

Localizing the M-theory Path Integral

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with

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[arXiv:2607.07255](#)

[arXiv:2607.07621](#)

[arXiv:2609.31829](#)

Programme: **Equivariant Localization in Supergravity**

[Benetti Genolini, JPG, Sparks 23]

Compute BPS observables for susy solutions **without solving the supergravity equations of motion**. Instead use:

1. Susy Killing vector and equivariantly closed forms
2. Input global data
3. Assume solution exists

For example: compute the on-shell action for Euclidean sugra solutions on M . In a holographic context, for SCFT on ∂M

$$\exp [-I_{\text{SUSY}}] \sim Z_{\text{SCFT}}$$

Until recently: computed for two derivative gravity and also with some higher derivative corrections

Can we do more and localize the M-theory path integral?

$$Z_{\text{M-theory}}^{\text{pert}} = \int [d\nu] \exp[-I_{\text{SUSY}}] Z_{1\text{-loop}} = Z_{\text{SCFT}}^{\text{pert}}$$

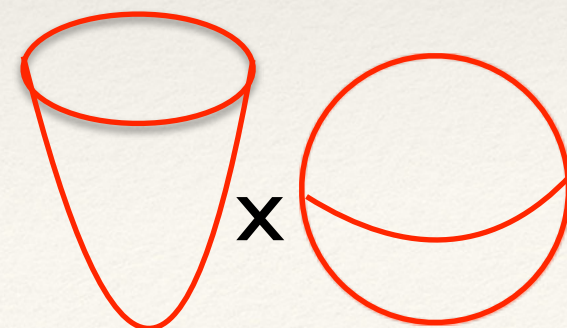
- Provide a precise proposal for particular classes of M-theory backgrounds. Captures the **full** perturbative expansion in ℓ_p
- Specifically for a broad class of M2-brane SCFTs holographically dual to $AdS_4 \times Y_7$ where Y_7 is Sasaki-Einstein
- Main example: $Y_7 = S^7/\mathbb{Z}_k$ dual to ABJM $U(N)_k \times U(N)_{-k}$

Two applications

ABJM theory on $\partial M = S^3$

ABJM on squashed $\partial M = S^1 \times S^2$ compute TTI:

Entropy for rotating, magnetically charged black holes



Results from localization in susy field theory

Place ABJM on a three sphere

[Marino, Putrov I I; Fuji, Hirano, Moriyama I I]

$$J_{\text{pert}}(\mu) = \frac{\mathcal{C}}{3}\mu^3 + \mathcal{B}\mu + \mathcal{A} \quad \text{where } \mu \text{ is conjugate to } N$$

After an inverse Laplace transform

$$Z^{\text{pert}}(N) = \mathcal{C}^{-1/3} e^{\mathcal{A}} \text{Ai}\left[\mathcal{C}^{-1/3}(N - \mathcal{B})\right],$$

Large N: infinite number of perturbative corrections $\left(\frac{L}{\ell_p}\right)^6 \sim kN$

Can place ABJM on squashed three sphere and also consider deforming SCFT by mass deformations captured by flavour chemical potentials $\Delta_I \quad I = 1, \dots, 4$

Goal: compute $\mathcal{C}, \mathcal{B}, \mathcal{A}$ from M-theory as a function of b, Δ_I

Field theory results also exist for TTI. Goal: match with M-theory

Susy Solutions of Supergravity and Equivariant Localization

- Supersymmetry $\Leftrightarrow$ Killing spinors $\epsilon \Rightarrow K = \bar{\epsilon} \Gamma^\mu \epsilon \partial_\mu$
Killing vector
- Construct polyforms from spinor bilinears and sugra fields:

$$\Phi = \Phi_{2n} + \Phi_{2n-2} + \dots \Phi_0$$

$$d_K \Phi = 0 \quad d_K = d - \iota_K$$

- BPS observables given by $\int \Phi$. Determined by fixed point data:

BVAB theorem [Berline, Vergne 82; Atiyah, Bott 84]

- Consider closed M_{2n} ; fixed point set F is where $K = 0$

$$\int_{M_{2n}} \Phi = \int_F \frac{\Phi}{e_K(NF)} = \sum_A (2\pi)^n \frac{\Phi_0|_A}{\prod_{\alpha=1}^n w_{A,\alpha}} \quad K|_A = \sum_{i=1}^n w_{A,\alpha} \frac{\partial}{\partial \phi_\alpha}$$



F isolated fixed points

No need to solve SUGRA PDEs!

Apply to D=11 sugra and perturbative M-theory

D=11 effective perturbative action: $\mathcal{L} = \mathcal{L}^{GI} + \mathcal{L}^{CS}$

where $\mathcal{L}^{GI} = R * 1 - \frac{1}{2} G \wedge *G + \dots$ $G = dC$

$$\mathcal{L}^{CS} \sim \frac{1}{6} C \wedge G \wedge G + (2\pi\ell_p)^6 C \wedge X_8$$

$$X_8 = \frac{1}{192} (P_1^2 - 4P_2)$$

To define D=11 CS action on M use a D=12 filling W
with $M = \partial W$:

$$S^{CS} \sim \int_W \frac{1}{6} G \wedge G \wedge G + (2\pi\ell_p)^6 G \wedge X_8$$

Exponentiated action is independent of choice of W [Witten 96]

Relative equivariant localization of M theory action

Construct polyforms on W

$$\begin{aligned}\Phi^G &= G + \Omega_2 + \Phi_0^G && \text{with} && d_K \Phi^G = 0 \\ \Phi^{\text{anom}} &= \frac{i}{6} (\Phi_G)^3 + i(2\pi\ell_p)^6 \Phi_G \Phi_{X_8} && \text{with} && d_K \Phi^{\text{anom}} = 0 \\ &&& && d_K \Phi^{(\text{GI})} = -\Phi^{(\text{anom})}|_M\end{aligned}$$

$$I_{\text{SUSY}} \sim \int_M \Phi^{\text{GI}} + \int_W \Phi^{\text{anom}}$$

BVAB for closed $M \Rightarrow$

$$I_{\text{SUSY}} = \frac{1}{(2\pi)^8 \ell_p^9} \int_F \frac{\Phi^{\text{anom}}}{e_K(NF)}$$

assuming all fixed points of K lie in the interior of W

For isolated fixed points:

$$I_{\text{SUSY}} = \frac{1}{(2\pi)^2 \ell_p^9} \sum_A \frac{(\Phi^{\text{anom}})_0|_A}{\prod_{\alpha=1}^6 w_{A,\alpha}}$$

Established:

- Relative equivariant construction and fixed-point formula at two-derivative $D=11$ supergravity level

Conjectural:

- The same fixed-point expression gives the supersymmetric action to all orders in ℓ_p
- For holographic backgrounds it gives the fully renormalized supersymmetric action, including all boundary counterterms.

Evidence: agreement with field theory

Apply to ABJM theory dual to $AdS_4 \times S^7/\mathbb{Z}_k$

Consider ABJM on squashed S_b^3 $b = 1$ round

$$Q \equiv b + b^{-1}$$

Allow mass deformations: packaged into Δ_I and $\sum_I \Delta_I = 2$

First compute I_{susy} in M-theory (aiming for Airy $\mathcal{C}$ and $\mathcal{B}$)

Assume the all-orders corrected M-theory background admits the D=12 filling

$$W = \mathbb{C}^2 \times \mathbb{C}^4/\mathbb{Z}_k$$

With Killing vector K having a single fixed point

with weights $w_\alpha = (b_1, b_2, u_1, u_2, u_3, u_4)$

$$\text{UV/IR: } u_I = -\frac{Q}{2}\Delta_I \qquad b = \sqrt{\frac{b_1}{b_2}}$$

Computing $\mathcal{C}, \mathcal{B}$

Flux $\mu \equiv \frac{i}{(2\pi)^2 \ell_p^3} \int_{EA dS_4}^{\text{eq}} \Phi^G = \frac{i}{b_1 b_2 \ell_p^3} \Phi_0^G$

For this S_b^3 background we consider a single flux μ

Recall:

$$I_{\text{SUSY}} = \frac{1}{(2\pi)^2 \ell_p^9} \sum_A \frac{(\Phi^{\text{anom}})_0|_A}{\prod_{\alpha=1}^6 w_{A,\alpha}} \quad \Phi^{\text{anom}} = \frac{i}{6} (\Phi_G)^3 + i(2\pi \ell_p)^6 \Phi_G \Phi_{X_8}$$

gives

$$I_{\text{SUSY}} = -\frac{i(2\pi)^6}{k \prod_{\alpha=1}^6 w_{\alpha}} \left[\frac{(ib_1 b_2 \mu)^3}{6(2\pi)^8} + \frac{ib_1 b_2 \mu}{(2\pi)^2} (\Phi_{X_8})_0 \right]$$

$$-I_{\text{SUSY}}(\mu) = \frac{\mathcal{C}}{3} \mu^3 + \mathcal{B} \mu$$

Thus from M-theory we compute $J_{\text{pert}}(\mu) = \frac{\mathcal{C}}{3}\mu^3 + \mathcal{B}\mu + \mathcal{A}$
and from I_{SUSY} we obtain $\mathcal{C}, \mathcal{B}$

$$\mathcal{C} = \frac{2}{\pi^2 k Q^4 \prod_{I=1}^4 \Delta_I} \quad Q = b + b^{-1}$$

$$\mathcal{B} = \frac{1}{24k} \left[1 - 2 \sum_I \frac{1}{\Delta_I} + \frac{4}{Q^2} \sum_{I < J} \frac{1}{\Delta_I \Delta_J} \right] + \frac{k^2 - 1}{24k}$$

Note: $\mathcal{B}$ comes from X_8 plus orbifold M2 charge [Bergman, Hirano 09]

Gravity: direct M-theory derivation $\mathcal{C}, \mathcal{B}$ for general b, Δ_I

Field theory: $\mathcal{C}, \mathcal{B}$ known exactly on special loci, conjectured for general b, Δ_I [Nosaka 15; Bobev, Hong, Reys 23; Kubo, Nosaka, Pang 24]

What is missing at this point is the Airy prefactor $\mathcal{A}$

Comes from $Z_{1\text{-loop}}$ in $Z_M^{\text{pert}} = \int [d\nu] \exp[-I_{\text{SUSY}}] Z_{1\text{-loop}}$

In field theory localization: most bosonic and fermionic fluctuations pair up under action of Q : $Z_{1\text{-loop}} \sim \det(\text{unpaired modes})$

Encoded in character of the Q -complex $\mathcal{K}_W(t) = \text{Str}_{\mathcal{H}} e^{-t\mathcal{L}_K}$

Motivated by [Nekrasov, Okounkov 14][Costello, Li 16][Costello, Gaiotto 18]
[Raghavendran, Saberi, Williams 21]

we propose the local complex:

$$\Omega^{0,\bullet}(W, TW) - \Omega^{0,\bullet}(W, T^*W)$$

where $TW = T^{1,0}W$, $T^*W = (T^{1,0}W)^*$

Promote to a relative equivariant character on W :

M-theory proposal: $\mathcal{K}_W(t) = \text{Ind}_{tK} [\bar{\partial}_W \otimes (TW - T^*W)]$

Key point: $\mathcal{K}_W(t)$ is computable using BVAB:

For isolated fixed points we have

$$\mathcal{K}_W(t) = \sum_{A \in F} \frac{\sum_{\alpha=1}^6 (q_{A,\alpha}^{-1} - q_{A,\alpha})}{\prod_{\alpha=1}^6 (1 - q_{A,\alpha})}$$

$$q_{A,\alpha} \equiv e^{-tw_{A,\alpha}}$$

Once we have the character we can obtain the $Z_{1\text{-loop}}$ determinant in the usual way via Mellin transform and zeta function regularization:

$$\zeta_{\mathcal{K}}(s) = \frac{1}{\Gamma(s)} \int_0^\infty dt t^{s-1} \mathcal{K}_W(t)$$

$$\log Z_{1\text{-loop}} = \zeta'_{\mathcal{K}}(0)$$

For ABJM we can carry this out on $W = \mathbb{C}^2 \times \mathbb{C}^4 / \mathbb{Z}_k$

(the orbifold requires a Molien average for $k \neq 1$)

Explicit analytic expressions for $\mathcal{A}_{\text{ABJM}}(k, b, \Delta_I)$ in terms of Barnes functions

e.g. $b = 1, \quad \Delta_I = \frac{1}{2}$ round S^3 SCFT point

$$K_k(t) = \frac{\coth(kt/4)}{2 \sinh^2(t/2)}$$

$$\xrightarrow{\zeta \text{ reg.}} \mathcal{A}_{\text{ABJM}} = A(k) = \frac{2\zeta(3)}{\pi^2 k} \left(1 - \frac{k^3}{16}\right) + \frac{k^2}{\pi^2} \int_0^\infty dx \frac{x \log(1 - e^{-2x})}{e^{kx} - 1}$$

Exactly reproduces the known Airy constant $A(k)$

Reproduces all field theory results

[Nosaka 15][Kubo,Nosaka,Pang 24]
[Bobev,Smet,Hong,Reys,Zhang 25]

Topologically Twisted Index

$$Z_M^{\text{pert}} = \int [d\nu] \exp[-I_{\text{SUSY}}] Z_{1\text{-loop}}$$

For sphere partition functions

$$M_4 = \mathbb{C}^2 \quad \partial M_4 = S^3$$

$$W = M_4 \times C^4 / \mathbb{Z}_k$$

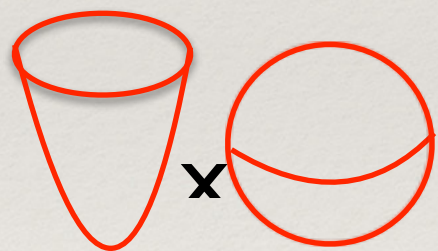
There was a single flux parameter μ on M_4

Apply the formalism to the TTI associated with magnetically charged, rotating AdS_4 black holes:

[Benini, Zaffaroni 15]

$$M_4 = \mathbb{C} \times S^2$$

$$\partial M_4 = S^2 \times S^1$$



Magnetic fluxes

n_I

$$\sum_I n_I = 2$$

Chemical potentials

Δ_I

$$\sum_I \Delta_I = 2$$

$$\mathbb{C}^4 / \mathbb{Z}_k \longrightarrow W \longrightarrow M_4$$

Now two fixed points on M_4 , $\sigma = \pm$, at north and south poles

This gives two flux parameters $\mu_{\pm}$, one at each pole

$$\mu = \mu_+ + \mu_-$$

Total flux through M_4

Fixed boundary data conjugate to N

$$\mu_+ = \nu, \quad \mu_- = \mu - \nu \quad \nu \quad \text{not fixed at boundary; integrate over it}$$

$$\int \frac{d\nu}{2\pi i} \quad \text{remnant of M-theory path integral}$$

Computing as before we get two gravitational blocks:

$$I_{\text{SUSY}} = - \sum_{\sigma} \left(\frac{\mathcal{C}_{\sigma}}{3} \mu_{\sigma}^3 + \mathcal{B}_{\sigma} \mu_{\sigma} \right)$$

$$\log Z_{1\text{-loop}} = \sum_{\sigma=\pm} \mathcal{A}_{\text{ABJM}}(k, b_{\sigma}, \Delta_{\sigma}) .$$

$$Z^{\text{pert}}[N] = Z_{1\text{-loop}} \prod_{\sigma} \int \frac{d\mu_{\sigma}}{2\pi i} \exp[-I_{\text{SUSY}}^{\sigma}(\mu_{\sigma}) - N \mu_{\sigma}]$$

Two fixed points $\Rightarrow$ two Airy blocks

Unrefined TTI: turn off rotation parameter after combining the two blocks:

$$\log Z_{\text{TTI}}^{\text{pert}} = -\frac{\pi \sqrt{2k} \prod_I \Delta_I}{3} \sum_I \frac{n_I}{\Delta_I} \left(\hat{N}_{k,\Delta}^{3/2} - \frac{c_I}{k} \hat{N}_{k,\Delta}^{1/2} \right) - \frac{1}{2} \log \hat{N}_{k,\Delta} + \hat{f}_0(k, \Delta, n)$$

$$\hat{N}_{k,\Delta} = N - \frac{k}{24} + \frac{1}{12k} \sum_I \Delta_I^{-1} \quad c_I(\Delta) = \frac{\prod_{J \neq I} (\Delta_I + \Delta_J)}{8\Delta_1 \Delta_2 \Delta_3 \Delta_4} \sum_{J \neq I} \Delta_J$$

N -dependent terms reproduce the all-orders field-theory conjecture, supported by high-precision numerics. [Bobev,Hong,Reys 22]

Our $Z_{1\text{-loop}}$ also determines the N^0 constant $\hat{f}_0(k, \Delta, n)$ for generic Δ_I, n_I

Agrees with numerical field theory results [Bobev,Hong,Reys 23]

Agrees analytically at $\Delta_I = n_I = \frac{1}{2}$ [Hong 26, Hosseini 26]

Final Comments

- Localised perturbative M-theory path integral for M2-branes:

$$Z_M^{\text{pert}} = \int [d\nu] \exp[-I_{\text{SUSY}}] Z_{1\text{-loop}}$$

- ABJM theory on S_b^3 : complete Airy data for general k, b, Δ_I
- AdS_4 black holes/TTI: general $k, \Delta_I, \mathfrak{n}_I$ with new N^0 one loop term
- Extended to toric Sasaki-Einstein spaces with crepant resolution
- Extended to $V^{5,2} = \frac{SO(5)}{SO(3)}$: non-toric example
- More general black holes: eg baryonic fluxes? SCl?
- Beyond M2-backgrounds - needs modification