

Cuts of Feynman Integrals

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- Loop integration of Feynman diagrams is a challenge!
- On-shell “cuts” have been very useful. They allow reconstruction from an ansatz or basis.
- Three related approaches to cuts, with different pictures and purposes.
 - ▶ Delta functions — directly “on shell”
 - ▶ Residues — for more detailed structure
 - ▶ Feynman parameters — for relations and analysis of singularities
- Differential-equations methods and other parametrizations (e.g. Baikov) are useful in a complementary way.

Cuts and discontinuities

A “cut” takes virtual particles to be physical (“on shell”), giving the imaginary part of the amplitude, or discontinuity across a branch cut.

[Cutkosky; Veltman]

$$\vec{p} \rightarrow \text{loop} = \int \text{cut loop} = 2\text{Im } A^{1\text{-loop}} = \text{Disc}_{p^2} A^{1\text{-loop}}$$

Unitarity of the scattering matrix: $S = 1 + i T$

If $S^\dagger S = 1$, then $-i(T - T^\dagger) = T^\dagger T$.

Classic application: reconstruction by dispersion relations.

Cuts and discontinuities

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[Cutkosky; Veltman]

$$\vec{p} \text{ --- } \bullet \text{ --- } \text{---} \bullet \text{ ---} = \int \text{---} \bullet \text{ ---} \times \text{---} \bullet \text{ ---} = 2\text{Im } A^{1\text{-loop}} = \text{Disc}_{p^2} A^{1\text{-loop}}$$

Traditional cutting rules [Veltman]:

$$\bullet = i$$

$$\circ = -i$$

$$\bullet \xrightarrow{p} \bullet = \frac{i}{p^2 + i\varepsilon}$$

$$\circ \xrightarrow{p} \circ = \frac{-i}{p^2 - i\varepsilon}$$

$$\bullet \xrightarrow{p} \circ = 2\pi \delta(p^2) \theta(p_0)$$

Cuts are built from on-shell ingredients. Exploit their simplicity.

Cuts and discontinuities

A “cut” takes virtual particles to be physical (“on shell”), giving the imaginary part of the amplitude, or discontinuity across a branch cut.

[Cutkosky; Veltman]

$$\vec{p} \text{ --- } \bullet \text{ --- } \text{---} \bullet \text{ ---} = \int \text{---} \bullet \text{ ---} \times \text{---} \bullet \text{ ---} = 2\text{Im } A^{1\text{-loop}} = \text{Disc}_{p^2} A^{1\text{-loop}}$$

The diagram shows a bubble diagram with two external lines on the left, labeled with a vector $\vec{p}$. The bubble has two vertices, each represented by a black dot. A vertical dashed red line is drawn through the bubble, representing a branch cut. The equation states that this is equal to the integral of two tree-level diagrams (each with a single vertex and multiple external lines) multiplied together, which is equal to $2\text{Im } A^{1\text{-loop}}$ and $\text{Disc}_{p^2} A^{1\text{-loop}}$.

Example: massless bubble integral

$$\text{---} \bullet \text{ ---} \text{---} \bullet \text{ ---} = \int d^{4-2\epsilon} k \frac{1}{k^2} \frac{1}{(p-k)^2} = \frac{1}{\epsilon} - \log(-p^2) + \mathcal{O}(\epsilon)$$

$$\text{---} \bullet \text{ ---} \text{---} \circ \text{ ---} = \int d^{4-2\epsilon} k \delta(k^2) \delta((p-k)^2) = -2\pi i + \mathcal{O}(\epsilon)$$

The second diagram is similar to the first, but the right vertex is an open circle. A vertical dashed red line is drawn through the bubble, representing a branch cut. The equation states that this is equal to the integral of $d^{4-2\epsilon} k \delta(k^2) \delta((p-k)^2)$, which is equal to $-2\pi i + \mathcal{O}(\epsilon)$.

Cuts and discontinuities

A “cut” takes virtual particles to be physical (“on shell”), giving the imaginary part of the amplitude, or discontinuity across a branch cut.

[Cutkosky; Veltman]

$$\vec{p} \rightarrow \text{bubble diagram} = \int \text{cut bubble} \times \text{cut bubble} = 2\text{Im } A^{1\text{-loop}} = \text{Disc}_{p^2} A^{1\text{-loop}}$$

Example: triangle integral

$$\begin{aligned} \text{triangle diagram} &= \int d^{4-2\epsilon} \ell \frac{1}{\ell^2} \frac{1}{(\ell+p_3)^2} \frac{1}{(\ell-p_1)^2} = \frac{1}{p_3^2} \text{Li}_2(z) + \dots \\ \text{cut triangle diagram} &= \int d^{4-2\epsilon} \ell \frac{\delta(\ell^2) \delta((\ell+p_3)^2)}{(\ell-p_1)^2} = \frac{2\pi i}{p_3^2} \log(z) + \dots \end{aligned}$$

where $z = \frac{p_1^2 - p_2^2 + p_3^2 + \sqrt{p_1^4 + p_2^4 + p_3^4 - 2p_1^2 p_2^2 - 2p_1^2 p_3^2 - 2p_2^2 p_3^2}}{2p_3^2}$.

The unitarity method for one-loop amplitudes

There is a well-known basis of 1-loop **master integrals**: boxes, triangles, bubbles, tadpoles.



$$A^{1\text{-loop}} = \sum_i c_i I_i + r,$$

c_i, r are rational functions. [Passarino, Veltman; Van Neerven, Vermaseren]

e.g.:

$$A^{1\text{-loop}} = c_1 \text{ (box with 6 external lines) } + c_2 \text{ (box with 5 external lines) } + c_3 \text{ (triangle with 5 external lines) } + \dots$$

The image shows an example of a one-loop amplitude expansion. It starts with $A^{1\text{-loop}} =$ followed by a sum of terms. The first term is c_1 multiplied by a box diagram with six external lines. The second term is $+ c_2$ multiplied by a box diagram with five external lines. The third term is $+ c_3$ multiplied by a triangle diagram with five external lines. The expansion ends with $+ \dots$.

The unitarity method for one-loop amplitudes

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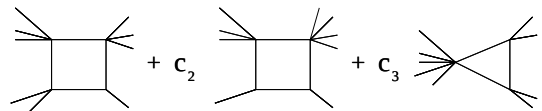
c_i, r are rational functions.

e.g.:

$$\begin{aligned}
 \text{Box Diagram} &= I_4^{2mh} = \frac{2r_\Gamma}{st} \frac{1}{\epsilon^2} \left[\frac{1}{2}(-s)^{-\epsilon} + (-t)^{-\epsilon} - \frac{1}{2}(-p_1^2)^{-\epsilon} - \frac{1}{2}(-p_2^2)^{-\epsilon} \right] \\
 &\quad - \frac{2r_\Gamma}{st} \left[-\frac{1}{2} \log\left(\frac{s}{p_1^2}\right) \log\left(\frac{s}{p_2^2}\right) + \frac{1}{2} \log^2\left(\frac{s}{t}\right) \right. \\
 &\quad \left. + \text{Li}_2\left(1 - \frac{p_1^2}{t}\right) + \text{Li}_2\left(1 - \frac{p_2^2}{t}\right) \right] + \mathcal{O}(\epsilon).
 \end{aligned}$$

Amplitudes from unitarity cuts

$$A^{1\text{-loop}} = \sum c_i l_i + r$$

$$A^{1\text{-loop}} = c_1 \text{ (box diagram) } + c_2 \text{ (crossed box diagram) } + c_3 \text{ (triangle diagram) } + \dots$$


The image shows three Feynman diagrams representing one-loop amplitudes. The first diagram is a box diagram with four external lines. The second diagram is a crossed box diagram, also with four external lines. The third diagram is a triangle diagram with three external lines. Each diagram is preceded by a coefficient c_i , and the sum is followed by an ellipsis indicating further terms.

Amplitudes from unitarity cuts

$$\text{Cut } A^{1\text{-loop}} = \sum c_i \text{Cut } I_i$$

$$\text{Cut} \sim \text{Disc}$$

Tree-level input.

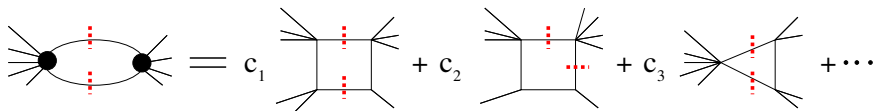
$$\text{Cut } A^{1\text{-loop}} = c_1 \text{Cut } I_1 + c_2 \text{Cut } I_2 + c_3 \text{Cut } I_3 + \dots$$

Amplitudes from unitarity cuts

$$\text{Cut } A^{1\text{-loop}} = \sum c_i \text{Cut } I_i$$

$$\text{Cut} \sim \text{Disc}$$

Tree-level input.

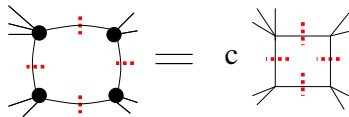


Matching 4-dimensional cuts can suffice to determine reduction coefficients!
Logarithms with unique arguments. “cut-constructibility”

[Bern, Dixon, Dunbar, Kosower]

But: we get several coefficients together in the same equation.

Box Coefficients from Quadruple Cuts



Generalized unitarity: get other singularities by applying more delta functions.

This operation isolates any given box!

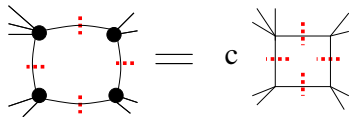
$$\Delta_4 A^{1\text{-loop}} \equiv \int d^4 \ell \, \delta(\ell_1^2) \delta(\ell_2^2) \delta(\ell_3^2) \delta(\ell_4^2) A_1^{\text{tree}} A_2^{\text{tree}} A_3^{\text{tree}} A_4^{\text{tree}}$$

$$\Delta_4 A^{1\text{-loop}} = c_\star \Delta_4 I_\star$$

In four dimensions, four delta functions **localize** the integral completely.

[RB, Cachazo, Feng]

Box Coefficients from Quadruple Cuts



The box coefficients computed from quadruple cuts are given by

$$c_{\star} = \frac{1}{2} \sum_S A_1^{\text{tree}} A_2^{\text{tree}} A_3^{\text{tree}} A_4^{\text{tree}}$$

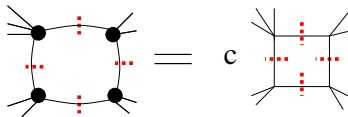
S is the set of all solutions of the on-shell conditions for the internal lines.

$$S = \{\ell \mid \ell^2 = 0, (\ell - K_1)^2 = 0, (\ell - K_1 - K_2)^2 = 0, (\ell + K_4)^2 = 0\}$$

In **complexified momentum space**, there are exactly 2 solutions.

Complex analysis is key.

Box Coefficients from Quadruple Cuts



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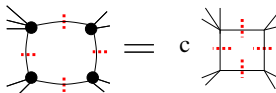
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In **complexified momentum space**, there are exactly 2 solutions.

Complex analysis is key.

...or alternatively, twistor space with its natural (2,2) signature. Twistor geometry is still important for amplitude calculations.



Generalized cuts are used in numerical packages designed for 1-loop QCD at LHC:

- **CutTools** [Ossola, Papadopoulos, Pittau]
- **GoSam** [Cullen, Greiner, Heinrich, Luisoni, Mastrolia, Ossola, Reiter, Tramontano]
- **Ninja** [Mastrolia, Mirabella, Ossola, Peraro]
- **BlackHat** [Berger, Bern, Dixon, Febres Cordero, Forde, Hoeche, Ita, Kosower, Lo Presti, Maître]
- **aMC@NLO** [Alwall, Artoisenet, Conte, Degrande, Frederix, Frixione, Fuks, Hirschi, Lauyers, Maltoni, Mattelaer, Pittau, Stelzer, Torrielli, Zaro]
- **Open Loops** [Cascioli, Maierhöfer, Pozzorini]

and beyond 1-loop:

- **Caravel** [Abreu, Dormans, Febres Cordero, Ita, Kraus, Page, Pascual, Ruf, Sotnikov]

More loops!

Conceptual challenges at two loops and beyond:
master integrals more numerous, not canonical, functions are more complicated,...

Cuts still probe analytic properties of the functions and can be used for reconstruction.

Also: since cuts can reduce loop order all the way to tree level, they can be used as a tool in proving symmetries and relations.

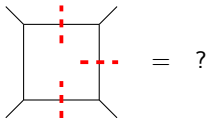
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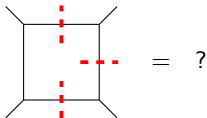
Cuts still probe analytic properties of the functions and can be used for reconstruction.

Also: since cuts can reduce loop order all the way to tree level, they can be used as a tool in proving symmetries and relations.
All under the umbrella of “on-shell methods.”

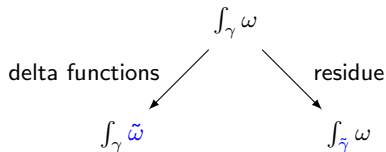
A little puzzle



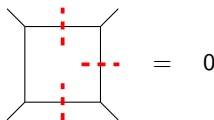
A little puzzle



Resolution: interpret cuts as **residues**.



- Consistent with prior expectations (unitarity)
- Fits into deeper algebraic structure (coaction, and then differential equations)
- Freedom to explore all contours
- New linear relations are revealed


$$= 0$$

in the massless case.

Generalized cuts as residues and determinants

1-loop cuts defined as **residues**:

$$\mathcal{C}_C[I_n] = \frac{e^{\gamma_E \epsilon}}{i\pi^{\frac{D}{2}}} \int_{\Gamma_C} d^D k \prod_{j \notin C} \frac{1}{(k - q_j)^2 - m_j^2 + i\epsilon} \quad \text{mod } i\pi,$$

- C is the set of cut propagators
- Contour Γ_C encircles poles of cut propagators

Cut integrals give **discontinuities** of their uncut counterparts.

Generalized cuts as residues and determinants

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- C is the set of cut propagators
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Cut integrals give **discontinuities** of their uncut counterparts.
Some results:

$$\mathcal{C}_C I_n = \frac{e^{\gamma_E \epsilon}}{\sqrt{Y_C}} \left(\frac{Y_C}{G_C} \right)^{(D-c)/2} \int \frac{d\Omega_{D-c+1}}{i\pi^{D/2}} \left[\prod_{j \notin C} \frac{1}{(k - q_j)^2 - m_j^2} \right]_C$$

where we often find Gram and modified Cayley determinants:

$$G_C = \det(q_i \cdot q_j)_{i,j \in C \setminus *}$$

$$Y_C = \det \left(\frac{1}{2} (m_i^2 + m_j^2 + (q_i - q_j)^2) \right)_{i,j \in C}$$

Explicit cuts from residues

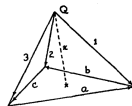
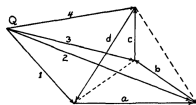
Some special cases: **maximal** and **next-to-maximal** cuts.

$$\begin{aligned}C_{2k}[l_{2k}] &= \frac{\Gamma(1-\epsilon)}{\Gamma(1-2\epsilon)} \frac{e^{\gamma_E \epsilon}}{\sqrt{Y_{2k}}} \left(\frac{Y_{2k}}{G_{2k}} \right)^{-\epsilon} \\C_{2k+1}[l_{2k+1}] &= \frac{e^{\gamma_E \epsilon}}{\Gamma(1-\epsilon) \sqrt{G_{2k+1}}} \left(\frac{Y_{2k+1}}{G_{2k+1}} \right)^{-\epsilon}.\end{aligned}$$

$$C_{2k-1}[l_{2k}] = -\frac{\Gamma(1-2\epsilon)^2}{\Gamma(1-\epsilon)^3} \frac{e^{\gamma_E \epsilon}}{\sqrt{Y_{2k}}} \left(\frac{Y_{2k-1}}{G_{2k-1}} \right)^{-\epsilon} {}_2F_1\left(\frac{1}{2}, -\epsilon; 1-\epsilon; \frac{G_{2k} Y_{2k-1}}{Y_{2k} G_{2k-1}}\right).$$

1-line cuts are **mass discontinuities**.

These determinants are **volumes** of simplices. [Cutkosky]



A technically advanced argument shows that cuts are related to the uncut integral by a linear relation.

[Fotiadi, Pham; Hwa, Teplitz; Federbusch; Eden, Fairlie, Landshoff, Nuttall, Olive, Polkinghorne,...]

An uncut 1-loop integral is equivalent to the **sum of its cuts**:

The diagram shows an equation between four Feynman diagrams. On the left is an uncut bubble diagram with two external lines and two internal lines labeled e_1 (top) and e_2 (bottom). This is equal to the sum of three cut bubble diagrams. Each cut diagram has a red vertical line segment on one of its internal lines. The first cut is on the top line e_1 , the second is on the bottom line e_2 , and the third is on both lines. To the right of the sum is the text $\text{mod } i\pi$.

$$\epsilon \text{ --- } \text{bubble}(e_1, e_2) = \text{cut_bubble}(e_1) + \text{cut_bubble}(e_2) + \text{cut_bubble}(e_1, e_2) \pmod{i\pi}$$

[Abreu, RB, Duhr, Gardi]

A sort of residue theorem, that requires understanding the residue at infinite momentum.

Cuts of the scalar bubble

The diagram shows an equation for the scalar bubble. On the left, a bubble with two external lines labeled ϵ , top arc labeled e_1 , and bottom arc labeled e_2 . This is followed by a tilde symbol $\sim$. On the right, there are three terms separated by plus signs. The first term is the bubble with a red vertical line (cut) on the top arc e_1 . The second term is the bubble with a red vertical line (cut) on the bottom arc e_2 . The third term is the bubble with red vertical lines (cuts) on both arcs e_1 and e_2 . The equation ends with a period.

Now in $D = 2 - 2\epsilon$.

$$l_2 = \frac{1}{p^2(w - \bar{w})} \log \frac{\bar{w}(1 - w)}{w(1 - \bar{w})} + \mathcal{O}(\epsilon).$$

$$C_1 l_2 = \frac{1}{p^2(w - \bar{w})} + \mathcal{O}(\epsilon),$$

$$C_2 l_2 = \frac{1}{p^2(w - \bar{w})} + \mathcal{O}(\epsilon),$$

$$C_{12} l_2 = -\frac{2}{p^2(w - \bar{w})} + \mathcal{O}(\epsilon).$$

(factors of $i\pi$ have been stripped)

Cuts of the scalar bubble

$$\epsilon \text{ (bubble)} \sim \text{ (bubble with cut on } e_1) + \text{ (bubble with cut on } e_2) + \text{ (bubble with cuts on } e_1 \text{ and } e_2) .$$

Now in $D = 2 - 2\epsilon$.

$$I_2 = - \frac{e^{\gamma E \epsilon} \Gamma(1 + \epsilon)}{\epsilon} \frac{(-p^2)^{-1-\epsilon}}{(w - \bar{w})^{1+\epsilon}} \left[w^{-\epsilon} {}_2F_1 \left(-\epsilon, 1 + \epsilon; 1 - \epsilon; \frac{w}{w - \bar{w}} \right) - (w - 1)^{-\epsilon} {}_2F_1 \left(-\epsilon, 1 + \epsilon; 1 - \epsilon; \frac{w - 1}{w - \bar{w}} \right) \right]$$

$$C_1 I_2 = - \frac{e^{\gamma E \epsilon}}{\Gamma(1 - \epsilon)} \frac{(-p^2)^{-1-\epsilon}}{w(w\bar{w})^\epsilon} {}_2F_1 \left(1, 1 + \epsilon; 1 - \epsilon; \frac{\bar{w}}{w} \right) \quad \sim \text{Disc}_{m_1^2} I_2$$

$$C_2 I_2 = - \frac{e^{\gamma E \epsilon}}{\Gamma(1 - \epsilon)} \frac{(-p^2)^{-1-\epsilon} (1 - w)^{-\epsilon}}{(w - \bar{w})^{1+\epsilon}} {}_2F_1 \left(-\epsilon, 1 + \epsilon; 1 - \epsilon; \frac{w - 1}{w - \bar{w}} \right) \quad \sim \text{Disc}_{m_2^2} I_2$$

$$C_{12} I_2 = - 2 \frac{e^{\gamma E \epsilon} \Gamma(1 - \epsilon)}{\Gamma(1 - 2\epsilon)} (p^2)^{-1-\epsilon} (w - \bar{w})^{-1-2\epsilon} \quad \sim \text{Disc}_{p^2} I_2$$

There is a simpler explanation for the relation!

Feynman/Schwinger trick:

$$\frac{1}{\prod_i A_i} = \int_{\alpha_i \geq 0} d^E \alpha \frac{\delta(1 - \sum_i \alpha_i)}{(\sum_i \alpha_i A_i)^E},$$

where $A_i = (k - q_i)^2 - m_i^2 + i\epsilon$.

The Feynman integral becomes an integral over these parameters as well as loop momenta.

The cut conditions are some subset of $A_i = 0$.

If we then perform the loop momentum integral, we find the following parametric representation:

$$I = \Gamma\left(E - \frac{LD}{2}\right) \int_{\alpha_i \geq 0} d^E \alpha \delta\left(1 - \sum_{i \in S} \alpha_i\right) \frac{\mathcal{U}^{E-(L+1)D/2}}{\mathcal{F}^{E-LD/2}}.$$

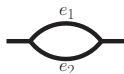
$\mathcal{U}$ and $\mathcal{F}$ are graph polynomials in α_i .

$$\mathcal{U} = \sum_{T \in \mathcal{T}_1} \prod_{e_i \notin T} \alpha_i,$$

$$\mathcal{F} = \mathcal{U} \sum_{i=1}^E \alpha_i m_i^2 - \sum_{(T_1, T_2) \in \mathcal{T}_2} \left(\prod_{e_i \notin (T_1, T_2)} \alpha_i \right) q_{(T_1, T_2)}^2.$$

$\mathcal{T}_1$ is the set of spanning trees, $\mathcal{T}_2$ is the set of spanning 2-forests.

Example: bubble



$$\begin{aligned} I &= \Gamma\left(2 - \frac{D}{2}\right) \int_{\alpha_i \geq 0} \frac{d\alpha_1 d\alpha_2 \delta(1 - \alpha_1 - \alpha_2) (\alpha_1 + \alpha_2)^{2-D}}{[-\alpha_1 \alpha_2 p^2 + (\alpha_1 + \alpha_2)(\alpha_1 m_1^2 + \alpha_2 m_2^2)]^{2-D/2}} \\ &= \Gamma\left(2 - \frac{D}{2}\right) \int_0^1 dx \left[-x(1-x)p^2 + (1-x)m_1^2 + xm_2^2\right]^{D/2-2} \\ &= \Gamma\left(2 - \frac{D}{2}\right) \int_0^1 dx \left[p^2(x-w)(x-\bar{w})\right]^{D/2-2} \end{aligned}$$

What are the cuts? The on-shell condition is inaccessible!

Cuts are discontinuities

$$I = \Gamma \left(E - \frac{LD}{2} \right) \int_{\alpha_i \geq 0} d^E \alpha \, \delta \left(1 - \sum_{i \in S} \alpha_i \right) \frac{\mathcal{U}^{E-(L+1)D/2}}{\mathcal{F}^{E-LD/2}},$$

$$\mathcal{U} = \sum_{T \in \mathcal{T}_1} \prod_{e_i \notin T} \alpha_i,$$

$$\mathcal{F} = \mathcal{U} \sum_{i=1}^E \alpha_i m_i^2 - \sum_{(T_1, T_2) \in \mathcal{T}_2} \left(\prod_{e_i \notin (T_1, T_2)} \alpha_i \right) q_{(T_1, T_2)}^2.$$

Kinematic dependence is all in $\mathcal{F}$.

Discontinuities appear when $\mathcal{F}$ becomes negative. The change happens at $\mathcal{F} = 0$.

$\mathcal{F} = 0$ can be chosen as a **boundary** of the integration region for a cut.

$$I = \Gamma\left(E - \frac{LD}{2}\right) \int_{\alpha_i \geq 0} d^E \alpha \, \delta\left(1 - \sum_{i \in S} \alpha_i\right) \frac{\mathcal{U}^{E-(L+1)D/2}}{\mathcal{F}^{E-LD/2}},$$

The original integral has boundaries $\alpha_i = 0$.

Cuts can exchange $\alpha_i = 0 \longleftrightarrow k_i^2 - m_i^2 = 0$.

We also include $\mathcal{F} = 0$ in the boundaries of the integration regions.

Example: bubble

$$\text{bubble}(e_1, e_2) \sim \int_0^1 dx \left[p^2(x-w)(x-\bar{w}) \right]^{D/2-2}.$$

$$\mathcal{F}=0 \Leftrightarrow x=w \text{ or } x=\bar{w}. \quad \alpha_1=0 \Leftrightarrow x=1. \quad \alpha_2=0 \Leftrightarrow x=0.$$

$$\text{bubble}(e_1, e_2) \sim \int_0^{\bar{w}} dx \left[p^2(x-w)(x-\bar{w}) \right]^{D/2-2}. \quad \text{no } \alpha_1=0$$

$$\text{bubble}(e_1, e_2) \sim \int_w^1 dx \left[p^2(x-w)(x-\bar{w}) \right]^{D/2-2}. \quad \text{no } \alpha_2=0$$

$$\text{bubble}(e_1, e_2) \sim \int_{\bar{w}}^w dx \left[p^2(x-w)(x-\bar{w}) \right]^{D/2-2}. \quad \text{neither } \alpha_1=0 \text{ nor } \alpha_2=0$$

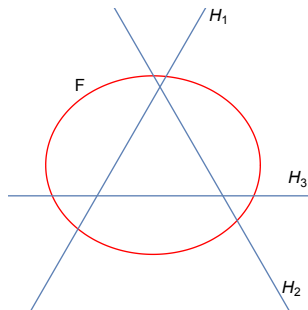
Therefore

$$\text{bubble}(e_1, e_2) \sim \text{bubble}(e_1, e_2)_{\alpha_1=0} + \text{bubble}(e_1, e_2)_{\alpha_2=0} + \text{bubble}(e_1, e_2)_{\text{neither}}.$$

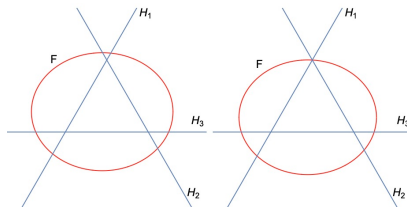
Example: triangle (or any $n \geq 3$)

$$\mathcal{F} = \alpha_1 m_1^2 + \alpha_2 m_2^2 + \alpha_3 m_3^2 - \alpha_1 \alpha_2 p_3^2 - \alpha_1 \alpha_3 p_2^2 - \alpha_2 \alpha_3 p_1^2$$

shown in the plane $\alpha_1 + \alpha_2 + \alpha_3 = 1$,
for Euclidean kinematics with $m_i^2 > 0$, $p_i^2 < 0$.

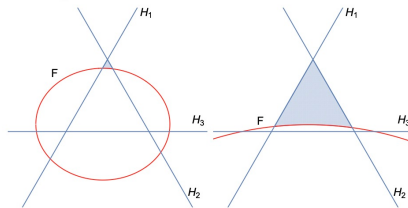


Mass discontinuity of the triangle



(a) m_3^2 is small and positive.

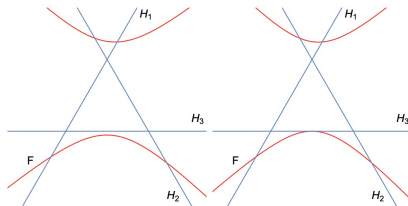
(b) $m_3^2 = 0$.



(c) m_3^2 is small and negative.

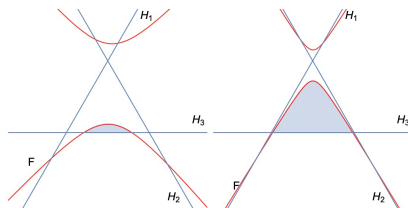
(d) m_3^2 is large and negative.

Momentum discontinuity of the triangle



(a) p_3^2 just below threshold.

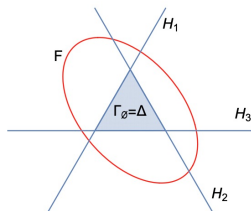
(b) $p_3^2 = (m_1 + m_2)^2$.



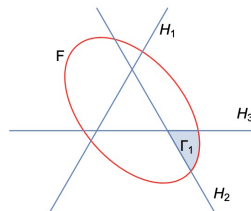
(c) p_3^2 just above threshold.

(d) p_3^2 large compared to other scales.

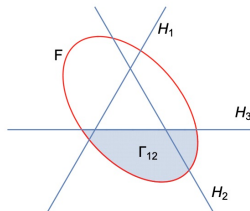
Integration regions of the triangle



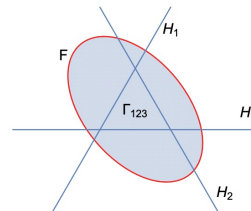
(a) $\Gamma_\emptyset$



(b) Γ_1

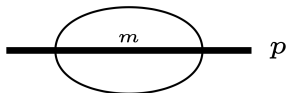


(c) Γ_{12}



(d) Γ_{123}

The linear relation among cuts is just the **inclusion-exclusion principle** here.



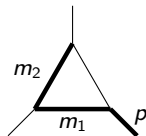
$$\alpha_1 = x, \quad \alpha_2 = 1 - x, \quad \alpha_3 = y$$

$$\mathcal{U} = x(1 - x) + y$$

$$\mathcal{F} = y \left[m^2 y + (m^2 - p^2)x(1 - x) \right]$$

Here $\mathcal{F} = 0$ has two components: a variable parabola, and the coordinate hyperplane $y = 0$.

$\Rightarrow$ **Demonstration**



We can interpret sequences of discontinuities to reconstruct the ‘symbol’ of the function

$$m_1^2 \otimes \left(\frac{m_1^2 - m_2^2}{m_2^2} \right) + m_2^2 \otimes \left(\frac{m_1^2 - m_2^2 - p^2}{m_1^2 - m_2^2} \right) - (m_1^2 - p^2) \otimes \left(\frac{m_1^2 - m_2^2 - p^2}{m_2^2} \right)$$

by following the shapes of the regions as we travel through phase space.

[RB, Hannesdottir]

⇒ **Animations**

- Cuts are a key tool in computing Feynman integrals and amplitudes.
- Three approaches to cuts: delta functions, residues, Feynman parameters.
- Different setups expose relations and geometry differently.
- Introduced parametric approach
 - ▶ to discover relations
 - ▶ to probe sequential discontinuities (symbol bootstrap...)
 - ▶ to expose hypergeometric and coaction properties
 - ▶ for efficient numerical computation?
 - ▶ with new angles on geometric interpretation?
- Seeking the general structure of the underlying functions, in order to tame computation!

The hypergeometric function ${}_2F_1$

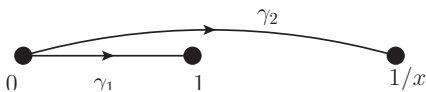
Gauss's original hypergeometric function. Series definition:

$${}_2F_1(\alpha, \beta; \gamma; z) = \sum_{n=0}^{\infty} \frac{(\alpha)_n (\beta)_n}{(\gamma)_n} \frac{z^n}{n!}, \quad \text{where } (x)_n \equiv \frac{\Gamma(x+n)}{\Gamma(x)}.$$

Euler's integral representation:

$${}_2F_1(\alpha, \beta; \gamma; x) = \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\gamma-\alpha)} \int_0^1 du u^{\alpha-1} (1-u)^{\gamma-\alpha-1} (1-ux)^{-\beta}$$

Contours can be extended to a set of two by integrating between different branch points: $\gamma_1 = [0, 1]$, $\gamma_2 = [0, 1/x]$.



$${}_2F_1(\alpha, \beta; \gamma; x) \sim \int_0^1 du u^{\alpha-1} (1-u)^{\gamma-\alpha-1} (1-ux)^{-\beta}$$

$${}_2F_1\left(\alpha, 1+\alpha-\gamma; 1-\beta+\alpha; \frac{1}{x}\right) \sim \int_0^{1/x} du u^{\alpha-1} (1-u)^{\gamma-\alpha-1} (1-ux)^{-\beta}$$