Cuts of Feynman Integrals

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Trinity College Dublin
Coláiste na Tríonóide, Baile Átha Cliath
The University of Dublin

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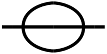
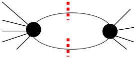
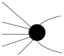
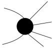
- Loop integration of Feynman diagrams is a challenge! 
- On-shell "cuts" have been very useful. They allow reconstruction from an ansatz or basis.
- Three related approaches to cuts, with different pictures and purposes.
  - Delta functions — directly "on shell"
  - Residues — for more detailed structure
  - Feynman parameters — for relations and analysis of singularities
- Differential-equations methods and other parametrizations (e.g. Baikov) are useful in a complementary way.

Cuts and discontinuities

A "cut" takes virtual particles to be physical ("on shell"),
giving the imaginary part of the amplitude, or discontinuity across a branch cut.

[Cutkosky; Veltman]

$\vec{p}$  $= \int$  x  $= 2\mathrm{Im}\, A^{1-\mathrm{loop}} = \mathrm{Disc}_{p^2} A^{1-\mathrm{loop}}$

Unitarity of the scattering matrix: S = 1 + i T

If $S^\dagger S = 1$, then $-i(T - T^\dagger) = T^\dagger T$.

Classic application: reconstruction by dispersion relations.

Cuts and discontinuities

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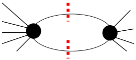
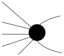
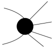
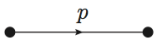
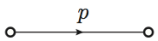
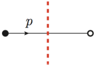
$\vec{p}$  $= \int$  x  $= 2\mathrm{Im}\, A^{1-\mathrm{loop}} = \mathrm{Disc}_{p^2} A^{1-\mathrm{loop}}$

Traditional cutting rules [Veltman]:

$$\bullet = i \qquad\qquad \circ = -i$$

 $= \dfrac{i}{p^2 + i\varepsilon}$  $= \dfrac{-i}{p^2 - i\varepsilon}$

 $= 2\pi\, \delta\left(p^2\right) \theta\left(p_0\right)$

Cuts are built from on-shell ingredients. Exploit their simplicity.

Cuts and discontinuities

A "cut" takes virtual particles to be physical ("on shell"),
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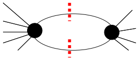
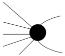
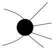
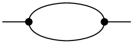
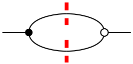
[Cutkosky; Veltman]

$\vec{p}$  $= \int$  **x**  $= 2\mathrm{Im}\, A^{1-\mathrm{loop}} = \mathrm{Disc}_{p^2} A^{1-\mathrm{loop}}$

Example: massless bubble integral

 $= \int d^{4-2\epsilon}k \frac{1}{k^2}\frac{1}{(p-k)^2} = \frac{1}{\epsilon} - \log(-p^2) + \mathcal{O}(\epsilon)$

 $= \int d^{4-2\epsilon}k \; \delta(k^2)\delta((p-k)^2) = -2\pi i + \mathcal{O}(\epsilon)$

A "cut" takes virtual particles to be physical ("on shell"),
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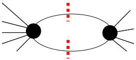
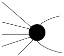
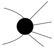
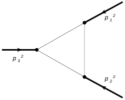
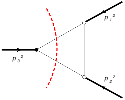
[Cutkosky; Veltman]

$\vec{p}$  $= \int$  **x**  $= 2\mathrm{Im}\, A^{1-\mathrm{loop}} = \mathrm{Disc}_{p^2} A^{1-\mathrm{loop}}$

Example: triangle integral



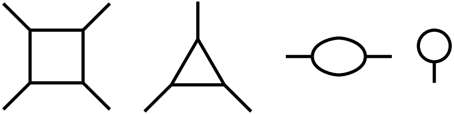
$$= \int d^{4-2\epsilon}\ell \frac{1}{\ell^2} \frac{1}{(\ell+p_3)^2} \frac{1}{(\ell-p_1)^2} = \frac{1}{p_3^2} \mathrm{Li}_2(z) + \cdots$$



$$= \int d^{4-2\epsilon}\ell \frac{\delta(\ell^2)\delta((\ell+p_3)^2)}{(\ell-p_1)^2} = \frac{2\pi i}{p_3^2} \log(z) + \cdots$$

where $z = \frac{p_1^2 - p_2^2 + p_3^2 + \sqrt{p_1^4 + p_2^4 + p_3^4 - 2p_1^2 p_2^2 - 2p_1^2 p_3^2 - 2p_2^2 p_3^2}}{2p_3^2}$.

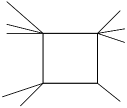
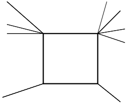
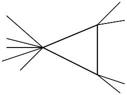
There is a well-known basis of 1-loop master integrals: boxes, triangles, bubbles, tadpoles.



$$A^{1-\mathrm{loop}} = \sum_i c_i I_i + r,$$

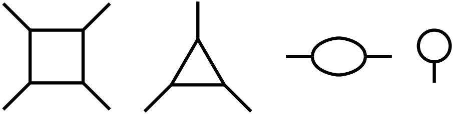
c_i, r are rational functions. [Passarino, Veltman; Van Neerven, Vermaseren]

e.g.:

$A^{1-\mathrm{loop}} = c_1$  $+ c_2$  $+ c_3$  $+ \cdots$

The unitarity method for one-loop amplitudes

There is a well-known basis of 1-loop master integrals: boxes, triangles, bubbles, tadpoles.



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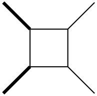


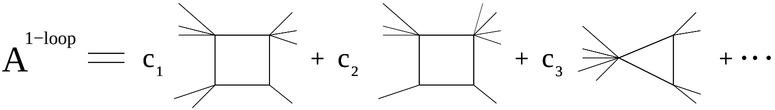
$$= I_4^{2m\,h} \quad = \quad \frac{2r_\Gamma}{st}\,\frac{1}{\epsilon^2}\left[\frac{1}{2}(-s)^{-\epsilon} + (-t)^{-\epsilon} - \frac{1}{2}(-p_1^2)^{-\epsilon} - \frac{1}{2}(-p_2^2)^{-\epsilon}\right]$$

$$-\frac{2r_\Gamma}{st}\left[-\frac{1}{2}\log\left(\frac{s}{p_1^2}\right)\log\left(\frac{s}{p_2^2}\right) + \frac{1}{2}\log^2\left(\frac{s}{t}\right)\right.$$

$$\left.+\mathrm{Li}_2\left(1-\frac{p_1^2}{t}\right) + \mathrm{Li}_2\left(1-\frac{p_2^2}{t}\right)\right] + \mathcal{O}(\epsilon).$$

Amplitudes from unitarity cuts

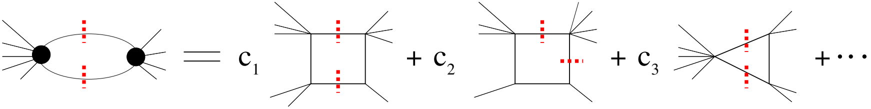
$$A^{1-\text{loop}} = \sum c_i I_i + r$$



Amplitudes from unitarity cuts

$$\text{Cut } A^{1-\text{loop}} = \sum c_i \text{ Cut } I_i \qquad\qquad \text{Cut} \sim \text{Disc}$$

Tree-level input.



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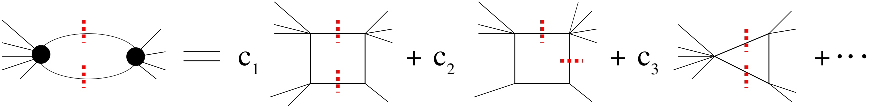
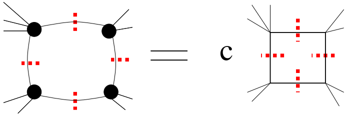


Matching 4-dimensional cuts can suffice to determine reduction coefficients!
Logarithms with unique arguments. "cut-constructibility"

[Bern, Dixon, Dunbar, Kosower]

But: we get several coefficients together in the same equation.



Generalized unitarity: get other singularities by applying more delta functions.

This operation isolates any given box!

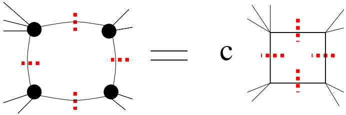
$$\Delta_4 A^{1-\mathrm{loop}} \equiv \int d^4\ell \; \delta(\ell_1^2)\; \delta(\ell_2^2)\; \delta(\ell_3^2)\; \delta(\ell_4^2)\; A_1^{\mathrm{tree}} A_2^{\mathrm{tree}} A_3^{\mathrm{tree}} A_4^{\mathrm{tree}}$$

$$\Delta_4 A^{1-\mathrm{loop}} = c_\star \; \Delta_4 I_\star$$

In four dimensions, four delta functions localize the integral completely.

[RB, Cachazo, Feng]

Box Coefficients from Quadruple Cuts



The box coefficients computed from quadruple cuts are given by

$$c_\star = \frac{1}{2} \sum_{\mathcal{S}} A_1^{\text{tree}} A_2^{\text{tree}} A_3^{\text{tree}} A_4^{\text{tree}}$$

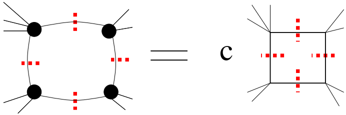
$\mathcal{S}$ is the set of all solutions of the on-shell conditions for the internal lines.

$$\mathcal{S} = \{\ell |\ \ell^2 = 0,\ \ (\ell - K_1)^2 = 0,\ \ (\ell - K_1 - K_2)^2 = 0,\ \ (\ell + K_4)^2 = 0\}$$

In complexified momentum space, there are exactly 2 solutions.

Complex analysis is key.

Box Coefficients from Quadruple Cuts



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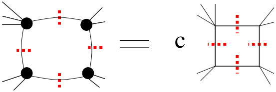
In complexified momentum space, there are exactly 2 solutions.

Complex analysis is key.

...or alternatively, twistor space with its natural (2,2) signature. Twistor geometry is still important for amplitude calculations.



Generalized cuts are used in numerical packages designed for 1-loop QCD at LHC:

- CutTools [Ossola, Papadopoulos, Pittau]
- GoSam [Cullen, Greiner, Heinrich, Luisoni, Mastrolia, Ossola, Reiter, Tramontano]
- Ninja [Mastrolia, Mirabella, Ossola, Peraro]
- BlackHat [Berger, Bern, Dixon, Febres Cordero, Forde, Hoeche, Ita, Kosower, Lo Presti, Maître]
- aMC@NLO [Alwall, Artoisenet, Conte, Degrande, Frederix, Frixione, Fuks, Hirschi, Lauyers, Maltoni, Mattelaer, Pittau, Stelzer, Torrielli, Zaro]
- Open Loops [Cascioli, Maierhöfer, Pozzorini]

and beyond 1-loop:

- Caravel [Abreu, Dormans, Febres Cordero, Ita, Kraus, Page, Pascual, Ruf, Sotnikov]

More loops!

Conceptual challenges at two loops and beyond:
master integrals more numerous, not canonical, functions are more complicated,...

Cuts still probe analytic properties of the functions and can be used for reconstruction.

Also: since cuts can reduce loop order all the way to tree level, they can be used as a tool in proving symmetries and relations.

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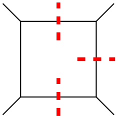
Cuts still probe analytic properties of the functions and can be used for reconstruction.

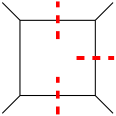
Also: since cuts can reduce loop order all the way to tree level, they can be used as a tool in proving symmetries and relations.
All under the umbrella of "on-shell methods."

A little puzzle



$= \quad ?$

A little puzzle



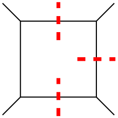
$= \ ?$

Resolution: interpret cuts as residues.

$$\begin{tikzcd}[column sep=small]
 & \int_\gamma \omega \arrow[dl, "\text{delta functions}"'] \arrow[dr, "\text{residue}"] & \\
\int_\gamma \tilde{\omega} & & \int_{\tilde{\gamma}} \omega
\end{tikzcd}$$

Cuts as residues

- Consistent with prior expectations (unitarity)
- Fits into deeper algebraic structure (coaction, and then differential equations)
- Freedom to explore all contours
- New linear relations are revealed



$= \quad 0$

in the massless case.

Generalized cuts as residues and determinants

1-loop cuts defined as residues:

$$\mathcal{C}_C[I_n] = \frac{e^{\gamma_E \epsilon}}{i\pi^{\frac{D}{2}}} \int_{\Gamma_C} d^D k \prod_{j \notin C} \frac{1}{(k-q_j)^2 - m_j^2 + i\varepsilon} \quad \text{mod } i\pi \,,$$

- C is the set of cut propagators
- Contour Γ_C encircles poles of cut propagators

Cut integrals give discontinuities of their uncut counterparts.

Generalized cuts as residues and determinants

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Cut integrals give discontinuities of their uncut counterparts.
Some results:

$$\mathcal{C}_C I_n = \frac{e^{\gamma_E \epsilon}}{\sqrt{Y_C}} \left(\frac{Y_C}{G_C}\right)^{(D-c)/2} \int \frac{d\Omega_{D-c+1}}{i\pi^{D/2}} \left[\prod_{j \notin C} \frac{1}{(k-q_j)^2 - m_j^2}\right]_C$$

where we often find Gram and modified Cayley determinants:

$$\begin{aligned} G_C &= \det\left(q_i \cdot q_j\right)_{i,j \in C \setminus *} \\ Y_C &= \det\left(\frac{1}{2}(m_i^2 + m_j^2 + (q_i - q_j)^2)\right)_{i,j \in C} \end{aligned}$$

[Abreu, RB, Duhr, Gardi 2017]

Explicit cuts from residues

Some special cases: maximal and next-to-maximal cuts.

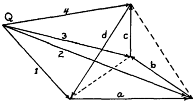
$$\mathcal{C}_{2k}[I_{2k}] = \frac{\Gamma(1-\epsilon)}{\Gamma(1-2\epsilon)} \frac{e^{\gamma_E \epsilon}}{\sqrt{Y_{2k}}} \left(\frac{Y_{2k}}{G_{2k}}\right)^{-\epsilon}$$

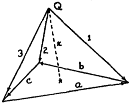
$$\mathcal{C}_{2k+1}[I_{2k+1}] = \frac{e^{\gamma_E \epsilon}}{\Gamma(1-\epsilon)\sqrt{G_{2k+1}}} \left(\frac{Y_{2k+1}}{G_{2k+1}}\right)^{-\epsilon} .$$

$$\mathcal{C}_{2k-1}[I_{2k}] = -\frac{\Gamma(1-2\epsilon)^2}{\Gamma(1-\epsilon)^3} \frac{e^{\gamma_E \epsilon}}{\sqrt{Y_{2k}}} \left(\frac{Y_{2k-1}}{G_{2k-1}}\right)^{-\epsilon} {}_2F_1\left(\frac{1}{2}, -\epsilon; 1-\epsilon; \frac{G_{2k}\, Y_{2k-1}}{Y_{2k}\, G_{2k-1}}\right) .$$

1-line cuts are mass discontinuities.

These determinants are volumes of simplices. [Cutkosky]

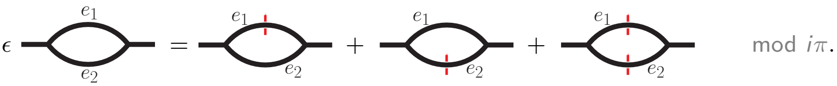




A technically advanced argument shows that cuts are related to the uncut integral by a linear relation.

[Fotiadi, Pham; Hwa, Teplitz; Federbusch; Eden, Fairlie, Landshoff, Nuttall, Olive, Polkinghorne,...]

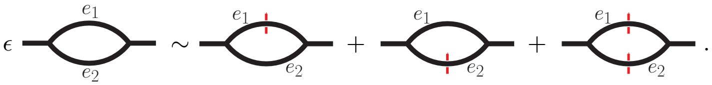
An uncut 1-loop integral is equivalent to the sum of its cuts:



[Abreu, RB, Duhr, Gardi]

A sort of residue theorem, that requires understanding the residue at infinite momentum.

Cuts of the scalar bubble



Now in $D = 2 - 2\epsilon$.

$$I_2 = \frac{1}{p^2(w-\bar{w})} \log \frac{\bar{w}(1-w)}{w(1-\bar{w})} + \mathcal{O}(\epsilon).$$

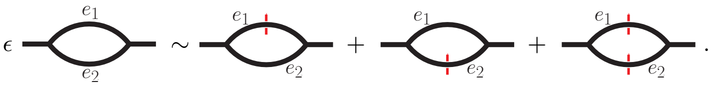
$$\mathcal{C}_1 I_2 = \frac{1}{p^2(w-\bar{w})} + \mathcal{O}(\epsilon),$$

$$\mathcal{C}_2 I_2 = \frac{1}{p^2(w-\bar{w})} + \mathcal{O}(\epsilon),$$

$$\mathcal{C}_{12} I_2 = -\frac{2}{p^2(w-\bar{w})} + \mathcal{O}(\epsilon).$$

(factors of $i\pi$ have been stripped)

Cuts of the scalar bubble



Now in $D = 2 - 2\epsilon$.

$$I_2 = -\frac{e^{\gamma_E \epsilon}\Gamma(1+\epsilon)}{\epsilon}\frac{\left(-p^2\right)^{-1-\epsilon}}{(w-\bar{w})^{1+\epsilon}}\left[w^{-\epsilon}\,{}_2F_1\left(-\epsilon,1+\epsilon;1-\epsilon;\frac{w}{w-\bar{w}}\right)-(w-1)^{-\epsilon}\,{}_2F_1\left(-\epsilon,1+\epsilon;1-\epsilon;\frac{w-1}{w-\bar{w}}\right)\right]$$

$$\mathcal{C}_1 I_2 = -\frac{e^{\gamma_E \epsilon}}{\Gamma(1-\epsilon)}\frac{\left(-p^2\right)^{-1-\epsilon}}{w(w\bar{w})^{\epsilon}}\,{}_2F_1\left(1,1+\epsilon;1-\epsilon;\frac{\bar{w}}{w}\right) \qquad \sim \mathrm{Disc}_{m_1^2} I_2$$

$$\mathcal{C}_2 I_2 = -\frac{e^{\gamma_E \epsilon}}{\Gamma(1-\epsilon)}\frac{\left(-p^2\right)^{-1-\epsilon}(1-w)^{-\epsilon}}{(w-\bar{w})^{1+\epsilon}}\,{}_2F_1\left(-\epsilon,1+\epsilon;1-\epsilon;\frac{w-1}{w-\bar{w}}\right) \qquad \sim \mathrm{Disc}_{m_2^2} I_2$$

$$\mathcal{C}_{12} I_2 = -2\frac{e^{\gamma_E \epsilon}\Gamma(1-\epsilon)}{\Gamma(1-2\epsilon)}\left(p^2\right)^{-1-\epsilon}(w-\bar{w})^{-1-2\epsilon} \qquad \sim \mathrm{Disc}_{p^2} I_2$$

There is a simpler explanation for the relation!

Cuts from Feynman parameters

Feynman/Schwinger trick:

$$\frac{1}{\prod_i A_i} = \int_{\alpha_i \geq 0} \mathrm{d}^E \alpha \, \frac{\delta\left(1 - \sum_i \alpha_i\right)}{\left(\sum_i \alpha_i A_i\right)^E},$$

where $A_i = (k - q_i)^2 - m_i^2 + i\varepsilon$.

The Feynman integral becomes an integral over these parameters as well as loop momenta.

The cut conditions are some subset of $A_i = 0$.

Feynman parameter space

If we then perform the loop momentum integral, we find the following parametric representation:

$$I = \Gamma\left(E - \frac{LD}{2}\right) \int_{\alpha_i \geq 0} \mathrm{d}^E \alpha \, \delta\left(1 - \sum_{i \in S} \alpha_i\right) \frac{\mathcal{U}^{E-(L+1)D/2}}{\mathcal{F}^{E-LD/2}} \, .$$

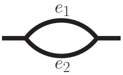
$\mathcal{U}$ and $\mathcal{F}$ are graph polynomials in α_i.

$$\mathcal{U} = \sum_{T \in \mathcal{T}_1} \prod_{e_i \notin T} \alpha_i \, ,$$

$$\mathcal{F} = \mathcal{U} \sum_{i=1}^{E} \alpha_i m_i^2 - \sum_{(T_1, T_2) \in \mathcal{T}_2} \left(\prod_{e_i \notin (T_1, T_2)} \alpha_i \right) q^2_{(T_1, T_2)} \, .$$

$\mathcal{T}_1$ is the set of spanning trees , $\mathcal{T}_2$ is the set of spanning 2-forests.

Example: bubble



$$
\begin{aligned}
I &= \Gamma\left(2-\frac{D}{2}\right)\int_{\alpha_i\geq 0}\frac{\mathrm{d}\alpha_1\,\mathrm{d}\alpha_2\,\delta(1-\alpha_1-\alpha_2)\,(\alpha_1+\alpha_2)^{2-D}}{[-\alpha_1\alpha_2 p^2+(\alpha_1+\alpha_2)(\alpha_1 m_1^2+\alpha_2 m_2^2)]^{2-D/2}}\\
&= \Gamma\left(2-\frac{D}{2}\right)\int_0^1 \mathrm{d}x\,\left[-x(1-x)p^2+(1-x)m_1^2+xm_2^2\right]^{D/2-2}\\
&= \Gamma\left(2-\frac{D}{2}\right)\int_0^1 \mathrm{d}x\,\left[p^2(x-w)(x-\bar{w})\right]^{D/2-2}
\end{aligned}
$$

What are the cuts? The on-shell condition is inaccessible!

Cuts are discontinuities

$$I = \Gamma\left(E - \frac{LD}{2}\right) \int_{\alpha_i \geq 0} \mathrm{d}^E\alpha\, \delta\left(1 - \sum_{i \in S} \alpha_i\right) \frac{\mathcal{U}^{E-(L+1)D/2}}{\mathcal{F}^{E-LD/2}},$$

$$\mathcal{U} = \sum_{T \in \mathcal{T}_1} \prod_{e_i \notin T} \alpha_i\,,$$

$$\mathcal{F} = \mathcal{U} \sum_{i=1}^{E} \alpha_i m_i^2 - \sum_{(T_1,T_2) \in \mathcal{T}_2} \left(\prod_{e_i \notin (T_1,T_2)} \alpha_i \right) q^2_{(T_1,T_2)}\,.$$

Kinematic dependence is all in $\mathcal{F}$.

Discontinuities appear when $\mathcal{F}$ becomes negative. The change happens at $\mathcal{F} = 0$.

$\mathcal{F} = 0$ can be chosen as a boundary of the integration region for a cut.

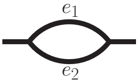
$$I = \Gamma\left(E - \frac{LD}{2}\right) \int_{\alpha_i \geq 0} \mathrm{d}^E\alpha\, \delta\left(1 - \sum_{i \in S} \alpha_i\right) \frac{\mathcal{U}^{E-(L+1)D/2}}{\mathcal{F}^{E-LD/2}},$$

The original integral has boundaries $\alpha_i = 0$.

Cuts can exchange $\alpha_i = 0 \longleftrightarrow k_i^2 - m_i^2 = 0$.

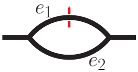
We also include $\mathcal{F} = 0$ in the boundaries of the integration regions.

Example: bubble



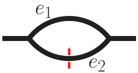
$$\sim \int_0^1 \mathrm{d}x \left[p^2 (x-w)(x-\bar{w}) \right]^{D/2-2} .$$

$\mathcal{F} = 0 \Leftrightarrow x = w$ *or* $x = \bar{w}$. $\quad \alpha_1 = 0 \Leftrightarrow x = 1. \quad \alpha_2 = 0 \Leftrightarrow x = 0.$



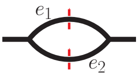
$$\sim \int_0^{\bar{w}} \mathrm{d}x \left[p^2 (x-w)(x-\bar{w}) \right]^{D/2-2} .$$

no $\alpha_1 = 0$



$$\sim \int_w^1 \mathrm{d}x \left[p^2 (x-w)(x-\bar{w}) \right]^{D/2-2} .$$

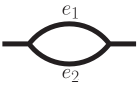
no $\alpha_2 = 0$

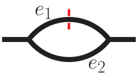


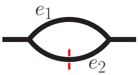
$$\sim \int_{\bar{w}}^w \mathrm{d}x \left[p^2 (x-w)(x-\bar{w}) \right]^{D/2-2} .$$

neither $\alpha_1 = 0$ nor $\alpha_2 = 0$

Therefore



$\sim$


$+$


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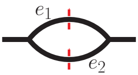
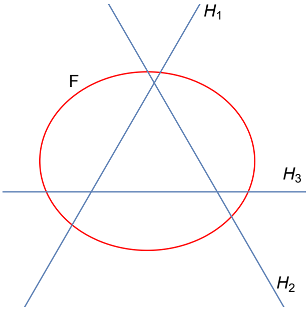


.

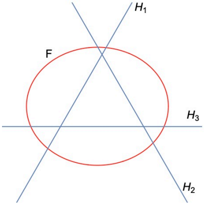
Example: triangle (or any $n \geq 3$)

$$\mathcal{F} = \alpha_1 m_1^2 + \alpha_2 m_2^2 + \alpha_3 m_3^2 - \alpha_1\alpha_2 p_3^2 - \alpha_1\alpha_3 p_2^2 - \alpha_2\alpha_3 p_1^2$$

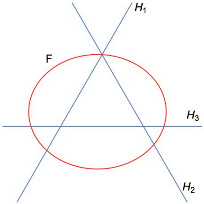
shown in the plane $\alpha_1 + \alpha_2 + \alpha_3 = 1$,
for Euclidean kinematics with $m_i^2 > 0, p_i^2 < 0$.



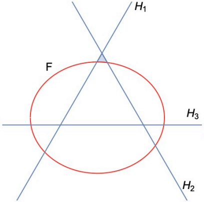
Mass discontinuity of the triangle



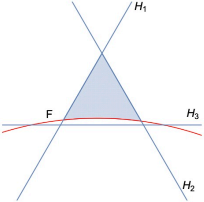
(a) m_3^2 is small and positive.



(b) $m_3^2 = 0$.

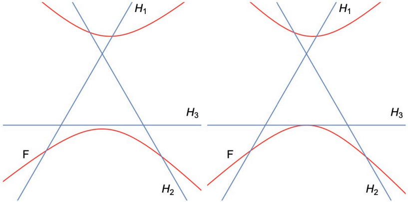


(c) m_3^2 is small and negative.



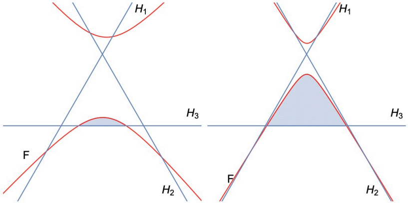
(d) m_3^2 is large and negative.

Momentum discontinuity of the triangle



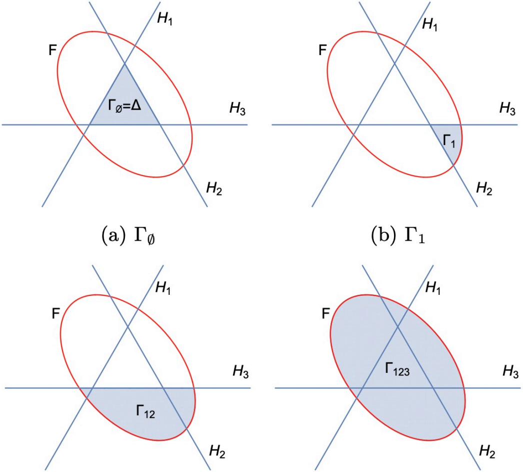
(a) p_3^2 just below threshold.

(b) $p_3^2 = (m_1 + m_2)^2$.



(c) p_3^2 just above threshold.

(d) p_3^2 large compared to other scales.

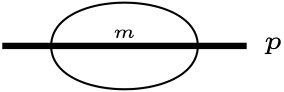


(a) $\Gamma_\emptyset$ (b) Γ_1

(c) Γ_{12} (d) Γ_{123}

The linear relation among cuts is just the inclusion-exclusion principle here.

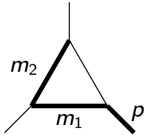
One-mass sunrise



$$\alpha_1 = x,\ \alpha_2 = 1 - x,\ \alpha_3 = y$$
$$\mathcal{U} = x(1-x) + y$$
$$\mathcal{F} = y\left[m^2 y + (m^2 - p^2)x(1-x)\right]$$

Here $\mathcal{F} = 0$ has two components: a variable parabola, and the coordinate hyperplane $y = 0$.

$\Rightarrow$ **Demonstration**



We can interpret sequences of discontinuities to reconstruct the 'symbol' of the function

$$m_1^2 \otimes \left(\frac{m_1^2 - m_2^2}{m_2^2}\right) + m_2^2 \otimes \left(\frac{m_1^2 - m_2^2 - p^2}{m_1^2 - m_2^2}\right) - (m_1^2 - p^2) \otimes \left(\frac{m_1^2 - m_2^2 - p^2}{m_2^2}\right)$$

by following the shapes of the regions as we travel through phase space.

[RB, Hannesdottir]

$\Rightarrow$ **Animations**

Summary & Outlook

- Cuts are a key tool in computing Feynman integrals and amplitudes.
- Three approaches to cuts: delta functions, residues, Feynman parameters.
- Different setups expose relations and geometry differently.
- Introduced parametric approach
  - to discover relations
  - to probe sequential discontinuities (symbol bootstrap...)
  - to expose hypergeometric and coaction properties
  - for efficient numerical computation?
  - with new angles on geometric interpretation?
- Seeking the general structure of the underlying functions, in order to tame computation!

The hypergeometric function ${}_2F_1$

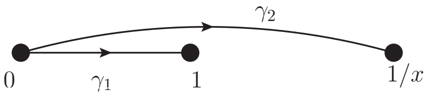
Gauss's original hypergeometric function. Series definition:

$$ {}_2F_1(\alpha,\beta;\gamma;z) = \sum_{n=0}^{\infty} \frac{(\alpha)_n(\beta)_n}{(\gamma)_n}\frac{z^n}{n!}\,, \qquad \text{where } (x)_n \equiv \frac{\Gamma(x+n)}{\Gamma(x)}. $$

Euler's integral representation:

$$ {}_2F_1(\alpha,\beta;\gamma;x) = \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\gamma-\alpha)} \int_0^1 du\, u^{\alpha-1}(1-u)^{\gamma-\alpha-1}(1-ux)^{-\beta} $$

Contours can be extended to a set of two by integrating between different branch points: $\gamma_1 = [0,1], \quad \gamma_2 = [0, 1/x]$.



$$ {}_2F_1(\alpha, \beta; \gamma; x) \sim \int_0^1 du\, u^{\alpha-1}(1-u)^{\gamma-\alpha-1}(1-ux)^{-\beta} $$

$$ {}_2F_1\left(\alpha, 1+\alpha-\gamma; 1-\beta+\alpha; \frac{1}{x}\right) \sim \int_0^{1/x} du\, u^{\alpha-1}(1-u)^{\gamma-\alpha-1}(1-ux)^{-\beta} $$