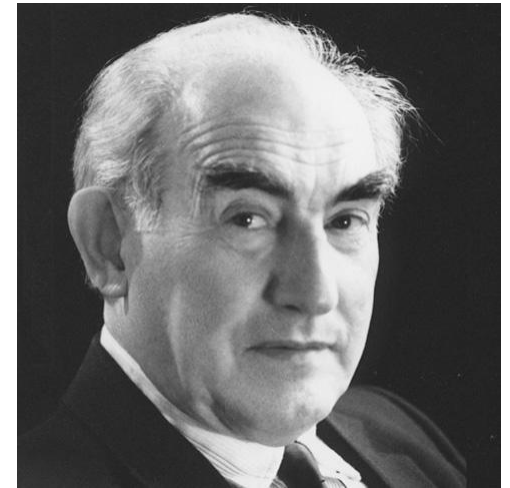
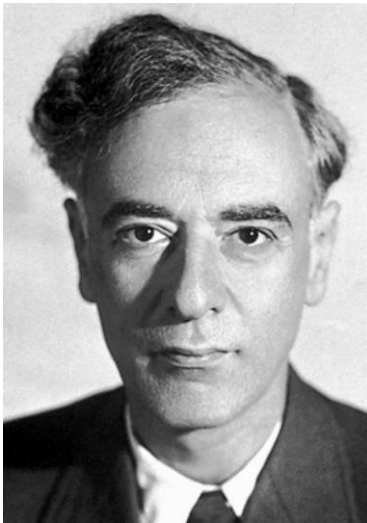


Landau-Ginzburg Description of Some Minimal Conformal Models

Igor R. Klebanov



Princeton University



Talk at LITP Michigan workshop
October 11, 2026

Based mainly on

- L. Fei, S. Giombi, IRK, G. Tarnopolsky, arXiv:1411.1099
- IRK, V. Narovlansky, Z. Sun, G. Tarnopolsky, arXiv:2211.07029
- A. Katsevich, IRK, Z. Sun, arXiv:2410.11714
- E. Arguello Cruz, IRK, G. Tarnopolsky, Y. Xin, arXiv:2505.06369 (published in PRX)
- Y. Nakayama, A. Katsevich, IRK, Z. Sun, arXiv:2609.10972

Minimal Conformal Theories in 2D

- Introduced in the landmark 1983 paper by Belavin, Polyakov and Zamolodchikov, who found an exact solution of an infinite class of “massless interacting 2D field theories.”
- $M(p,q)$ where p and q are coprime positive integers. The Virasoro central charges are

$$c(p, q) = 1 - 6 \frac{(p-q)^2}{pq}$$

- Contain $(p-1)(q-1)/2$ primary fields $\phi_{r,s}$ of holomorphic dimensions

$$h_{r,s} = \frac{(rq - sp)^2 - (p - q)^2}{4pq}, \quad r = 1, \dots, p-1; \quad s = 1, \dots, q-1$$

- BPZ identified $M(3,4)$ with the critical Ising model and commented that $M(5,6)$ may be related to the 3-state Potts model.
- The unitary theories $M(m+1, m+2)$ describe the multi-critical behavior in Landau-Ginzburg scalar field theories Zamolodchikov (1986)

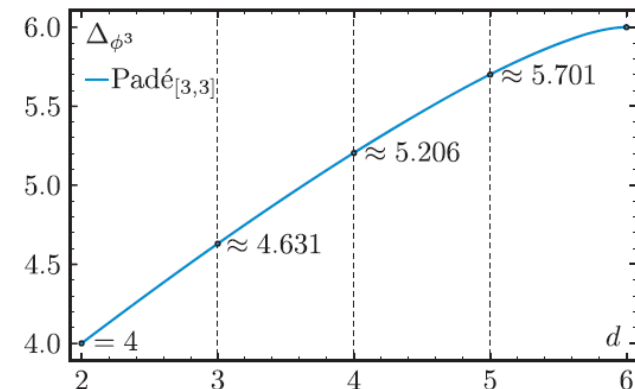
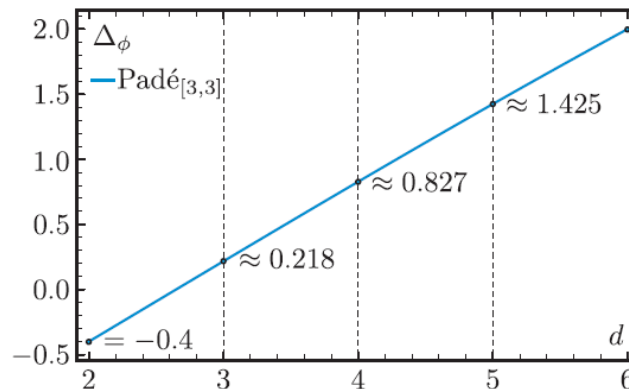
$$S_{m+1,m+2} = \int d^d x \left(\frac{1}{2} (\partial_\mu \phi)^2 + \frac{g}{(2m)!} \phi^{2m} \right)$$

- Most minimal theories are non-unitary, and the simplest one $M(2,5)$ was identified by Cardy (1985) as the Yang-Lee critical point whose GL description is Michael Fisher (1978)

$$S_{2,5} = \int d^d x \left(\frac{1}{2} (\partial_\mu \phi)^2 + \frac{g}{6} \phi^3 \right)$$

Yang-Lee universality class

- The coupling constant g is **imaginary**, as required by the fixed point in $d = 6 - \epsilon$
- The elementary field becomes the only Virasoro primary in $d=2$; it has dimension $-2/5$.
- Operator ϕ^3 becomes the quasiprimary $T\bar{T}$
- Two-sided Padé continuations of the $6 - \epsilon$ expansions



D-series M(3,10)

- A product of two Yang-Lee CFTs $M(2,5) \times M(2,5)$ is equivalent to the D_6 modular invariant of $M(3,10)$.
- Its Landau-Ginzburg description is provided by the theory of two scalar fields

$$S_{3,10} = \int d^d x \left(\frac{1}{2} (\partial_\mu \phi_1)^2 + \frac{1}{2} (\partial_\mu \phi_2)^2 + \frac{g}{6} (\phi_1^3 + \phi_2^3) \right)$$

- Later on, we will use define

$$\phi_1 + \phi_2 \sim \sigma \qquad \phi_1 - \phi_2 \sim \phi$$

- Here are the spin-zero primary fields contained in the modular invariant

$$Z_{3,10}^{D_6} = |\chi_{1,1} + \chi_{1,9}|^2 + |\chi_{1,3} + \chi_{1,7}|^2 + 2|\chi_{1,5}|^2$$

$(3, 10)$	$\phi_{1,1}$	$\phi_{1,3}$	$\phi_{1,5}^+$	$\phi_{1,5}^-$	$\phi_{1,7}$	$\phi_{1,9}$
h	0	$-\frac{2}{5}$	$-\frac{1}{5}$	$-\frac{1}{5}$	$\frac{3}{5}$	2
$\mathbb{Z}_2$	even	even	even	odd	even	even
$\mathcal{PT}$	even	even	odd	even	even	even
GL	1	$\phi_1\phi_2$	$\phi_1 + \phi_2$	$\phi_1 - \phi_2$	$i\phi_1\phi_2(\phi_1 + \phi_2)$	$T_{1\mu\nu}T_2^{\mu\nu}$

- There is also one primary field of spin 1 corresponding to $\phi_1\partial_\mu\phi_2 - \phi_2\partial_\mu\phi_1$
- And one of spin 2: $T_1^{\mu\nu} - T_2^{\mu\nu}$

O(N) Model in $4 < d < 6$

- The model with $\frac{\lambda}{4}(\phi^i \phi^i)^2$ interaction has a UV fixed point in $4 < d < 6$ with a good $1/N$ expansion. Parisi (1975)
- In $4 + \epsilon$ dimensions $\beta_\lambda = \epsilon\lambda + \frac{N+8}{8\pi^2}\lambda^2 + \dots$
- So, the UV fixed point is at a negative coupling
$$\lambda_* = -\frac{8\pi^2}{N+8}\epsilon + O(\epsilon^2)$$
- Another weakly coupled description in $d = 6 - \epsilon$
- O(N) symmetric field theory of N+1 scalar fields

$$S = \int d^d x \left(\frac{1}{2}(\partial\phi^i)^2 + \frac{1}{2}(\partial\sigma)^2 + \frac{g_1}{2}\sigma\phi^i\phi^i + \frac{g_2}{6}\sigma^3 \right)$$

Perturbative IR Fixed Points

The one-loop beta functions Fei, Giombi, IK (2014)

$$\beta_1 = -\frac{\epsilon g_1}{2} + \frac{(N-8)g_1^3 - 12g_1^2 g_2 + g_1 g_2^2}{12(4\pi)^3}$$
$$\beta_2 = -\frac{\epsilon g_2}{2} + \frac{-4N g_1^3 + N g_1^2 g_2 - 3g_2^3}{4(4\pi)^3}$$

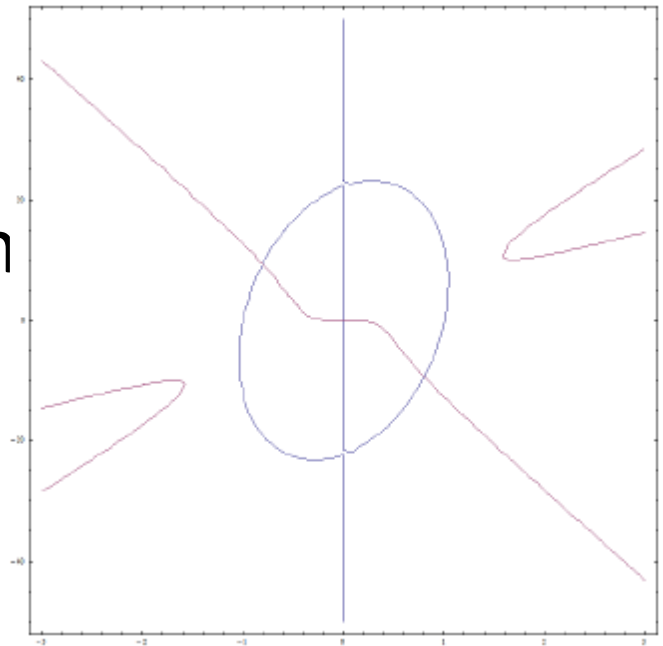
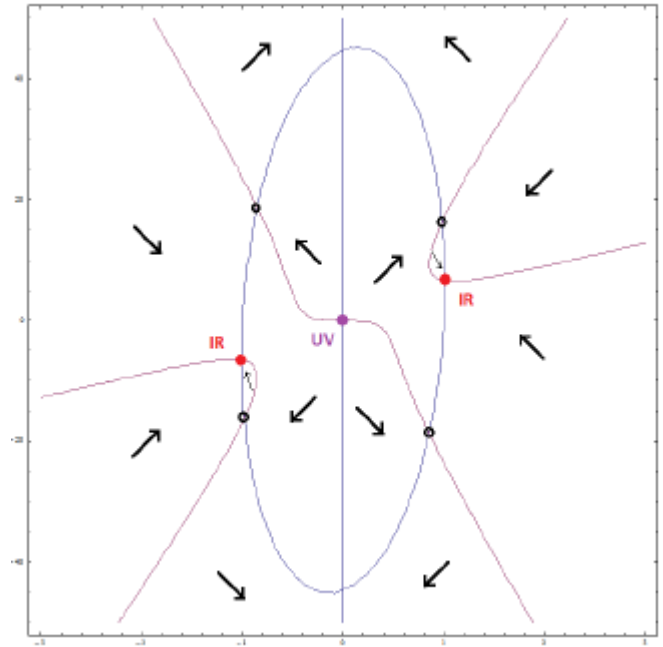
- For large N , the IR stable fixed point is at **real** couplings

$$g_{1*} = \sqrt{\frac{6\epsilon(4\pi)^3}{N}} \qquad g_{2*} = 6g_{1*}$$

- Non-perturbative instability.

RG Flows

- Here is the flow pattern for $N=2000$
- The IR stable fixed points go off to complex couplings for $N < 1039$. Large N expansion breaks down very early!



Three Loop Analysis

- The beta functions are found to be Fei, Giombi, IRK, Tarnopolsky (2014)

$$\begin{aligned}
 \beta_1 = & -\frac{\epsilon}{2}g_1 + \frac{1}{12(4\pi)^3}g_1 \left((N-8)g_1^2 - 12g_1g_2 + g_2^2 \right) \\
 & - \frac{1}{432(4\pi)^6}g_1 \left((536 + 86N)g_1^4 + 12(30 - 11N)g_1^3g_2 + (628 + 11N)g_1^2g_2^2 + 24g_1g_2^3 - 13g_2^4 \right) \\
 & + \frac{1}{62208(4\pi)^9}g_1 \left(g_2^6(5195 - 2592\zeta(3)) + 12g_1g_2^5(-2801 + 2592\zeta(3)) \right. \\
 & - 8g_1^2g_2^4(1245 + 119N + 7776\zeta(3)) + g_1^4g_2^2(-358480 + 53990N - 3N^2 - 2592(-16 + 5N)\zeta(3)) \\
 & + 36g_1^5g_2(-500 - 3464N + N^2 + 864(5N - 6)\zeta(3)) \\
 & \left. - 2g_1^6(125680 - 20344N + 1831N^2 + 2592(25N + 4)\zeta(3)) + 48g_1^3g_2^3(95N - 3(679 + 864\zeta(3))) \right) \\
 \beta_2 = & -\frac{\epsilon}{2}g_2 + \frac{1}{4(4\pi)^3} \left(-4Ng_1^3 + Ng_1^2g_2 - 3g_2^3 \right) \\
 & + \frac{1}{144(4\pi)^6} \left(-24Ng_1^5 - 322Ng_1^4g_2 - 60Ng_1^3g_2^2 + 31Ng_1^2g_2^3 - 125g_2^5 \right) \\
 & + \frac{1}{20736(4\pi)^9} \left(-48N(713 + 577N)g_1^7 + 6272Ng_1^2g_2^5 + 48Ng_1^3g_2^4(181 + 432\zeta(3)) \right. \\
 & - 5g_2^7(6617 + 2592\zeta(3)) - 24Ng_1^5g_2^2(1054 + 471N + 2592\zeta(3)) \\
 & \left. + 2Ng_1^6g_2(19237N - 8(3713 + 324\zeta(3))) + 3Ng_1^4g_2^3(263N - 6(7105 + 2448\zeta(3))) \right)
 \end{aligned}$$

- The epsilon expansions of scaling dimensions agree in detail with the large N expansion at the UV fixed point of the quartic O(N) model:

$$\begin{aligned}\Delta_\phi &= \frac{d}{2} - 1 + \gamma_\phi \\ &= 2 - \frac{\epsilon}{2} + \left(\frac{1}{N} + \frac{44}{N^2} + \frac{1936}{N^3} + \dots \right) \epsilon + \left(-\frac{11}{12N} - \frac{835}{6N^2} - \frac{16352}{N^3} + \dots \right) \epsilon^2 \\ &\quad + \left(-\frac{13}{144N} + \frac{6865}{72N^2} + \frac{54367/2 - 3672\zeta(3)}{N^3} + \dots \right) \epsilon^3,\end{aligned}$$

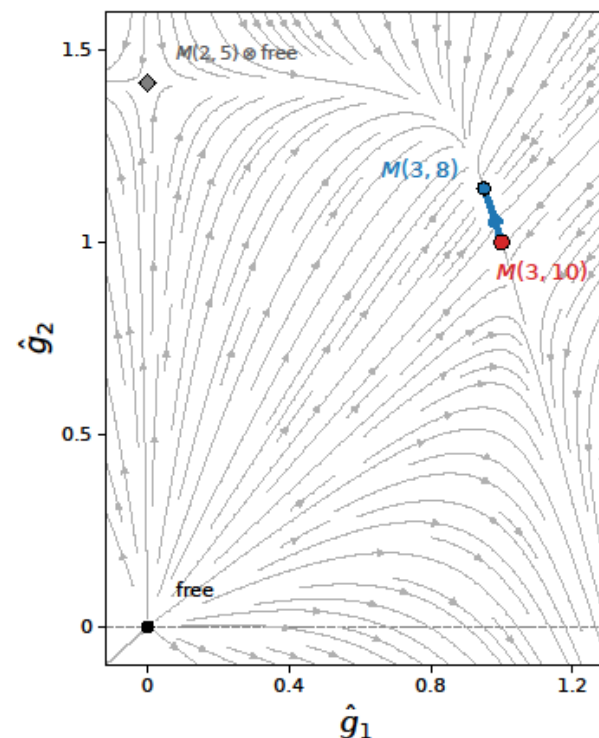
$$\begin{aligned}\Delta_\sigma &= \frac{d}{2} - 1 + \gamma_\sigma \\ &= 2 + \left(\frac{40}{N} + \frac{6800}{N^2} + \dots \right) \epsilon + \left(-\frac{104}{3N} - \frac{34190}{3N^2} + \dots \right) \epsilon^2 \\ &\quad + \left(-\frac{22}{9N} + \frac{47695/18 - 2808\zeta(3)}{N^2} + \dots \right) \epsilon^3.\end{aligned}$$

- This provides checks of higher loop diagrams.

Borinsky, Gracey, Kompaniets, Schnetz

- Fixed points in $d = 6 - \epsilon$ with purely imaginary coupling constants appear for $N < 1.02$, and we noticed a curious thing at $N=1$.
- In addition to the decoupled fixed point $g_1^* = g_2^*$ which describes $M(3,10)$, there is another one

$$g_2^* = 6g_1^*/5 + O(\epsilon)$$
- We made a (in retrospect good) guess that, upon continuation to $d=2$ this fixed point describes the $M(3,8)$ non-unitary model.



RG flow $M(3,10) \rightarrow M(3,8)$

- In $d = 6 - \epsilon$ there is flow from the product model to another imaginary fixed point of

$$S_3 = \int d^d x \left(\frac{1}{2} (\partial_\mu \phi)^2 + \frac{1}{2} (\partial_\mu \sigma)^2 + \frac{g_1}{2} \sigma \phi^2 + \frac{g_2}{6} \sigma^3 \right)$$

- PT symmetry under $\sigma \rightarrow -\sigma, \quad i \rightarrow -i$
- Continuation of the coupled fixed point to 2D provides a GL description of the D_5 modular invariant of $M(3,8)$:

$$Z_{3,8}^{D_5} = |\chi_{1,1}|^2 + |\chi_{1,3}|^2 + |\chi_{1,5}|^2 + |\chi_{1,7}|^2 + |\chi_{1,4}|^2 + \chi_{1,2} \bar{\chi}_{1,6} + \chi_{1,6} \bar{\chi}_{1,2}$$

- The epsilon expansions of operator dimensions are in good agreement with the values in $M(3,8)$.

c-theorem for non-unitary flows

- In $d=2$, this RG flow satisfies the inequality

$$c_{\text{eff}}^{\text{UV}} > c_{\text{eff}}^{\text{IR}} \qquad c_{\text{eff}} = c - 24h_{\text{min}}$$

which applies to the PT invariant flows.

Castro-Alvaredo, Doyon, Ravaninini

- For the minimal models $c_{\text{eff}}(p, q) = 1 - \frac{6}{pq}$
- For non-unitary models, this positive quantity determines the thermal free energy and the Entanglement Entropy. Itzykson, Saleur, Zuber
- For our flow, it decreases from $4/5$ to $3/4$

LG/BPZ correspondence

- $M(3,8)$ provides **exact solution** of a strongly interacting CFT with the known $\mathcal{PT}$ -symmetric cubic Euclidean action. There are 5 spin-zero primary fields

$(3, 8)$	$\phi_{1,1}$	$\phi_{1,3}$	$\phi_{1,4}^-$	$\phi_{1,5}$	$\phi_{1,7}$
h	0	$-\frac{1}{4}$	$-\frac{3}{32}$	$\frac{1}{4}$	$\frac{3}{2}$
$\mathbb{Z}_2$	even	even	odd	even	even
$\mathcal{PT}$	even	odd	even	odd	even
GL	1	σ	ϕ	$i\sigma^2 + i\phi^2$	$i\phi^2\sigma + i\sigma^3$

Spin 1

- The only Virasoro primary field with spin has 2D dimension $h_{1,2} + h_{1,6} = \frac{9}{16} \approx 0.563$

- We identify it with the vector operator

$$J_\mu = \sigma \partial_\mu \phi - \phi \partial_\mu \sigma$$

- One-loop calculation of its dimension gives

$$5 - \frac{1089}{998}\epsilon + O(\epsilon^2)$$

- In 2D this leads to $\Delta_J \approx 0.635$ which is only 13% off the exact result.

New families of RG flows

- In 2024, Nakayama and Tanaka proposed an infinite set of families of RG flows

$$M(kq + I, q) + \phi_{1,2k+1} \rightarrow M(kq - I, q)$$

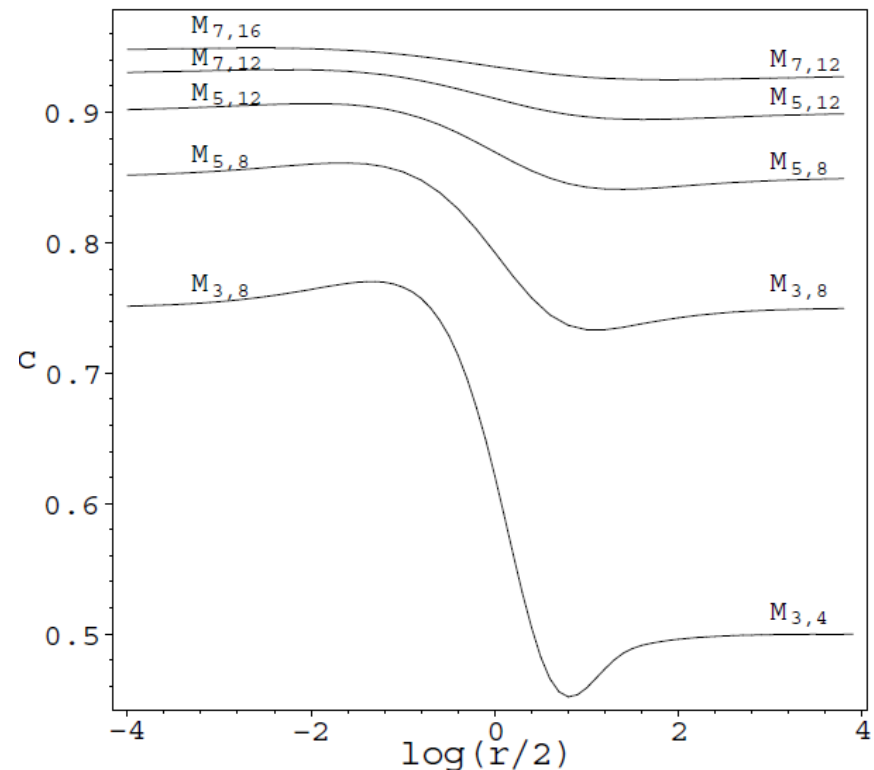
- The $k=1$ and $k=2$ families were known long ago. The Zamolodchikov unitary flows are $k=l=1$.
- The $k=2$ flows are non-unitary and were studied in the 1990s. Martins; Ravanini, Stanishkov, Tateo
- $M(3,10) \rightarrow M(3,8)$ turned out to be the first $k=3$ flow corresponding to $q=3$ and $l=1$.

- Directly in 2D, the Hamiltonian truncation is a useful approach to this RG flow. Delouche, Elias-Miro, Ingoldby
- Nakayama and Tanaka provided evidence for all the new families based on matching of invariants for topological line operators.

- Adding the mass terms in the GL description of $M(3,8)$ provides a realization of the $k=2$ flow

$$M(3,8) + \phi_{1,5} \rightarrow M(3,4)$$
- It was studied using a non-linear integral equation approach

Dorey, Dunning, Tateo (2000)



Emergent Supersymmetry

- Renormalization Group flow can lead to IR fixed points with enhanced symmetry.
- The minimal 3-d Yukawa theory for one Majorana fermion and one real pseudo-scalar was conjectured to have “emergent supersymmetry.”
Scott Thomas (unpublished seminar at KITP); Grover, Shen, Vishwanath
- The fermion mass is forbidden by the time reversal symmetry.
- After tuning the pseudo-scalar mass to zero, the theory is conjectured to flow to a $\mathcal{N}=1$ supersymmetric 3-d CFT.

Super Ising Model

- The UV lagrangian may be taken as

$$\mathcal{L}_{\mathcal{N}=1} = \frac{1}{2}(\partial_\mu \sigma)^2 + \frac{1}{2}\bar{\psi} \not{\partial} \psi + \frac{\lambda}{2}\sigma \bar{\psi} \psi + \frac{\lambda^2}{8}\sigma^4$$

- Has cubic superpotential $W \sim \lambda \Sigma^3$ in terms of the superfield $\Sigma = \sigma + \bar{\theta}\psi + \frac{1}{2}\bar{\theta}\theta f$
- This also provides the Landau-Ginzburg description of the simplest 2D $\mathcal{N}=1$ superconformal minimal model SM(3), which has central charge 7/10.

$\mathcal{N}=1$ minimal models

- SM(m) with $m=3, 4, \dots$ Friedan, Qiu, Shenker; Bershadsky, Knizhnik, Teitelman
- Have central charges $c(m) = \frac{3}{2}(1 - \frac{8}{m(m+2)})$
- For $m=10$ and $m=12$, in addition to the diagonal modular invariants, there are theories with exceptional modular invariants.
- For $m=10$, this theory is a product of two SM(3) models. Kastor, Martinec, Shenker It has $c=7/5=1.4$
- For $m=12$, the exceptional theory is especially interesting. It has $c=10/7$ and is the only unitary theory which exhibits the super- W_3 symmetry.

The Gross-Neveu-Yukawa Model

- The GNY model is the UV completion of the GN model in $d < 4$ Zinn-Justin; Hasenfratz, Hasenfratz, Jansen, Kuti, Shen

$$\mathcal{L}_{\text{GNY}} = \frac{1}{2}(\partial_\mu \sigma)^2 + \bar{\psi}_j \not{\partial} \psi^j + g_1 \sigma \bar{\psi}_j \psi^j + \frac{1}{24} g_2 \sigma^4$$

- IR stable fixed point in $4-\epsilon$ dimensions

$$\beta_{g_1} = -\frac{\epsilon}{2} g_1 + \frac{N+6}{2(4\pi)^2} g_1^3 + \frac{1}{(4\pi)^4} \left(-\frac{3}{4} (4N+3) g_1^5 - 2g_1^3 g_2 + \frac{g_1 g_2^2}{12} \right)$$

$$\beta_{g_2} = -\epsilon g_2 + \frac{1}{(4\pi)^2} \left(3g_2^2 + 2N g_1^2 g_2 - 12N g_1^4 \right) + \frac{1}{(4\pi)^4} \left(96N g_1^6 + 7N g_1^4 g_2 - 3N g_1^2 g_2^2 - \frac{17g_2^3}{3} \right)$$

$$\frac{(g_1^*)^2}{(4\pi)^2} = \frac{1}{N+6} \epsilon + \frac{(N+66) \sqrt{N^2 + 132N + 36} - N^2 + 516N + 882}{108(N+6)^3} \epsilon^2$$

$$\frac{g_2^*}{(4\pi)^2} = \frac{-N+6 + \sqrt{N^2 + 132N + 36}}{6(N+6)} \epsilon$$

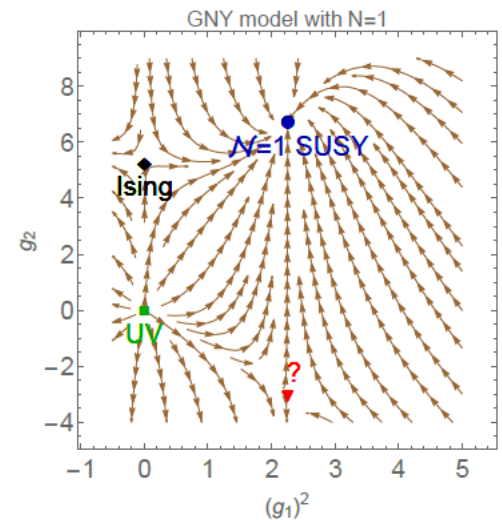
The Minimal Case: N=1

- For a single Majorana doublet, N=1
- We have developed the 4- ϵ expansion by continuing to N=1 (half of 4 component Majorana fermion)

- $\sqrt{N^2 + 132N + 36}$ equals 13.

$$\frac{(g_1^*)^2}{(4\pi)^2} = \frac{1}{7}\epsilon + \frac{3}{49}\epsilon^2 + \mathcal{O}(\epsilon^3)$$

$$\frac{g_2^*}{(4\pi)^2} = \frac{3}{7}\epsilon + \frac{9}{49}\epsilon^2 + \mathcal{O}(\epsilon^3)$$



- Consistent with the emergent SUSY

$$3g_1^2 = g_2 = 3\lambda^2$$

Evidence of emergent SUSY

$$\Delta_\sigma = 1 - \frac{3}{7}\epsilon + \frac{1}{49}\epsilon^2 + \mathcal{O}(\epsilon^3)$$

$$\Delta_\psi = \frac{3}{2} - \frac{3}{7}\epsilon + \frac{1}{49}\epsilon^2 + \mathcal{O}(\epsilon^3)$$

$$\Delta_{\sigma^2} = 2 - \frac{3}{7}\epsilon + \frac{1}{49}\epsilon^2 + \mathcal{O}(\epsilon^3)$$

- Consistent with the SUSY relation

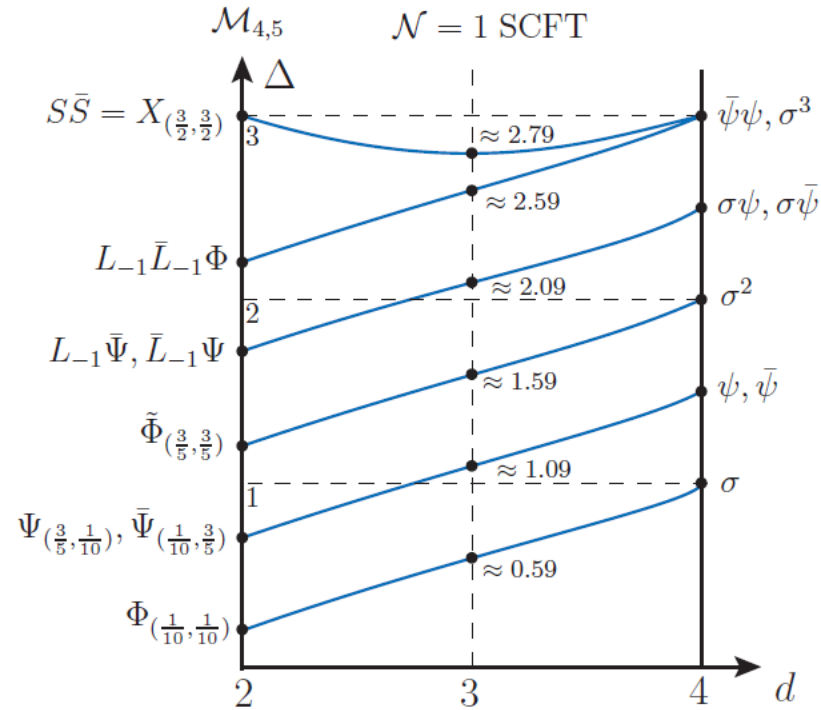
$$\Delta_{\sigma^2} = \Delta_\psi + \frac{1}{2} = \Delta_\sigma + 1$$

- Has been tested at higher loops
- Pade continuation to d=3 gives $\Delta_\sigma \approx 0.588$
- Our prediction, obtained in 2016, agrees to better than 1% with the latest bootstrap result 0.5844...

Atanasov, Hillman, Poland; Rong, Su

Continuation to d=2

- Gives $\mathcal{N}=1$ minimal model SM(3).
- Pade continuation gives
- $\Delta_\sigma \approx 0.217$ which is close to the exact dimension $1/5$
- This 2D CFT is the fermionic version of the tricritical Ising model M(4,5).



- In a recent paper, Nakayama, Katsevich, Sun, and I proposed that both the $m=12$ model with (E_6, D_8) modular invariant and the $m=10$ model with (D_6, E_6) modular invariant are described by the $\mathcal{N}=1$ supersymmetric Landau-Ginzburg theory with superpotential

$$\mathcal{W} = \frac{g_1}{2}XY^2 + \frac{g_2}{6}X^3$$

- In $d=2$, the superfields are $\Phi(\theta, \tilde{\theta}) = \phi_\Phi + i\theta\psi_\Phi + i\tilde{\theta}\tilde{\psi}_\Phi + \theta\tilde{\theta}F_\Phi$
- In $d = 4 - \epsilon$ the $m=10$ fixed point has $g_1 = g_2$
- It is decoupled after introducing

$$X = \frac{1}{\sqrt{2}}(\Phi_1 + \Phi_2), \quad Y = \frac{1}{\sqrt{2}}(\Phi_1 - \Phi_2)$$

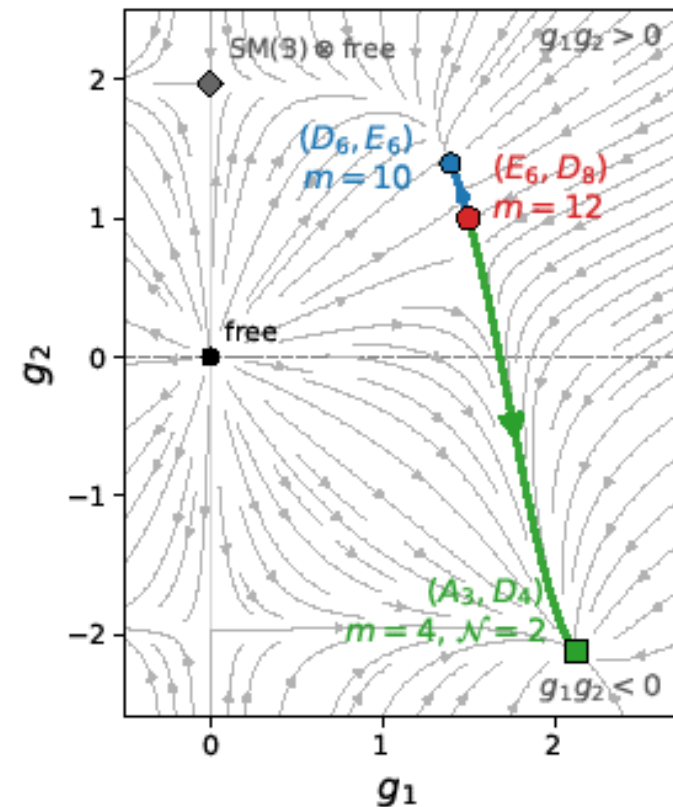
- The coupled fixed point upon continuation to $d=2$ describes the **exceptional** $m=12$ minimal model. It is the only one with super- W_3 symmetry

$$\frac{g_{1,\star}}{g_{2,\star}} = \frac{3}{2} + \frac{37}{109}\epsilon + \mathcal{O}(\epsilon^2)$$

- There is also a fixed point at

$$g_1 = -g_2$$

- It describes the $\mathcal{N}=2$ minimal model with $c=1$.



- There is emergent supersymmetry in GNY formulation, and anomalous dimensions

$$\gamma_X = \frac{13}{218}\epsilon + \frac{14836}{109^3}\epsilon^2 + \mathcal{O}(\epsilon^3), \quad \gamma_Y = \frac{9}{109}\epsilon + \frac{72753}{2 \cdot 109^3}\epsilon^2 + \mathcal{O}(\epsilon^3)$$

- Pade extrapolation to $d=2$ gives results not far from exact dimensions $\Delta_X = \frac{1}{7}$ $\Delta_Y = \frac{2}{7}$
- Performing Pade with this boundary condition, provides a prediction for the new 3D superconformal theory

$$\Delta_X \approx 0.567, \quad \Delta_Y \approx 0.612$$

Comments

- Can look for the new superconformal 3D theory using bootstrap and fuzzy sphere constructions.
- Description of $M(2, 2n+1)$ was proposed using PT symmetric LG theory with $i\phi^{2n-1}$ interaction.
Katsevich, IRK, Sun, Tarnopolsky
- There are many other massless field theories that are **exactly solved** in 2D by some BPZ minimal models. **Look for additional explicit examples.**
- Need to describe better the critical theories and RG flows directly in 2D and 3D.