

Wilson spools and de Sitter quantum gravity

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Castro, Coman, Fliss, CZ [2302.12281], Castro, Coman, Fliss, CZ [2304.02668]

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WIP w/ Sam Hauptfear, Cynthia Keeler, Victoria Martin, Rahul Poddar

WIP w/ Lars Aalsma

Outline

- Introduction
- Wilson spools
- The Fredholm-based trace formula: A new algorithm for deriving one-loop determinants
- Further generalizations
 - Broken symmetry
 - Product spaces
- Conclusions

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Introduction

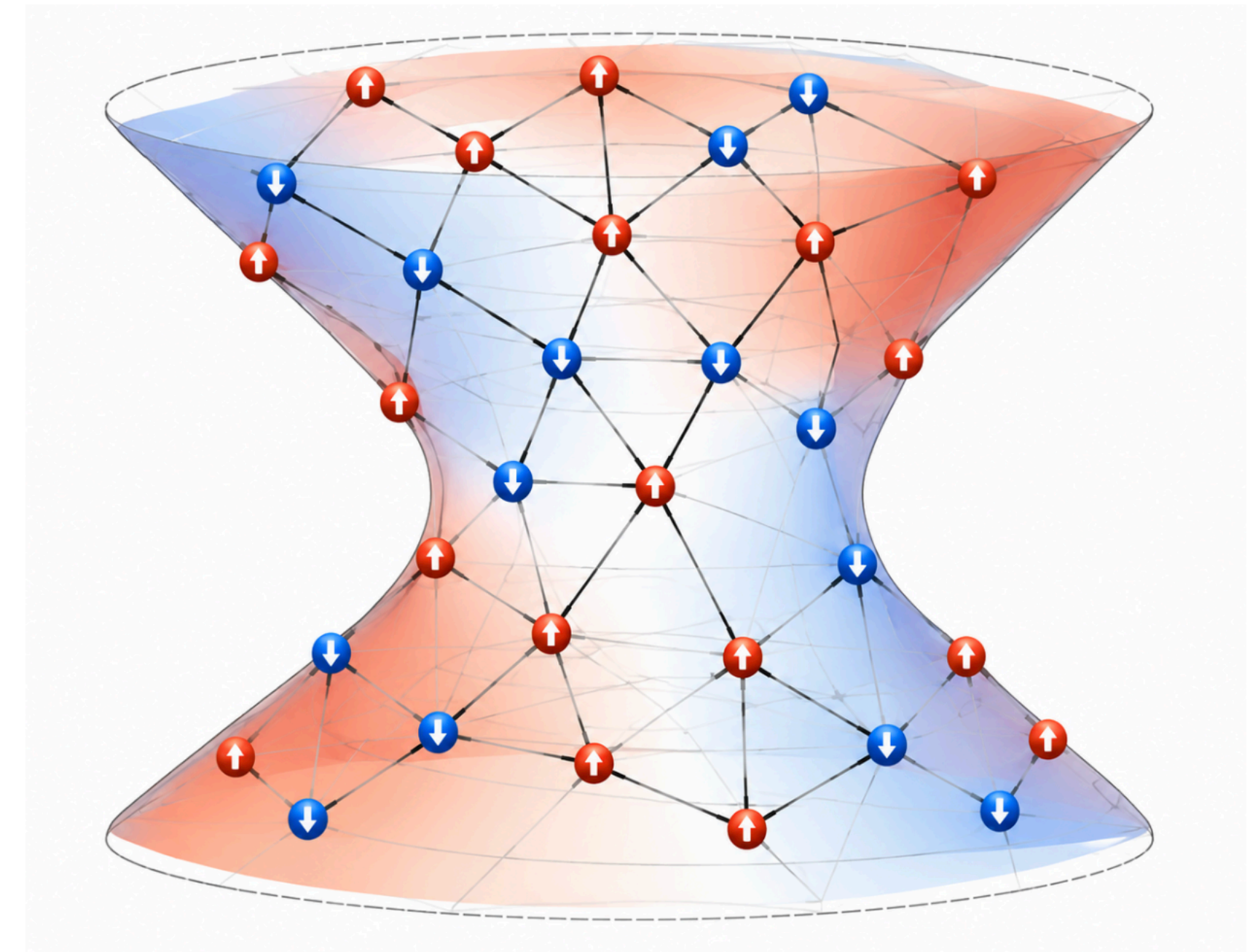
One-loop physics is a testing ground for any toy model of quantum gravity.

$$\mathcal{Z}_E = \int [\mathcal{D}g_{\mu\nu}][\mathcal{D}\phi] e^{-I_E[g_{\mu\nu}, \phi]}$$

$$I_E[g, \phi] = I_E^{\text{on-shell}}[g_c, \phi_c] + \underbrace{I_E^{(2)}[h, \psi]}_{\text{quadratic in fields}} + \dots$$

$$S = \log \mathcal{Z}_E = \frac{A}{4G_N} + \text{corrections}$$


 Fixed by one-loop physics



Introduction

We have more control over calculations, and a mechanism for quantization, using the relationship between gravity and Chern-Simons theory specific to 3d.

Goal 1: Gather data about one-loop physics in 3d de Sitter, including for metric fluctuations.

Goal 2: Move beyond 3d pure de Sitter.

The Chern-Simons formulation of 3d gravity

Gravity in 3d

$\longleftrightarrow$

Chern-Simons theory

$$S_{\text{CS}}[A] = \frac{k}{4\pi} \int_M \left(A \wedge dA + \frac{2}{3} A^3 \right)$$

isometry algebra

gauge algebra

metric

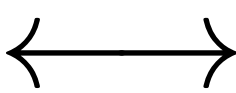
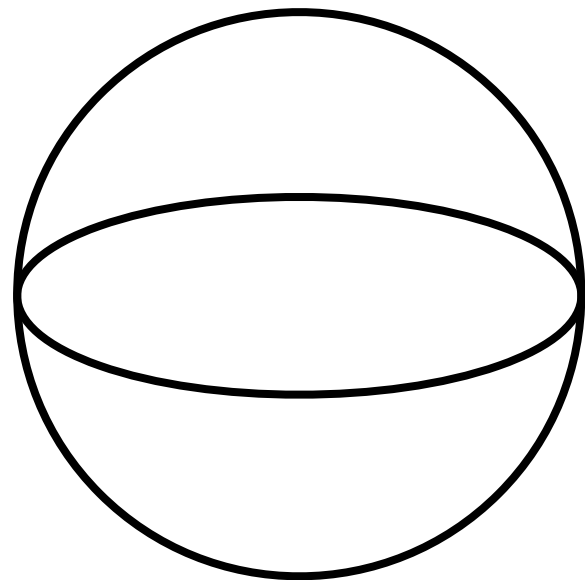
connections $A, \bar{A}$

The Chern-Simons formulation for de Sitter

Witten (1988)

Witten (1990)

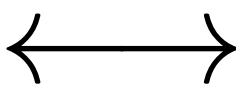
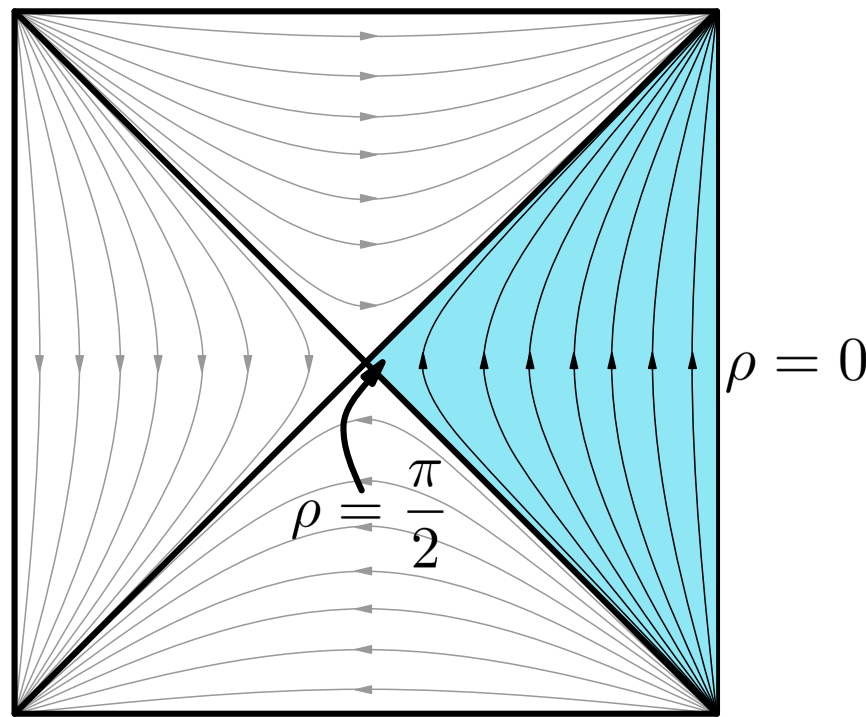
Gravity in $\text{EdS}_3 (= S^3)$



$$SU(2) \times SU(2)$$

Chern-Simons theory

Gravity in dS_3



$$SL(2, \mathbb{C})$$

Chern-Simons theory

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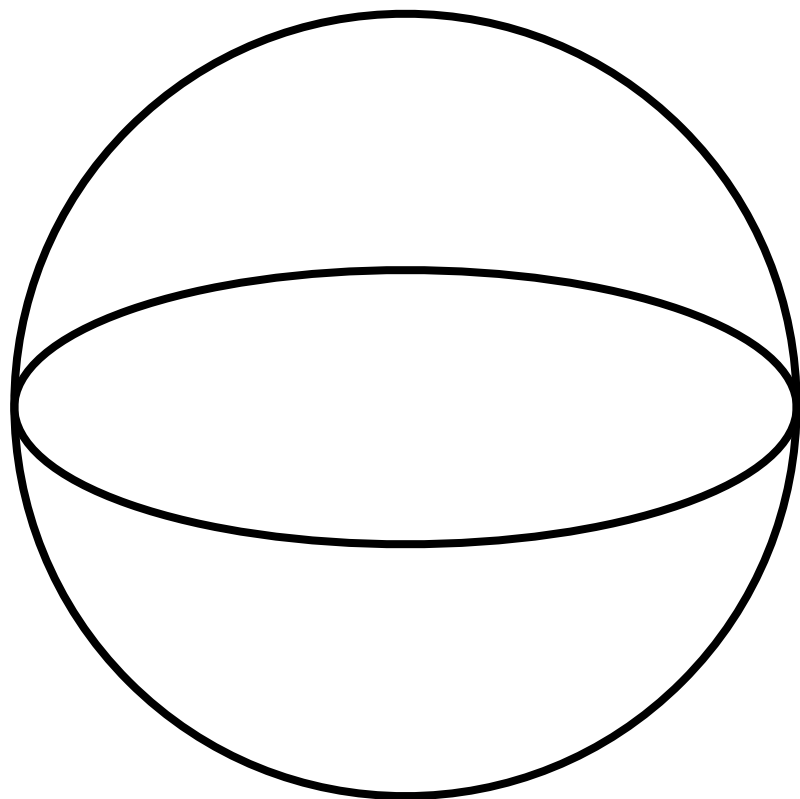
Gravitational Path Integral Coupled to Matter

Gravitational path integral: $\int [\mathcal{D}g_{\mu\nu}] e^{-I_{\text{EH}}[g_{\mu\nu}]}$

How to couple to matter? $\int [\mathcal{D}g_{\mu\nu}] e^{-I_{\text{EH}}[g_{\mu\nu}]} Z_{\text{matter}}[g_{\mu\nu}]$

Euclidean $S^3 \simeq \text{EdS}_3$

$$ds^2 = dr^2 + \cos^2 r \, d\tau^2 + \sin^2 r \, d\phi^2$$

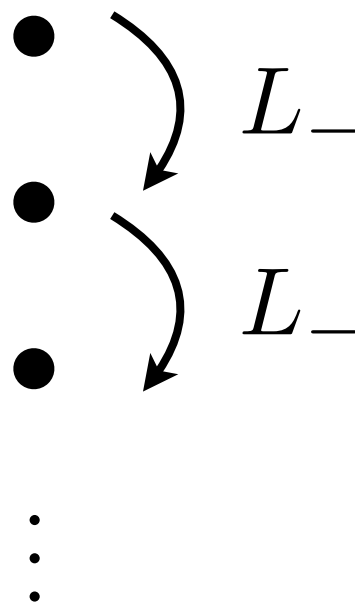


$\mathfrak{su}(2)$ algebra: $[L_i, L_j] = i\epsilon_{ijk} L_k \quad i, j = 1, 2, 3$

$$L_0 = L_3$$
$$L_{\pm} = L_1 \pm iL_2$$

(Non-standard) highest weight representation:

$$L_1^{\dagger} = -L_1$$
$$L_2^{\dagger} = -L_2$$



The Wilson Spool

Proposal:

$$Z_{\text{matter}}[g_{\mu\nu}] = \exp \left(\frac{1}{4} \mathbb{W}[g_{\mu\nu}] \right) \quad \text{“Wilson spool” in CS theory}$$

$$\mathbb{W}_j := \int_{\mathcal{C}} \frac{d\alpha}{\alpha} \frac{\cos \alpha/2}{\sin \alpha/2} \text{Tr}_{R_j} \left(\mathcal{P} \exp \frac{\alpha}{2\pi} \oint A \right) \text{Tr}_{R_j} \left(\mathcal{P} \exp \frac{-\alpha}{2\pi} \oint \bar{A} \right)$$

Integral implements “winding”
sum over a Wilson loop that
wraps n times

Holonomies:

$$\mathcal{P} \exp \oint_{\gamma} A = g_{\rho}^{-1} e^{i2\pi L_3 \mathfrak{h}_L} g_{\rho}$$

$$\mathcal{P} \exp \oint_{\gamma} \bar{A} = g_{\rho} e^{i2\pi L_3 \mathfrak{h}_R} g_{\rho}^{-1}$$

For cycles wrapping the causal horizon:

$$\mathfrak{h}_L = 1, \quad \mathfrak{h}_R = -1$$

The Wilson Spool for de Sitter

$$\mathbb{W}_j := \int_{\mathcal{C}} \frac{d\alpha}{\alpha} \frac{\cos \alpha/2}{\sin \alpha/2} \text{Tr}_{R_j} \left(\mathcal{P} \exp \frac{\alpha}{2\pi} \oint A \right) \text{Tr}_{R_j} \left(\mathcal{P} \exp \frac{-\alpha}{2\pi} \oint \bar{A} \right)$$

Evaluate for non-standard representations R_j with $j = -\frac{1}{2} \left(1 + \sqrt{1 \ominus m^2 \ell^2} \right)$

$$\begin{aligned} \log Z_{\text{scalar}}[S^3] &\stackrel{?}{=} \frac{\mathbb{W}_j}{4} = \frac{i}{4} \int_{\mathcal{C}} \frac{d\alpha}{\alpha} \frac{\cos \alpha/2}{\sin \alpha/2} \chi_j \left(\frac{\alpha}{2\pi} \mathfrak{h}_L \right) \chi_j \left(-\frac{\alpha}{2\pi} \mathfrak{h}_R \right) \\ &= -\frac{i}{16} \int_{\mathcal{C}} \frac{d\alpha}{\alpha} \frac{\cos \alpha/2}{\sin^3 \alpha/2} e^{i(2j+1)\alpha} \\ &= \sum_{\pm} \left(-\frac{1}{8\pi^2} \text{Li}_3 \left(e^{\mp 2\pi\mu} \right) \mp \frac{\mu}{4\pi} \text{Li}_2 \left(e^{\mp 2\pi\mu} \right) - \frac{\mu^2}{4} \text{Li}_1 \left(e^{\mp 2\pi\mu} \right) \right) \end{aligned}$$



$$\mu^2 \equiv m^2 \ell^2 - 1 \qquad \text{Li}_q(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^q}$$

One-Loop Determinants Summary

Thus, in the $G_N \rightarrow 0$ limit, the spool reproduces:

$$\lim_{G_N \rightarrow 0} \frac{\langle \mathbb{W} \rangle_{\text{grav}}}{\mathcal{Z}_{\text{grav}}[S^3]} = 4 \log Z_{\text{scalar}}[S^3] = 4 \log \det(-\nabla_{S^3}^2 + \mathfrak{m}^2 \ell^2)^{-1/2}$$

Perturbative Corrections/Quantum Gravity Predictions

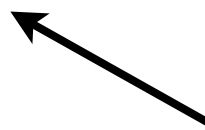
The spool can be inserted into the path integral, evaluated using localization/Abelianization, and expanded to any order in G_N to incorporate quantum corrections.

$$\int [\mathcal{D}g_{\mu\nu}] e^{-I_{\text{EH}}[g_{\mu\nu}]} e^{\frac{1}{4}\mathbb{W}[g_{\mu\nu}]}$$

We find quantum corrections can be naturally identified with a renormalization of mass:

$$\frac{\langle \log Z_{\text{scalar}}[S^3] \rangle_{\text{grav}}}{\mathcal{Z}_{\text{grav}}[S^3]} = \log Z_{\text{scalar}}[S^3] + \sum_{m=1}^{\infty} \left(\frac{G_N}{\ell} \right)^{2m} (\log Z)_{(2m)}$$

Depends on the mass

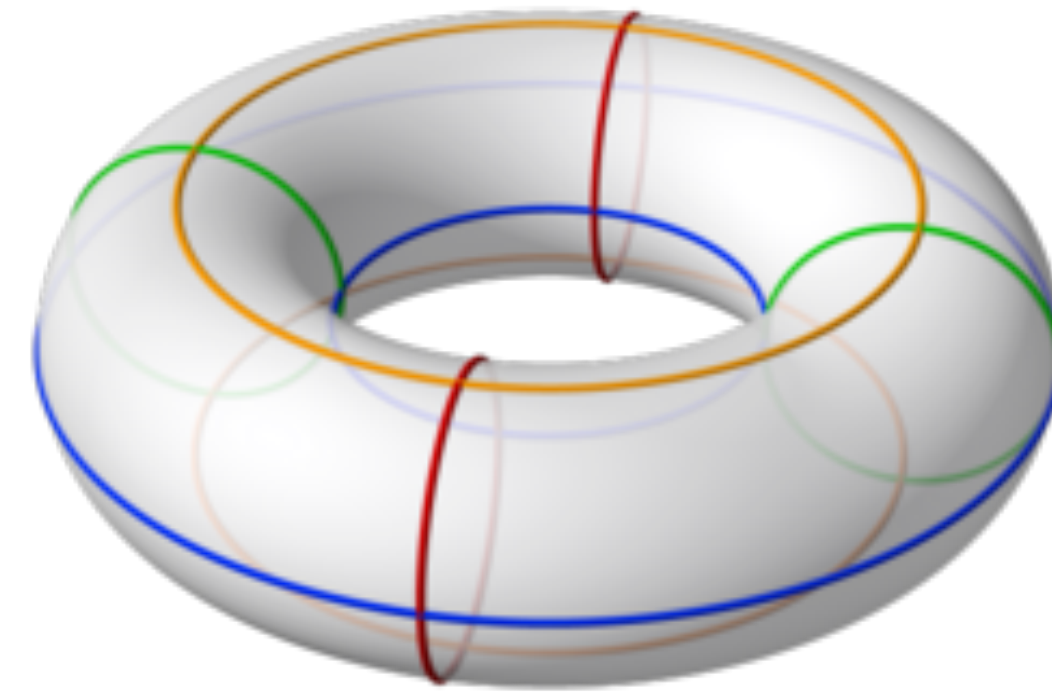
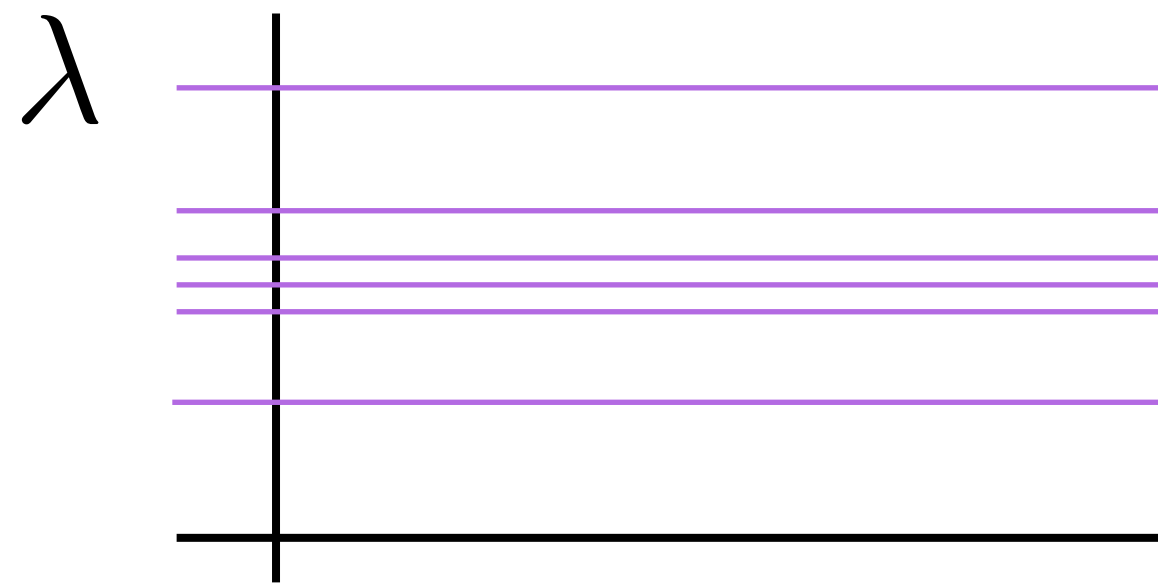


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Trace formulae

A “trace formula” relates spectral information about an operator to geometric data.



Selberg zeta function

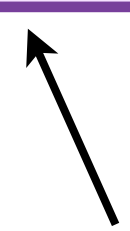
Consider e.g. $\mathbb{H}^2/\Gamma$:

Riemann zeta: $\zeta(s) = \prod_{p \text{ prime}} \frac{1}{1 - p^{-s}}$

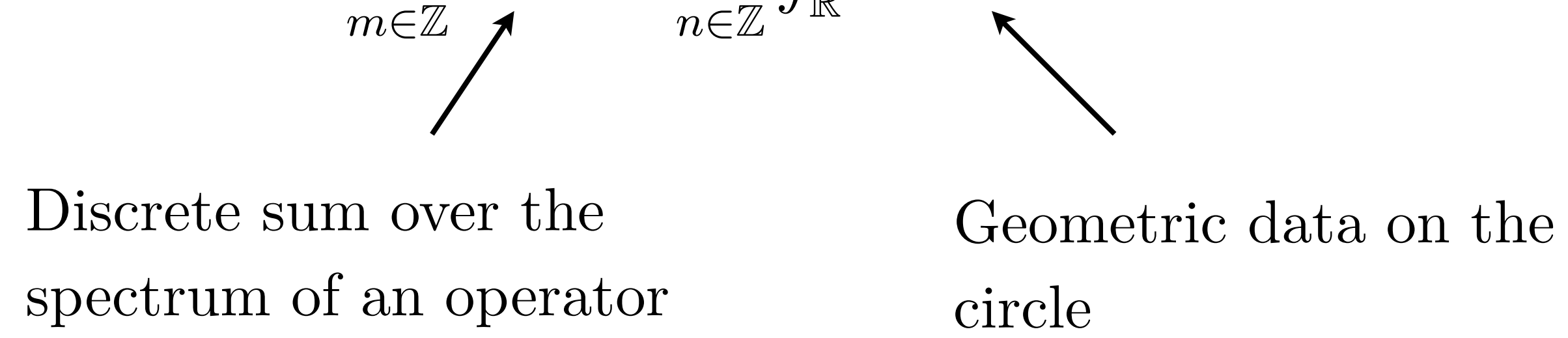
Selberg zeta:

$$\zeta_{\Gamma}(s) = \prod_p \prod_{k=0}^{\infty} (1 - \ell(p)^{-s-k})$$

Length of a “primitive” geodesic



Poisson resummation formula

$$\sum_{m \in \mathbb{Z}} h(m) = \sum_{n \in \mathbb{Z}} \int_{\mathbb{R}} h(z) e^{2\pi i n z} dz$$
The diagram consists of two arrows. One arrow starts from the text 'Discrete sum over the spectrum of an operator' and points diagonally upwards and to the right, ending at the summation term $\sum_{m \in \mathbb{Z}} h(m)$ in the equation. The other arrow starts from the text 'Geometric data on the circle' and points diagonally upwards and to the left, ending at the integral term $\int_{\mathbb{R}} h(z) e^{2\pi i n z} dz$ in the equation.

Discrete sum over the
spectrum of an operator

Geometric data on the
circle

Fredholm determinant

Consider the operator: $\mathcal{O} = -\nabla^2 - \lambda$

Functional determinant: $D(\lambda) = \det(\mathcal{O}(\lambda)) = \prod_l (\lambda_l - \lambda)^{d_l}$ Generally divergent!

Fredholm determinant: $\Delta(\lambda) \equiv \prod_{\ell=0}^{\infty} \left(1 - \frac{\lambda}{\lambda_{\ell}}\right)^{d_{\ell}} e^{\frac{\lambda}{\lambda_{\ell}} d_{\ell}}$ Canonically regularized!

Relation to one-loop determinant: $D(\lambda) \equiv \det(\hat{\mathcal{O}}) = \Delta(\lambda) e^{-\zeta'_{\hat{\mathcal{O}}}(0)} e^{-\lambda \zeta_{\hat{\mathcal{O}}}(1)}$

Fredholm determinant test function

$$R(\lambda) \equiv -\frac{d}{d\lambda} \log \Delta(\lambda) \qquad R'(\lambda) = \frac{1}{2} \sum_{n \in \mathbb{Z}} d(n) h(n) = \frac{1}{2} \sum_{n \in \mathbb{Z}} \int_{-\infty}^{\infty} d(z) h(z) e^{2\pi i n z} dz$$

Test function determined from Fredholm/resolvent



Samuel Haupfear

New method to compute one-loop determinants

$$R'(\lambda) = \frac{1}{2} \sum_{n \in \mathbb{Z}} d(n) h(n) = \frac{1}{2} \sum_{n \in \mathbb{Z}} \int_{-\infty}^{\infty} d(z) h(z) e^{2\pi i n z} dz$$

Laplace transform
Double integration

$$\Rightarrow \log D(\lambda) = \frac{i}{8} \int_{\mathcal{C}} \frac{dt}{t} \frac{\cos(t/2)}{\sin^3(t/2)} e^{-it\sqrt{1+\lambda}}$$

Wilson spool for S^3 !!!

$$j = -\frac{1}{2} \left(1 \pm \sqrt{1+\lambda} \right)$$

Selberg zeta function for de Sitter?

- The spool itself can be taken as an extended representation theoretic definition for a Selberg-like function:

Primitive geodesic $\longleftrightarrow$ Holonomy

Length $\longleftrightarrow$ Wilson loop

Higher-dimensional spool

The trace formula method allows us to generalize the structure of the spool to higher dimensions:

$$\mathbb{W}^{\text{higher-d}} = 2 \left(\frac{i}{2} \right)^d \int_{\mathcal{C}} \frac{dt}{t} \frac{\cos(t/2)}{\sin^d(t/2)} e^{-it\sqrt{\frac{1}{4}(d-1)^2 + \lambda}}$$

$$j = -\frac{1}{2} \left(1 \pm \sqrt{1 + \lambda} \right)$$

Lens space geometries

Lens spaces are quotients of the three-sphere by a discrete subgroup of $SO(4)$, which constitute additional saddles of the Euclidean path integral for de Sitter.

The trace formula method allows us to derive a candidate spool for lens spaces:

$$\mathbb{W}^{\text{lens}} = -\frac{i}{p} \sum_{k \in \mathbb{Z}_p} \int_{\mathcal{C}} \frac{dt \cos(t/2)}{t \sin(t/2)} \frac{\sin^2(t/2) e^{-it\sqrt{1+\lambda}}}{\left(\cos\left(\frac{2\pi k}{p}\right) - \cos(t)\right) \left(\cos\left(\frac{2\pi kq}{p}\right) - \cos(t)\right)}$$

$$j = -\frac{1}{2} \left(1 \pm \sqrt{1+\lambda}\right)$$

New method to compute one-loop determinants

- We found it useful to define a new trace formula for the sphere where the test function is obtained from the canonically regularized Fredholm determinant.
- With this choice, and some manipulations, what naturally falls out of the trace formula is the de Sitter Wilson spool.
- This approach provides an algorithm that gives a new method for computing one loop-determinants, and is generalizable beyond 3d. It gives the analogous object to the Wilson spool in different situations where Chern-Simons is unavailable.

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Warped AdS

$$\text{AdS}_3 : \quad ds^2 = \frac{L^2}{4} (-\cosh^2 \sigma d\tau^2 + d\sigma^2 + (du + \sinh \sigma d\tau)^2) \quad \text{SL}(2, \mathbb{R}) \times \text{SL}(2, \mathbb{R})$$



$$\text{Warped AdS}_3 : \quad ds^2 = \frac{L^2}{\nu^2 + 3} \left(-\cosh^2 \sigma d\tau^2 + d\sigma^2 + \frac{4\nu^2}{\nu^2 + 3} (du + \sinh \sigma d\tau)^2 \right) \quad \text{SL}(2, \mathbb{R}) \times U(1)$$

Wilson spools for warped AdS

Warped AdS_3 : $ds^2 = \frac{L^2}{\nu^2 + 3} \left(-\cosh^2 \sigma d\tau^2 + d\sigma^2 + \frac{4\nu^2}{\nu^2 + 3} (du + \sinh \sigma d\tau)^2 \right)$ $\text{SL}(2, \mathbb{R}) \times U(1)$

2) Lower-spin gravity

$\text{SL}(2, \mathbb{R}) \times U(1)$ Chern-Simons theory

Connections $\mathcal{A}, \mathcal{C}$

Interchange of a
sum and integral

$$\mathbb{W}_\Delta [\mathcal{C}, \mathcal{A}] = i \int_\Gamma \frac{dz}{z} \frac{\cos(z/2)}{\sin(z/2)} \text{Tr}_{\mathcal{R}} \left[\mathcal{P} e^{-\frac{z}{2\pi} \oint_\gamma (\mathcal{C} - \mathcal{A})} \right]$$
$$\zeta(\Delta) = \prod_{m=0}^\infty \prod_{k=0}^\infty \left(1 - e^{2bm - 2bk - 2a(m+k+\Delta)} \right) = \frac{1}{Z}$$

$$\log Z = \frac{1}{4} \mathbb{W}_\Delta$$

$$a = \frac{\pi r_+ (\nu^2 + 3) (2\nu r_+ - r_- \sqrt{\nu^2 + 3})}{8L\nu(r_+ - r_-)}$$

$$b = \frac{\pi r_- (\nu^2 + 3) (-2\nu r_- + r_+ \sqrt{\nu^2 + 3})}{8L\nu(r_+ - r_-)}$$

Wilson spools for warped AdS

Warped AdS_3 :
$$ds^2 = \frac{L^2}{\nu^2 + 3} \left(-\cosh^2 \sigma d\tau^2 + d\sigma^2 + \frac{4\nu^2}{\nu^2 + 3} (du + \sinh \sigma d\tau)^2 \right) \quad \text{SL}(2, \mathbb{R}) \times U(1)$$

1) Topologically massive gravity

$$S_{TMG} = -\frac{k}{2} \left(1 - \frac{1}{\mu} \right) S_{CS}[A_L] + \frac{k}{2} \left(1 + \frac{1}{\mu} \right) S_{CS}[A_R] \quad \nu = \frac{\mu L}{3}$$

$$G_{\mu\nu} - \frac{1}{\ell^2} g_{\mu\nu} + \frac{\ell}{3\nu} C_{\mu\nu} = 0 \quad C_{\mu\nu} = \varepsilon_\mu^{\alpha\beta} \nabla_\alpha \left(R_{\beta\nu} - \frac{1}{4} R g_{\beta\nu} \right)$$

Connections are not flat! To get the correct answer for the one-loop determinant, we need to average over Wilson loops with a measure set by the wordline path integral formalism.

Tentatively, a saddle point calculation singles out a preferred geodesic.

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One-loop determinants for product spaces

Two motivations for studying one-loop determinants on product spaces:

- How 1-loop data is modified by the effect of extra dimensions.
- What are the low-temperature corrections to black hole thermodynamics beyond AdS (e.g. Nariai black hole in various limits).

One-loop determinants for product spaces

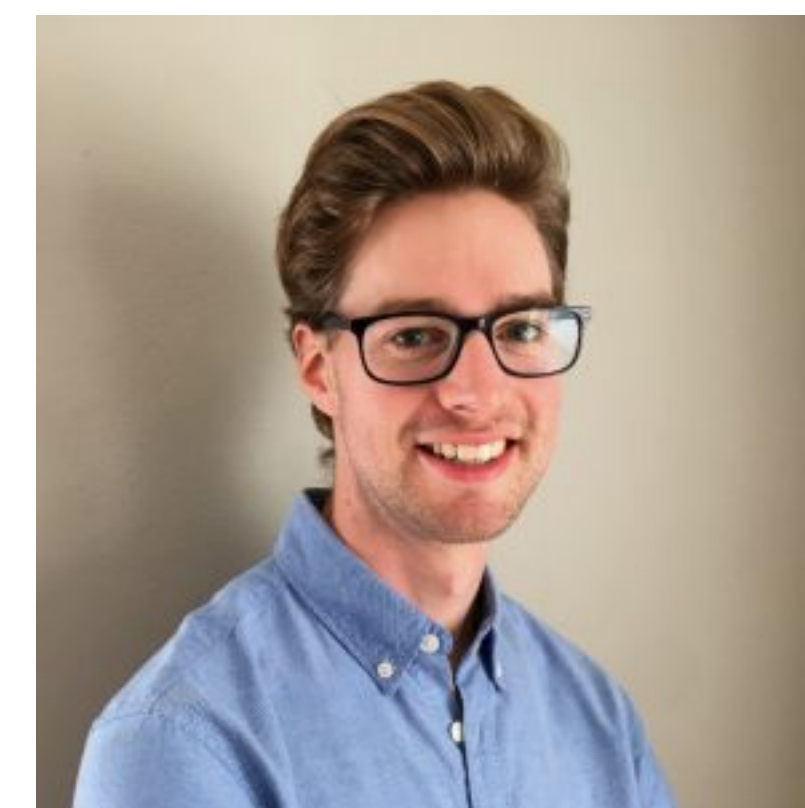
Fredholm techniques allow one to regularize the one-loop determinant on product spaces, e.g. $S^p \times S^q$.

(Unregulated expressions/overall phase can be obtained from Kaluza-Klein theory.)

e.g. $S^1 \times S^1$

$$D(\lambda) = \prod_{(n_1, n_2) \in \mathbb{Z}^2} (n_1^2 + \tau^2 n_2^2 - \lambda)$$

$$= D_1(\lambda) e^{2\pi \tilde{Z}_2(-\frac{1}{2}, -\lambda)} \prod_{n_2 \neq 0} \left(1 - e^{-2\pi \sqrt{\tau^2 n_2^2 - \lambda}}\right)^2$$



Lars Aalsma

One-loop determinants for product spaces

Our Fredholm trace formula algorithm lands on the Denef-Hartnoll-Sachdev-like form of the one-loop determinant:

e.g. $S^1 \times S^1$

$$\log D(\lambda) = \int_0^\infty \frac{dt}{t} \frac{1 + e^{-t}}{1 - e^{-t}} \sum_{m \in \mathbb{Z}} \left(e^{t\sqrt{\lambda - \tau^2 m^2}} + e^{-t\sqrt{\lambda - \tau^2 m^2}} \right)$$

$$= \prod_{(m,n) \in \mathbb{Z}^2} (|n| + i\omega_m^+) (|n| + i\omega_m^-) \quad \omega_m^\pm = \pm i\sqrt{\lambda - \tau^2 m^2}$$

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Conclusions

- We defined a “Wilson spool” that, in the $G_N \rightarrow 0$ limit, computes one-loop determinants for a massive scalar field on a Euclidean de Sitter background. The spool can be systematically expanded to any order in G_N .
- One can obtain the Wilson spool from a trace formula using the Fredholm determinant to determine the test function. This generalizes the structure of the spool to other situations (lens spaces, higher dimensions).
- The methods continue to apply in situations with broken symmetry (e.g. warped AdS).
- They also give new approaches to regularizing one-loop determinants on product spaces (for e.g. quantum corrections to BH entropy beyond AdS).

Thank you!

Extra Slides

Fredholm determinant

Consider a sequence of operators

$$\hat{\mathcal{O}} = -\nabla^2 - \lambda$$

Fredholm determinant:

$$\Delta(\lambda) \equiv \det(\mathbb{1} + \lambda (\nabla^2)^{-1}) = \prod_{\ell=0}^{\infty} \left(1 - \frac{\lambda}{\lambda_{\ell}}\right)^{d_{\ell}} e^{\frac{\lambda}{\lambda_{\ell}} d_{\ell}} \quad \text{Canonically regularized!}$$

Relation to one-loop determinant:

$$D(\lambda) \equiv \det(\hat{\mathcal{O}}) = \Delta(\lambda) e^{-\zeta'_{\hat{\mathcal{O}}}(0)} e^{-\lambda \zeta_{\hat{\mathcal{O}}}(1)}$$

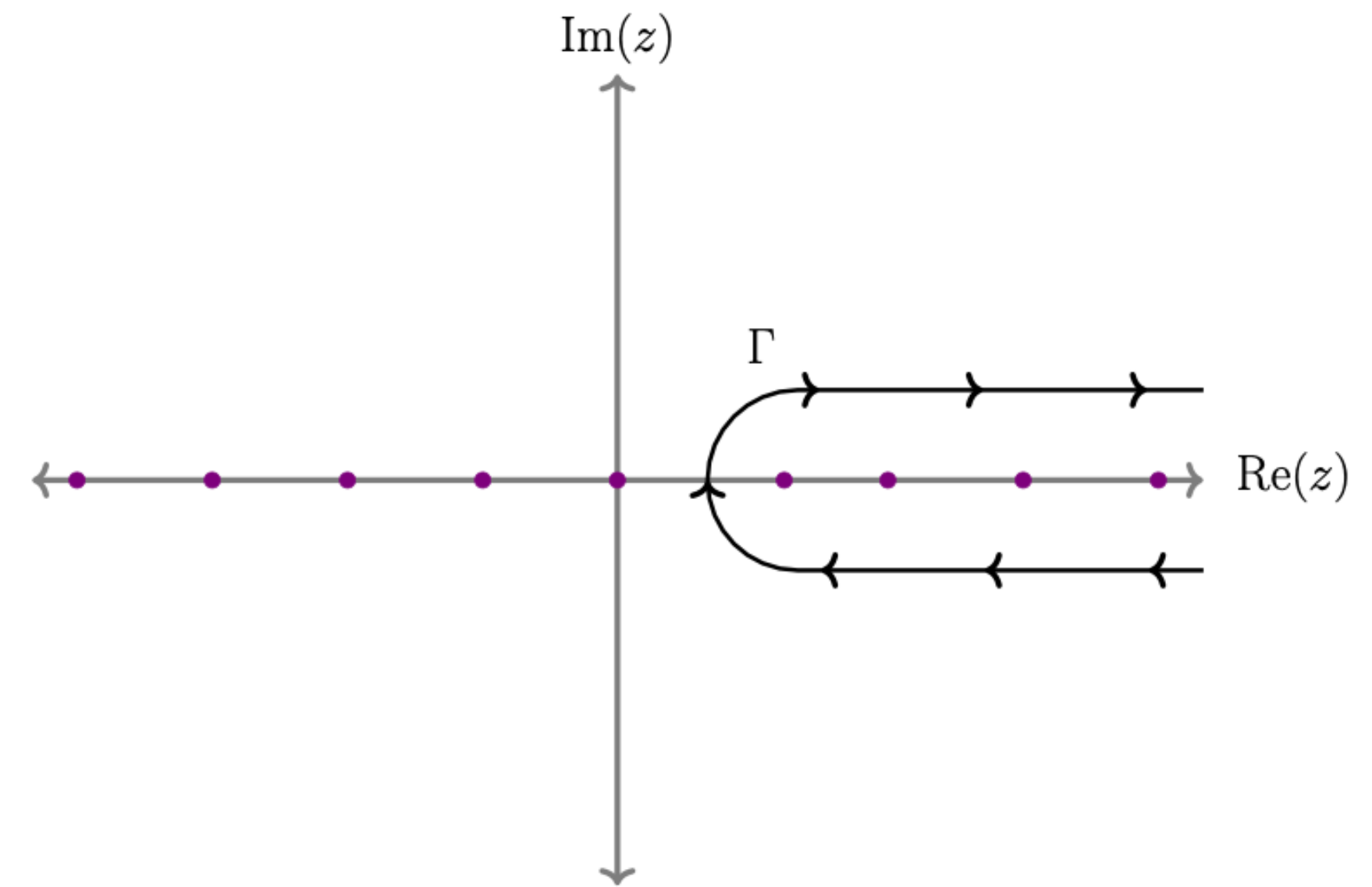
$\left. S^3 \right $	$\lambda_{\ell} = l(l+2)$	$\xrightarrow{\hspace{1.5cm}}$	$\lambda_n = n^2 - 1$	Shift ensures an even test function
	$d_{\ell} = (\ell+1)^2$	$n = \ell + 1$	$d(n) = n^2$	

New method to compute one-loop determinants

$$R'(\lambda) = \frac{1}{2} \sum_{n \in \mathbb{Z}} d(n) h(n) = \frac{1}{2} \sum_{n \in \mathbb{Z}} \int_{-\infty}^{\infty} d(z) h(z) e^{2\pi i n z} dz$$

1) Re-express and deform contour, interchange sum and integral

$$\text{LHS} = \frac{i}{2} \int_{\Gamma} d(z) h(z) \cot(\pi z) dz$$



New method to compute one-loop determinants

$$R'(\lambda) = \frac{1}{2} \sum_{n \in \mathbb{Z}} d(n) h(n) = \frac{1}{2} \sum_{n \in \mathbb{Z}} \int_{-\infty}^{\infty} d(z) h(z) e^{2\pi i n z} dz$$

2) Perform Hadamard factorization of Fredholm determinant/test function

$$\Delta(\lambda) = Q(\ell) \Delta^+(\sqrt{1+\lambda}) \Delta^+(-\sqrt{1+\lambda})$$

$$\Rightarrow h(n) = h_+(n) + h_-(n)$$

New method to compute one-loop determinants

$$R'(\lambda) = \frac{1}{2} \sum_{n \in \mathbb{Z}} d(n) h(n) = \frac{1}{2} \sum_{n \in \mathbb{Z}} \int_{-\infty}^{\infty} d(z) h(z) e^{2\pi i n z} dz$$

3) Re-express test function in terms of a Laplace transform

$$\mathcal{L}\left\{\left(t\sqrt{1+\lambda}\pm 1\right)e^{\mp t\sqrt{1+\lambda}}\right\}(z)=\int_0^\infty\left(t\sqrt{1+\lambda}\pm 1\right)e^{-t(z\pm\sqrt{1+\lambda})}dt=\frac{2\sqrt{1+\lambda}\pm z}{(z\pm\sqrt{1+\lambda})^2}$$

$\nearrow$
 $h_{\pm}(z)$ for S^3

New method to compute one-loop determinants

$$R'(\lambda) = \frac{1}{2} \sum_{n \in \mathbb{Z}} d(n) h(n) = \frac{1}{2} \sum_{n \in \mathbb{Z}} \int_{-\infty}^{\infty} d(z) h(z) e^{2\pi i n z} dz$$

4) Integrate $R'(\lambda)$ from trace formula twice, interchanging to evaluate λ integrals:

$$\log D(\lambda) = - \int_{-\infty}^{\lambda} \int_{-\infty}^{\lambda'} d\lambda' d\lambda'' R'(\lambda'')$$

New method to compute one-loop determinants

$$R'(\lambda) = \frac{1}{2} \sum_{n \in \mathbb{Z}} d(n) h(n) = \frac{1}{2} \sum_{n \in \mathbb{Z}} \int_{-\infty}^{\infty} d(z) h(z) e^{2\pi i n z} dz$$

5) Use Cauchy's theorem to evaluate the z integral

$$\Rightarrow \log D(\lambda) = \frac{i}{8} \int_{\mathcal{C}} \frac{dt}{t} \frac{\cos(t/2)}{\sin^3(t/2)} e^{-it\sqrt{1+\lambda}}$$

Wilson spool for S^3 !!!

$$j = -\frac{1}{2} (1 \pm \sqrt{1+\lambda})$$

Representation-theoretic DHS construction

DHZ Computation of One-Loop Determinants

Denef, Hartnoll, Sachdev (2010)
 Castro, Coman, Fliss, CZ (2023a)
 Castro, Coman, Fliss, CZ (2023b)
 Bourne, Castro, Fliss (2024)

$$Z_{\text{scalar}} = \det(-\nabla^2 + m^2 \ell^2)^{-1/2}$$

View as a meromorphic function of $\Delta = 1 + \sqrt{1 - m^2 \ell^2}$

Can be completely determined (together with a boundary condition at infinity) by the poles at $-\nabla^2 + m^2 \ell^2 = 0 \longrightarrow$ quasinormal modes

$$2c_2^L + 2c_2^R$$

Poles at $j(j+1) = -\frac{m^2}{4} \implies$ non-standard representations
 (and standard reps, upon analytic continuation of mass)

Some additional technical requirements (see Bourne et al. 2024) force the poles to lie at the finite reps

DHZ-Inspired Construction of the Spool

$$Z_{\text{scalar}} = \prod_{\substack{(\lambda_L, \lambda_R) \\ \in \mathbf{R}_j \otimes \mathbf{R}_j}} \prod_{N \in \mathbb{Z}} (|N| - \lambda_L - \lambda_R)^{-1/2} \times \prod_{\substack{(\bar{\lambda}_L, \bar{\lambda}_R) \\ \in \mathbf{R}_{\bar{j}} \otimes \mathbf{R}_{\bar{j}}}} \prod_{\bar{N} \in \mathbb{Z}} (|\bar{N}| - \bar{\lambda}_L - \bar{\lambda}_R)^{-1/2}$$

Poles set from single-valued condition:

$$\lambda_L \mathfrak{h}_L - \lambda_R \mathfrak{h}_R \in \mathbb{Z}$$

Use various tricks:

$$\log M = - \int_{\times}^{\infty} \frac{\mathrm{d}\alpha}{\alpha} e^{-\alpha M}$$

$$\sum_{n \in \mathbb{Z}} e^{-|n|\alpha} = \frac{\cosh \alpha/2}{\sinh \alpha/2}$$

$$\sum_{(\lambda_L, \lambda_R) \in R_j \otimes R_j} e^{\alpha \lambda_L \mathfrak{h}_L - \alpha \lambda_R \mathfrak{h}_R} = \mathrm{Tr}_{R_j} \left(e^{\alpha \mathfrak{h}_L L_3} \right) \mathrm{Tr}_{R_j} \left(e^{-\alpha \mathfrak{h}_R L_3} \right)$$

$\Rightarrow$

$$\log Z_{\text{scalar}}[S^3] = \frac{i}{4} \int_{\mathcal{C}} \frac{\mathrm{d}\alpha}{\alpha} \frac{\cos \alpha/2}{\sin \alpha/2} \mathrm{Tr}_{R_j} \left(\mathcal{P} e^{\frac{\alpha}{2\pi} \oint a_L} \right) \mathrm{Tr}_{R_j} \left(\mathcal{P} e^{-\frac{\alpha}{2\pi} \oint a_R} \right)$$