

Electron spin polarization preservation in the Electron-Ion Collider

Matt Signorelli

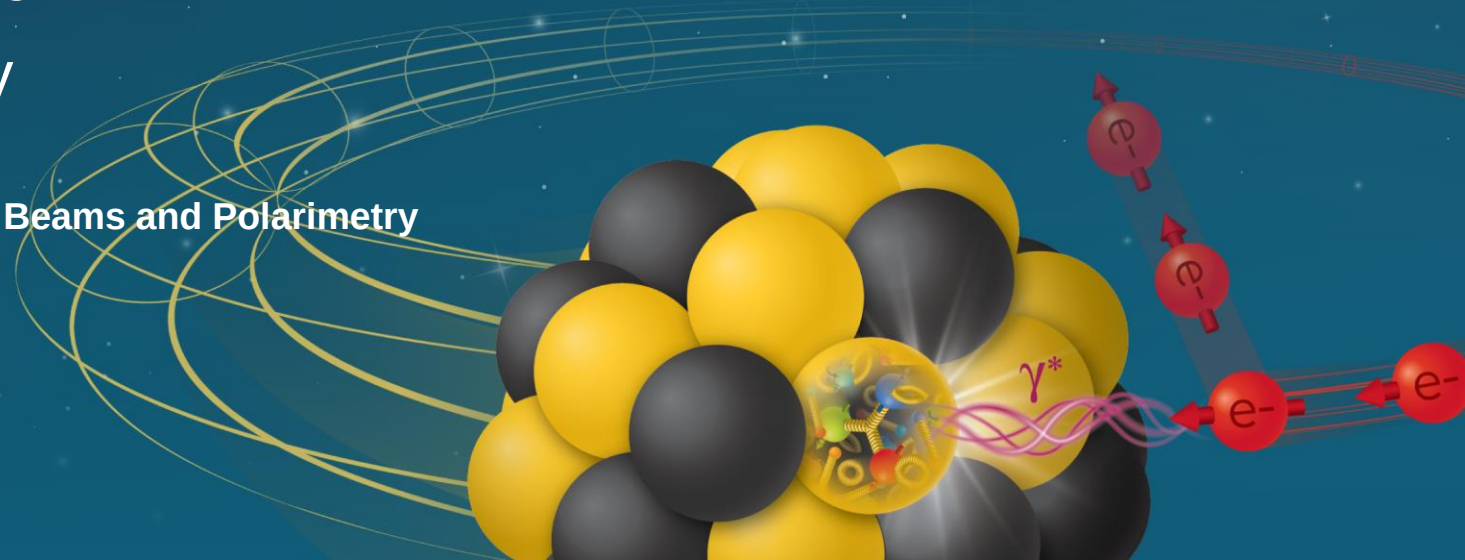
Postdoctoral Research Associate

Brookhaven National Laboratory

EIC Accelerator Collaboration Meeting on Polarized Beams and Polarimetry

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Electron-Ion Collider



Overview

- **Background**
- **Electron Storage Ring (ESR) of the Electron-Ion Collider (EIC)**
- **Best Adjustment Groups for Electron Spin (BAGELS)**
- **Conclusions**
- **New Software Plug: SciBmad**

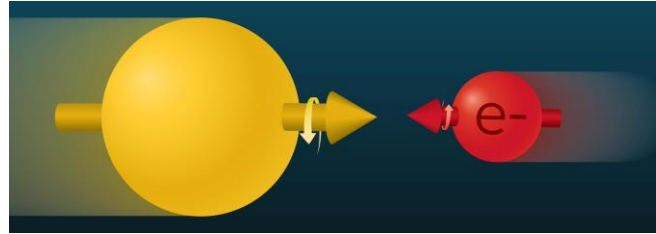
Relevant papers cited at bottom of slides

Also see my PhD thesis “Electron spin polarization preservation in the Electron-Ion Collider”

Background: What is the EIC?

▪ Electron-Ion Collider (EIC)

- Collides **spin-polarized hadrons** (e.g. protons) with **spin-polarized electrons** for a **wide range of energies**
- Study origin of **proton spin**, among other outstanding Standard Model questions

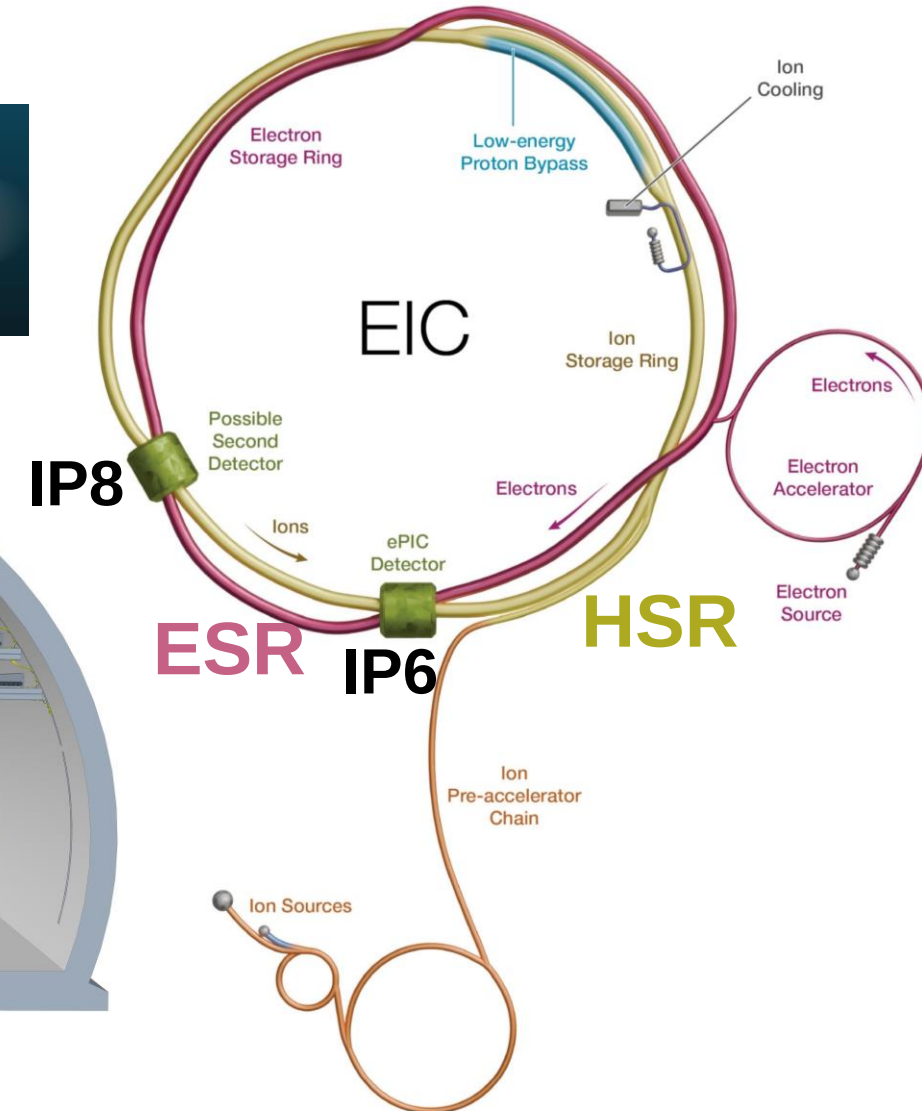
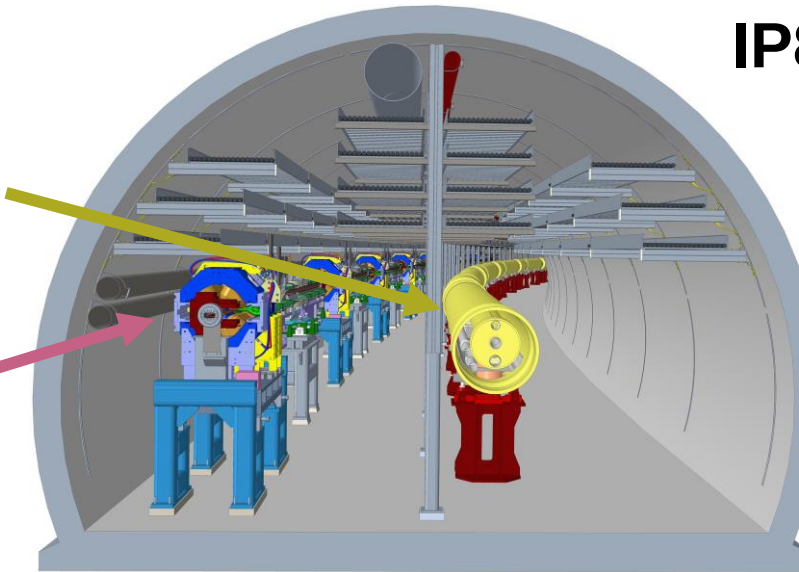


▪ Hadron Storage Ring (HSR)

- 41, 100, and 275 GeV

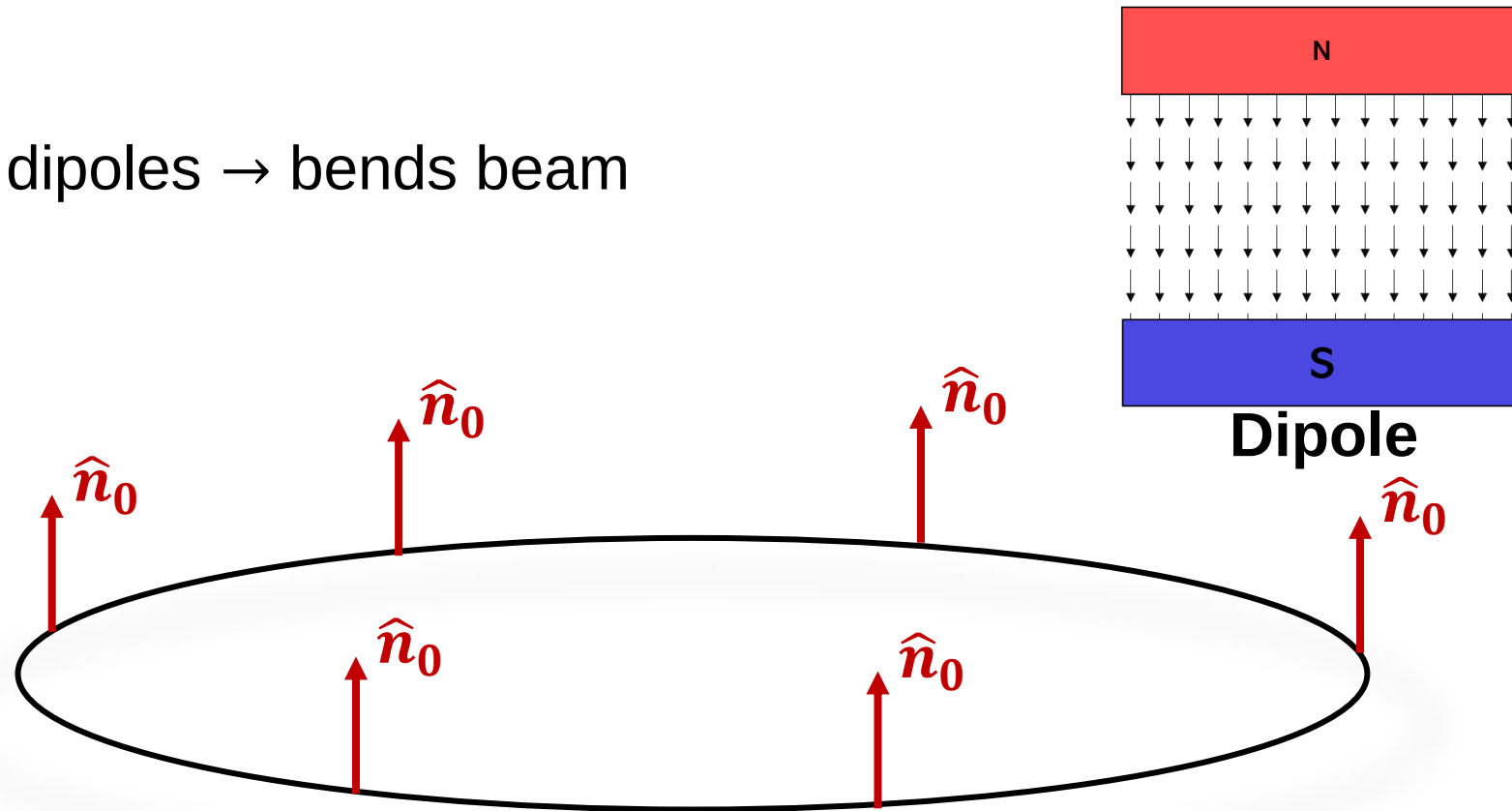
▪ Electron Storage Ring (ESR)

- 5, 9, and 18 GeV



Background: Electron Spin Polarization

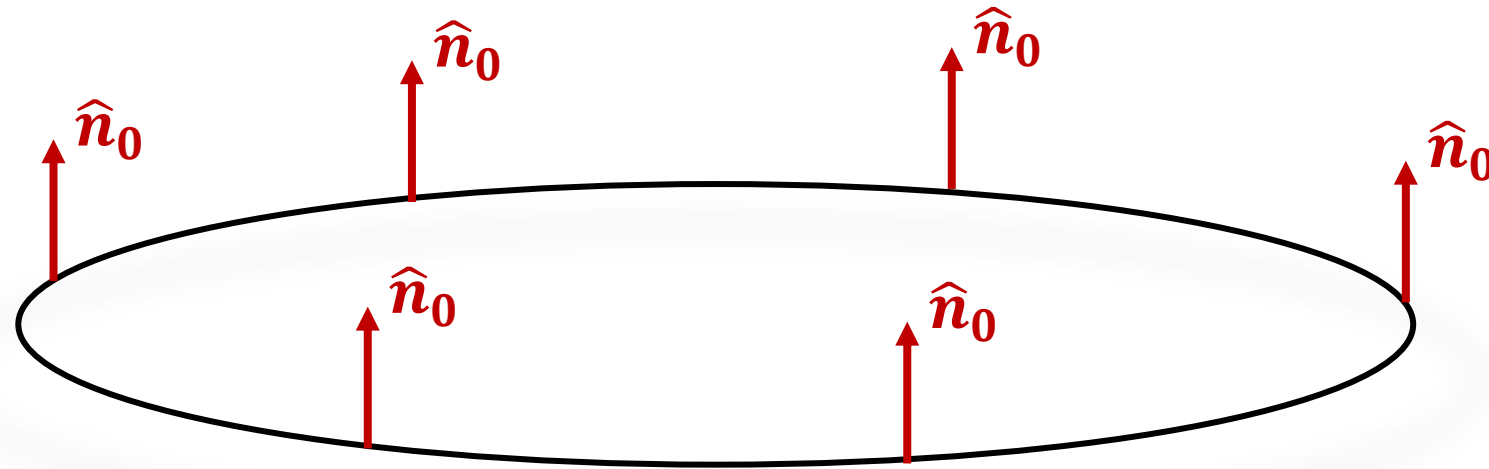
- Ring with only dipoles → bends beam



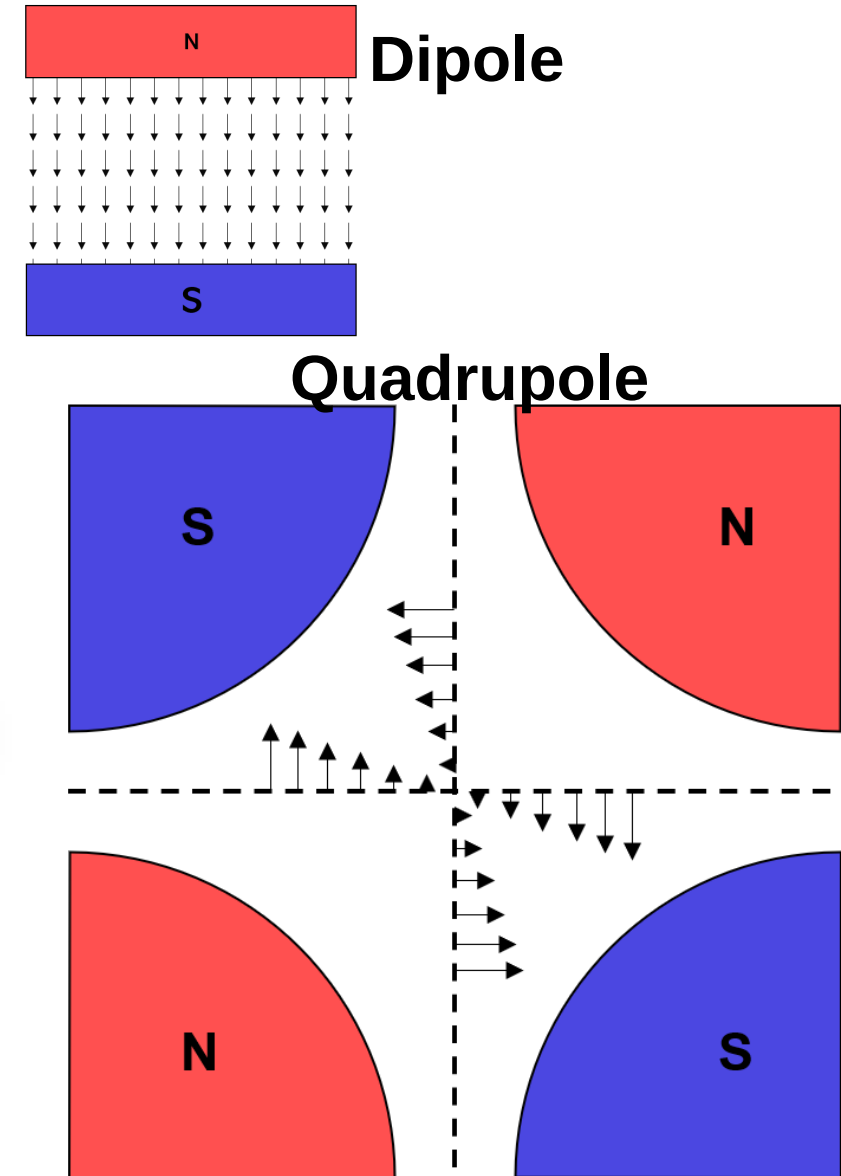
- Periodic spin direction on closed orbit $\hat{n}_0$

Background: Electron Spin Polarization

- Ring with only dipoles & quadrupoles (focusing)

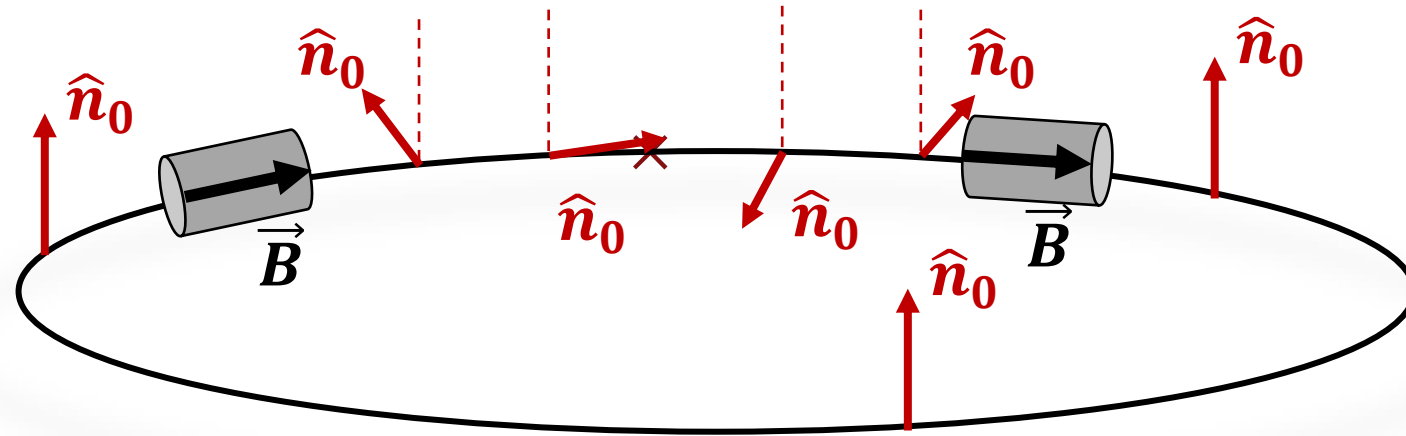
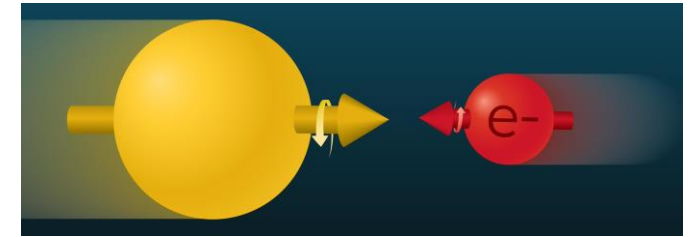


- Periodic spin direction on closed orbit $\hat{n}_0$



Background: Electron Spin Polarization

- What if we want longitudinal spin at the interaction point?
- **Use solenoids:**



Background: Electron Spin Polarization

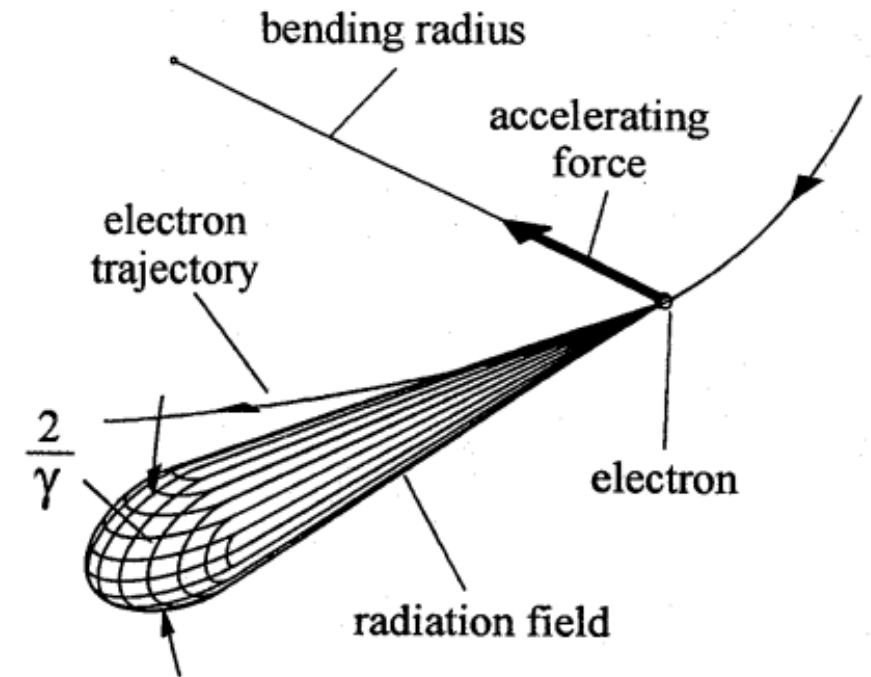
Is that it?

For hadrons, mostly. But...

Electrons emit a lot of synchrotron radiation!

Effects on polarization:

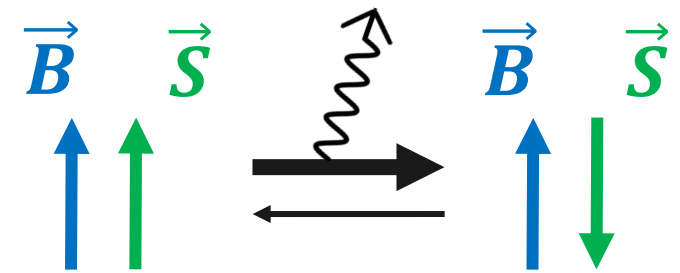
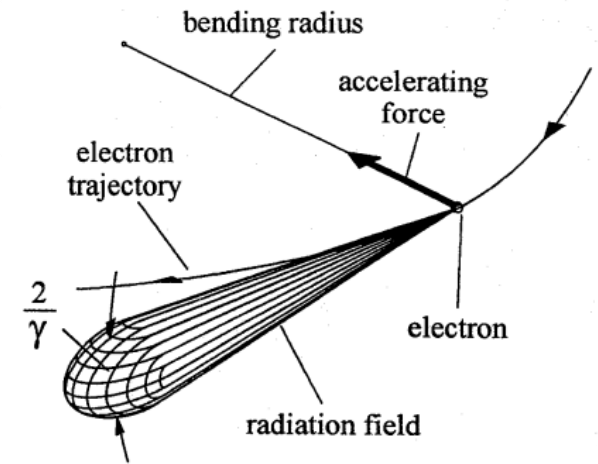
1. Sokolov-Ternov (spin flip) effect
2. Radiative Depolarization
3. Kinetic effects (usually small)



Background: Electron Spin Polarization

Sokolov-Ternov Effect

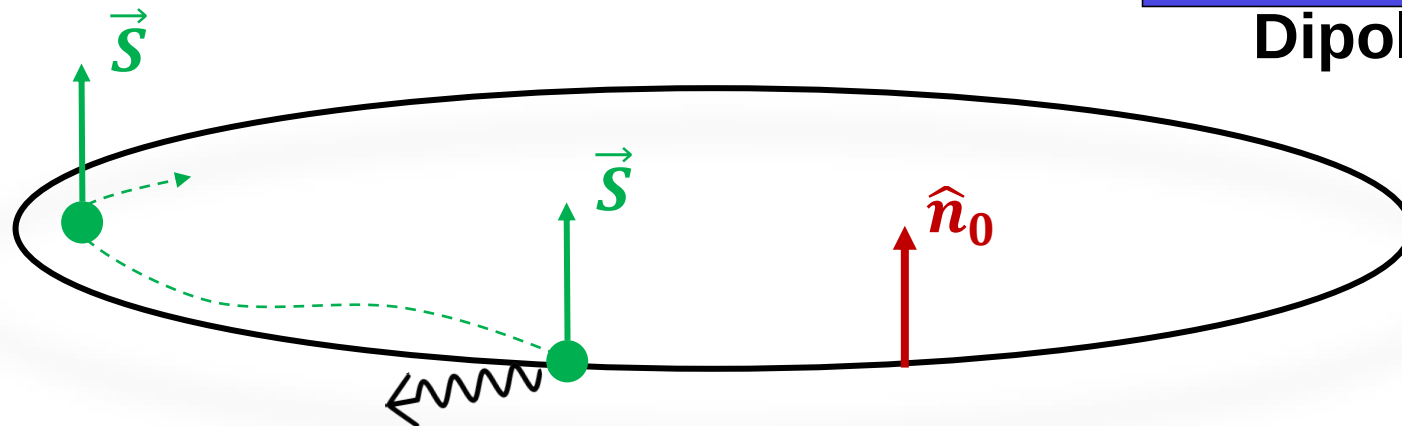
- Photons emitted when bending in a magnetic field
- Sometimes, spin will **flip** when a photon is emitted
 - Relativistic quantum mechanical effect
- **More likely to flip anti-parallel than parallel!**
- In a perfectly homogenous field, **builds up polarization to $\approx 92.38\%$!**



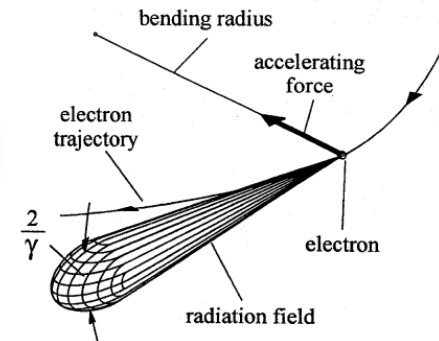
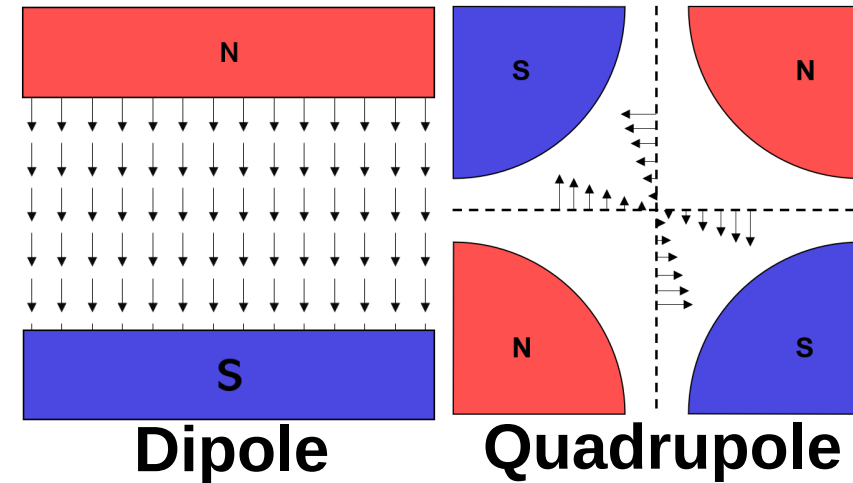
Background: Electron Spin Polarization

Radiative Depolarization (a simplistic picture)

- Ring with only dipoles and quadrupoles

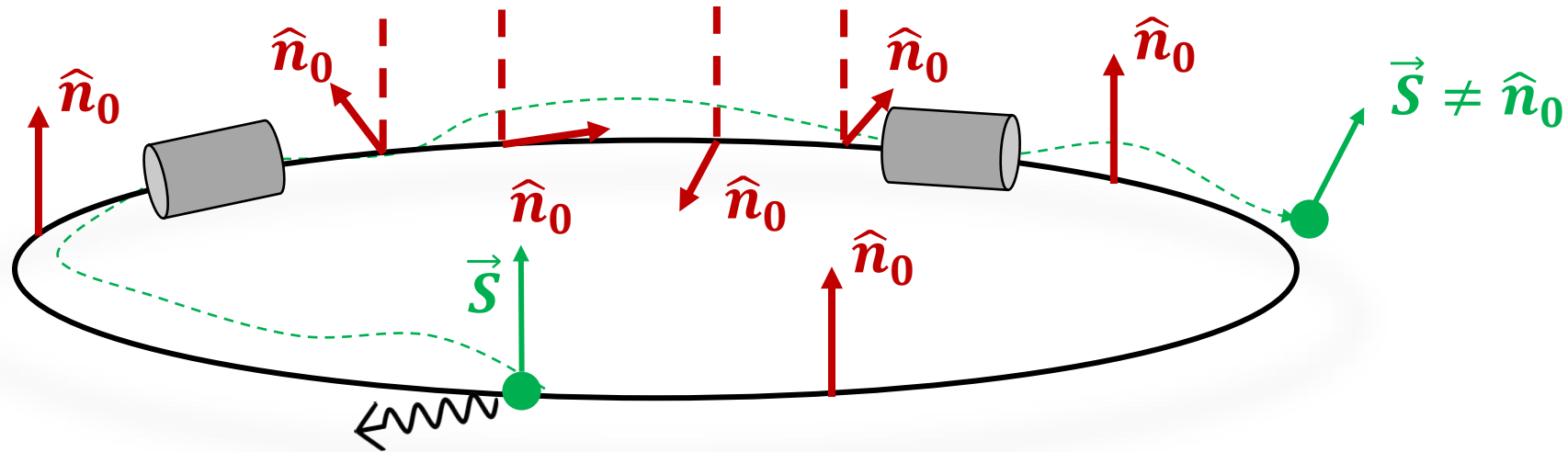


- No depolarizing effects in a perfectly planar ring



Background: Electron Spin Polarization

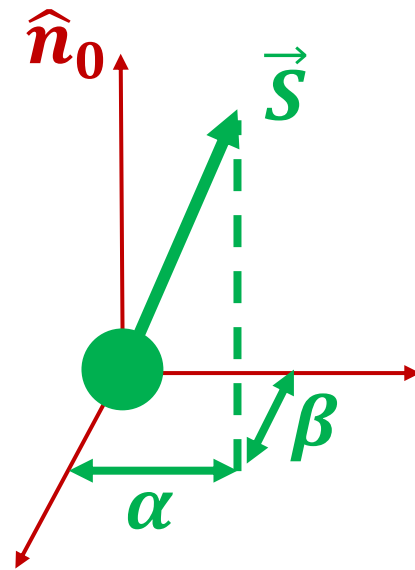
Radiative Depolarization (a simplistic picture)



- Electron stochastically sees different magnetic fields
- **Depolarizes an electron bunch!**
- How to remedy this? **Traditionally, “spin matching” is used...**

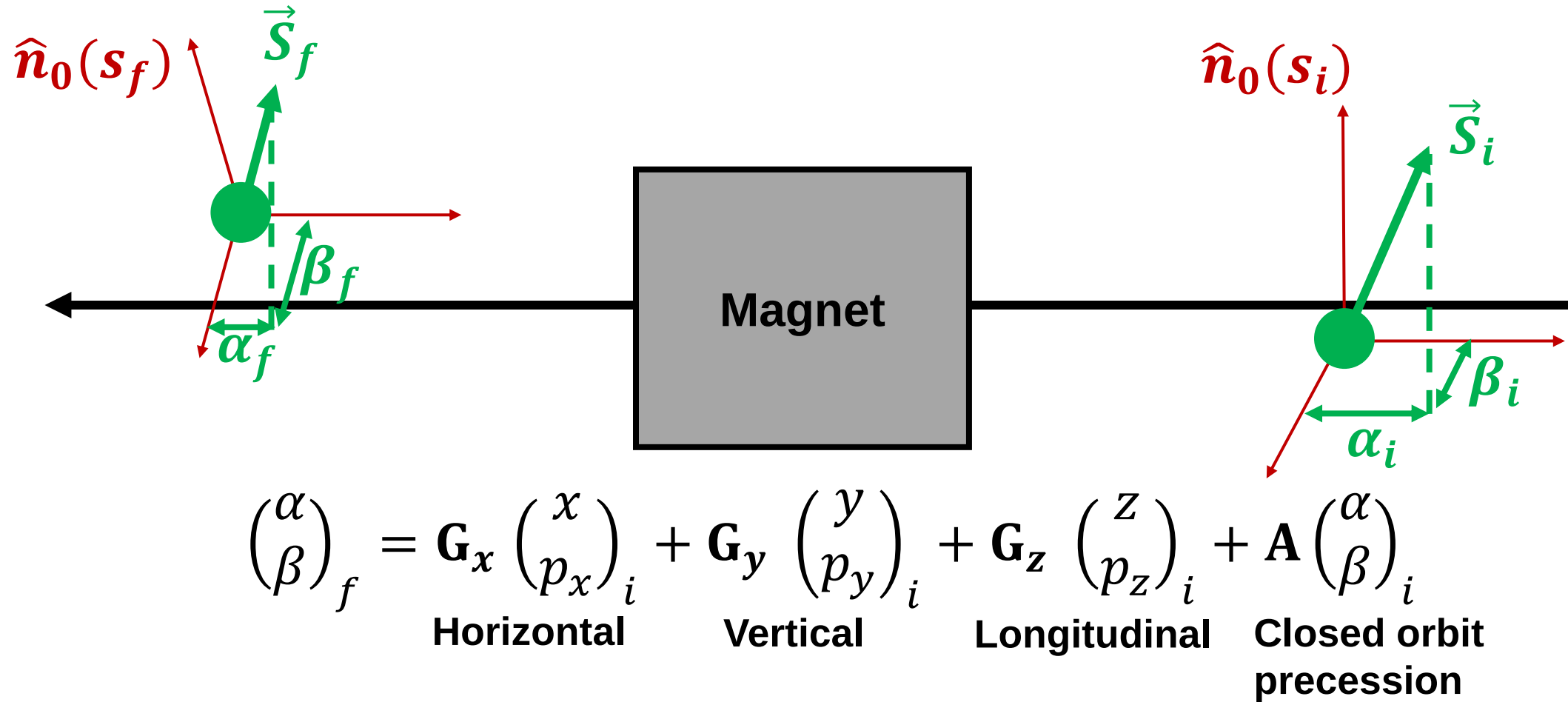
Background: Electron Spin Polarization

“Spin Matching”



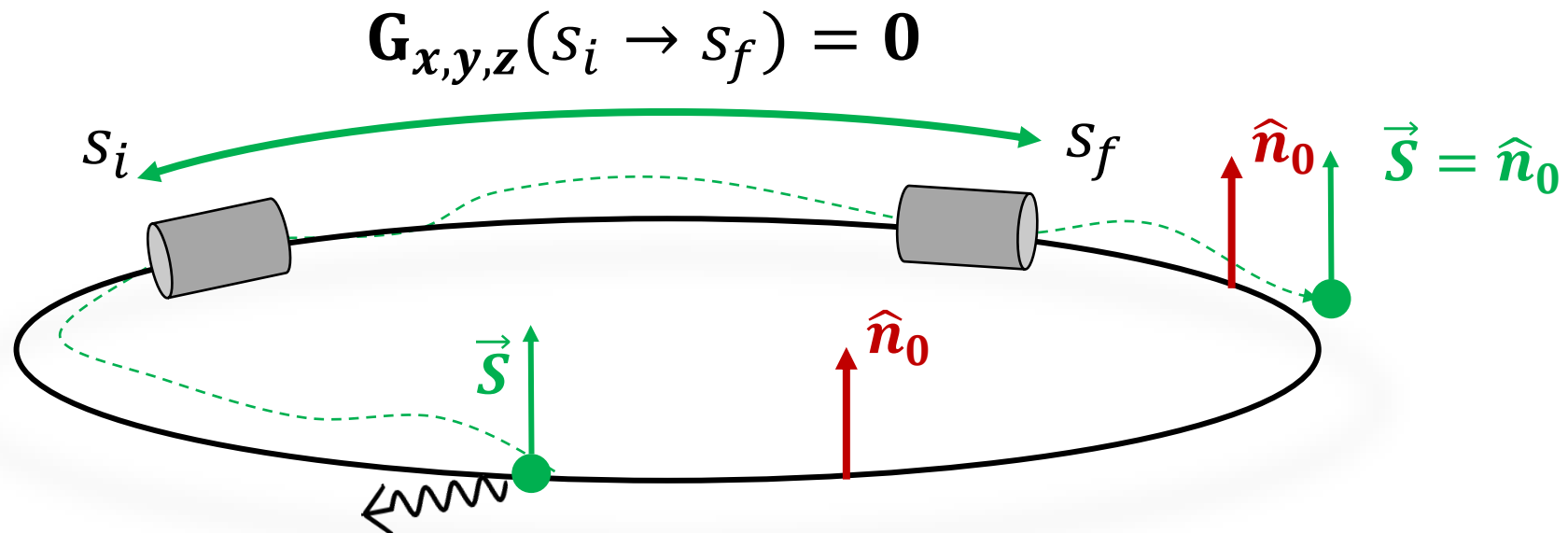
Background: Electron Spin Polarization

“Spin Matching”



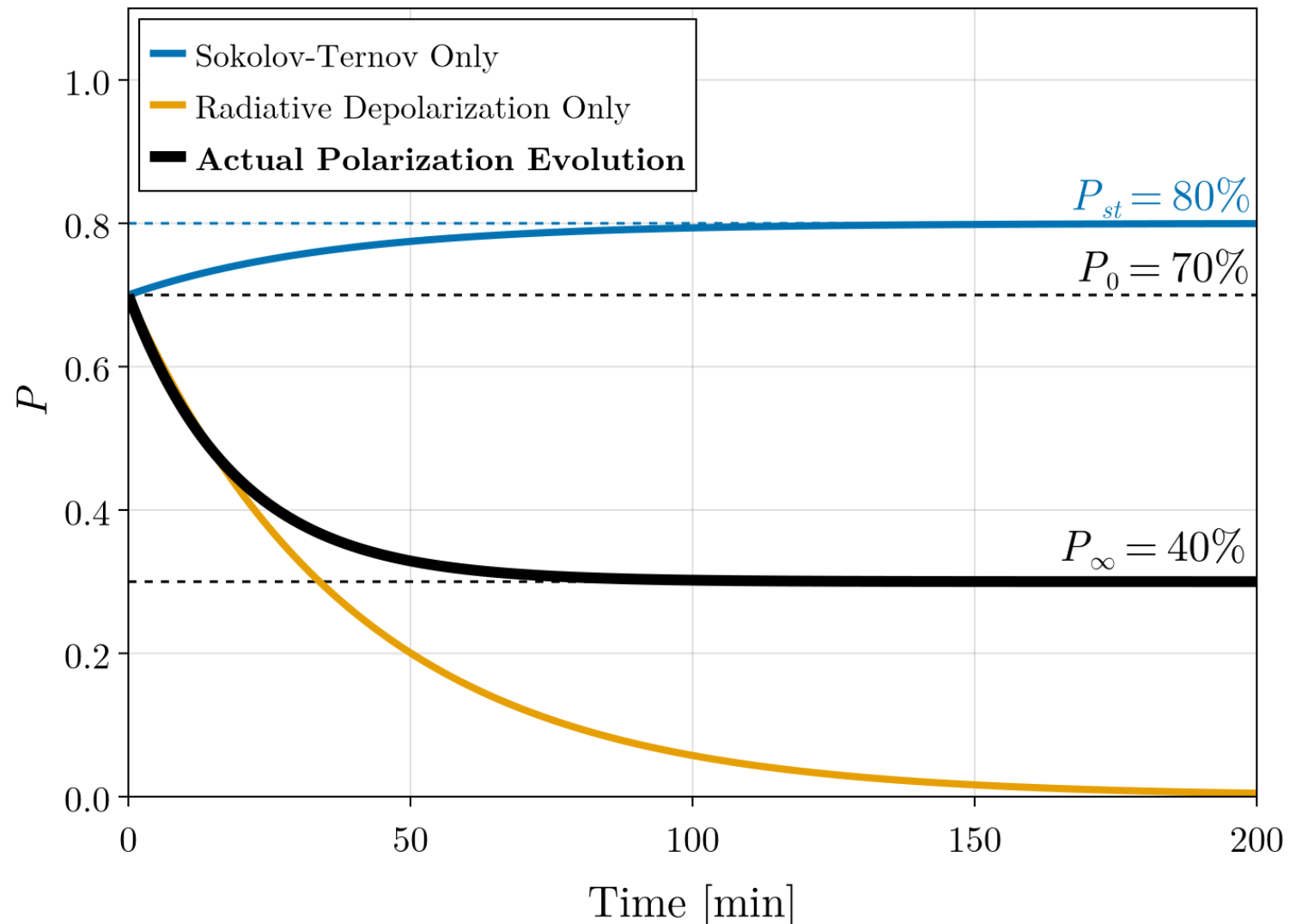
Background: Electron Spin Polarization

“Spin Matching”



Background: Electron Spin Polarization

- **Sokolov-Ternov effect**
 - Builds up polarization at a rate τ_{st}^{-1}
- **Radiative depolarization**
 - Depolarizes at a rate τ_{dep}^{-1} , where $\tau_{dep}^{-1} \gg \tau_{st}^{-1}$
- **Both effects balance out!**
 - Builds up to P_∞ at a rate $\tau_{eq}^{-1} = \tau_{dep}^{-1} + \tau_{st}^{-1}$



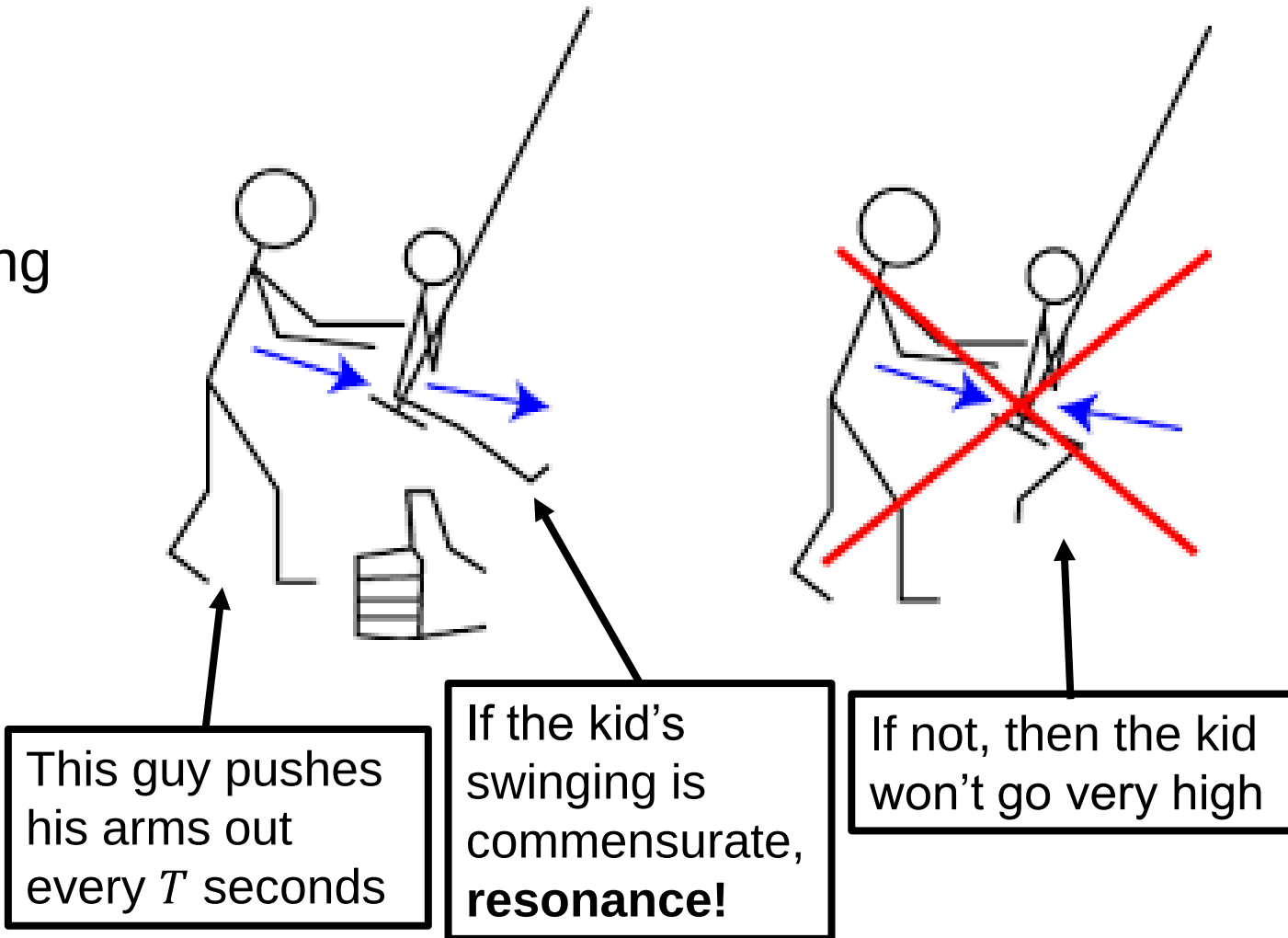
Background: Resonances

- **We want to avoid resonances!**
 - Beam would “blow up”!
- **“Tune”**: # of oscillations/turn in a ring
 - $Q_x \rightarrow$ Horizontal
 - $Q_y \rightarrow$ Vertical
 - $Q_z \rightarrow$ Longitudinal
 - $\nu_s \rightarrow$ Spin

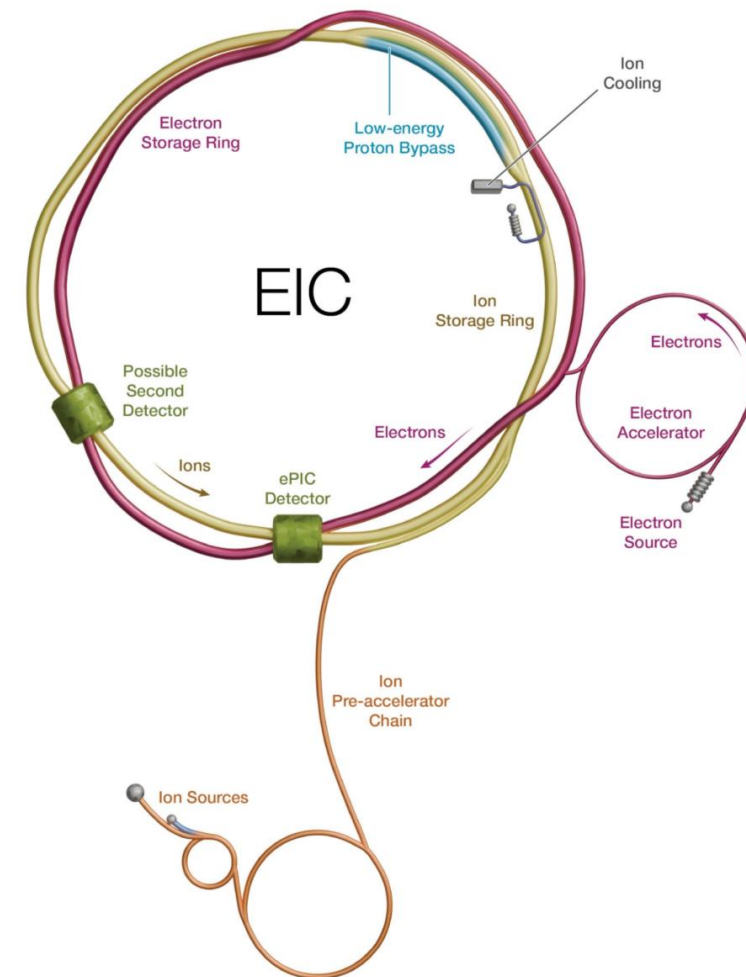
Resonance:

$$m_x Q_x + m_y Q_y + m_z Q_z + m_s \nu_s = n$$

where $m_x, m_y, m_z, m_s, n \in \mathbb{Z}$

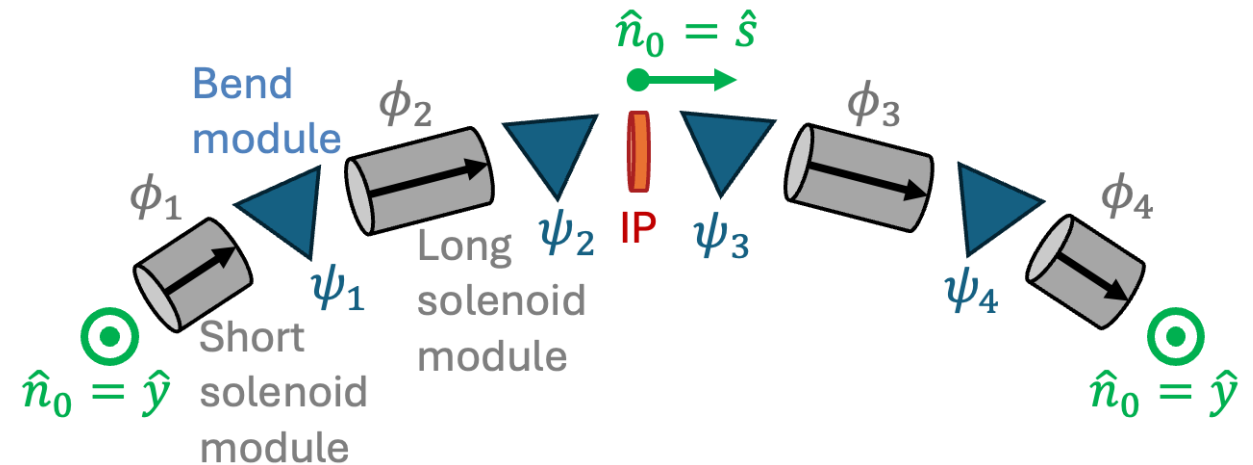


Electron Storage Ring of the EIC



Electron Storage Ring of the EIC

- Rotates $\hat{n}_0$ to longitudinal at interaction point(s) for ~ 5 , ~ 9 , and ~ 18 GeV beams
- Exact energies chosen to avoid **spin resonances**, specifically $\nu_s = \text{integer}$
- At 5 and 18 GeV, spin tune precession across rotator is equal to that if rotator is turned off $\rightarrow \nu_s = a\gamma_0$
- For 9 GeV, this is NOT the case, so $\nu_s \neq a\gamma_0$ and 1-IP different from 2-IP¹**



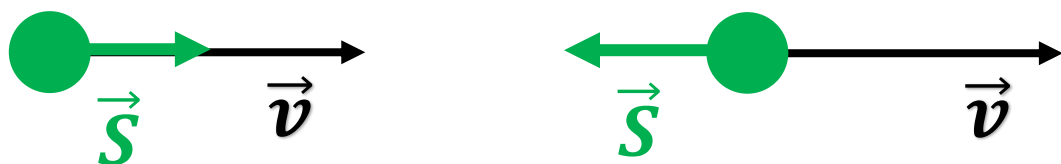
	Solenoids		Bends	
	$\phi_{1,4}$	$\phi_{2,3}$	$\psi_{1,4}$	$\psi_{2,3}$
17.846 GeV 1-IP/2-IP	0°	−90°	N/A	−90°
8.897 GeV 1-IP	−50.4°	−115.4°	−113.1°	−44.9°
8.759 GeV 2-IP	−50.4°	−113.8°	−111.4°	−44.2°
5.068 GeV 1-IP/2-IP	−90°	0°	−64.4°	−25.6°

- [M. G. Signorelli et al, “Polarization preservation methods for the Electron Storage Ring of the EIC”, MOPC83 IPAC ‘24](#)

Electron Storage Ring of the EIC

- Polarized bunches injected and **swapped out** when sufficiently depolarized

- Simultaneously store bunches with **both positive and negative helicity** at the IP



- Half of bunches polarized parallel to arc field, half anti-parallel
 - Sokolov-Ternov works FOR anti-parallel
 - Sokolov-Ternov works AGAINST parallel

$P_0 = 77.5\%$ $\langle |P| \rangle = 70\%$
 $T_{pos} = 7.83 \text{ min}$, $T_{neg} = 2.25 \text{ min}$

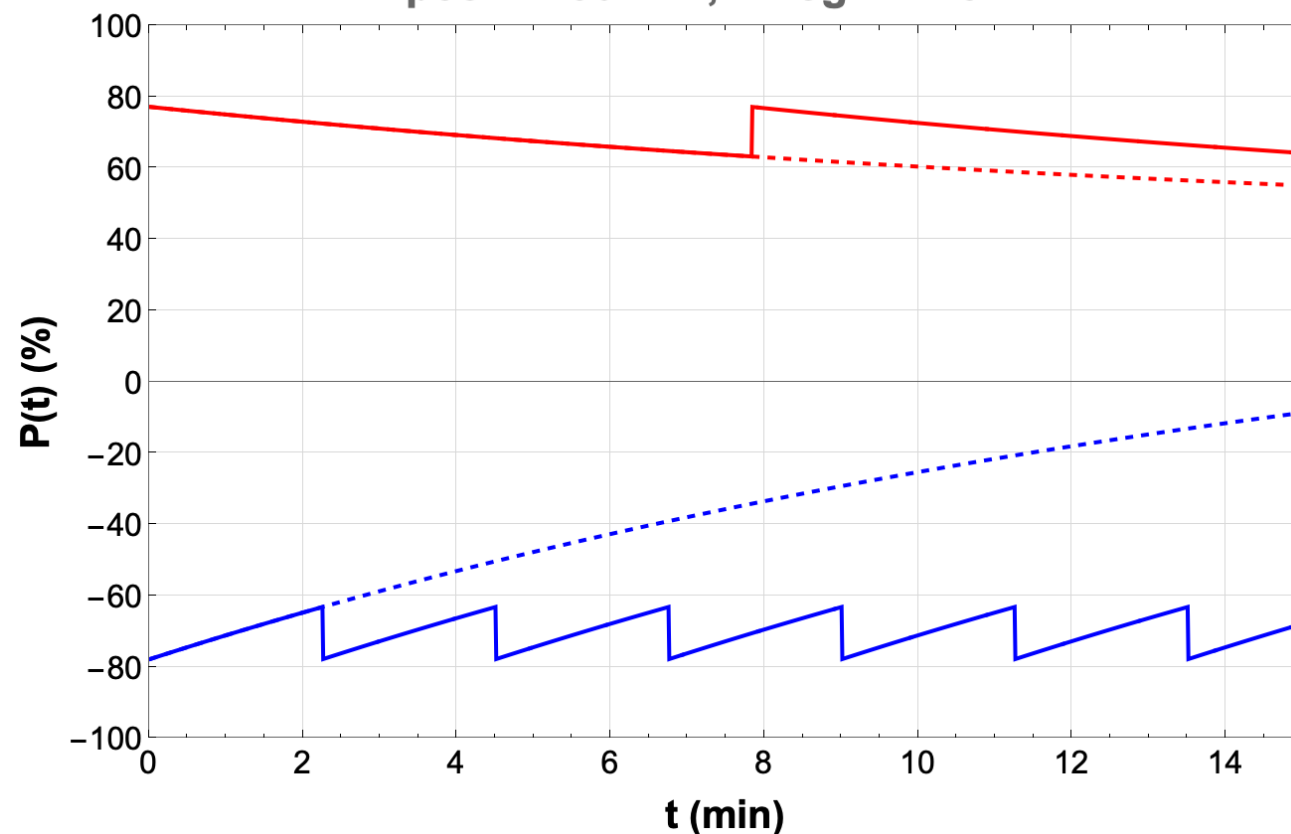
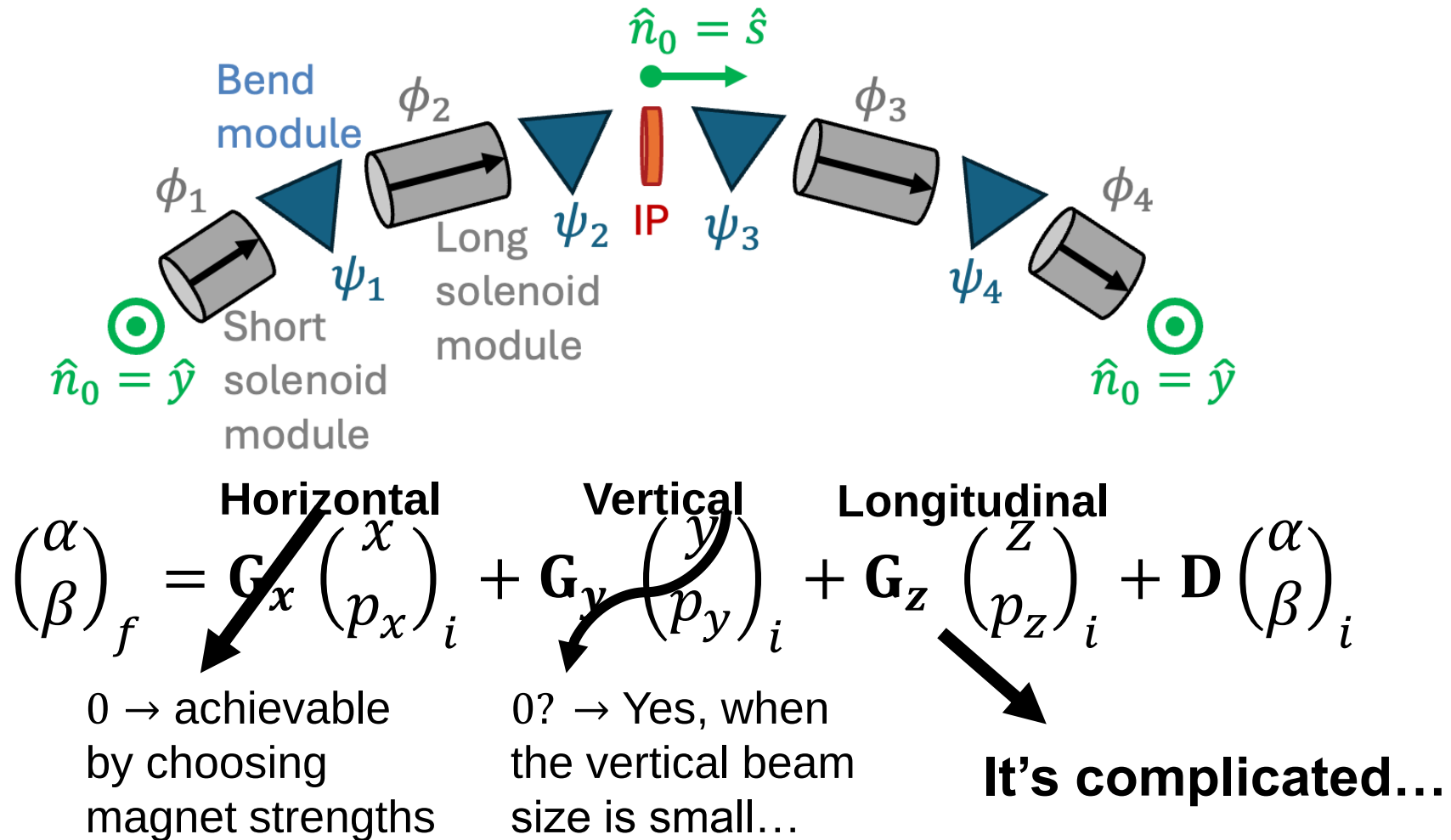


Figure credit: Christoph Montag

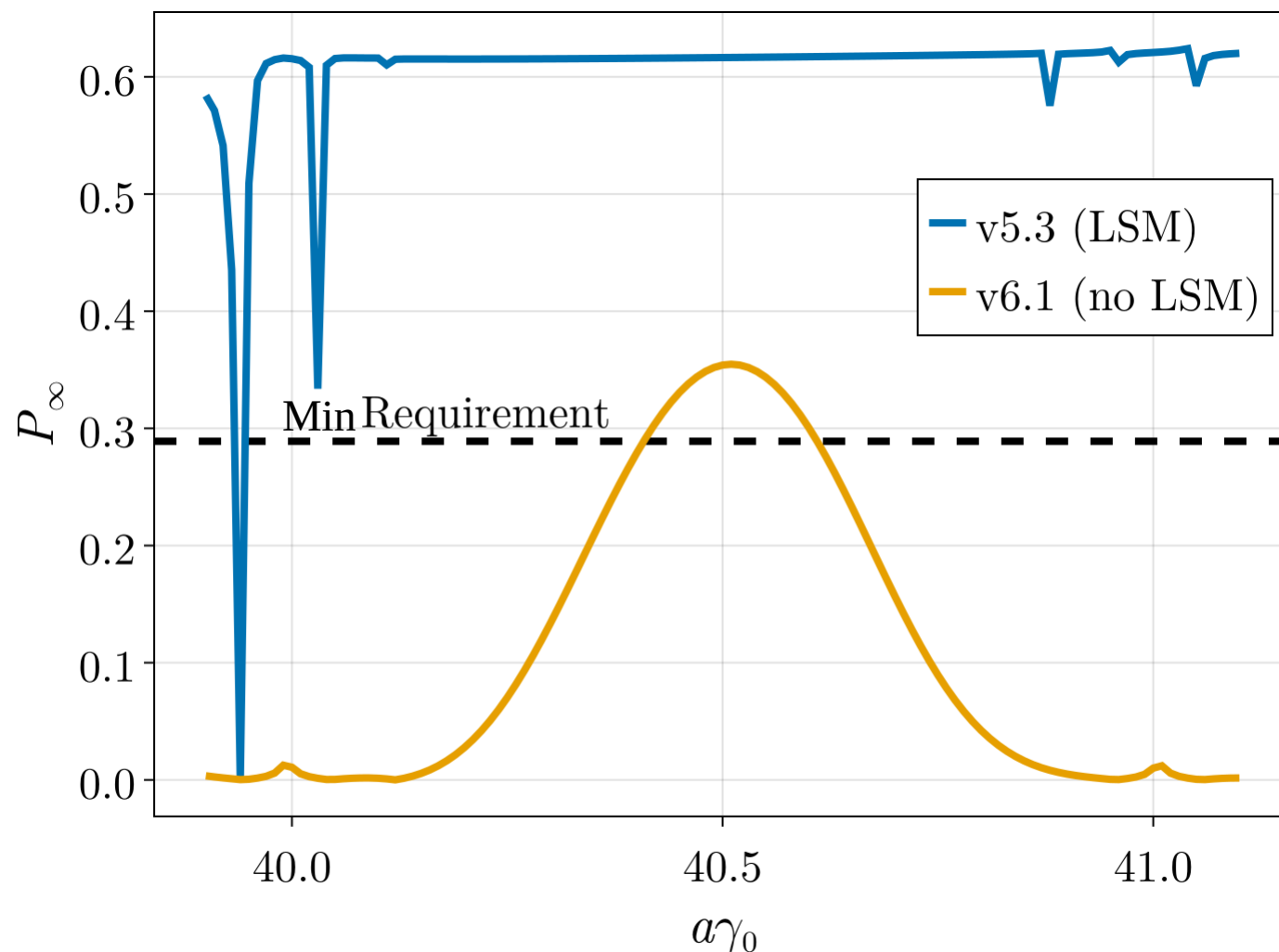
Electron Storage Ring of the EIC

- **Spin match?**



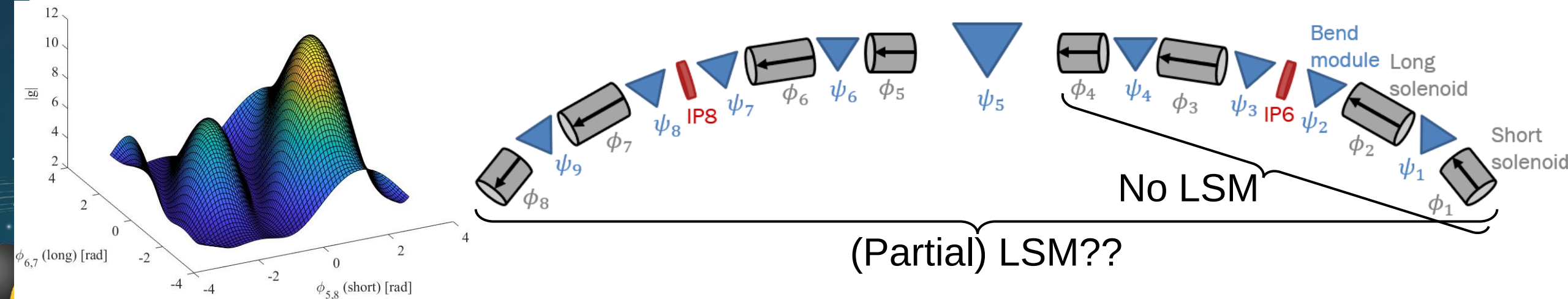
Electron Storage Ring of the EIC

- 5 and 10 GeV cases do not require a “longitudinal spin match” (LSM)
- **At 18 GeV, the effect on polarization is *significant***
- For an LSM, we need **solenoids that produce 11 T fields!**
 - Not feasible to engineer
- **A solution is needed...**



Electron Storage Ring of the EIC

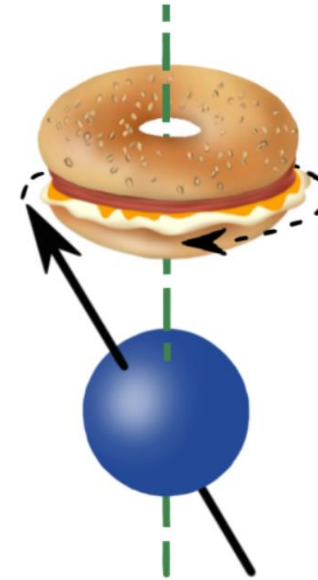
- ESR has two interaction points $\rightarrow$ two spin rotators
- Can we use the second rotator to achieve some *partial* LSM?



- In-depth study performed, but improvement only marginal³: $\Delta P_\infty \approx +2\%$
- Is there any other way to restore high polarization?

3. [M. G. Signorelli et al, "Electron Polarization Preservation in the EIC", MOPA052, IPAC '23](#)

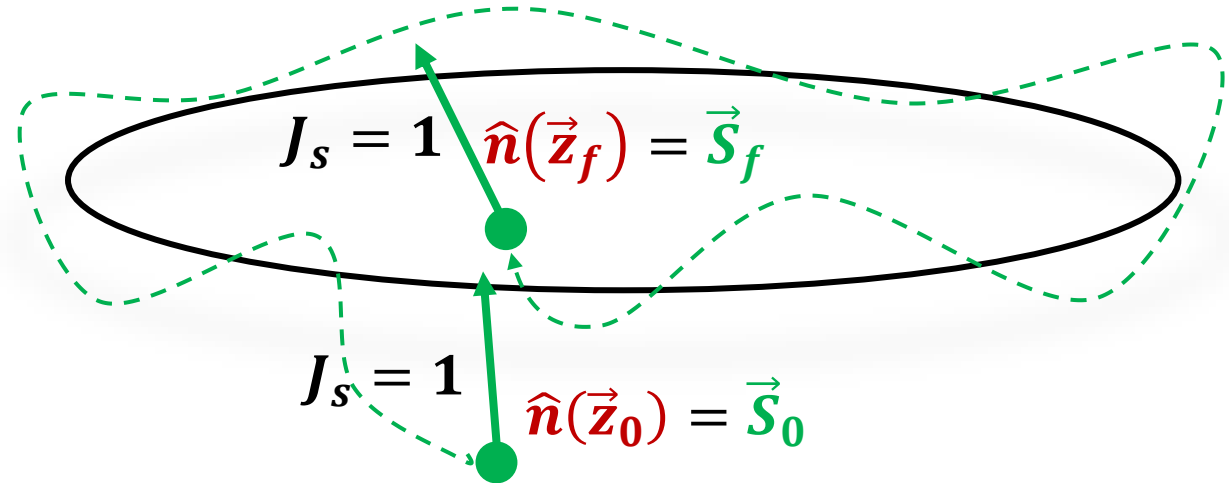
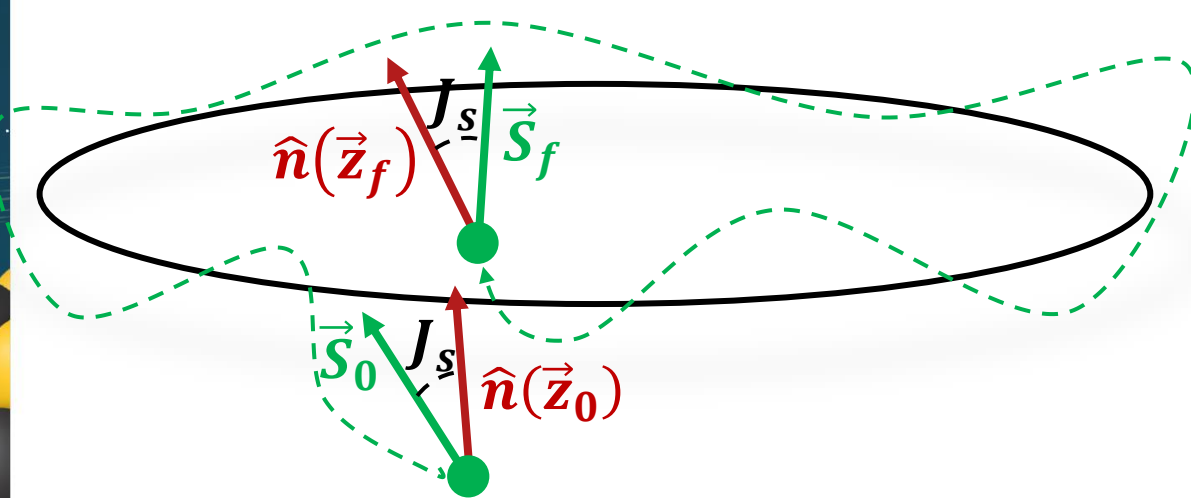
Best Adjustment Groups for Electron Spin (BAGELS)



All work published in [M. G. Signorelli and G. H. Hoffstaetter, “BAGELS for simultaneous polarization, orbit, and optics control in electron storage rings”, Phys. Rev. Accel. Beams **28**, 031002](#) , *Editor’s Suggestion*

Best Adjustment Groups for ELelectron Spin

- To best solve this, we need a deeper understanding of spin dynamics
- At every position around ring, attach a **special** spin 3-vector to each (x, p_x, y, p_y, z, p_z)
- “Invariant spin field” $\hat{n}(x, p_x, y, p_y, z, p_z)$: Projection of spin along $\hat{n}$ is **invariant!**

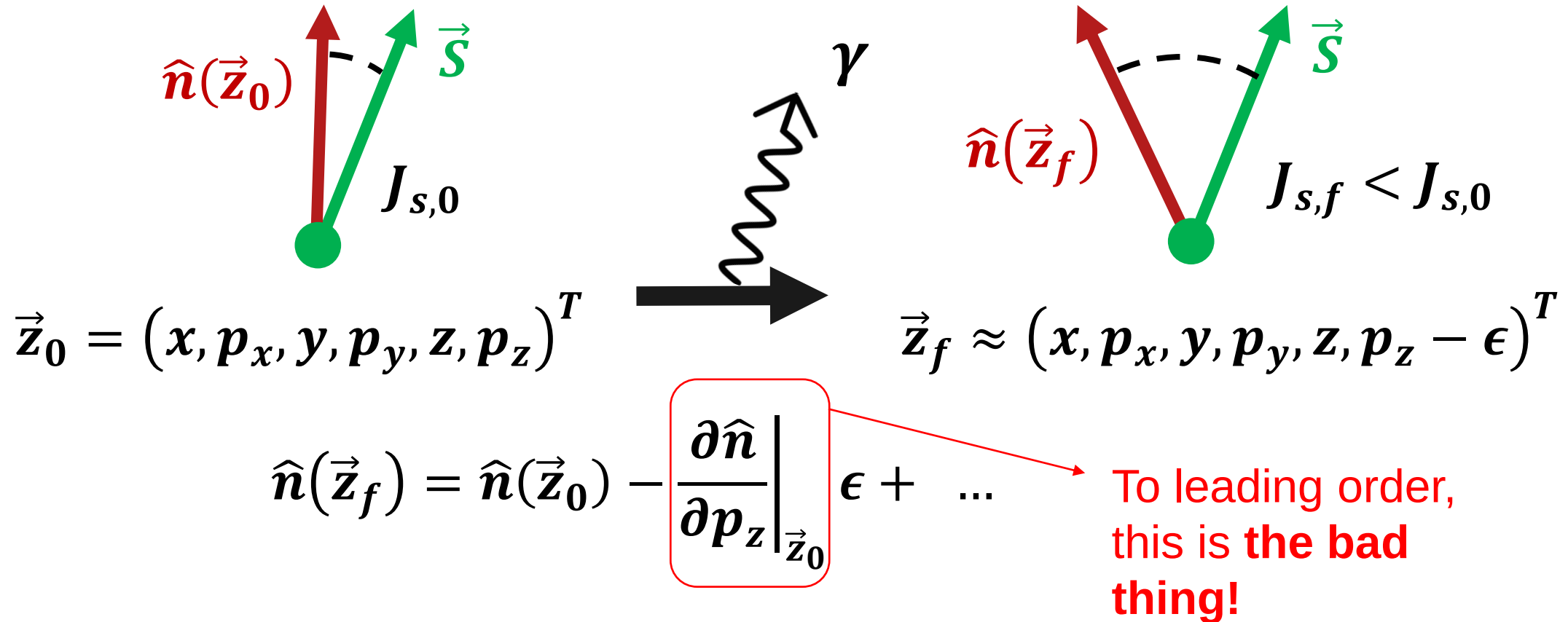


- $J_s = \hat{n} \cdot \vec{S}$ = spin action (adiabatic invariant)
- The invariant spin field (ISF) defines polarization properties of a ring

Best Adjustment Groups for ELelectron Spin

Radiative depolarization (not simplistic)

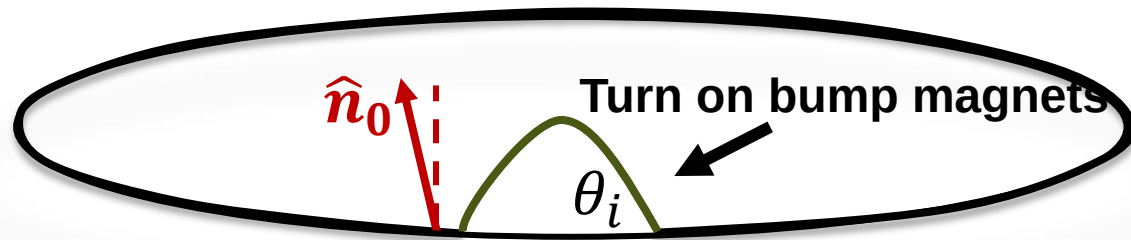
- Consider what happens to the spin action with photon emission



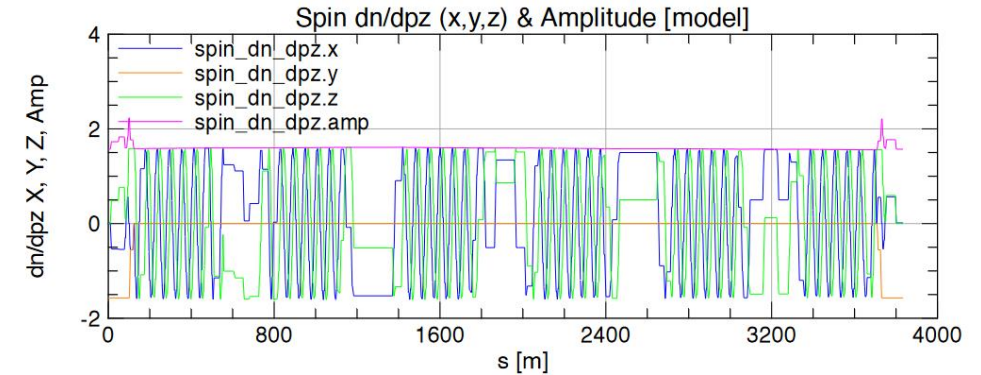
Best Adjustment Groups for EElectron Spin

- $\vec{d} = \frac{\partial \hat{n}}{\partial p_z}$ = "spin-orbit coupling function" = **the bad thing!**

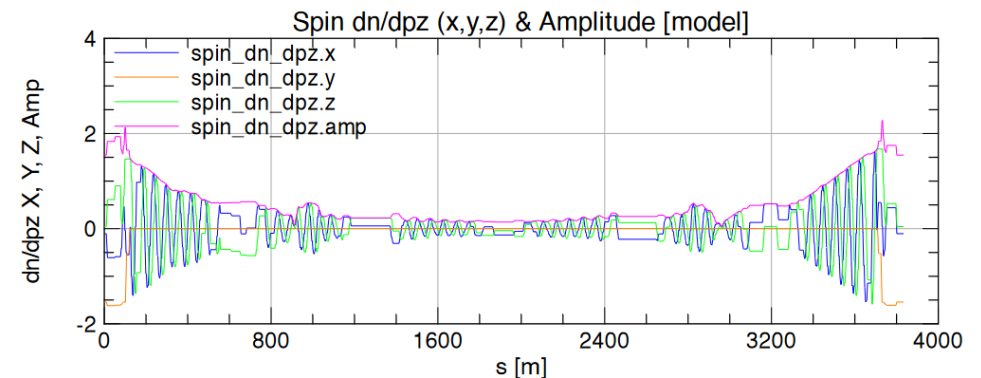
- Without a longitudinal spin match, 
- **Vertical closed orbit bumps** tilt $\hat{n}_0$, causing small impacts on $\vec{d}$ around the ring:



- On a computer, we can **optimize many (~50+)** vertical bumps to **reduce $\vec{d}$ globally !**



 **Vertical bumps + optimizer**



Best Adjustment Groups for ELelectron Spin

~~“We solved the problem! We just have to optimize 50+ bumps in the control room!”~~

- Polarization measurements are costly, ~50+ variables will take way too long

How can we reduce the number of knobs to turn in the control room?

Consider how $\vec{d}$ at m positions around ring depends on all n bumps (expand in $\vec{\theta}$):

$$\begin{pmatrix} \vec{d}_1 \\ \vdots \\ \vec{d}_m \end{pmatrix} \approx \begin{pmatrix} \vec{d}_1 \\ \vdots \\ \vec{d}_m \end{pmatrix}_0 + \underbrace{R_{\vec{d}}}_{\text{“Linear response matrix”}} \begin{pmatrix} \theta_1 \\ \vdots \\ \theta_n \end{pmatrix} + \dots, \quad R_{\vec{d}} = \begin{pmatrix} \frac{\partial \vec{d}_1}{\partial \theta_1} & \dots & \frac{\partial \vec{d}_1}{\partial \theta_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial \vec{d}_m}{\partial \theta_1} & \dots & \frac{\partial \vec{d}_m}{\partial \theta_n} \end{pmatrix}$$

Which $\vec{\theta}'$ s make this term really big?

Seek orthogonal, linear combinations of bumps $\vec{\theta}$ that maximize $|\mathbf{R}_{\vec{d}} \vec{\theta}|^2$

- Each linear combination is a knob that maximally impacts polarization

Best Adjustment Groups for EElectron Spin

$$\operatorname{argmax}_{\vec{\theta}_1, \dots, \vec{\theta}_k} \frac{|\mathbf{R}_{\vec{d}} \vec{\theta}|^2}{|\vec{\theta}|^2}$$



Lagrange
multipliers



$$(\mathbf{R}_{\vec{d}}^T \mathbf{R}_{\vec{d}}) \vec{\theta} = \lambda \vec{\theta}$$

First k eigenvectors with
largest eigenvalues

- Instead of ~50+ knobs, **only k most effective, orthogonal groups of bumps!**
- Also want **minimal impacts** on other things, e.g. the orbit $y \rightarrow |\mathbf{R}_y \vec{\theta}|^2$ small

$$\operatorname{argmax}_{\vec{\theta}_1, \dots, \vec{\theta}_k} \frac{|\mathbf{R}_{\vec{d}} \vec{\theta}|^2}{|\mathbf{R}_y \vec{\theta}|^2}$$



Lagrange
multipliers



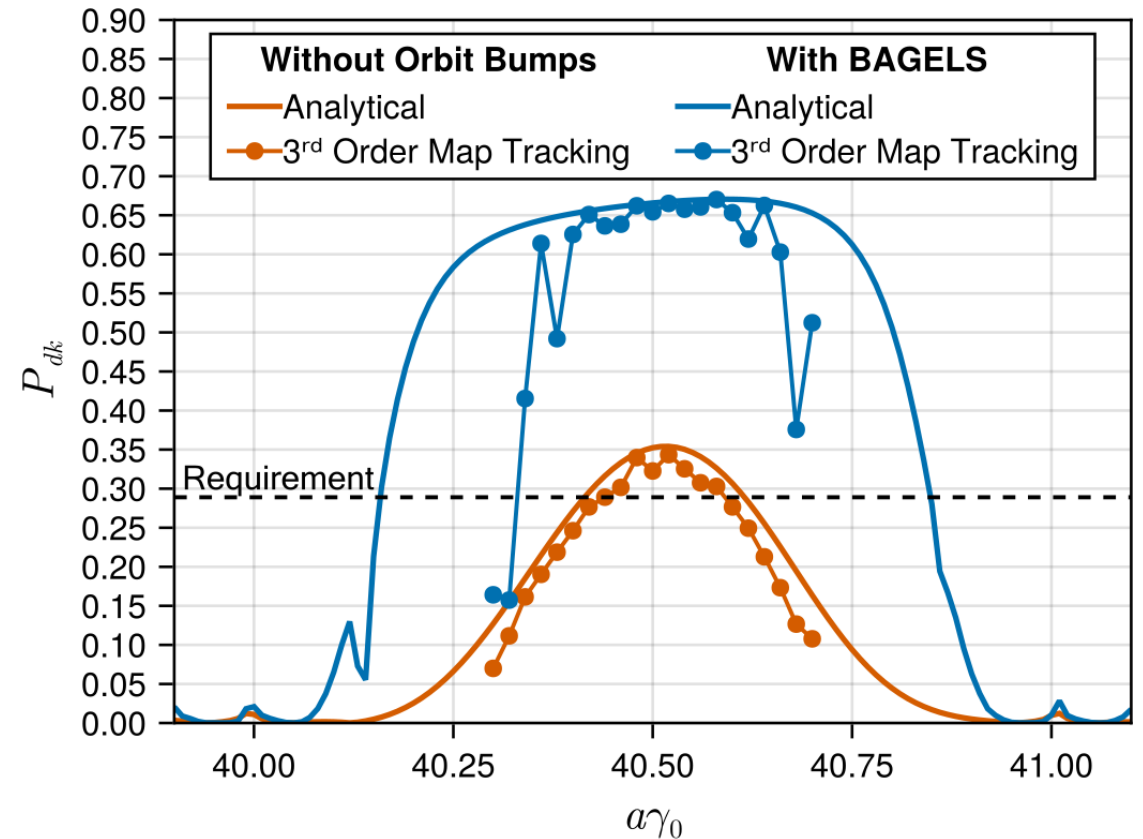
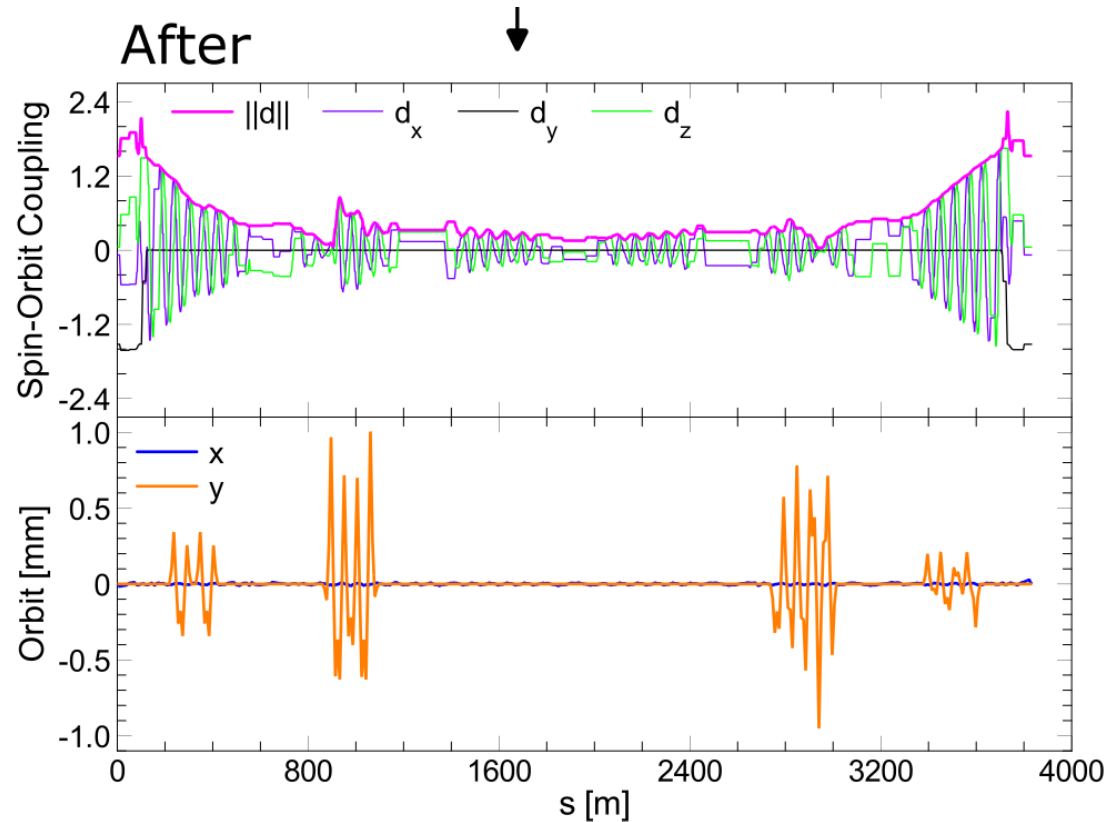
$$(\mathbf{R}_{\vec{d}}^T \mathbf{R}_{\vec{d}}) \vec{\theta} = \lambda (\mathbf{R}_y^T \mathbf{R}_y) \vec{\theta}$$

First k *generalized eigenvectors*
with largest eigenvalues

- These bump groups have **maximal impacts on spin, minimal on the orbit**
- **They are the Best Adjustment Groups for EElectron Spin!**

BAGELS: 1-IP 18 GeV ESR

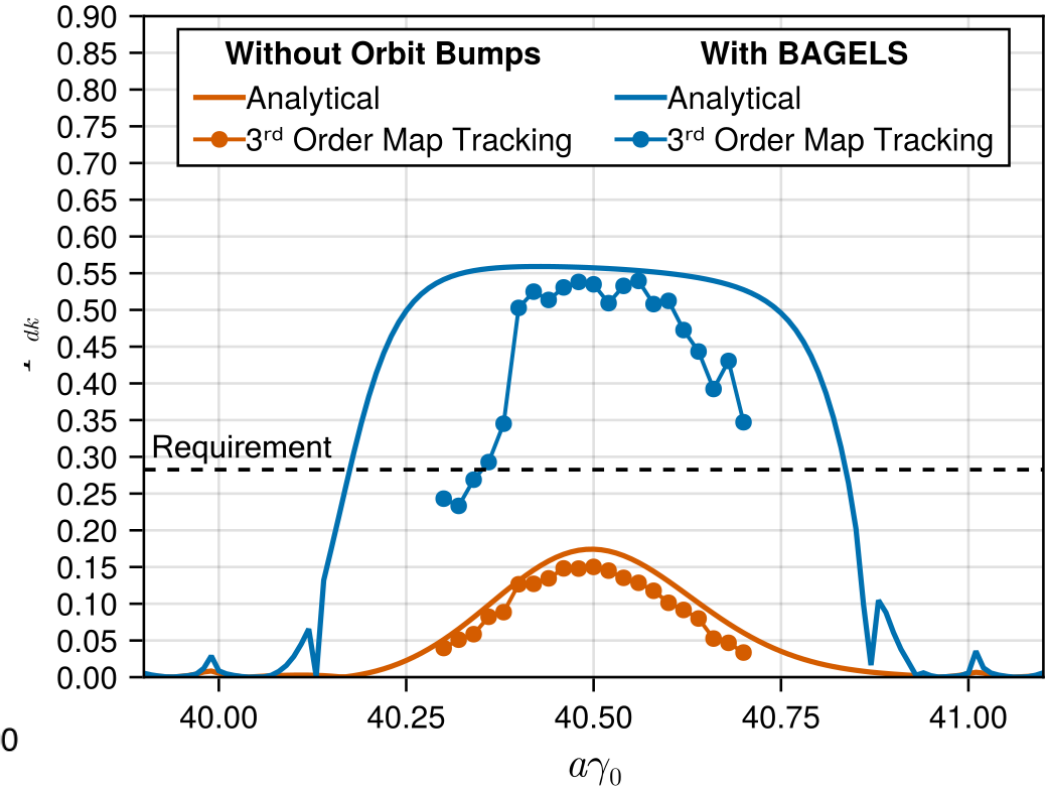
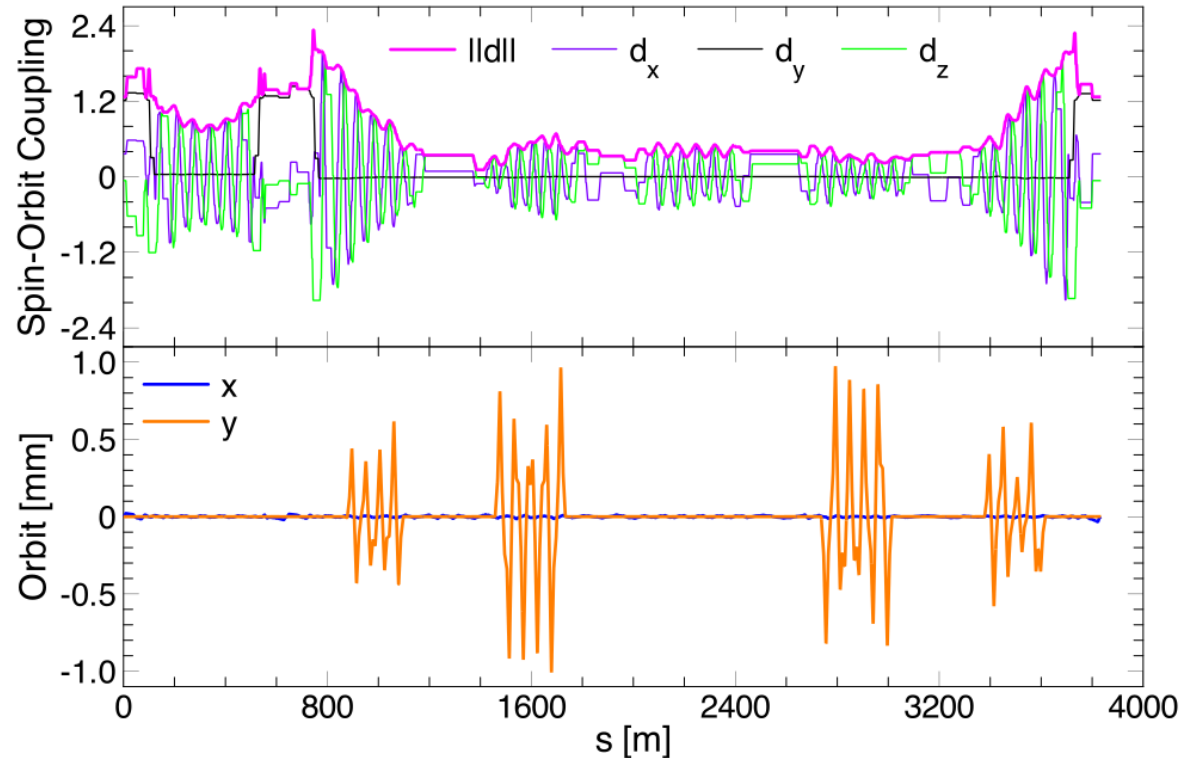
- Turning only 4 BAGELS knobs in the 1-IP 18 GeV ESR:
 - “Analytical” = first-order, Map Tracking = Monte Carlo simulation (nonlinear)



- Max orbit < 1mm everywhere, P_{∞} nearly doubled!

BAGELS: 2-IP 18 GeV ESR

- Turning only 4 BAGELS knobs in the 2-IP 18 GeV ESR:



- Max orbit < 1mm everywhere, P_∞ more than tripled!
- BAGELS saves the polarization without a longitudinal spin match!

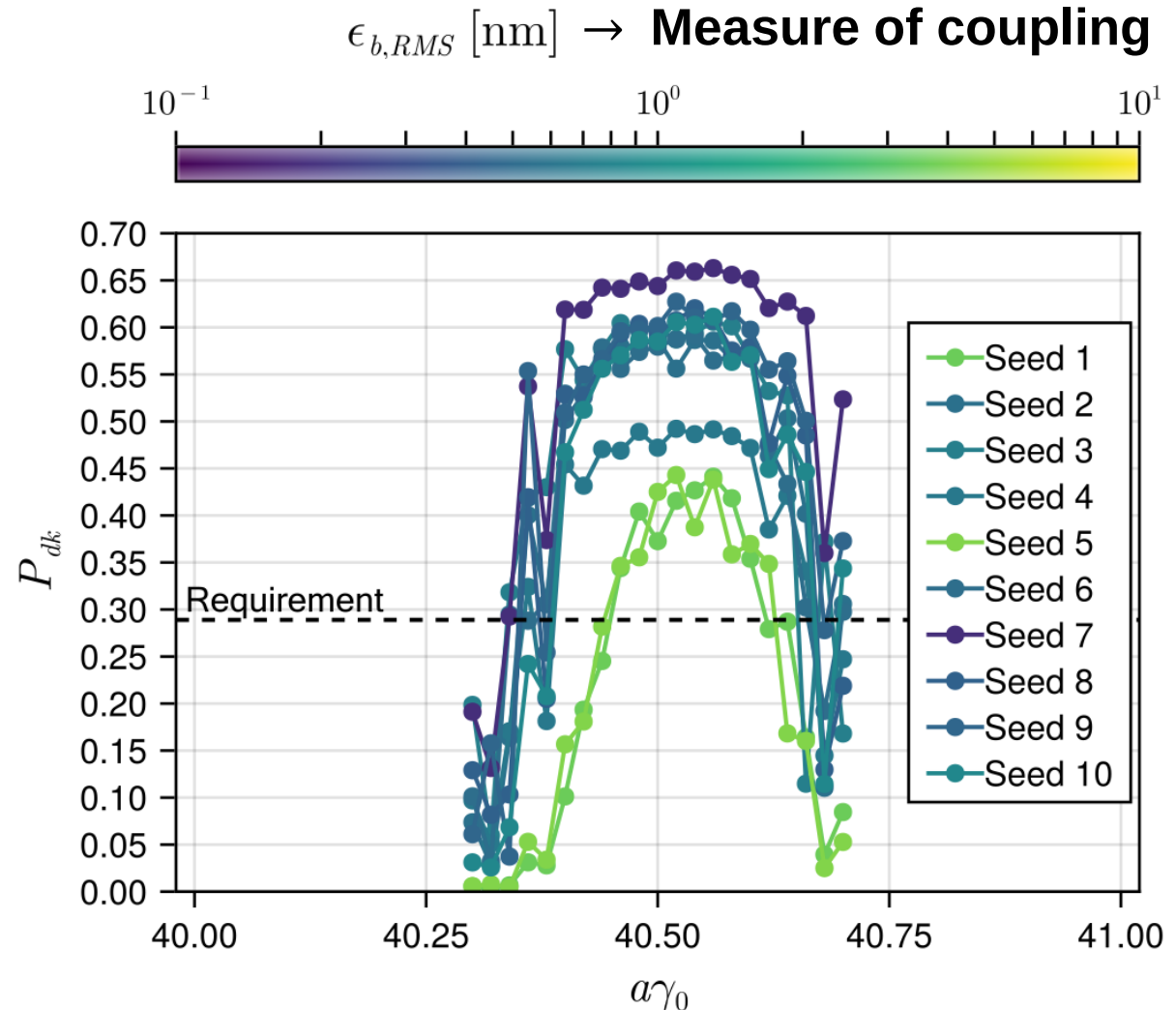
BAGELS: Polarization-Safe Error Correction

We can also use BAGELS to do something else and **NOT** affect polarization

- E.g., magnet errors will **couple horizontal & vertical motion** → requires correction

$$\arg \max_{\theta_1, \dots, \theta_k} \frac{||R_A \theta||^2}{||R_B \theta||^2} \quad R_A = \text{coupling matrix} \\ R_B = \text{orbit } \mathbf{y} \text{ AND } \vec{d}$$

- 4 BAGELS coupling correction knobs**
 - Minimal impacts to polarization + orbit
- Tested on 10 random 1-IP ESR lattices**
- All exceed polarization requirements!**
 - Excellent linear coupling control
 - 2 green seeds show *nonlinear coupling* → outside of BAGELS's current scope



BAGELS: Polarization-Safe Beam Size Match

What else can BAGELS do?

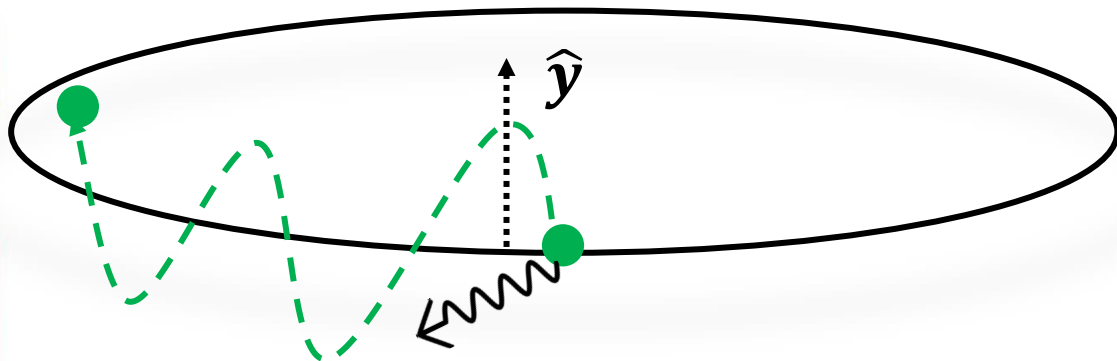
- Remember the vertical spin match in the ESR? →
- Electron vertical beam size σ_y is naturally $\sim$ zero, so we ignored this

$$+ G_y \left(\frac{y}{p_y} \right)_i + G$$

0? → Yes, when the vertical beam size is small...

To maximize collisions with the ions, we must increase σ_y to match the ion beam

- One way is to create **vertical dispersion** η_y



- **Vertical orbit excited with photon emission**
- Can be made using vertical orbit bumps
- **Will damage polarization if not careful**
- **Must have $\eta_y = \eta_y' = 0$ at the IP**

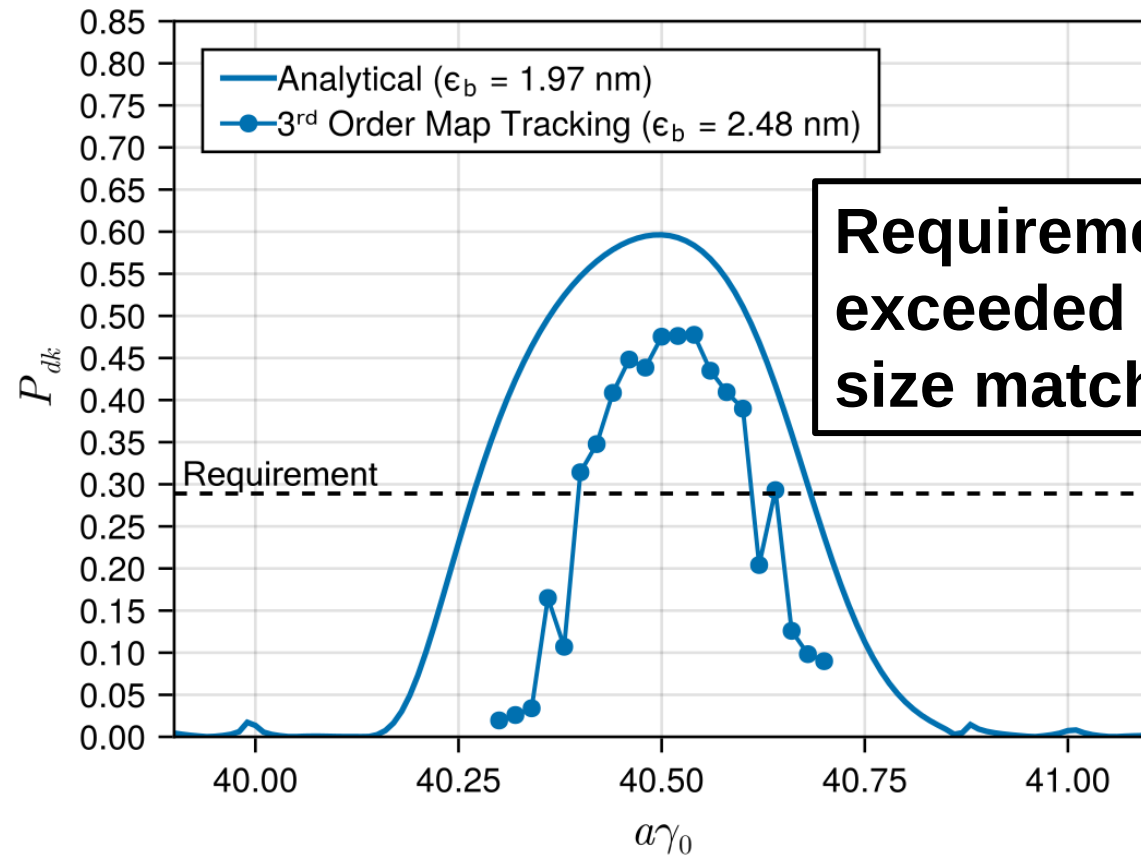
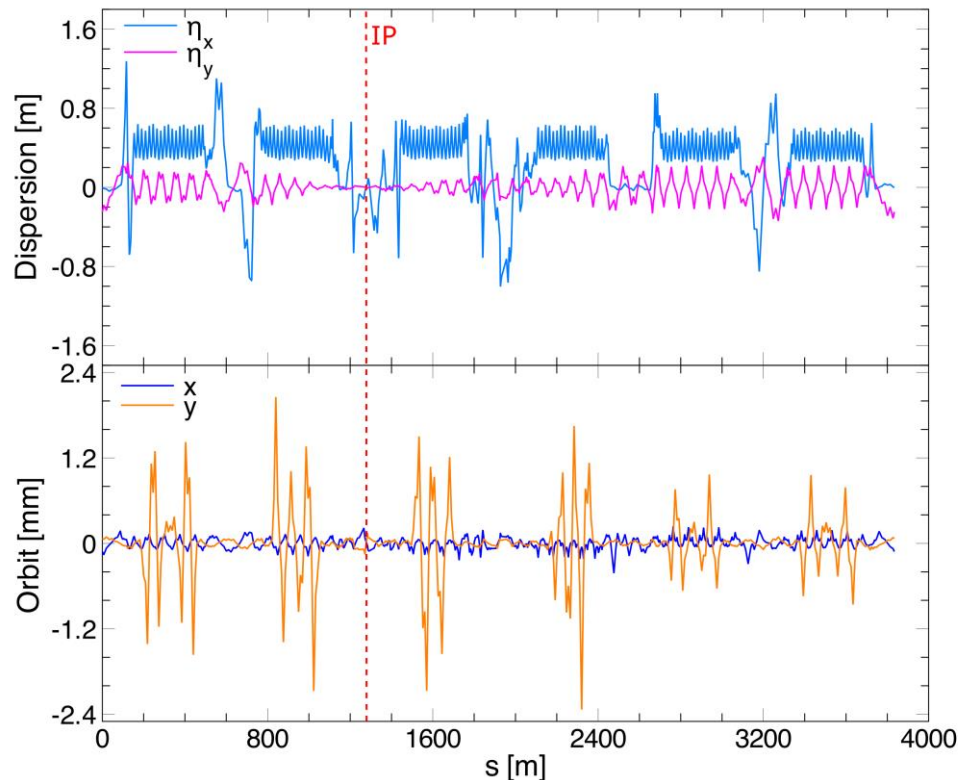
BAGELS: Polarization-Safe Beam Size Match

Let's apply BAGELS here:

$$\arg \max_{\theta_1, \dots, \theta_k} \frac{||R_A \theta||^2}{||R_B \theta||^2} \quad R_A = \text{vertical dispersion } \eta_y \text{ around the ring}$$

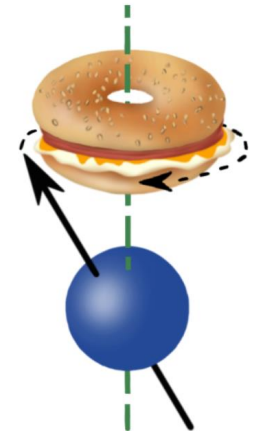
$R_B = \text{orbit } y, \quad \text{spin-orbit coupling } \vec{d}, \quad \eta_y, \eta_y' \text{ at the IP}$

- Obtain **1 BAGELS polarization-safe beam-size matching knob**, turn to



Conclusions

- EIC-ESR will provide 5, 9, 18 GeV spin-polarized electrons
- Without longitudinal spin match, 18 GeV polarization is marginal, but...
- Novel Best Adjustment Groups for ELelectron Spin (BAGELS) method:
 - ✓ Nearly doubles P_{∞} in the 1-IP
 - ✓ More than triples P_{∞} in the 2-IP
 - ✓ Provides knobs for **correcting polarization in real life**
 - ✓ Provides **polarization-safe knobs** to correct **coupling** due to errors
 - ✓ Provides **polarization-safe knobs** for **beam-size matching**



New Software: SciBmad

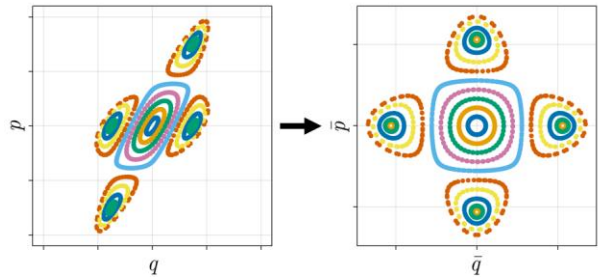
- New, **CPU/GPU-accelerated**, forwards-/ backwards-/ **Taylor-differentiable** accelerator physics code



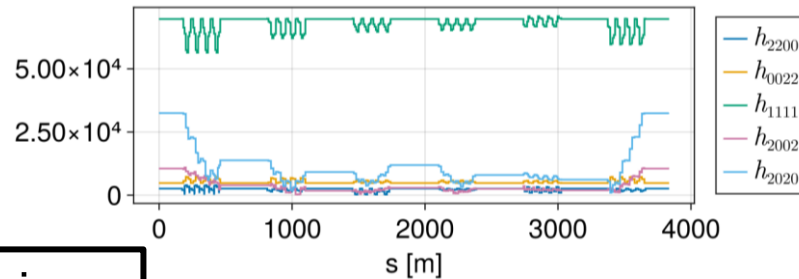
Cornell University

- **Full suite of nonlinear analysis tools for design, e.g:**

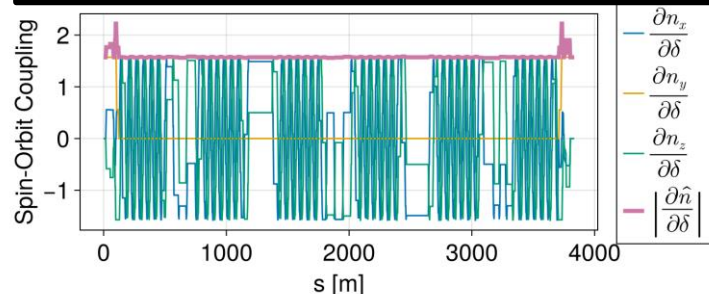
One resonance normal form



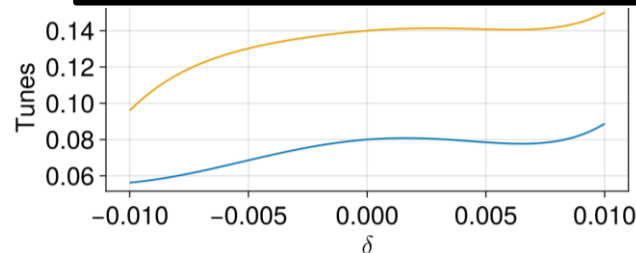
n^{th} order resonance driving terms / detune coefficients



Invariant spin field expansions



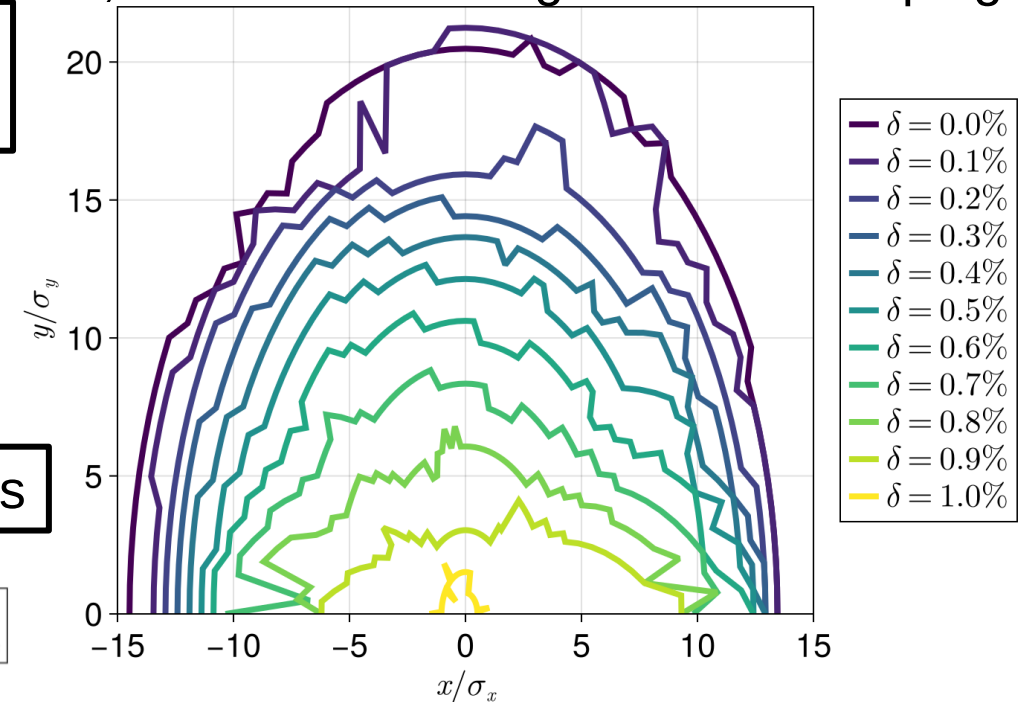
Amplitude-dependent tunes



And super fast symplectic tracking:

18 GeV EIC-Electron Storage Ring DA

- 6,271 lattice elements
- 20,746 particles (64-bit precision)
- 2,000 turns including radiation damping



8.2 minutes on one GPU

New Software: SciBmad

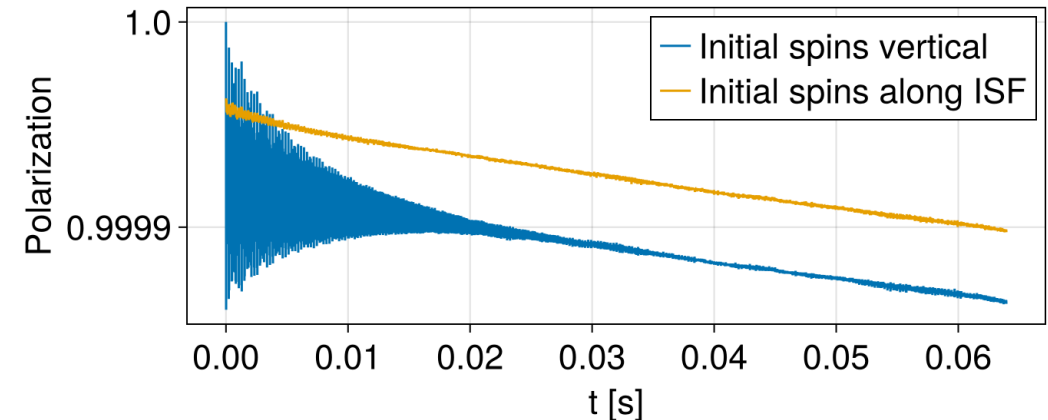
- Parallelized over **particles AND parameters (“batch” parameters)**:
 - E.g. set a sextupole strength to be 100,000 different strengths:
 - **Currently being used for dynamic aperture optimization in the EIC-ESR**
- **Arbitrary time-dependent parameters**
 - E.g., an AC-kicker (perhaps for RDP)
- **18 GeV EIC-ESR Radiative Depolarization**
 - 6,271 lattice elements
 - 25,000 particles (64-bit precision)
 - 5,000 turns including radiation

```
using CUDA  
s=Sextupole(Kn2L=BatchParam(CUDA.rand(100000)))
```

100,000 different lattices on one GPU!

```
v = VKicker(L=0.5, Ks0=1e-4*sin(1e6*Time()))
```

Also GPU-parallelized!

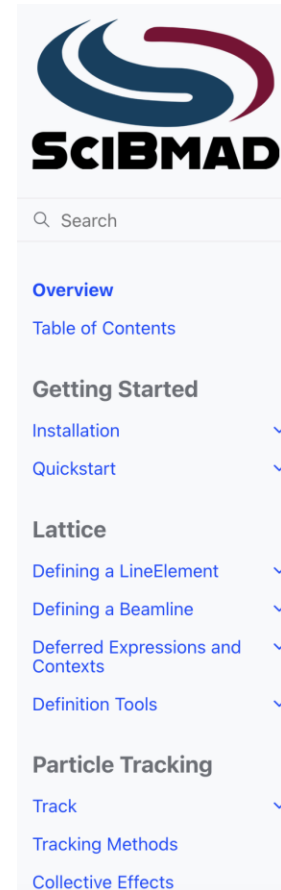


36.8 minutes on one GPU

New Software: SciBmad



- SciBmad is fully documented and in a production-ready state
 - <https://bmad-sim.github.io/SciBmad.jl/>
- Project status is kept up-to-date
- Currently being used in EIC design



SciBmad

docs stable

[Installation](#)

[Quickstart Guide](#)

Overview

SciBmad is a new open source, high-performance, CPU/GPU compatible, polymorphic, and forwards-/backwards-/Taylor-differentiable accelerator physics simulation ecosystem.

Project Status

SciBmad development is progressing rapidly. Features included in the current state of the project (0.5.0) and currently being planned for later releases are:

Current Release 0.5

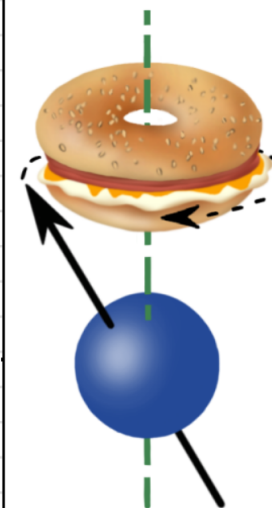
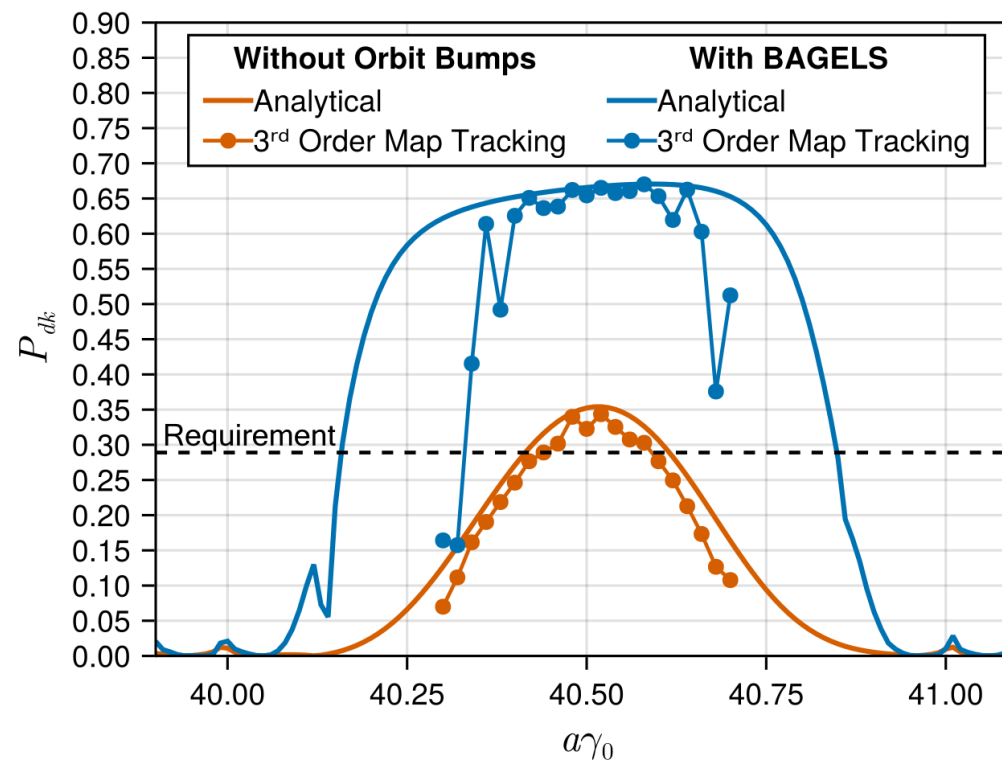
- CPU/GPU parallelized 6D symplectic particle tracking including spin and radiation
- Fully forwards-/backwards-/Taylor differentiable to extract gradients w.r.t. anything
- Taylor series nonlinear normal form analysis (i.e. nonlinear periodic Twiss functions) including spin and radiation
- Arbitrary time-dependent accelerator parameters (e.g. magnet strengths, reference energy, misalignments)
- CPU/GPU parallelized tracking and analysis over differing accelerator parameters (batch parameter evaluation)
- Arbitrarily-interdependent accelerator parameters with lazily-evaluated deferred expressions
- CPU/GPU parallelized dynamic aperture scans

Acknowledgements

- Excited to have contributed significantly to the EIC during my PhD
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 - Oleksii Beznozov
 - Laurent Deniau
 - Joe Devlin
 - Cornell ERL/EIC Group
 - All SciBmad contributors
 - And the rest of the EIC team

Thank you!

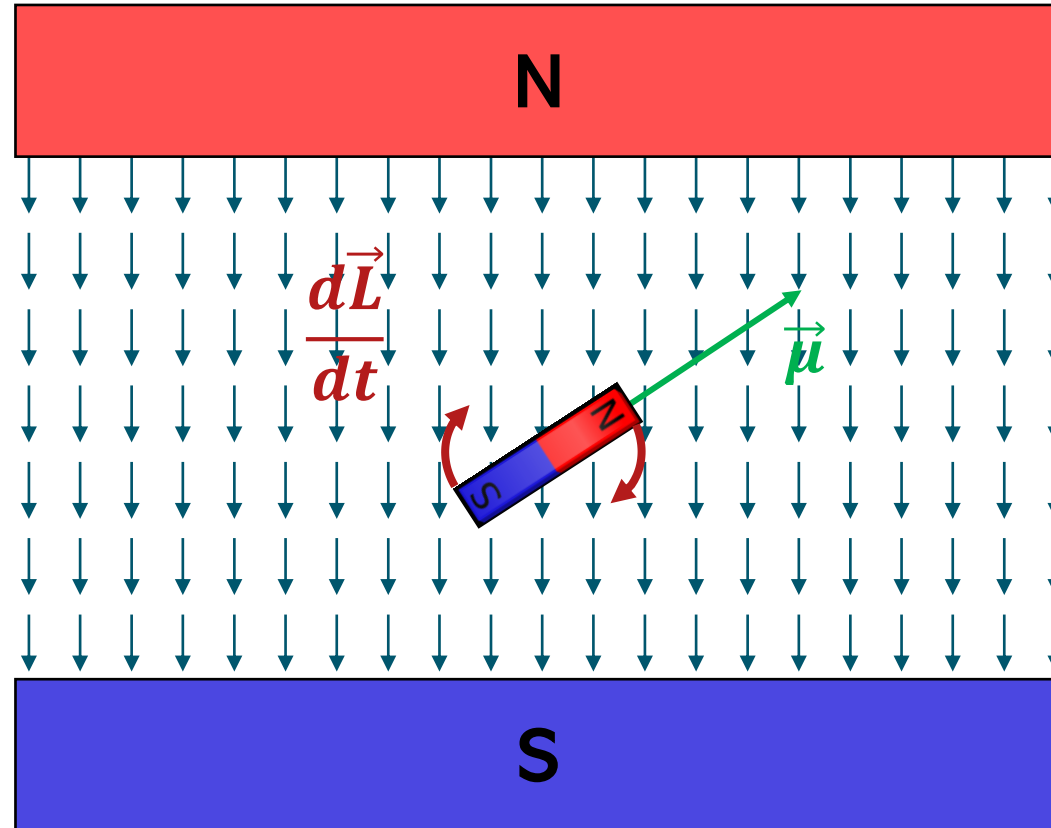
Questions?



Backup Slides

$$\frac{d\vec{L}}{dt} = \vec{\mu} \times \vec{B}$$

Magnet will rotate



$\vec{\mu}$ = magnetic moment

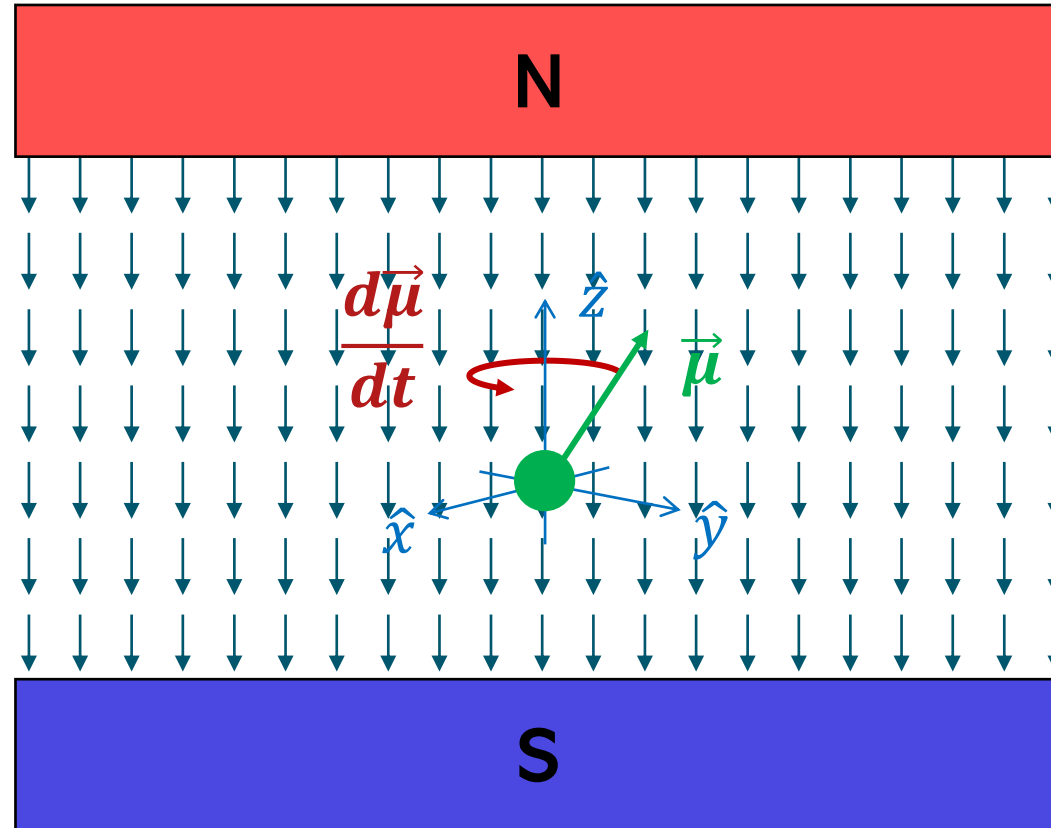
Backup Slides

$$\frac{d\vec{L}}{dt} = \vec{\mu} \times \vec{B}$$

For single particles,

$$\vec{L} \propto \vec{\mu}$$

$$\frac{d\vec{\mu}}{dt} \propto \vec{\mu} \times \vec{B}$$



Spin precesses around magnetic fields

Backup Slides

- Intentionally tilt $\hat{n}_0$ to cancel left-over spin-orbit coupling from rotator and random errors
- Use least-invasive vertical orbit bump: “**Opposing π pair**”
 - Delocalized $\hat{n}_0$ tilt, but localized D_y and localized coupling

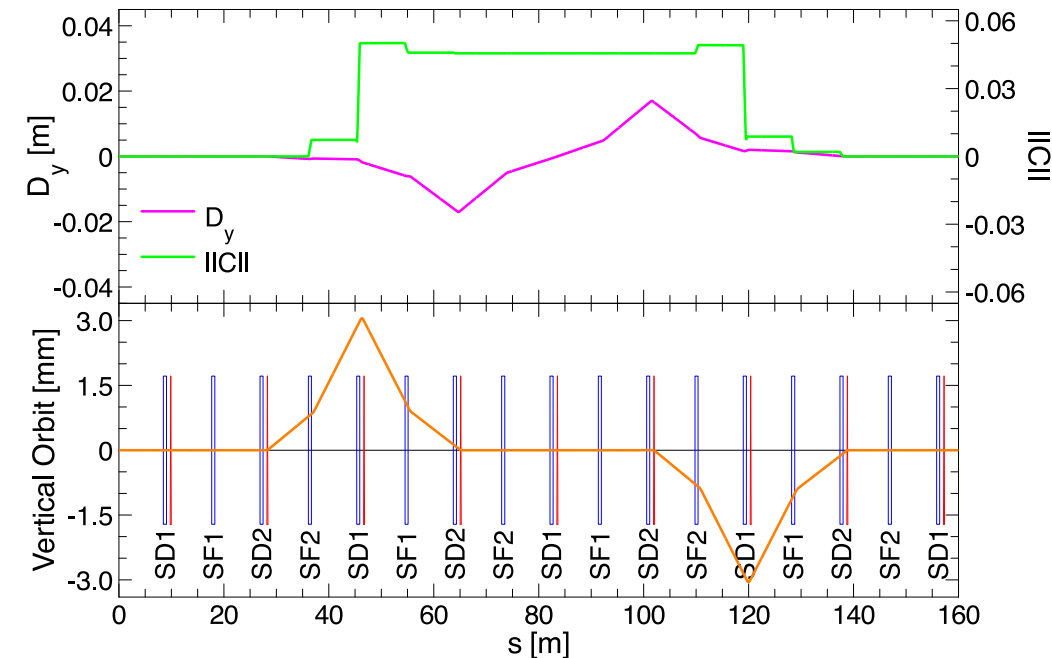
$$\mathbf{f}_d \approx (\mathbf{f}_d)_0 + \mathbf{R}_d \boldsymbol{\theta}$$

$$\mathbf{f}_d = \begin{pmatrix} \mathbf{d}_1 \\ \vdots \\ \mathbf{d}_m \end{pmatrix}, \quad \mathbf{R}_d = \begin{pmatrix} \frac{\partial \mathbf{d}_1}{\partial \theta_1} & \dots & \frac{\partial \mathbf{d}_1}{\partial \theta_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial \mathbf{d}_m}{\partial \theta_1} & \dots & \frac{\partial \mathbf{d}_m}{\partial \theta_n} \end{pmatrix}, \quad \boldsymbol{\theta} = \begin{pmatrix} \theta_1 \\ \vdots \\ \theta_n \end{pmatrix}$$

d at all m bends

Linear response matrix $3m \times n$

All opposing π pair strengths



- Can we just solve $-(\mathbf{f}_d)_0 = \mathbf{R}_d \boldsymbol{\theta}$ for least squares?
 - **NO! Values for some θ_i with little impact will exceed linear response regime**

Backup Slides

- PCA works well (as shown in all “old” BAGELS method results), but what if we also want to find knobs that *minimize* the orbit excursions at all elements *while* maximizing spin-orbit coupling in the bends? E.g., if $\mathbf{R}_y$ is the response matrix of the orbit at all elements, we seek:

$$\operatorname{argmax}_{\theta_1, \dots, \theta_k} \frac{||\mathbf{R}_d \boldsymbol{\theta}||^2}{||\mathbf{R}_y \boldsymbol{\theta}||^2} = \operatorname{argmax}_{\theta_1, \dots, \theta_k} \frac{\boldsymbol{\theta}^T \mathbf{R}_d^T \mathbf{R}_d \boldsymbol{\theta}}{\boldsymbol{\theta}^T \mathbf{R}_y^T \mathbf{R}_y \boldsymbol{\theta}}$$

- The quotient is called a *generalized Rayleigh quotient*. This problem can be equivalently expressed as maximizing $\boldsymbol{\theta}^T \mathbf{R}_d^T \mathbf{R}_d \boldsymbol{\theta}$ subject to $\boldsymbol{\theta}^T \mathbf{R}_y^T \mathbf{R}_y \boldsymbol{\theta} = 1$. Letting $\mathbf{A} = \mathbf{R}_d^T \mathbf{R}_d$ and $\mathbf{B} = \mathbf{R}_y^T \mathbf{R}_y$

$$\mathcal{L} = \boldsymbol{\theta}^T \mathbf{A} \boldsymbol{\theta} + \lambda (\boldsymbol{\theta}^T \mathbf{B} \boldsymbol{\theta} - 1). \quad \rightarrow \quad \frac{\partial \mathcal{L}}{\partial \boldsymbol{\theta}} = 2\mathbf{A} \boldsymbol{\theta} - 2\lambda \mathbf{B} \boldsymbol{\theta} \quad \rightarrow \quad \mathbf{A} \boldsymbol{\theta} = \lambda \mathbf{B} \boldsymbol{\theta}$$

- **Best Adjustment Groups for ELection Spin are those generalized eigenvectors v_i of A and B with largest eigenvalues!** Use these to solve $-(f_d)_0 = \widetilde{\mathbf{R}}_d \mathbf{w}$

Backup Slides

- One way to correct global delocalized coupling caused by random errors is with vertical orbits through the sextupoles, to create delocalized coupling which cancels out that leftover
- Using the BAGELS method, we can calculate bumps which **maximally impact the coupling $\bar{C}$ matrix components at all elements, while minimally impacting the spin-orbit coupling function at the bends and orbit excursions**

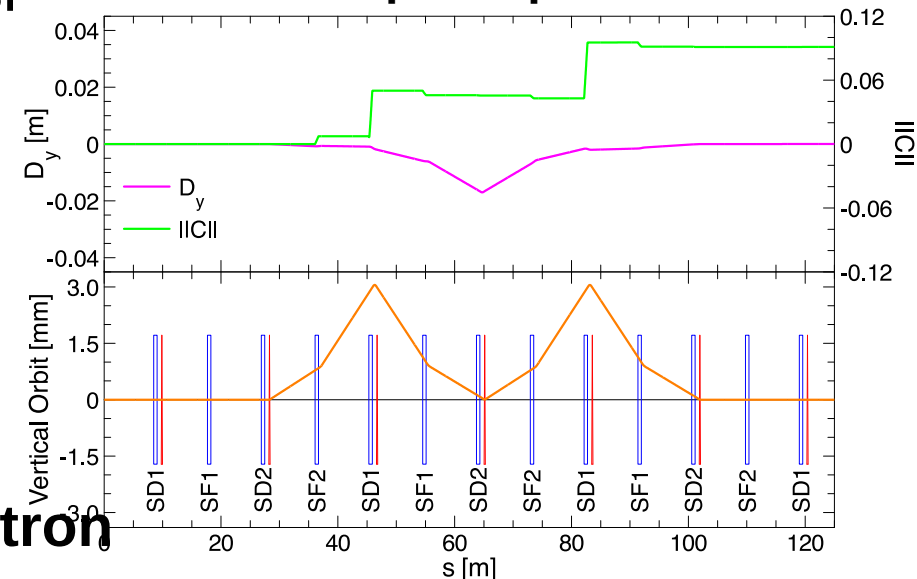
- *Basis bump* which generates delocalized coupling only

- “Equal π pair” Now seek:

$$\operatorname{argmax}_{\theta_1, \dots, \theta_k} \frac{\theta^T \mathbf{R}_A \mathbf{R}_A \theta}{\theta^T \mathbf{R}_B^T \mathbf{R}_B \theta} \quad \text{where} \quad \mathbf{R}_A = \mathbf{R}_{\bar{C}}, \quad \mathbf{R}_B =$$

$$\begin{pmatrix} \mathbf{R}_d / \|\mathbf{R}_d\| \\ \mathbf{R}_s / \|\mathbf{R}_s\| \end{pmatrix}$$

These knobs are the Best Adjustment Groups for Electron Spin for global coupling correction using vertical orbit bumps!



Backup Slides

- Another way to generate vertical emittance is with delocalized vertical dispersion so that a vertical amplitude is excited when a photon is emitted
- Specifically we seek to maximize $\mathcal{H}_b = \bar{\eta}_{b1}^2 + \bar{\eta}_{b2}^2$, $\bar{\eta}_{b1} = \frac{\eta_b}{\sqrt{\beta_b}}$, $\bar{\eta}_{b2} = \frac{\alpha_b \eta_b + \beta_b \eta'_b}{\sqrt{\beta_b}}$ in the bends
- Using the BAGELS method, we can calculate bumps which **maximally create $\bar{\eta}_{b1}, \bar{\eta}_{b2}$ in the bends while minimally impacting the spin-orbit coupling function at the bends and the orbit excursion at all elements, and generating no delocalized coupling**
- *Basis bump* which generates delocalized η_b only
- “ 2π pair”. Now seek:

$$\operatorname{argmax}_{\theta_1, \dots, \theta_k} \frac{\theta^T \bar{\mathbf{R}}_A^T \bar{\mathbf{R}}_A \theta}{\theta^T \bar{\mathbf{R}}_B^T \bar{\mathbf{R}}_B \theta} \quad \text{where} \quad \mathbf{R}_A = \mathbf{R}_{(\bar{\eta}_{b1}, \bar{\eta}_{b2})}, \quad \mathbf{R}_B = \begin{pmatrix} \mathbf{R}_d / ||\mathbf{R}_d|| \\ \mathbf{R}_y / ||\mathbf{R}_y|| \end{pmatrix}$$

