

Single minus amplitudes from the matrix model of M-theory

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We will discuss the computation of a very special scattering amplitude in the BFSS matrix model of M-theory.

BFSS =Banks, Fischler, Shenker, Susskind, 1996

These are the so-called single minus amplitudes, which are nonzero only on a special kinematic submanifold, away from standard Minkowski space.

So, they are not physical Lorentzian amplitudes. But they are amplitude-like objects.

Simplest example: the three point amplitude.

It has proven to be a very useful seed for recursion relations.

e. g. Britto, Cachazo, Feng, Witten

In the process, we will find that the amplitude will be related to the number of $\frac{1}{4}$ BPS states in **N=4** SYM in the Coulomb branch.

Outline

- Review of single minus amplitudes.
- Review of the matrix model for M-theory.
- Basic idea for the computation
- More details about the computation.

Based on work with:



Alfredo Guevara



Alex Lupsasca



Andy Strominger

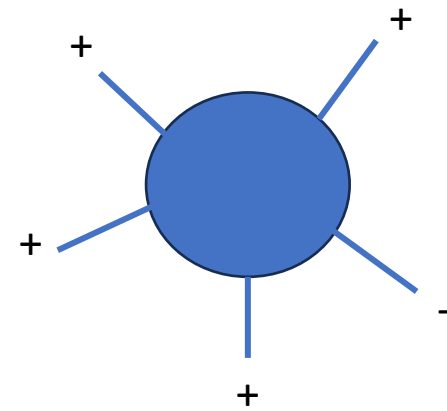
[ArXiv: 2608.25239](#)

Note on AI use: It was not crucially used for our paper.

Also previous work with A. Herderschee on 3pt amplitudes [ArXiv 2312.12592](#)

Four dimensional massless particles are characterized by their helicity, ± 1 , ± 2 , or just simply $\pm$.

Use all incoming notation



For tree level or for SUSY amplitudes at all loops:

- All plus amplitudes vanish.
- Single minus amplitudes generically vanish.
- Two minus amplitudes = MHV amplitudes are non vanishing and particularly simple.

Single minus amplitudes are non-zero on a particular kinematic subspace.

Guevara, Lupsasca, Skinner, Strominger, Weil

(for the special $n=3$ case, there is no restriction, beyond momentum conservation.)

To describe the subspace, it is useful to use spinor helicity notation

$$p_{\alpha\dot{\beta}} = \lambda_{\alpha}\tilde{\lambda}_{\dot{\beta}}$$

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The subspace consists of choosing λ_{α}^i so that they are all aligned $\rightarrow$ choose a gauge and frame where they are all equal:

$$\lambda_{\alpha}^i = (1,0).$$

$$p_{\alpha\dot{\beta}} = \lambda_{\alpha}\tilde{\lambda}_{\dot{\beta}}$$

$$\lambda_{\alpha}^i = (1,0).$$

The $\tilde{\lambda}_{\dot{\beta}}^i$ are all independent, up to the momentum conservation condition

$$0 = \sum_i \widetilde{\lambda}_{\dot{\beta}}^i$$

Another way to say this:

$$ds^2 = dx^+ dx^- + dz d\bar{z}$$

$$p_+^i = p_z^i = 0 \ , \quad p_-^i \neq 0 \ , \quad p_{\bar{z}}^i \neq 0$$

$\bar{z}$ is not the complex conjugate of z ,
they are two independent real variables, like $x^\pm$.
(2,2) signature or complex momenta in (1,3) signature.

An important feature:

these amplitudes preserve some supersymmetries.
(Susy annihilates the amplitude with the given
polarizations)

There is a single spinor ϵ obeying

$$\not{p}^i \epsilon = \Gamma^\mu p_\mu^i \epsilon = 0 \ , \quad \text{for all } i = 1, \dots, n$$

For $n > 3$, this is a non-trivial kinematic restriction.

A quick calculation shows that this implies that

$$p^i \cdot p^j = 0 \text{ , for all } i, j$$

In particular, if all λ_{α}^i are equal, this indeed implies

$$p^i . p^j = \langle ij \rangle [ij] = 0 \ , \quad \text{for all } i, j$$

Conclusion:

In a supersymmetric theory, the single minus scattering amplitudes are supersymmetric in the subspace where they are non-vanishing.

Supersymmetric observables are usually more computable than more generic observables.

So far, we have discussed amplitudes in $d=4$. However, we can also consider amplitudes in higher dimensions, restricted to this particular subspace. In fact, any amplitude that preserves exactly $\frac{1}{4}$ of SUSY in any dimension can be Lorentz transformed to this subspace of four dimensional kinematics.

For this reason, we can use 4d kinematics to describe them.

For the case of graviton amplitudes we can use the 4d $\mathbf{N}=8$ supergravity notation.

Brandhuber, Pichini, Travaglini, Wen

Function of $[\tilde{\lambda}^i, \tilde{\lambda}^j] = [i, j]$

Momentum conservation

“broken” susy

$$\mathcal{A}_n(1, 2, \dots, n) = 2\kappa_{11}^{n-2} (2\pi)^2 \mathcal{M}_n^{\text{grav}} \prod_{a=1}^{n-1} [2\pi \delta(\langle an \rangle)] \delta^2 \left(\sum_{i=1}^n \tilde{\lambda}_{\dot{\beta}}^i \right) \delta^8 \left(\sum_{i=1}^n \tilde{\eta}_I^i \right)$$

Extra delta functions. Set all λ_{α}^i “equal”

Generate multiplet

Examples

$$\mathcal{M}_3^{\text{grav}}|_{\mathcal{R}_{3,2}} = [21], \quad \mathcal{M}_4^{\text{grav}}|_{\mathcal{R}_{4,3}} = ([21] + [31])[32]$$

Special kinematic regions.

$$\left. \begin{array}{l} \mathcal{R}_{n,n-1} : \end{array} \right\} \quad \tilde{z}_1 < \cdots < \tilde{z}_{n-2} < \tilde{z}_n < \tilde{z}_{n-1}, \quad \text{where} \quad \tilde{z}_i := \frac{\tilde{\lambda}_2^i}{\tilde{\lambda}_1^i}, \quad \tilde{\lambda}_{\dot{\beta}}^i = \begin{pmatrix} \tilde{\lambda}_1^i \\ \tilde{\lambda}_2^i \end{pmatrix}$$

Examples

$$\mathcal{M}_3^{\text{grav}} = |[12]| ,$$

$$\mathcal{M}_4^{\text{grav}}|_{\mathcal{R}} = ([21] + [31])[32]$$



Special kinematic region.

$$\mathcal{R}_{n,n-1} : \quad \tilde{z}_1 < \cdots < \tilde{z}_{n-2} < \tilde{z}_n < \tilde{z}_{n-1}, \quad \text{where} \quad \tilde{z}_i := \frac{\tilde{\lambda}_2^i}{\tilde{\lambda}_1^i}, \quad \tilde{\lambda}_{\beta}^i = \begin{pmatrix} \tilde{\lambda}_1^i \\ \tilde{\lambda}_2^i \end{pmatrix}$$

$\mathcal{M}_n^{\text{grav}}$ is locally a polynomial in $[i,j]$ of degree $n-2$.

Change at walls.

End of the amplitude discussion.

Our goal is to reproduce these amplitudes from
the matrix model.

Review of the BFSS conjecture

Bank, Fischler, Shenker, Susskind:

“M- theory as a matrix model, a conjecture”,

[hep-th/9610043](#)

The matrix model

- Matrix quantum mechanics.
- Gauge group $U(N) \rightarrow$ restricted to $U(N)$ invariant states (gauged).
- 9 adjoint scalars + 16 adjoint fermions, $N \times N$ matrices.
- 9 of $SO(9)$ + 16 of $SO(9)$.
- 16 supersymmetries.

$$L = \text{Tr} \left[\frac{1}{2R} (\dot{X}^I)^2 + \frac{R}{4(2\pi)^2} \sum_{I,J} [X^I, X^J]^2 + \psi^t \left(\dot{\psi} + \frac{R}{2\pi} \Gamma_I [X^I, \psi] \right) \right]$$

X as positions. All in 11d Planck units. $X = X/l_p$, $R = R/l_p$.

A large N limit of a scattering problem in the matrix model
=
scattering amplitude in M-theory.

Simplest example:

$$U(N) \sim U(1) \times SU(N)$$

The $SU(N)$ part has a single bound state at threshold, at $E=0$.

Piljin Yi, Sethi, Stern,
OpenAI?

Interpret it as a single graviton with momentum $-p_- = \frac{N}{R}$

Many gravitons?

Expand around well separated $\vec{x}_i$:

$$\vec{X} \sim \begin{pmatrix} \vec{x}_1 \mathbf{1}_{N_1} & 0 & 0 \\ 0 & \vec{x}_2 \mathbf{1}_{N_2} & 0 \\ 0 & 0 & \vec{x}_3 \mathbf{1}_{N_3} \end{pmatrix}$$

Three bound states with $SU(N_i) \rightarrow$ three gravitons, with momenta: $-p_-^i = \frac{N_i}{R}$

Each center of mass degree of freedom, each $U(1)_i$ gives superparticles.

Matrix block bound states to matrix block bound states scattering

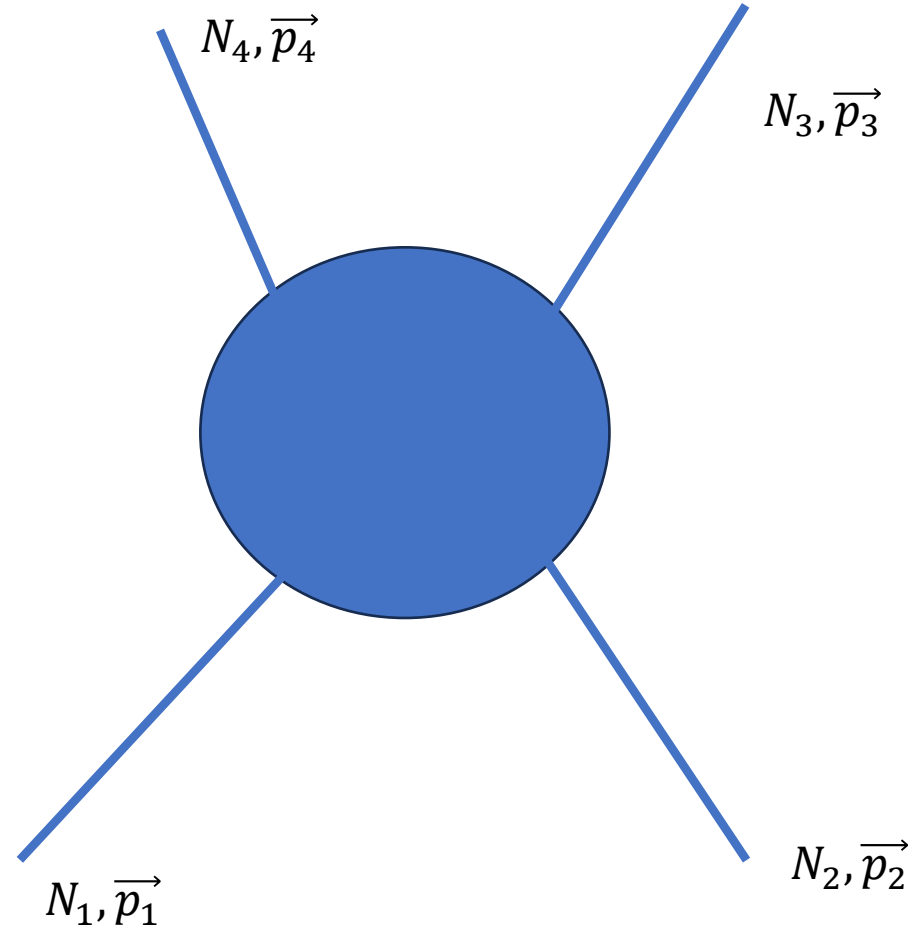
$$\begin{pmatrix} x_3 \mathbf{1}_{N_3} & 0 \\ 0 & x_4 \mathbf{1}_{N_4} \end{pmatrix}$$



$$\begin{pmatrix} * & * \\ * & * \end{pmatrix}$$



$$\begin{pmatrix} x_1 \mathbf{1}_{N_1} & 0 \\ 0 & x_2 \mathbf{1}_{N_2} \end{pmatrix}$$



$$N_1 + N_2 = N_3 + N_4$$

Large N limit

$$N \rightarrow \infty , \quad \frac{N}{R} = \text{fixed} , \quad \vec{p} = \text{fixed} , \quad \rightarrow \quad -p_+ = \text{fixed}$$

This is a very low energy regime in the matrix model. (Strong coupling)

The conjecture

$$\lim_{N \rightarrow \infty} \mathcal{A}_{MM}(\vec{p}_i, N_i) = \mathcal{A}(p_i^\mu)$$

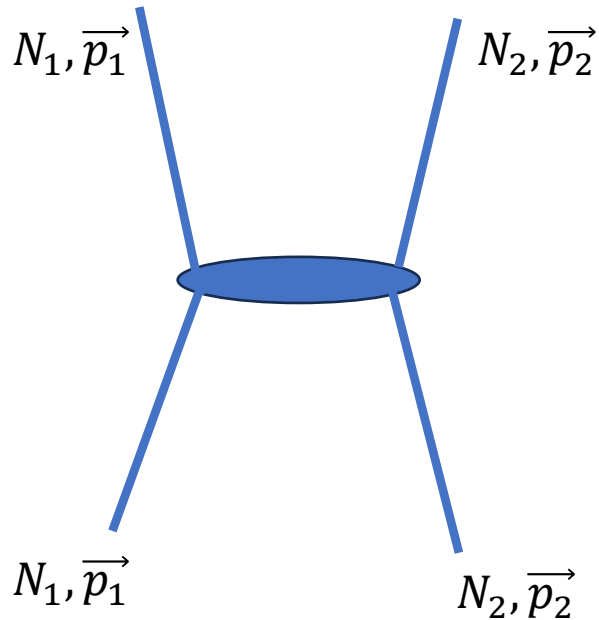
Scattering in matrix model

11d M-theory scattering amplitudes.

Sub-conjectures:

- 1) Limit exists
- 2) It is Lorentz invariant.
- 3) It is unitary (after taking the limit)
- 4) It has all the properties we expect from M theory.
(e.g. reproduces gravity at low energies, incorporates black holes, etc)

A few checks were done for very special limits of the four point amplitude



$$c_1 \frac{v^4}{r^7} + c_2 \frac{v^6}{r^{14}}$$

Explained by non-renormalization theorems.

Paban, Sethi, Stern
Becker, Becker, Polchinski

Many nice properties under compactification

Compactifying one dimension

- $x^9 = x^9 + 2 \pi R_9$
- Matrix QM $\rightarrow$ 1+1 dimensional field theory on a circle of size $\tilde{R}_9 = 1/R_9$ (T-duality)

It is hard to compute anything...

But, in principle, we do have a procedure for determining the S-matrix!

Strategy

- The special kinematics for the single minus amplitudes preserves $\frac{1}{4}$ of supersymmetry.
- We first compactify one extra spatial dimension to transform the problem to a 1+1 dimensional gauge theory.
- Viewing the extra compact dimension as Euclidean time, we relate the amplitude to a supersymmetric index.
- This index is computed using techniques such as wall crossing formulas, etc. Further tricks to relate it to a problem considered before.

Recall the kinematics for the single minus amplitudes

$$p_\mu = (p_+, p_-, p_z, p_{\bar{z}}) , \quad p_z = p_9 + ip_8 , \quad p_{\bar{z}} = p_9 - ip_8$$

$$p_\mu^i = (0, \frac{N_i}{R}, p_z^i, 0)$$

With p_9 real, this implies that p_8 is purely imaginary.

(Only four of the components of the momenta are shown, the rest are zero)

Compactifying the 9th dimension

Matrix QM to 1+1 dimensional matrix gauge theory.

Momentum along the 9th direction = $n/R_9 \rightarrow$

n = electric flux of the gauge field on the 1+1 dimensional worldvolume.

Similar to N D1 branes with n units of F1 string charge. $(p,q) = (N,n)$ strings.

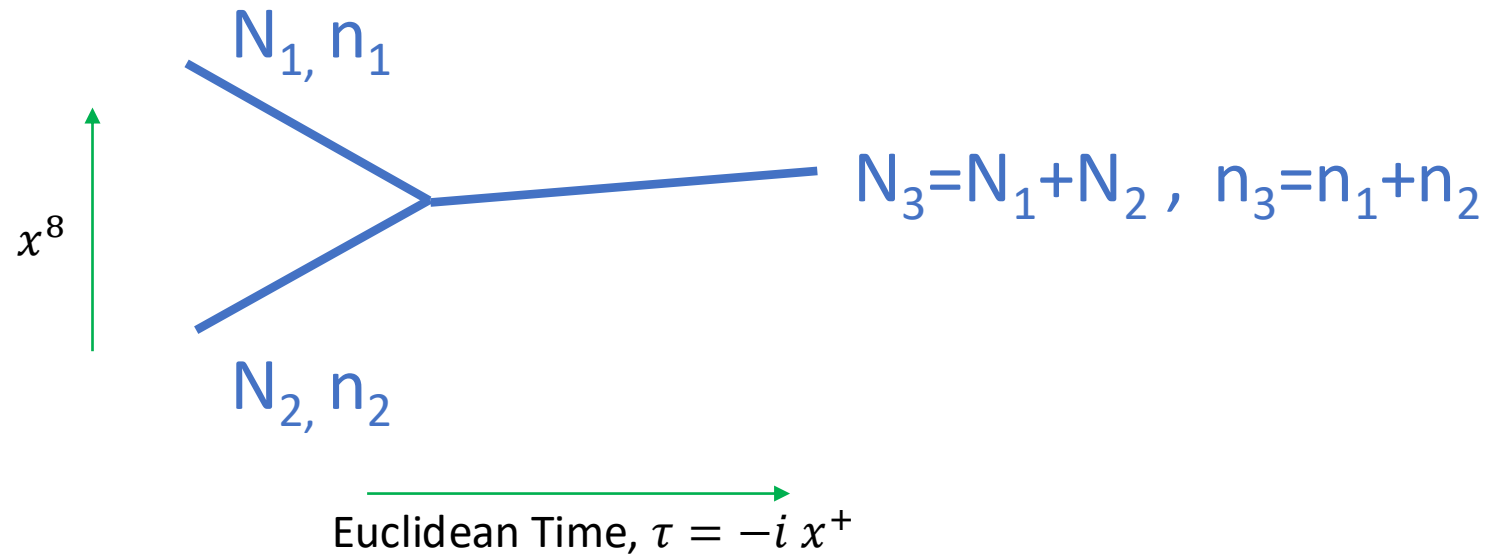
More precisely, $U(N)$ gauge theory with n units of electric flux under the $U(1)$.

N, n set the momenta on two of the dimensions.

p_8 is the momentum along another dimension; we choose it to be imaginary, so that $p_{\bar{z}} = p_9 - i p_8 = 0$

x^+ is the last relevant dimension; it is the time of the matrix model, or matrix string.

It is convenient to go to Euclidean values $x^+ = i \tau$. $p_8 \propto i$ (*real*) τ



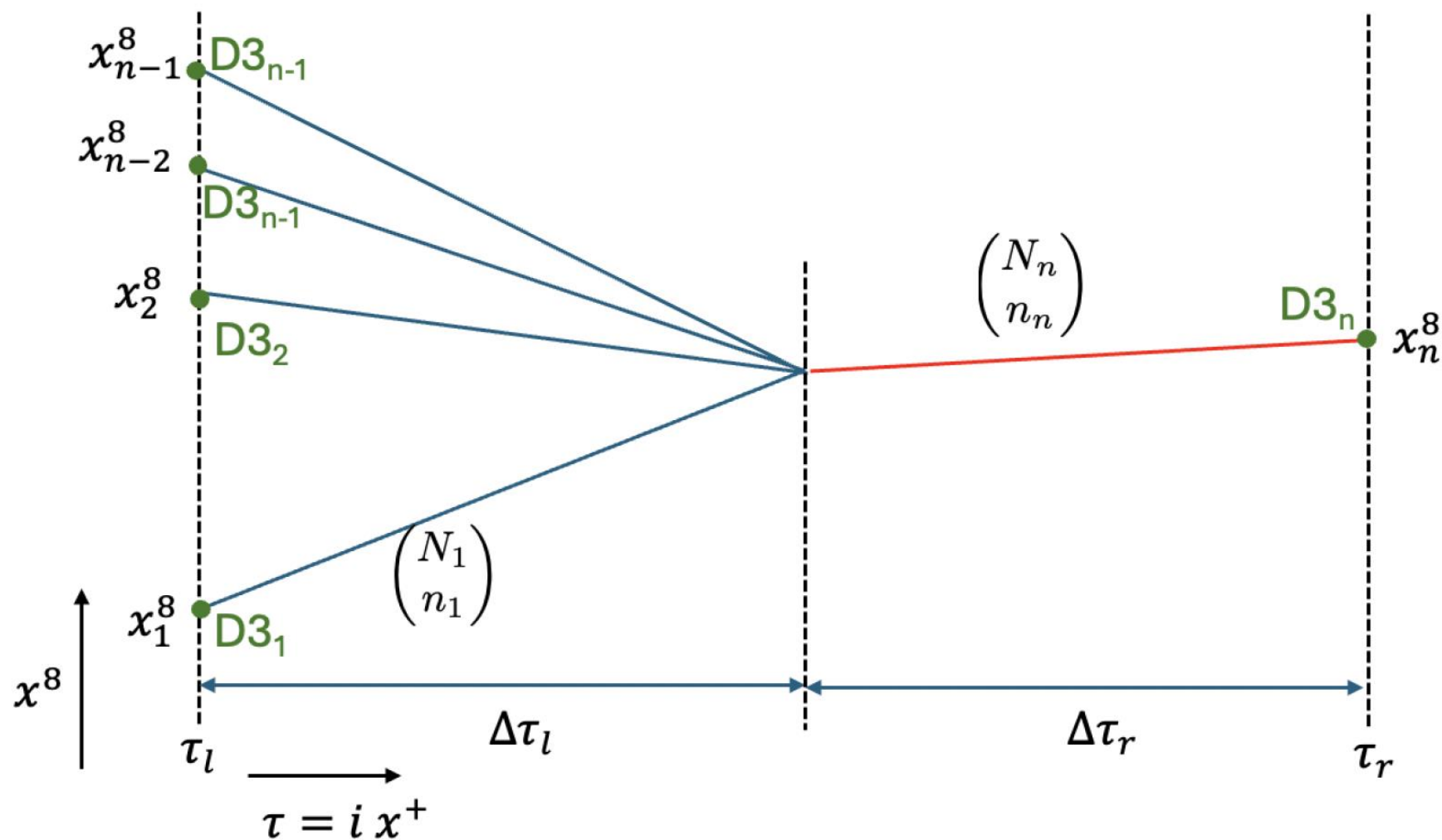
Similar to (N, n) string junction.

They are extended along the compact $\tilde{\theta}$ circle (perpendicular to the screen)

The angle in the τ, x^8 plane is set by the ratio n/N

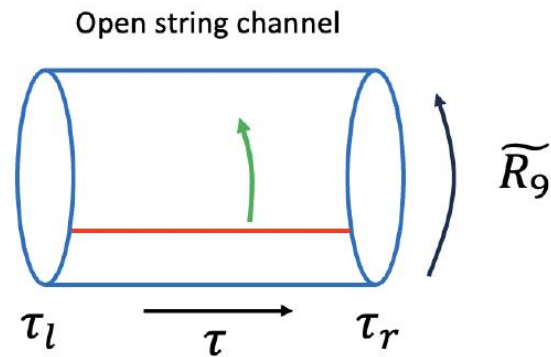
We will now add an extra element that helps
in the computation

Make them end on D3 branes. (still preserves SUSY). Set the D3 branes so that all lines meet close to a point.

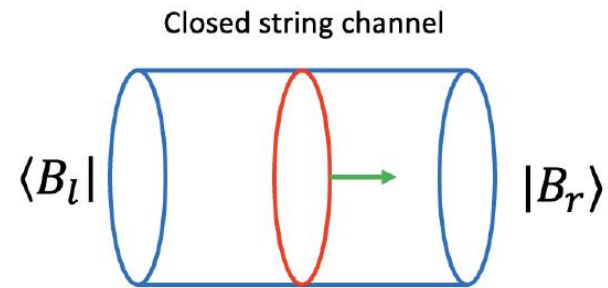


(susy fixes the angles)

The computation has an ``open string'' channel and a ``closed string'' channel.

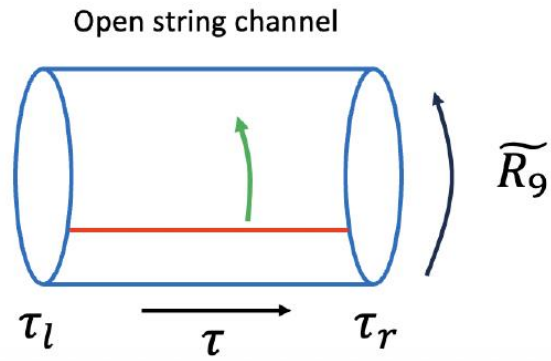


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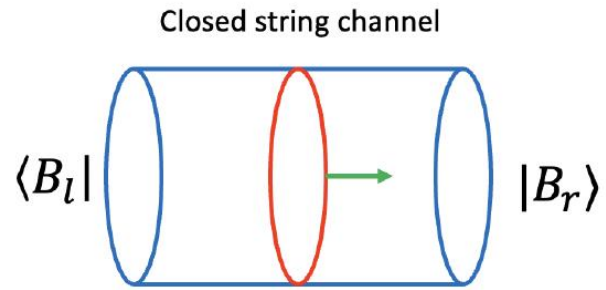


Index, $\propto Tr[(-1)^F]$

The boundary states creates the states we are scattering, with some amplitudes which are non-trivial functions.

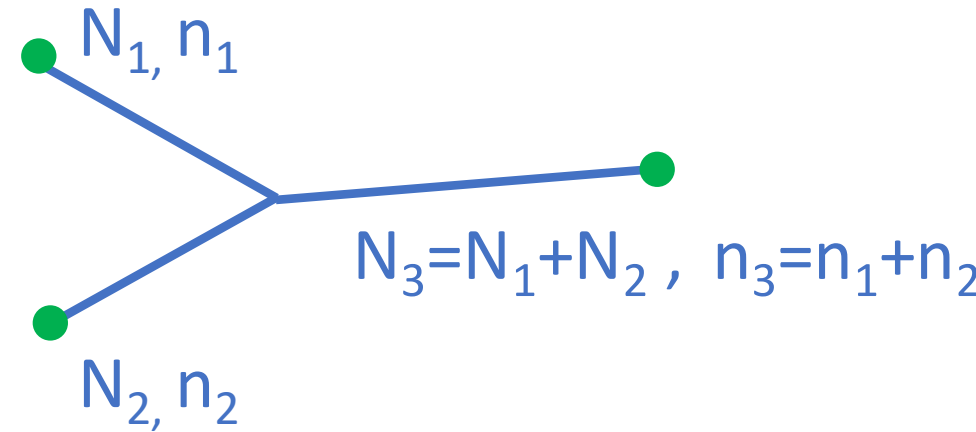


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Index, $\propto \text{Tr}[(-1)^F]$

Amplitude



Index = Amplitude

For a supersymmetric amplitude, the index is independent of continuous parameters.

There are techniques to compute it.

First, a kinematic comment

$[i,j]$ is proportional to the symplectic inner product

$$\langle \Gamma_i, \Gamma_j \rangle = N_i n_j - N_j n_i, \quad \text{with} \quad \Gamma_i = \begin{pmatrix} N_i \\ n_i \end{pmatrix} \quad \text{and} \quad [ij] = -\mu \langle \Gamma_i, \Gamma_j \rangle.$$

$$\tilde{\lambda}_{\dot{\alpha}}^i = \begin{pmatrix} N_i/R_- \\ \sqrt{2}n_i/R_9 \end{pmatrix}$$

$$\mu = \frac{\sqrt{2}}{R_- R_9}$$

The amplitude is then expected to be a polynomial in these inner products.

Note that electric magnetic duality for D3 branes, then implies Lorentz invariance for these amplitudes.

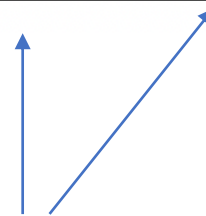
(Note that this is not a symmetry of the 1+1 dimensional matrix model beyond the index)

More precisely, we can argue that we have the identity

$$\mu^{2-n} \mathcal{M}_n^{\text{BFSS}} = \mathcal{I}_n = -\frac{1}{[2(n-1)]!} \text{Tr} \left[(-1)^F \frac{(2J)^2}{2} (2I_3)^{2(n-1)} \right]$$

$$\mu = \frac{\sqrt{2}}{R_- R_9}$$

Insertions that are needed to absorb some fermion zero modes.



After explicitly working out the index, we obtain precise agreement with the full functional form of the amplitude.

We could now take the large N and n limits and recover the amplitudes in completely uncompactified eleven dimensions.

Some details

A general comment on the normalization

We fix the amplitude $\langle B|p\rangle$ by looking at the two point function.



$$Tr[J^4(-1)^F] = 1$$

This involves again the relation between an index and an amplitude.

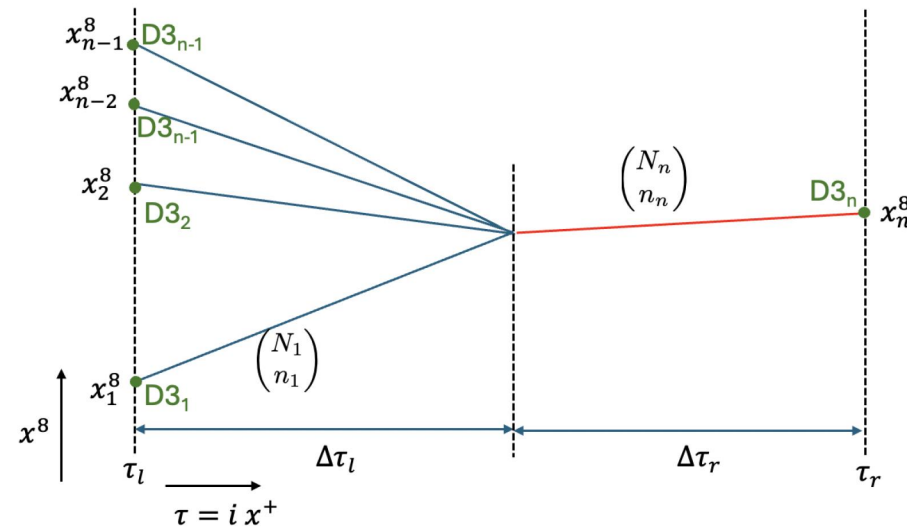
The fermion insertions in the open and closed string channels are normalized differently.

This leads to some $\tilde{R}_9$ dependence for the overlaps.

Fix $\langle B|p\rangle$

Details for computing the index

We are considering a problem in a 1+1 dimensional field theory with some boundary conditions.



Same as a problem on n D3 branes on the Coulomb branch.

The index is related to the problem of counting BPS states on the Coulomb branch for the theory living on D3 branes.

It was considered by Sen, who introduced an index

$$B_2(\nu) = -\text{Tr}\left[(-1)^F \frac{(2J)^2}{2} e^{\nu 2I_3}\right]$$

Rotation on worldvolume of D3

$$U(1) \subset SU(2)_L \subset SO(4) \subset SO(6)$$

Transverse to string network

$$B_2(\nu) = -\text{Tr}[(-1)^F \frac{(2J)^2}{2} e^{\nu 2I_3}]$$

Extracting the power $\nu^{2(n-1)}$ gives the index we want.

We will see that this is the leading dependence at small ν

This index can be computed by “Wall crossing”.

Kontsevich, Soibelman, Gaiotto, Moore, Neitzke, Denef, Manschot, Pioline, Sen

The index is independent of the position of the D3 branes within some regions. As you cross some codimension one “walls”, the index jumps in a clear way (given a “wall crossing formula”).

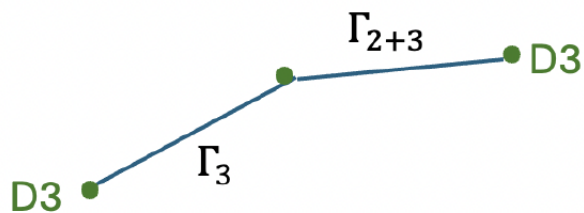
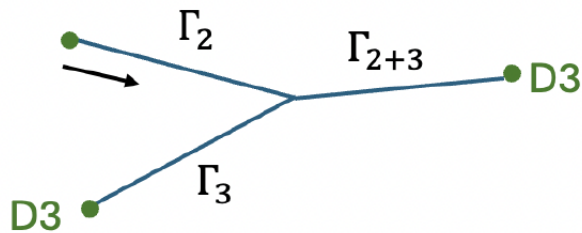
The walls can occur when BPS states can decay into other BPS states.

Example

N, n



$$B_2(\nu)_{\text{half}} = -\text{Tr}\left[(-1)^F \frac{(2J)^2}{2} e^{2\nu I_3}\right] = e^\nu + e^{-\nu} - 2 \equiv h(\nu) , \quad h(\nu) \sim \nu^2 \text{ as } \nu \rightarrow 0.$$

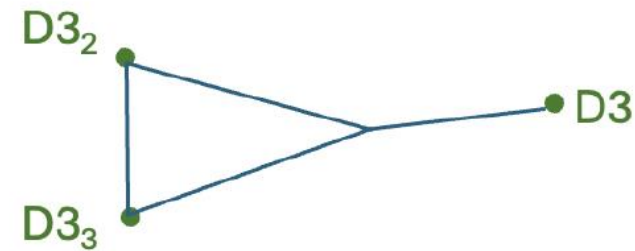
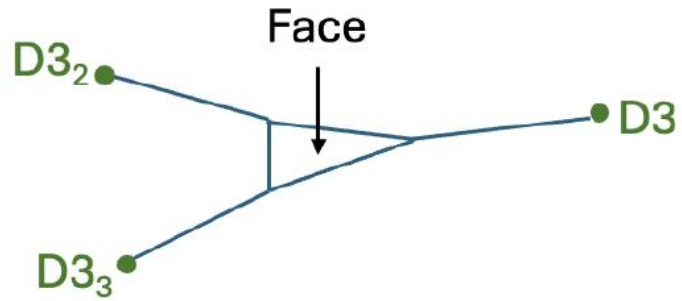


$$B_2(\nu) = h(\nu)^2 m_2 , \quad m_2 := -\langle \Gamma_3, \Gamma_2 \rangle = \frac{1}{\mu} [32] > 0.$$

A subtlety:

The faces (another source of wall crossing)

Facing the faces



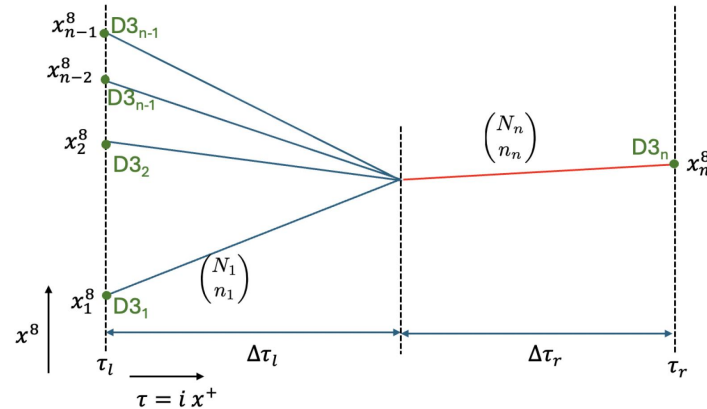
Lead to terms that are of higher order in the ν expansion, than what we are interested in.

→ We can ignore them

A tale of two walls:

Walls in the motion of D3 branes should not be confused with walls in the kinematic space of momenta of the amplitudes.

- We have discussed wall crossing as we change the position of the D3 branes.
- For our problem, we want these branes at specific positions.



- We move them purely as a technical convenience in order to compute the index.
- This whole story happens for one value of the N_i, n_i , one kinematic point of the scattering amplitude.

- Once we determine that, we can take the large N_i, n_i limit, and view these as continuous variables. Then in that space, we expect the walls of the gravity amplitude.
- They are a change in the polynomial form of the index.
- We have not understood clearly how to compute these second walls from the index.

Conclusions

- We computed the single minus amplitudes using the matrix model.
- We equated these amplitudes to a BPS state counting problem in supersymmetric field theory ($N = 4$ super Yang Mills).
- We computed the precise numerical coefficient for the three point amplitude and found agreement with the gravity answer.

Future

- Can we simplify the computation of indices or amplitudes?
- Further connection with other observables.
- Are these amplitudes useful for computations in the actual physical theory?

We have also argued for an identity that can be viewed as associated to a $w_{1+\infty}$ symmetry of the amplitude.

$$\mathcal{M}_n^{\text{BFSS}}(1, \dots, n-1, s) = \frac{1}{2} \sum_{j=1}^{n-1} |[sj]| \mathcal{M}_{n-1}^{\text{BFSS}}(1, \dots, j+s, \dots, n-1).$$

To all orders in the soft expansion, $s \ll j$