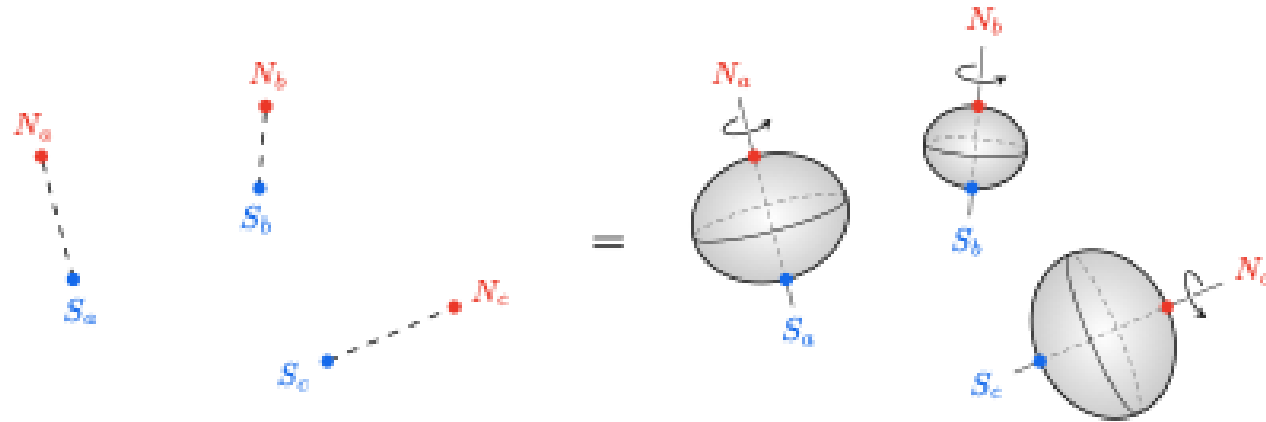
First few terms:

$$F_{\text{top}}(t, g_{\text{top}}) = \frac{(2\pi i)^3}{g_{\text{top}}^2} D_{ABC} t^A t^B t^C + \frac{\chi(\mathcal{Y}) \zeta(3)}{2g_{\text{top}}^2} - \frac{\pi i}{12} c_{2A} t^A + O(g_{\text{top}}^2)$$

The supersymmetric solutions, to all orders in derivatives

Supersymmetric solutions with a Killing spinor ϵ are specified by harmonic functions on $\mathbb{R}^3$:

$$H = h + \sum_{a=1}^n \left(\frac{\gamma_{N,a}}{|\vec{x} - \vec{x}_{N,a}|} + \frac{\gamma_{S,a}}{|\vec{x} - \vec{x}_{S,a}|} \right) \quad \text{where each pair of poles of } H \text{ is a BH with charges } \Gamma_a$$

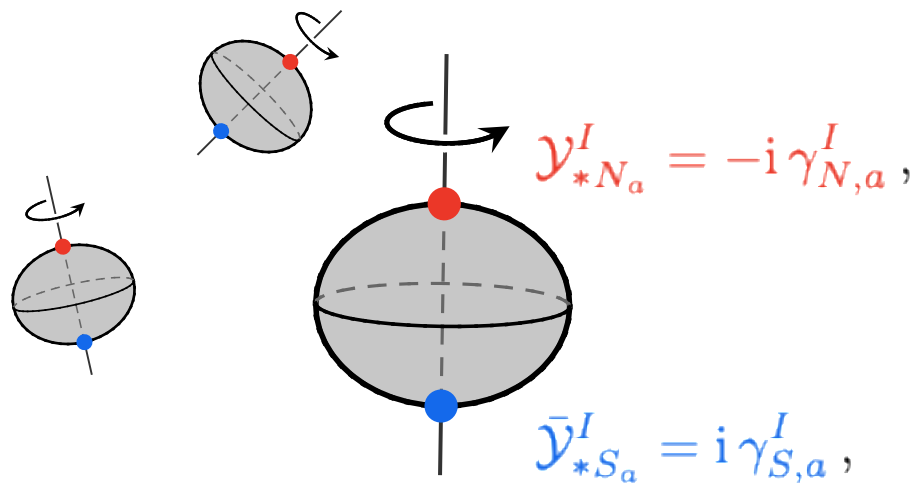


Determines $ds^2 = e^{2U} (d\tau + \omega)^2 + e^{-2U} d\vec{x}^2$, the gauge fields W , the scalars $\mathcal{Y}^I(X^I)$.

What determines $\gamma_{N,a}$ and $\gamma_{S,a}$?

The attractor mechanism to all orders in derivatives

At finite β each horizon has a **north** and a **south** pole, where the scalars are attracted to:



$$H = h + \sum_{a=1}^n \left(\frac{\gamma_{N,a}}{|\vec{x} - \vec{x}_{N,a}|} + \frac{\gamma_{S,a}}{|\vec{x} - \vec{x}_{S,a}|} \right)$$

Fixed charges:

$$\Gamma_a^I = \gamma_{N,a}^I + \gamma_{S,a}^I,$$

$$\Gamma_{a,I} = \mathcal{F}_I(\gamma_{N,a}^J, 64) + \bar{\mathcal{F}}_I(\gamma_{S,a}^J, 64).$$

Fixed electric chemical potential:

$$\gamma_{N,a}^I = n_{N,a}^I + \frac{i\beta}{2\pi} \left(\mu^I - \frac{1}{2} \phi^I \right),$$

$$\gamma_{S,a}^I = -n_{S,a}^I - \frac{i\beta}{2\pi} \left(\mu^I - \frac{1}{2} \phi^I \right), \quad n^I \in \Lambda^*$$

The scalars at the poles are fixed by the local charges alone — independent of the scalar moduli at infinity, the temperature and the other black holes.

Nevertheless, the black hole geometry is highly dependent on these parameters.

Computing the action with Dirichlet boundary conditions

Dirichlet boundary conditions for all fields — the natural boundary conditions of the OSV ensemble:

$$\partial M = S^1_\beta \times S^2_\infty : \quad W^I_\tau \rightarrow 2i\mu^I, \quad \frac{1}{4\pi} \int_{S^2_\infty} F^I = \Gamma^I, \quad \mathcal{Y}^I \rightarrow y^{I(0)} \text{ fixed}$$

In this ensemble,

$$I_{\text{mixed}} = I_{\mathbb{V}\mathbb{W}} + I_{\mathbb{H}} + I_{\text{bdy}}^{HD}.$$

Killing spinor bilinears:

$$S_{\pm} = \bar{\epsilon}_{i\pm} \epsilon_{\pm}^i, \quad \xi_a = 2i \bar{\epsilon}_{i+} \gamma_a \epsilon_-^i, \quad \Xi_{\pm i}^j = \frac{1}{2} (\bar{\epsilon}_{i\pm} \gamma_{ab} \epsilon_{\pm}^j) e^a \wedge e^b$$

Supersymmetry completes the Lagrangian of any weight-two chiral multiplet into an equivariantly closed polyform:

$$\Phi = (A, B^{ij}, F_{ab}^+, C), \quad \text{Weyl weight } w = 2$$

$$\Psi = \Psi_4 + \Psi_2 + \Psi_0, \quad (d - \iota_{\xi})\Psi = 0$$

$$\Psi_4 = \frac{1}{2} \left(C + \frac{1}{16} A (T_{ab}^-)^2 \right) \text{vol}_M$$

$$(\Psi_2)_{ab} = 2S_+ \left(F_{ab}^+ + \frac{1}{2} A T_{ab}^- \right) - B^{ij} \varepsilon_{jk} \Xi_{+i}^k{}_{ab}$$

$$\Psi_0 = -4A S_+^2$$

This follows from the off-shell supersymmetry transformations alone. For the action, the multiplet is built from the prepotential:

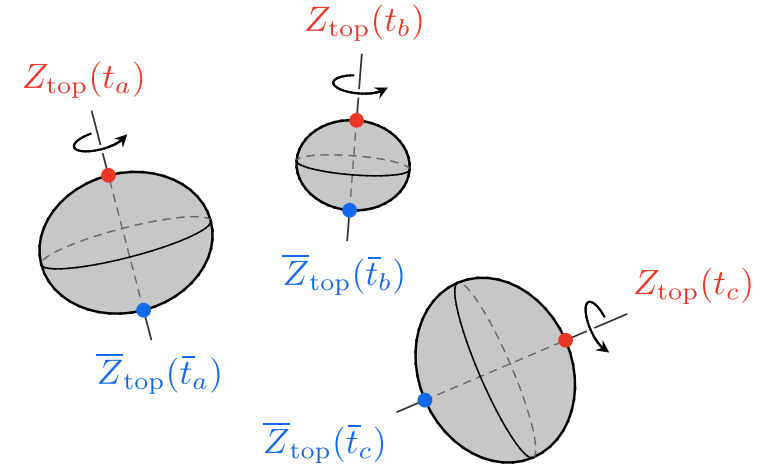
$$A|_{\mathcal{F}} = \mathcal{F}(X^I, A|_{\mathbb{W}^2}), \quad I_{\mathbb{W}\mathbb{W}} = -\frac{i}{8\pi} \int_M (\Psi_4 - \bar{\Psi}_4)$$

The BVAB fixed-point formula

Equivariantly closed forms localize onto the fixed points of $\xi = c_\xi \partial_\tau$ up to a boundary term:

$$\int_M \Psi = \sum_{\text{nuts } p} \frac{4\pi^2}{b_1^p b_2^p} \Psi_0|_p + I_{\partial M}$$

$$I_{\partial M} = - \int_{\partial M} \frac{\xi^\flat}{\|\xi\|^2} \wedge \left(\Psi_2 + \Psi_0 \frac{d\xi^\flat}{\|\xi\|^2} \right), \quad \xi^\flat = \xi_\mu dx^\mu$$



The fixed points are the **north** and **south** poles of each horizon.

$$\xi = c_\xi \partial_\tau : \quad b_1^{N_a} = b_2^{N_a} = -\frac{2\pi c_\xi}{\beta}, \quad b_1^{S_a} = -b_2^{S_a} = -\frac{2\pi c_\xi}{\beta}$$

Hence:

$$I_{\text{vw}} = -\frac{i}{8\pi} \left(I_{\text{nuts}} - \bar{I}_{\text{nuts}} + I_{\partial M} - \bar{I}_{\partial M} \right)$$

Contribution of the fixed points

Only **north** poles contribute to I_{nuts} and only **south** poles to $\bar{I}_{\text{nuts}}$:

$$-\frac{i}{8\pi} \left(I_{\text{nuts}} - \bar{I}_{\text{nuts}} \right) = 2\pi i \sum_{a=1}^n \left[\mathcal{F}(\gamma_{N,a}^I, 64) - \bar{\mathcal{F}}(\gamma_{S,a}^I, 64) \right]$$

Two copies of the prepotential, evaluated at the attractor values at the **north** and **south** poles.

The action in the mixed ensemble

Three contributions that still need to be computed: compensating hypermultiplets, BVAB boundary terms and GHY term.

Collecting the fixed points and boundary terms we find:

$$-I_{\text{mixed}}(\{\Gamma_a^I\}, \mu^I, \{n_a^I\}; \Omega_\infty) = \underbrace{\frac{\beta \phi_I \Gamma^I}{2}}_{\text{boundary terms} + \text{GHY}} - \underbrace{2\pi i \sum_{a=1}^n \left[\mathcal{F}(\gamma_{N,a}^I, 64) - \bar{\mathcal{F}}(\gamma_{S,a}^I, 64) \right]}_{\text{north and south poles}}$$

The second term is what OSV predicts.

The first term has **no interpretation within OSV**: it depends non-trivially on β through the potentials φ_I .

Why do we get an answer that disagrees with OSV?

Three ensembles

Dirichlet b.c. weigh BPS states by their mass: the **wrong ensemble** for OSV.

Mixed

Dirichlet b.c.

$$Z_{\text{BH}}(\Gamma^I, \mu^I; \Omega_\infty) = \text{Tr}_{\text{Fixed } \Gamma^I} (-1)^F (J_3)^2 e^{-\beta H + \beta \mu^I \Gamma_I}, \quad H = \{Q, Q^\dagger\} + |\mathcal{Z}|$$



Poisson resummation in n_a^I

Canonical

fixed charges

$$Z_{\text{BH}}(\Gamma^I, \Gamma_I; \Omega_\infty) = \text{Tr}_{\text{Fixed } (\Gamma^I, \Gamma_I)} (-1)^F (J_3)^2 e^{-\beta H} = \Omega(\Gamma^I, \Gamma_I; \Omega_\infty) e^{-\beta M_{\text{BPS}}}$$



$\times e^{\beta |\mathcal{Z}|}$, Fourier transform in $\Gamma_{I,a}$

OSV

the conjecture

$$Z_{\text{BH}}^{\text{OSV}}(\Gamma^I, \mu^I; \Omega_\infty) = \text{Tr}_{\text{Fixed } \Gamma^I} (-1)^F (J_3)^2 e^{-\beta \{Q, Q^\dagger\} + \beta \mu^I \Gamma_I}$$

From the mixed to the canonical ensemble

$$Z_{\text{BH}}(\{\Gamma_a^I\}, \{\Gamma_{I,a}\}) = (\text{Vol } \Lambda)^n \int \prod_{a,I} dn_a^I Z_{\text{BH}}(\{\Gamma_a^I\}, \mu^I, \{n_a^I\}) e^{-\sum_a (2\pi i n_a^I + \beta \mu^I) \Gamma_{I,a}}$$

Saddle point over the shifts (a Legendre transform at leading order in G_N):

$$\log Z_{\text{BH}}(\{\Gamma_a^I\}, \{\Gamma_{I,a}\}; \Omega_\infty) \approx -I_{\text{can}} = -\underbrace{\beta M_{\text{BPS}}}_{|\mathcal{Z}(\Gamma^I, \Gamma_I)|} + \underbrace{\sum_{a=1}^n S_a^{\text{Wald}}}_{\text{Not the area of each finite-T black hole}}$$

$$S_a^{\text{Wald}} = i\pi \left[\langle \gamma_{N,a}^I, \gamma_{S,a}^I \rangle - 128 \mathcal{F}_A(\gamma_{N,a}^J, 64) + 128 \bar{\mathcal{F}}_A(\gamma_{S,a}^J, 64) \right]$$

$$\Gamma_a^I = \gamma_{N,a}^I + \gamma_{S,a}^I, \quad \Gamma_{a,I} = \mathcal{F}_I(\gamma_{N,a}^J, 64) + \bar{\mathcal{F}}_I(\gamma_{S,a}^J, 64)$$

From the canonical to the OSV ensemble

Remove the BPS mass and Fourier transform in the electric charges:

$$Z_{\text{BH}}^{\text{OSV}}(\{\Gamma_a^I\}, \mu^I, \{n_a^I\}) = \frac{1}{(\text{Vol } \Lambda)^n} \int \prod_{a,I} d\Gamma_{I,a} \underbrace{Z_{\text{BH}}(\{\Gamma_a^I\}, \{\Gamma_{I,a}\})}_{\approx e^{-\tilde{I}_{\text{can}}}} e^{\beta |\mathcal{Z}(\Gamma, \Omega_\infty)|} e^{\sum_a (2\pi i n_a^I + \beta \mu^I) \Gamma_{I,a}}$$

$$\log Z_{\text{BH}}^{\text{OSV}}(\{\Gamma_a^I\}, \mu^I, \{n_a^I\}; \Omega_\infty) = -I_{\text{OSV}} = -2\pi i \sum_{a=1}^n \left[\mathcal{F}(\gamma_{N,a}^I, 64) - \bar{\mathcal{F}}(\gamma_{S,a}^I, 64) \right]$$

$$\gamma_{N,a}^I = \frac{\Gamma_a^I}{2} + \frac{i\beta\mu^I - 2\pi n_a^I}{2\pi}, \quad \gamma_{S,a}^I = \frac{\Gamma_a^I}{2} - \frac{i\beta\mu^I - 2\pi n_a^I}{2\pi}$$

The action depends only on the data at the **north** and **south** poles, and not on the moduli at infinity.

Proving OSV: focus on a single saddle

The prepotential is the topological string free energy:

$$F_{\text{top}}(t^A, g_{\text{top}}) = -2\pi i \mathcal{F}\left(\frac{2\pi i}{g_{\text{top}}}(1, t^A), 64\right), \quad \bar{F}_{\text{top}}(t^A, g_{\text{top}}) = +2\pi i \bar{\mathcal{F}}\left(\frac{2\pi i}{g_{\text{top}}}(1, t^A), 64\right)$$

Single-center saddle with $n^I = 0$ (continues to the extremal Lorentzian black hole as $\beta \rightarrow \infty$ at fixed $\beta\mu^I$):

$$\begin{aligned} t_N^A &= \frac{\Gamma^A + \frac{i\beta\mu^A}{\pi}}{\Gamma^0 + \frac{i\beta\mu^0}{\pi}}, & g_{\text{top},N} &= \frac{4\pi i}{\Gamma^0 + \frac{i\beta\mu^0}{\pi}}, \\ t_S^A &= \frac{\Gamma^A - \frac{i\beta\mu^A}{\pi}}{\Gamma^0 - \frac{i\beta\mu^0}{\pi}}, & g_{\text{top},S} &= \frac{-4\pi i}{\Gamma^0 - \frac{i\beta\mu^0}{\pi}} \end{aligned}$$

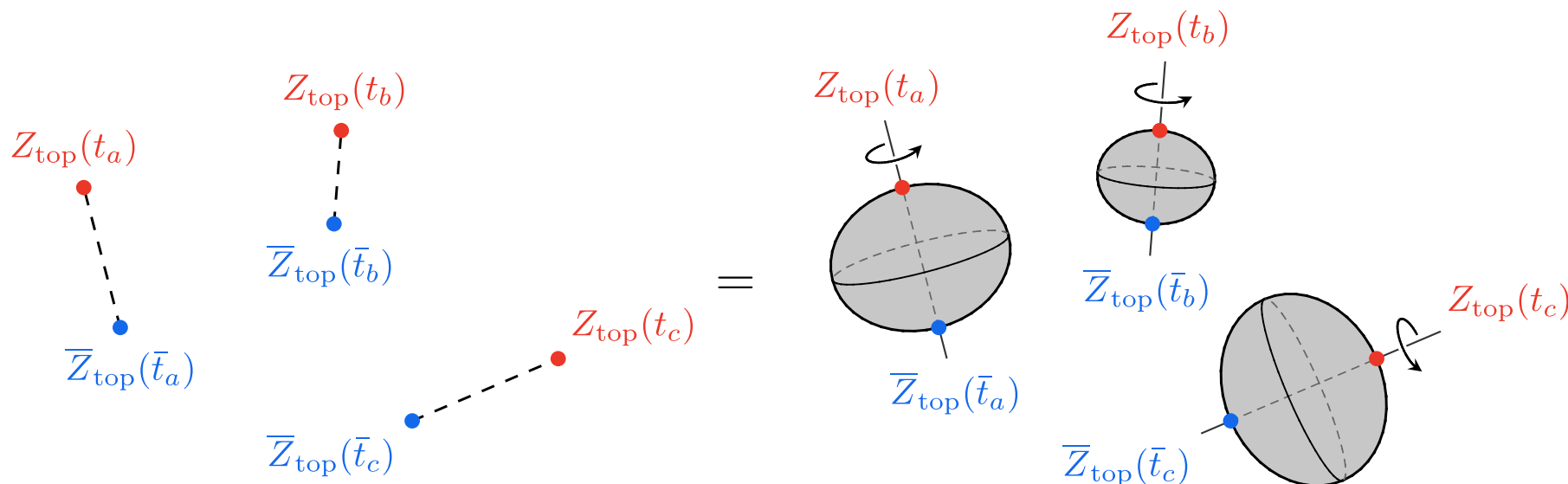
$$\boxed{e^{-I_{\text{OSV}}} = Z_{\text{top},N} \bar{Z}_{\text{top},S} = |Z_{\text{top}}(t, g_{\text{top}})|^2}, \quad Z_{\text{top}} = e^{F_{\text{top}}}$$

$\Rightarrow$ OSV holds to all orders in g_{top} for the unshifted single-center saddle, at leading order in G_N .

A non-perturbative refinement of OSV

The same localization formula applies to *every* saddle that contributes to the helicity supertrace:

$$Z_{\text{BH}}^{\text{OSV}}(\{\Gamma_a^I\}, \mu^I, \{n_a^I\}; \Omega_\infty) \underset{\text{saddle-point approx.}}{=} \prod_{a=1}^n Z_{\text{top}}(t_{N,a}, g_{\text{top},N,a}) \bar{Z}_{\text{top}}(t_{S,a}, g_{\text{top},S,a})$$



Two new kinds of saddles: **shifted saddles** and **multi-center black holes**.

Shifted saddles and the quantization of electric charges

The unshifted saddle alone is not periodic in $\beta\mu^I$, as quantized electric charges would require.
Single-center saddles with $n^I \neq 0$ have

$$\Gamma_I = \mathcal{F}_I\left(\frac{\Gamma^J}{2} + \frac{i\beta\mu^J - 2\pi n^J}{2\pi}, 64\right) + \bar{\mathcal{F}}_I\left(\frac{\Gamma^J}{2} - \frac{i\beta\mu^J - 2\pi n^J}{2\pi}, 64\right) \notin \Lambda \text{ in general}$$

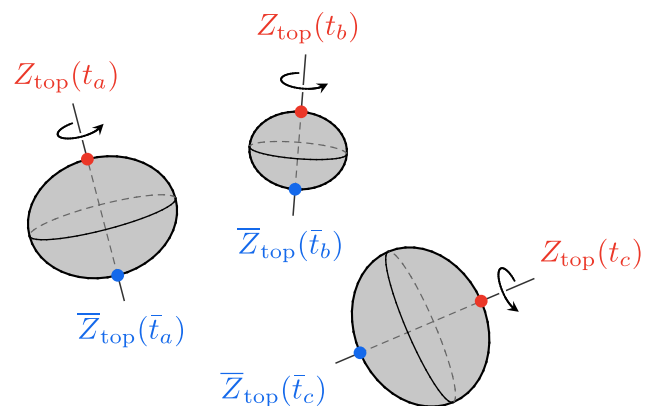
They still factorize into **north** $\times$ **south** pole contributions, now with $t_N \neq t_S$.

$$\beta\mu^I \rightarrow \beta\mu^I + 2\pi i n^I : \quad \text{saddle } \{n_a^I\} \mapsto \text{saddle } \{n_a^I + n^I\}$$

$\Rightarrow$ The sum over shifted saddles restores periodicity and charge quantization:

$$\sum_{n^I \in \Lambda^*} Z_{\text{top}}(t_N, g_{\text{top}, N}) \bar{Z}_{\text{top}}(t_S, g_{\text{top}, S}) \Big|_{n^I} \longleftrightarrow \sum_{\Gamma_I \in \Lambda} \Omega(\Gamma^I, \Gamma_I; \Omega_\infty) e^{\beta\mu^I \Gamma_I}$$

The sum is formal (the Wald entropy grows fast with the charges): it diverges, since both the OSV and mixed ensembles are technically ill-defined.



$$t_{N,a}^A = \frac{\Gamma_a^A + \frac{i\beta\mu^A}{\pi} - 2n_a^A}{\Gamma_a^0 + \frac{i\beta\mu^0}{\pi} - 2n_a^0},$$

$$g_{\text{top},N,a} = \frac{4\pi i}{\Gamma_a^0 + \frac{i\beta\mu^0}{\pi} - 2n_a^0},$$

$$t_{S,a}^A = \frac{\Gamma_a^A - \frac{i\beta\mu^A}{\pi} + 2n_a^A}{\Gamma_a^0 - \frac{i\beta\mu^0}{\pi} + 2n_a^0},$$

$$g_{\text{top},S,a} = \frac{-4\pi i}{\Gamma_a^0 - \frac{i\beta\mu^0}{\pi} + 2n_a^0}$$

Multi-center saddles have a moduli space, yet their action again localizes on all **north** and **south** poles:

$$Z_{\text{BH}}^{\text{OSV}}(\{\Gamma_a^I\}, \mu^I, \{n_a^I\}; \Omega_\infty) \underset{\text{saddle-point approx.}}{=} \prod_{a=1}^n Z_{\text{top}}(t_{N,a}, g_{\text{top},N,a}) \bar{Z}_{\text{top}}(t_{S,a}, g_{\text{top},S,a})$$

In the OSV ensemble, the total electric charge depends on how the magnetic charges are split.

A comment on AI: why human verification is still needed

While AI was not used to derive this result, I did use all frontier models to try to reproduce it for the purpose of verification.

The problem is technically difficult but well-posed and verifiable, so it should be in the wheelhouse of frontier models.

Prompt: Use the index saddles found by Boruch et al. together with equivariant localization for the on-shell action to prove the OSV conjecture to all orders in the topological string coupling.

Result: frontier models got this wrong even with adversarial agents, because they stopped at

$$-\frac{i}{8\pi} \left(I_{\text{nuts}} - \bar{I}_{\text{nuts}} \right) = 2\pi i \sum_{a=1}^n \left[\mathcal{F}(\gamma_{N,a}^I, 64) - \bar{\mathcal{F}}(\gamma_{S,a}^I, 64) \right]$$

with Dirichlet boundary conditions.

Likely culprit: ease of cheating and non-rigorous statements in training data.

AI was, however, useful in proving that the D-terms vanish.

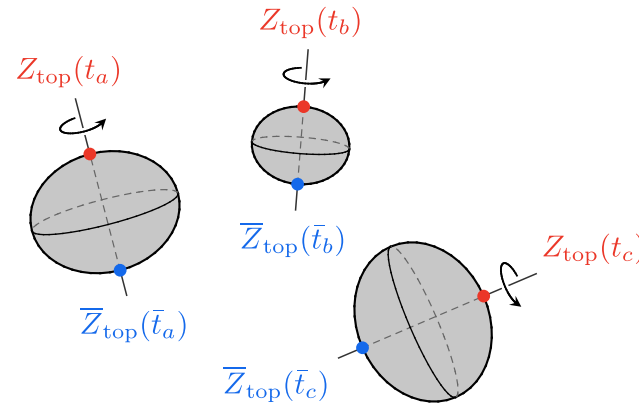
Summary

We can now find the index saddles to all orders in the derivative expansion; the attractor values at their poles are completely fixed by the prepotential.

Equivariant localization: the action localizes to the **north** and **south** poles of each black hole.

In the OSV ensemble, $e^{-I} = Z_{\text{top}}(\mathbf{N}) Z_{\text{top}}(\mathbf{S}) \Rightarrow$ OSV to all orders in g_{top} , at leading order in G_N .

We also defined a non-perturbative refinement for all saddles of the index.



Discussion: Can we do more and perform the GPI exactly for these geometries?

On-shell vs. off-shell

Equivariant localization applies to any configuration with a Killing spinor, not only to saddles.
The electromagnetic polyform is equivariantly closed on any ξ -invariant surface Σ , even off shell:

$$\mathcal{X} = \mathbb{F} - \mathrm{i}\Phi, \quad (\mathrm{d} - \iota_\xi)\mathcal{X} = \mathrm{d}\mathbb{F} = (\mathrm{d}F^I, \mathrm{d}G_I)$$

$$\int_\Sigma \mathbb{F} = -2\pi\mathrm{i} \sum_{\alpha \in \Sigma} \frac{\Phi_\alpha}{b_\alpha} + \mathrm{i} \int_{\partial\Sigma} \Phi \frac{\xi^\flat}{\|\xi\|^2}$$

Horizon Σ_a (compact):

$$4\pi \Gamma_a^{\text{horizon}} = \int_{\Sigma_a} \mathbb{F} = -2\pi\mathrm{i} \left(\frac{\Phi_{N,a}}{b_{N,a}} + \frac{\Phi_{S,a}}{b_{S,a}} \right)$$

Σ_α from a pole to infinity:

$$\beta\mu^I = \frac{\beta}{2}\phi^I - \frac{\pi}{b_\alpha}\Phi_\alpha^I + 2\pi\mathrm{i} n_\alpha^I \quad (\text{off shell})$$

$$\Gamma_a^I = \Gamma_a^{I,\text{horizon}} \quad (\text{off shell: Bianchi}), \quad \Gamma_{a,I} = \Gamma_{a,I}^{\text{horizon}} \quad (\text{only on shell: Maxwell})$$

In the OSV ensemble, the action is the same off shell

The action depends only on the pole data — fixed by the chemical potentials, the shifts and the magnetic charges through the Killing spinor equations alone:

$$\log Z_{\text{BH}}^{\text{OSV}}(\{\Gamma_a^I\}, \mu^I, \{n_a^I\}; \Omega_\infty) = -I_{\text{OSV}} = -2\pi i \sum_{a=1}^n \left[\mathcal{F}(\gamma_{N,a}^I, 64) - \bar{\mathcal{F}}(\gamma_{S,a}^I, 64) \right]$$

$$\gamma_{N,a}^I = \frac{\Gamma_a^I}{2} + \frac{i\beta\mu^I - 2\pi n_a^I}{2\pi}, \quad \gamma_{S,a}^I = \frac{\Gamma_a^I}{2} - \frac{i\beta\mu^I - 2\pi n_a^I}{2\pi}$$

$\Rightarrow I_{\text{OSV}}$ is unchanged under **any off-shell deformation that preserves the Killing spinor**.

In the canonical ensemble, by contrast, the fixed-point values are determined only on shell (Maxwell).

This suggests that, in the OSV ensemble, the contribution of each saddle is **one-loop exact**.

Why the answer looks one-loop exact

If the action is the same for all supersymmetric configurations, a Q-exact deformation localizes the GPI and leaves a one-loop determinant:

$$Z_{\text{BH}}^{\text{OSV}}(\Gamma^I, \mu^I; \Omega_\infty) \stackrel{?}{=} \sum_{\text{saddles}} Z_{\text{1-loop}} \prod_a Z_{\text{top}}(t_{N,a}, g_{\text{top},N,a}) \bar{Z}_{\text{top}}(t_{S,a}, g_{\text{top},S,a})$$

Evidence: in $\mathcal{N} = 4$ and $\mathcal{N} = 8$ string theories, Z^{OSV} can be computed from the exact microscopic degeneracies:

$$\mathcal{N} = 8 : \quad Z_{\text{BH}}^{\text{OSV}}(\Gamma^I, \mu^I) = \sum_{\beta\mu^I \rightarrow \beta\mu^I + 2\pi i n^I} |Z_{\text{top}}|^2 |g_{\text{top}}|^8 V_{T^6} + O(e^{-2\pi V_{T^6}/|g_{\text{top}}|^2})$$

$$\mathcal{N} = 4 : \quad Z_{\text{BH}}^{\text{OSV}}(\Gamma^I, \mu^I) = \sum_{\beta\mu^I \rightarrow \beta\mu^I + 2\pi i n^I} |Z_{\text{top}}|^2 V_{K3 \times T^2} + \dots$$

$$\Gamma^0 = 0 \text{ (no D6 charge)}, \quad \beta\mu^I \rightarrow \beta\mu^I + 2\pi i n^I \longleftrightarrow \text{shifted saddles}$$

$|Z_{\text{top}}|^2$ times a simple measure, summed over shifts — no further perturbative corrections.

What remains: the one-loop determinant

If factorization extends to quantum corrections, $Z_{1\text{-loop}}$ should factorize into **north** and **south** pole contributions.

The Atiyah–Bott formula localizes the equivariant index of the fluctuation complex onto the *same* nuts as the action:

$$\text{ind}_{D_{10}}(t) \equiv \text{Tr}_{\ker D_{10}} e^{-tQ^2} - \text{Tr}_{\text{coker } D_{10}} e^{-tQ^2} = \sum_{p \in \{\textcolor{red}{N}_a, \textcolor{blue}{S}_a\}} \left. \frac{\text{Tr}_{E_0} e^{-tQ^2} - \text{Tr}_{E_1} e^{-tQ^2}}{\det(1 - \partial\tilde{x}/\partial x)} \right|_p$$

$$\text{ind}_{D_{10}}(t) = \sum_k a(k) e^{-t\lambda_k} \quad \Rightarrow \quad Z_{1\text{-loop}} = \prod_k \lambda_k^{-a(k)/2} \stackrel{?}{=} \prod_{a=1}^n \textcolor{red}{Z}_{1\text{-loop}}^{N_a} \textcolor{blue}{Z}_{1\text{-loop}}^{S_a}$$

$$Q^2 = \mathcal{L}_\xi + \dots, \quad \xi = \partial_\tau \quad (\text{same Killing vector as in the BVAB formula})$$

Requires care with boundary conditions, boundary contributions and zero modes on these complex saddles — and fixes the measure of the Laplace transforms between ensembles.

Appendix

The vanishing of D-terms

D-terms (full-superspace integrals) can be rewritten with kinetic multiplets, i.e. as a $\mathbb{T}$ -log dependence of the prepotential:

$$\int d^4x d^4\theta d^4\bar{\theta} E f_n(\Phi) \bar{g}_n(\bar{\Phi}) = \int d^4x d^4\theta \mathcal{E} f_n(\Phi) \mathbb{T}(\bar{g}_n(\bar{\Phi}))$$

$$\mathcal{F} = \mathcal{F}_0(X, A|_{\mathbb{W}^2}) + \lambda \mathcal{F}_D(X, A|_{\mathbb{W}^2}, A|_{\mathbb{T}})$$

Following the family of saddles $\mathcal{S}_\lambda$ as the D-terms are turned on:

$$\frac{dI_\lambda[\mathcal{S}_\lambda]}{d\lambda} = I_D[\mathcal{S}_\lambda] + \underbrace{\int \frac{d\mathcal{S}_\lambda}{d\lambda} \frac{\delta I_\lambda}{\delta \mathcal{S}_\lambda}}_{=0} = I_D[\mathcal{S}_\lambda]$$

I_D localizes to the nuts, where

$$A|_{\mathbb{T}} \propto (b_1^p - b_2^p)^2 = 0 \text{ at north poles,} \quad \bar{A}|_{\mathbb{T}} \propto (b_1^p + b_2^p)^2 = 0 \text{ at south poles}$$

$$I_D = 0 \quad \Rightarrow \quad I_\lambda[\mathcal{S}_\lambda] = I_0[\mathcal{S}_0]$$