The soft waveform and the non-perturbative waveform

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Recent progress in Amplitudes

Foreword

This talk is based on two papers. One with

F. Alessio and C. Heissenberg, arXiv:2407.04128, JHEP **11** (2024) 124.

and the other with

D. Maskalaniec. arXiv:2609.07788.

Plan of the talk

1. Introduction
2. Soft graviton theorems (tree and loops)
3. Extended soft operator
4. Properties of Dp-branes
5. Uses of the boundary state
6. Elastic scattering of tachyon on a D0 brane
7. Moving D0-branes
8. 5-point amplitude with 4 D0- branes and one graviton
9. Elastic scattering
10. Field-theory calculation with world-line formalism
11. Outlook

Introduction

- In string theory we have four kinds of states.
- Perturbative open and closed string states with mass given by

$$1 + \alpha' M^2_{open} = N \longrightarrow M^2_{open} = \frac{N-1}{\alpha'} \quad ; \quad N = 0, 1 \ldots$$

$$2 + \frac{\alpha'}{2} M^2_{closed} = N \longrightarrow M^2_{closed} = \frac{2(N-2)}{\alpha'} \quad , \quad N = 0, 1, 2 \ldots$$

  They get infinite mass and disappear in the field theory limit ($\alpha' \to 0$) unless $N = 1$ or $N = 2$.
- Kaluza-Klein states with mass given by

$$M^2 = \frac{N^2}{R^2}$$

  when we compactify a number of extra dimensions and we take the gluon or graviton polarisations along these extra dimensions.
- They can be scalars or photons.

- Non-perturbative string states: Dp-branes.
- They have a mass per unit of p-volume given by

$$M_p = \frac{T_p}{\kappa_D} \;\; ; \;\; T_p = \frac{\sqrt{\pi}}{2^{\frac{D-10}{4}}} (2\pi\alpha')^{\frac{D}{2}-2-p} \;\; ; \;\; \kappa_D = (8\pi G_N^{(D)})^{1/2}$$

- If one is interested in computing amplitudes in field theory one can either perform the calculation directly in field theory writing all Feynman diagrams or using other techniques or write the amplitude in string theory and then perform the limit $\alpha' \to 0$.
- In the case of KK states the mass does not depend on α', while in the two other cases it does.
- In these two cases one must perform the field theory limit keeping the mass fixed.
- The two previous procedures can be used to compute amplitudes both at the tree and loop level in the bosonic string where the multiloop amplitudes are known.
- For computations in Einstein gravity one must eliminate the contribution of the dilaton.

- The amplitudes with four scalars and one graviton and with six scalars in the classical limit have been obtained in both ways.
- We are looking now at the amplitude with six scalars and one graviton and at the Compton amplitude at one loop.
- The hope is that, increasing the number of external legs, the string calculation becomes easier than the field theory one.
- The advantage of performing the calculation in string theory is that one gets also the string corrections.
- In the first part of my talk talk I will discuss the soft part of the waveform in the case of four scalar particles directly in field theory.
- In the second part of my talk I will compute the amplitude for the elastic scattering of two $D0$ branes with the production of a graviton both in field theory and in string theory.
- Then I will show that the field theory limit of the string result reproduces the field theory one in the limit of $\alpha' \to 0$.

- From N open tachyon amplitude get tree diagrams of ϕ^3 theory [Scherk, 1971]
- From 4 gluon amplitude get tree diagrams of gauge theory [Neveu and Scherk, 1972]
- From 4 graviton amplitude get the amplitude in Einstein gravity [Yoneya, 1973]
- Four-point amplitudes for gluons and gravitons obtained from superstring theory at one loop [Brink, Green and Schwarz, 1982]
- Some rules to obtain field theory amplitudes from string theory at tree and loop level [Bern and Kosower, 1992]
- Four-graviton at one loop from string theory [Dunbar and Norridge, 1995]
- One-loop β-function in Yang-Mills theory and two-loops diagrams in ϕ^3 theory from the bosonic string [Di Vecchia, Magnea, Lerda, Marotta and Russo, 1996]
- Two loops gluon amplitudes from superstring [Magnea, Playle, Russo and Sciuto, 2015].

Soft graviton theorems (tree and loops)

- At tree level gauge invariance requires that the amplitude for the emission of a soft graviton can always be written as follows:

$$\mathcal{M}^{\mu\nu}_{N+1}(p_i,p_f,k)=\kappa\left(\hat{\mathcal{S}}^{\mu\nu}_{(-1)}+\hat{\mathcal{S}}^{\mu\nu}_{(0)}+\hat{\mathcal{S}}^{\mu\nu}_{(1)}\right)\mathcal{M}_N(p_i,p_f)+\mathcal{O}(\omega^2)$$

where $\kappa=\sqrt{8\pi G_N}$, $\mathcal{M}_N(p_i,p_f)$ is the amplitude involving N scalar particles and

$$\hat{\mathcal{S}}^{\mu\nu}_{(-1)}=\sum_{j=1}^{N}\frac{p_j^\mu p_j^\nu}{p_j\cdot k}\;;\;\hat{\mathcal{S}}^{\mu\nu}_{(0)}=\sum_{j=1}^{N}\frac{p_j^{(\mu}k_\rho\hat{J}_j^{\nu)\rho}}{2p_j\cdot k}\;\;;\;\;\hat{\mathcal{S}}^{\mu\nu}_{(1)}=\frac{1}{2}\sum_{j=1}^{N}\frac{k_\rho k_\sigma\hat{J}_j^{\mu\rho}\hat{J}_j^{\nu\sigma}}{p_j\cdot k}$$

are the leading, subleading and sub-subleading soft factors, $\hat{J}_j^{\mu\nu}=p_j^\mu\frac{\partial}{\partial p_{j\nu}}-p_j^\nu\frac{\partial}{\partial p_{j\mu}}$ is the orbital angular momentum of the *jth* particle [Bern, Davies, Di Vecchia and Nohle, 1406.6987].

- The first two are universal, while the third contains only the universal part.
- Al one loop there are infrared divergences that were computed by Weinberg in 1965 and that modify the soft theorems.
- Weinberg showed that the infrared divergences exponentiate and one can write the amplitude:

$$\mathcal{M}_N^{\mu\nu}(p_i, p_f, k) = e^{-\mathcal{W}} \mathcal{M}_N^{\mu\nu\ I.F.}(p_i, p_f, k)$$

in terms of the one-loop amplitude $\mathcal{W}$ with a graviton exchanged between two external particles

$$W = G \sum_{n,m} w_{nm} = 4\pi i G$$
$$\times \sum_{n,m} \int \frac{d^D \ell}{(2\pi)^D} \frac{(p_n p_m)^2 - \frac{1}{2} m_n^2 m_m^2}{(\ell^2 - i\epsilon)(\eta_n p_n \ell - i\epsilon)(-\eta_m p_m \ell - i\epsilon)}$$

with $\eta_n = 1(-1)$ for outgoing (ingoing) particle.

- The integral can be computed getting:

$$\mathcal{W} = \frac{G}{\pi} \log \frac{\lambda}{\Lambda} \sum_{j,k=1}^{N} \frac{(p_j \cdot p_k)^2 - \frac{1}{2} m_j^2 m_k^2}{\sqrt{(p_j \cdot p_k)^2 - m_j^2 m_k^2}}$$

$$\times \left[\frac{\eta_j \eta_k}{2} \log \frac{-\eta_j \eta_k p_j \cdot p_k + \sqrt{(p_j \cdot p_k)^2 - m_j^2 m_k^2}}{-\eta_j \eta_k p_j \cdot p_k - \sqrt{(p_j \cdot p_k)^2 - m_j^2 m_k^2}} - i\pi \delta_{\eta_j \eta_k, 1} \right]$$

$$= \frac{G}{\pi} \log \frac{\lambda}{\Lambda}$$

$$\times \sum_{j,k=1}^{N} m_j m_k \eta_j \eta_k \left(\sigma_{jk}^2 - \frac{1}{2} \right) \frac{\log \left(\sigma_{jk} + \sqrt{\sigma_{jk}^2 - 1} \right) - i\pi \delta_{\eta_j \eta_k, 1}}{\sqrt{\sigma_{jk}^2 - 1}}$$

where $\sigma_{jk} = -\frac{p_j p_k}{m_j m_k}$, $-\eta_j \eta_k p_j p_k$ is always a positive quantity and $\lambda(\Lambda)$ is an (IR)(UV) regulator connected to the dimensional regularisation parameter $\epsilon = \frac{4-D}{2}$ by the following relation:

$$\log \frac{\lambda}{\Lambda} = \frac{1}{2\epsilon}$$

- For $j = k$ the imaginary part diverges. Neglect the divergent imaginary part and compute the real part with $\sigma_{jj} = 1$ getting $\frac{m_j^2}{2}$.
- If one of particles is massless we have to take the limit of zero mass of the previous expression getting:

$$\mathcal{W}_{\text{phase}} = \frac{G}{\pi} \log \frac{\lambda}{\Lambda} \sum_{j=1}^{N-1} 2(-p_j \cdot k) \left[\frac{1}{2} \log \frac{4(p_j \cdot k)^2}{m_j^2 \mu^2} - i\pi \delta_{\eta_j \eta_N, 1} \right].$$

  where the terms with $\log m_N$ cancel because of momentum conservation leaving behind an arbitrary energy scale μ.
- For more papers on this approach see
  [Z. Bern, S. Davies and J. Nohle, arXiv:1405.1015]
  [H. Krishna and B. Sahoo, arXiv:2308.16807]
  [S. Agrawal, L. Donnay, K. Nguyen, R. Ruzziconi, 2309.11220]
- The same results were obtained, with different methods, in a series of beautiful papers by
  [Laddha and Sen, 1806.01872, Sahoo and Sen, 1808.03288, Saha, Sahoo and Sen,1912.06413, Sahoo, 2008.04376, Sahoo and Sen, 2105.08739]

- In the classical limit one is left only with the terms without the logs:

$$\mathcal{W}_{\rm reg} \simeq -iG_N m_j m_k \sum_{j\neq k=1}^{N} \frac{\sigma_{jk}^2 - \frac{1}{2}}{\sqrt{\sigma_{jk}^2 - 1}} \delta_{\eta_j \eta_k, 1} \log(\omega L)$$

and

$$\mathcal{W}_{\rm phase} \simeq 2iG_N \log(\omega L) \sum_{j=n+1}^{m} p_j \cdot k$$

where the infrared cutoff has been chosen to be ω (the frequency of the emitted graviton).

- The classical limit is performed by writing the four-momentum of the graviton exchanged as $q^\mu = (\hbar\omega, \frac{\hbar\omega}{c})$ and sending $\hbar \to 0$ keeping ω fixed.
- Then we get

$$\mathcal{M}_N = e^{-\mathcal{W}_{reg}} \mathcal{M}_N^{I.F.} \;\; ; \;\; \mathcal{M}_{N+1} = e^{-\mathcal{W}_{phase}} e^{-\mathcal{W}_{reg}} \mathcal{M}_{N+1}^{I.F.}$$

and the soft theorem becomes:

$$\mathcal{M}_{N+1} = e^{-\mathcal{W}_{phase}} e^{-\mathcal{W}_{reg}} \hat{\mathcal{S}}^{\mu\nu} \mathcal{M}_N^{I.F.} = e^{-\mathcal{W}_{phase}} e^{-\mathcal{W}_{reg}} \hat{\mathcal{S}}^{\mu\nu} e^{\mathcal{W}_{reg}} \mathcal{M}_N$$

- Unlike QCD where the three-gluon interaction is proportional to the gauge coupling constant, in gravity the interaction is proportional to the momentum.
- Then the coupling is small when the graviton becomes soft.
- As a consequence, in gravity, the infrared divergences come only from one loop diagrams.
- They have been all computed in Weinberg paper from 1965.
- Computing the previous expression we get

$$\mathcal{M}^{\mu\nu}_{N+1}(p_i,p_f,k) = \kappa e^{-\mathcal{W}_{\text{phase}}}\Big\{\hat{\mathcal{S}}^{\mu\nu}_{(-1)} + \hat{\mathcal{S}}^{\mu\nu}_{(0)} + \hat{\mathcal{S}}^{\mu\nu}_{(1)} + (\hat{\mathcal{S}}^{\mu\nu}_{(0)}\mathcal{W}_{\text{reg}})$$
$$+ \sum_{j=1}^{N}\frac{k_\rho k_\sigma}{p_j\cdot k}(\hat{J}^{\mu\rho}_j\mathcal{W}_{\text{reg}})\hat{J}^{\nu\sigma}_j + \frac{1}{2}\sum_{j=1}^{N}\frac{k_\rho k_\sigma}{p_j\cdot k}(\hat{J}^{\mu\rho}_j\mathcal{W}_{\text{reg}})(\hat{J}^{\nu\sigma}_j\mathcal{W}_{\text{reg}})$$
$$+ (\hat{\mathcal{S}}^{\mu\nu}_{(1)}\mathcal{W}_{\text{reg}})\Big\}\mathcal{M}_N(p_i,p_f) + \mathcal{O}(\omega^2,\omega^2\log\omega)$$

- Because of the regularisation procedure the term proportional to ω^0 is not anymore universal.
- Actually, going to impact parameter space, already at the tree level in $D = 4$, there is a breakdown of the power series at subleading order and one gets instead non-analytic terms containing $\log \omega$.
- This is connected to the fact that the 4-dimensional eikonal also needs a regularisation procedure.
- We call universal terms those obtained by the previous procedure.
- In some cases as for $\frac{1}{\omega}$, $\log \omega$ and $\omega \log^2 \omega$ this is all that one gets, as stressed by [Krishna and Sahoo, 2308.16807].
- In other cases, as for the term $\omega \log \omega$, one gets additional terms that cannot be obtained by the previous procedure.
- In the following we will discuss the terms $\frac{1}{\omega}$, $\log \omega$ and $\omega \log^2 \omega$.....$\omega^{n-1} \log^n \omega$ that we are going to compute in impact parameter space and resum.

Extended soft operator

- An extended soft operator was constructed

$$\hat{S}_{\rm LLZ} = \sum_a \frac{\varepsilon_\mu \varepsilon_\nu}{p_a \cdot k} \left[p_a^\mu p_a^\nu + p_a^\mu k_\rho \hat{J}_a^{\nu\rho} + \sum_{\ell \geq 2} \frac{k_\rho \hat{J}_a^{\mu\rho} k_\sigma \hat{J}_a^{\nu\sigma}}{\ell!} \left(k^\alpha \frac{\partial}{\partial p_a^\alpha} \right)^{\ell-2} \right]$$

  up to gauge invariant terms
  [Z.Z. Li, H.H. Lin and Q.Q. Zhang,1802.03148]

- With the replacement:

$$k^\mu \frac{\partial}{\partial p_a^\mu} \mapsto -\frac{1}{2} \frac{F_{\mu\nu} \hat{J}_a^{\mu\nu}}{p_a \cdot \varepsilon}, \qquad F_{\mu\nu} = k_\mu \varepsilon_\nu - k_\nu \varepsilon_\mu$$

  one gets the new extended operator:

$$\hat{S}_{\rm ext} = \sum_a \frac{\varepsilon_\mu \varepsilon_\nu}{p_a \cdot k} \times \left[p_a^\mu p_a^\nu + p_a^\mu k_\rho \hat{J}_a^{\nu\rho} + \sum_{\ell \geq 2} \frac{k_\rho \hat{J}_a^{\mu\rho} k_\sigma \hat{J}_a^{\nu\sigma}}{\ell!} \left(-\frac{1}{2} \frac{F_{\mu\nu} \hat{J}_a^{\mu\nu}}{p_a \cdot \varepsilon} \right)^{\ell-2} \right]$$

- Change of operator required from the leading PN expansion.
- Using the identities

$$\varepsilon_\mu \varepsilon_\nu p_a^\mu k_\rho \hat{J}_a^{\nu\rho} = (p_a \cdot \varepsilon)^2 \left(-\frac{1}{2} \frac{F_{\mu\nu} \hat{J}_a^{\mu\nu}}{p_a \cdot \varepsilon} \right)$$

$$\varepsilon_\mu \varepsilon_\nu k_\rho \hat{J}^{\mu\rho} k_\sigma \hat{J}_a^{\nu\sigma} = (p_a \cdot \varepsilon)^2 \left(-\frac{1}{2} \frac{F_{\mu\nu} \hat{J}_a^{\mu\nu}}{p_a \cdot \varepsilon} \right)^2$$

the extended soft operator can be cast into the following exponential form (see also [Bautista and Guevara, 1903.12419])

$$\hat{S}_{\text{ext}} = \sum_a \frac{(p_a \cdot \varepsilon)^2}{p_a \cdot k} e^{\hat{R}_a}, \qquad \hat{R}_a = -\frac{1}{2} \frac{F_{\mu\nu} \hat{J}_a^{\mu\nu}}{p_a \cdot \varepsilon}.$$

- From it we can compute the "soft waveform"

$$\tilde{w} = -ie^{-\mathcal{W}_{\text{phase}}} e^{-\mathcal{W}_{\text{reg}}} \hat{S}_{\text{ext}} e^{\mathcal{W}_{\text{reg}}}$$

obtaining

$$\varepsilon \cdot \tilde{w} \cdot \varepsilon = -ie^{-\mathcal{W}_{\text{phase}}} \sum_a \frac{(p_a \cdot \varepsilon)^2}{p_a \cdot k} e^{\hat{R}_a \mathcal{W}_{\text{reg}}}$$

- Taking into account that

$$\hat{R}_{1,2}\mathcal{W}_{\rm reg} = iE\omega \log\omega G h(\sigma)\ ;\ h(\sigma_{12}) = \frac{\sigma_{12}(2\sigma_{12}^2-3)}{(\sigma_{12}^2-1)^{3/2}}$$

we arrive at the final result for the soft $2 \to 2$ scattering plus a graviton.

$$\varepsilon\cdot\tilde{w}\cdot\varepsilon = -i\,\omega^{2iGE\omega}\left[\sum_{a=1,2}\frac{(p_a\cdot\varepsilon)^2}{p_a\cdot k}\,\omega^{iGh(\sigma_{12})E\omega} + \sum_{a=3,4}\frac{(p_a\cdot\varepsilon)^2}{p_a\cdot k}\,\omega^{-iGh(\sigma_{34})E\omega}\right]\ ;\sigma_{12} = -\frac{p_1p_2}{m_1m_2}; \sigma_{34} = -\frac{p_3p_4}{m_1m_2}$$

provided that the condition $\varepsilon(p_1+p_2)=0$ is satisfied.

- This condition allows one to get rid of the factor $p_a\cdot\varepsilon$ in the denominator.
- The previous result in the probe limit agrees with the one obtained by solving the Teukolsky equation for spin 2 ($s=-2$, 1SF) [Fucito, Morales and Russo, 2408.07329].

- This result has been confirmed in a recent paper in the PN limit [Karan, Khatum, Mahajan, Sahoo and Sen, arxiv:2610.07134]
- They also propose an extension to the case of the scattering of n in m particles valid at all energies.
- It is natural to also include the spin by adding to the extended soft operator the spin part

$$\hat{S}_{\text{ext}} = \sum_a \frac{(p_a \cdot \varepsilon)^2}{p_a \cdot k} e^{-\frac{1}{2}\frac{F_{\mu\nu}\hat{J}_a^{\mu\nu}}{p_a\cdot\varepsilon}} e^{-\frac{1}{2}\frac{F_{\mu\nu}\hat{S}_a^{\mu\nu}}{p_a\cdot\varepsilon}}$$

- It is certainly consistent up the quadratic order in spin.
- One should check that it gives the correct PN expansion.
- For higher order in spin one gets factors of $p_a\varepsilon$ in the denominator that makes the all procedure inconsistent.
- How to cure it?

- One gets similar equations for the electromagnetic waveform

$$\varepsilon\cdot\tilde{w} = -i\,\omega^{2iGE\omega}\left[-\sum_{a=1,2} e_a\,\frac{p_a\cdot\varepsilon}{p_a\cdot k}\,\omega^{2if'(\zeta_{12})E\omega} + \sum_{a=3,4} e_a\,\frac{p_a\cdot\varepsilon}{p_a\cdot k}\,\omega^{-2if'(\zeta_{34})E\omega}\right]; 2f'(\zeta_{ab}) = \frac{e_a e_b}{4\pi}\,\frac{m_a^2 m_b^2}{(\zeta_{ab}^2 - m_a^2 m_b^2)^{3/2}}$$

[Karan, Khatun, Sahoo and Sen, 2501.07328]
and for the massless scalar:

$$\tilde{w} = -i\,\omega^{2iGE\omega}\left[\sum_{a=1,2}\frac{g_a}{p_a\cdot k}\,\omega^{2if'(\zeta_{12})E\omega} + \sum_{a=3,4}\frac{g_a}{p_a\cdot k}\,\omega^{-2if'(\zeta_{34})E\omega}\right]$$

where

$$-2f'(\zeta_{ab}) = \frac{g_a g_b}{4\pi}\,\frac{\zeta_{ab}}{(\zeta_{ab}^2 - m_a^2 m_b^2)^{3/2}}$$

[Duary and Ray, 2502.05941]

Properties of Dp branes

- The Dp-branes are non-perturbative states of string theory.
- They have the property of having open strings attached to them.
- The open string satisfies Neumann boundary conditions in the directions along the world-volume of the Dp-branes

$$\partial_\sigma X^\alpha|_{\sigma=0,\pi} = 0 \quad ; \quad \alpha = 0, 1 \dots p$$

- and Dirichlet boundary conditions in the remaining directions.

$$X^i|_{\sigma=o} = y^i \quad ; \quad X^i|_{\sigma=\pi} = w^i \quad ; \quad i = p+1 \dots D-1$$

y^i and w^i are the coordinates of the two Dp-branes .

- We take σ and τ in the two intervals

$$0 \leq \sigma \leq \pi \quad ; \quad 0 \leq \tau \leq T$$

- The interaction between two Dp-branes is given by one-loop diagram with the exchange of an open string: open-string channel

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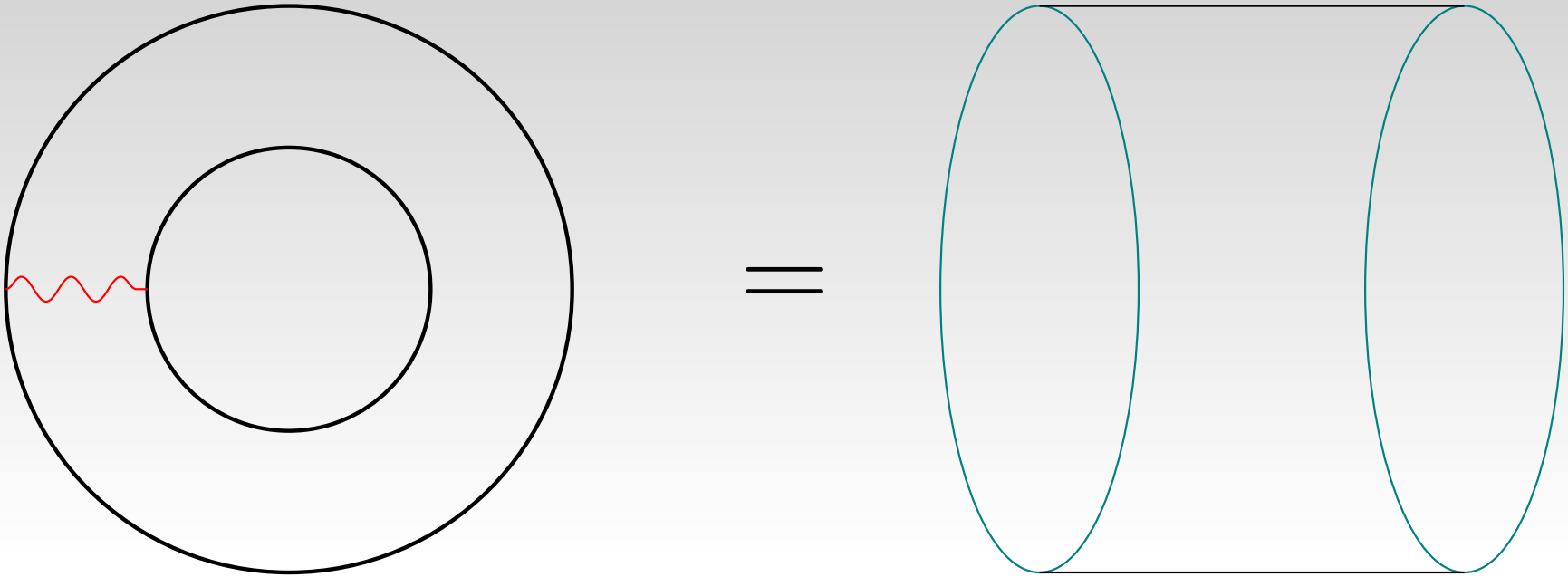

Open-string channel on the left and closed-string channel on the right
The wavy line is the open string exchanged between the two branes.
In the closed-string channel a closed string is exchanged between the two boundary states.

- It can also be computed in the closed-string channel with two boundary states and a closed string propagator.
- In the closed-string channel we have to use different boundary conditions.
- They are obtained from those in the open-string channel by a conformal transformation that essentially exchanges σ with τ.
- In the closed-string channel we have to impose the following conditions on the so-called boundary state

$$\partial_\tau X^\alpha|_{\tau=o}|B_X\rangle \quad , \quad \alpha = 0, 1 \dots p$$
$$X^i|_{\tau=0}|B_X\rangle = y^i \quad , \quad i = p+1 \dots D-1$$

  and similar conditions for $\tau = \hat{T}$.
- In the closed string channel σ, τ vary in the interval $0 \leq \sigma \leq \pi$ and $0 \leq \tau \leq \hat{T}$;$(\frac{\pi}{\hat{T}} = \frac{T}{\pi})$.

- In terms of oscillators and zero modes the previous conditions become

$$(\alpha_n^\alpha + \tilde{\alpha}_{-n}^\alpha)|B_X\rangle \quad , \quad (\alpha_n^i - \tilde{\alpha}_{-n}^i)|B_X\rangle = 0 \;\; ; \;\; n \neq 0$$
$$\hat{p}^\alpha|B_X\rangle = 0 \quad ; \quad (\hat{q}^i - y^i)|B_X\rangle = 0$$

- Introducing the reflection matrix

$$\mathcal{S}^{\mu\nu} = (\eta^{\alpha\beta}, -\delta^{ij})$$

the previous conditions on the oscillators can be grouped together

$$(\alpha_n^\mu + \mathcal{S}_\nu^\mu \tilde{\alpha}_{-n}^\nu)|B_X\rangle = 0 \quad ; \quad n \neq 0$$

- The solution of the previous equations for the bosonic string is given by

$$|B_X\rangle = \frac{T_p}{2}\delta^{D-p-1}(\hat{q}^i - y^i)\left(\prod_{n=1}^{\infty} e^{-\frac{1}{n}\alpha_{-n}\mathcal{S}\tilde{\alpha}_{-n}}\right)|0\rangle_\alpha|0\rangle_{\tilde{\alpha}}|p=0\rangle$$

- The one-loop free energy is given by

$$F = -\frac{1}{2} Tr \log(L_0 - 1) = \int_0^\infty \frac{d\tau}{2\tau} Tr[e^{-2\pi\tau(L_0-1)}]$$

where

$$L_0 = \alpha' k^2 + \frac{y^2}{2\pi\alpha'} + \sum_{n=1}^{\infty} \alpha_{-n}\alpha_n$$

- We get

$$F = V_{p+1}(8\pi^2\alpha')^{-\frac{p+1}{2}} \int_0^\infty \frac{d\tau}{\tau} \tau^{-\frac{p+1}{2}} e^{-\frac{y^2\tau}{2\pi\alpha'}} [f_1(e^{-\pi\tau})]^{-24}$$

where we have taken $D = 26$ and

$$f_1(e^{-\pi\tau}) = e^{-\frac{\pi\tau}{12}} \prod_{n=1}^{\infty}(1 - e^{-2\pi n\tau}) \qquad ; \qquad f_1(e^{-\frac{\pi}{\tau}}) = \sqrt{\tau} f_1(e^{-\pi\tau})$$

- Then we get

$$F = V_{p+1}(8\pi^2\alpha')^{-\frac{p+1}{2}} \int_0^\infty \frac{d\tau}{\tau} \tau^{-\frac{p+1}{2}+12} e^{-\frac{y^2\tau}{2\pi\alpha'}} [f_1(e^{-\frac{\pi}{\tau}})]^{-24}$$

- The same expression is obtained by computing

$$\langle B_X|D|B_X\rangle \;\;;\;\; D=\frac{\alpha'}{4\pi}\int_{|z|\leq 1}\frac{d^2z}{|z|^2}z^{L_0-1}\bar{z}^{\tilde{L}_0-1} \;\;;\;\; |z|=e^{-\pi t}$$

- This allows to compute the normalisation of the boundary state

$$T_p=\frac{\sqrt{\pi}}{2^{\frac{D-10}{4}}}(2\pi\sqrt{\alpha'})^{\frac{D}{2}-2-p}$$

- In superstring one must add the boundary state for the fermions and one gets the same expression with $D=10$.
- The ones for the NS-NS and R-R sectors are given by

$$|B_\psi,\eta\rangle_{NS-NS}=-i\prod_{r=\frac{1}{2}}^{\infty}\left(e^{i\eta\psi_{-r}\cdot S\cdot\tilde{\psi}_{-r}}\right)|0\rangle$$

$$|B_\psi,\eta\rangle_{R-R}=-\prod_{n=1}^{\infty}\left(e^{i\eta\psi_{-n}\cdot S\cdot\tilde{\psi}_{-n}}\right)\left(C\Gamma^0\ldots\Gamma^p\frac{1+i\eta\Gamma_{11}}{1+i\eta}\right)_{AB}|A\rangle|B\rangle|0\rangle_\varphi$$

- One must sum over the spin structures described by $\eta=\pm 1$ and one gets zero: no interaction between supersymmetric Dp-branes.

Uses of the boundary state

- The Dp-branes are described by classical solutions obtained from the following Lagrangian

$$S = \frac{1}{2\kappa^2}\int d^dx\sqrt{-g}\left[\hat{R} - \frac{1}{2}(\nabla\hat{\phi})^2 - \frac{1}{2(p+2)!}\hat{F}^2_{p+2}\right] + \textit{source terms}$$

- One can solve the equations derived from the previous Lagrangian getting

$$d\hat{s}^2 = H(r)^{2A}\eta_{\alpha\beta}dx^\alpha dx^\beta + H(r)^{2B}\eta_{ij}dx^i dx^j$$
$$e^{-\hat{\phi}} = [(H(r)]^\tau \qquad \hat{C}_{01\ldots p} = \sqrt{2\sigma}H^{-1}$$

with

$$A = -\frac{D-p-3}{2(D-2)} \;;\; B = \frac{p+1}{2(D-2)} \;;\; \tau = \frac{a}{2} \;;\; \sigma = \frac{1}{2}$$
$$4 = a^2 + \frac{2(p+1)(D-p-3)}{D-2} \;;\; r^2 = x_\perp^2$$

- H is only a function of $r^2 = x_\perp^2$ given by

$$H = 1 + 2\kappa T_p G(x_\perp)$$

$$G(x_\perp) = [(D-p-3)|x_\perp^{D-p-3}\Omega_{D-2-p}]^{-1} \;\; ; \;\; p < D-3$$

$$G(x_\perp) = \frac{1}{2\pi}\log|x_\perp| \;\; ; \;\; p = D-3 \;\; ; \;\; \mu_p = \sqrt{2}T_p$$

  T_p and μ_p are in the source term and Ω_q is the area of a unit q-dimensional sphere.
- We want to extract from the classical solution the behaviour of the fields at large distance.
- In order to compare with string theory we rescale the fields as follows

$$h_{\mu\nu} = \frac{1}{2\kappa}\hat{h}_{\mu\nu} \qquad \phi = \frac{1}{\kappa(D-2)}\hat{\phi} \qquad C_{01\ldots p} = \frac{\sqrt{2}}{\kappa}\hat{C}_{01\ldots p}$$

  and introduce the following quantity

$$\int dt d^p y d^{D_\perp} x e^{ik_\perp x_\perp} G(x_\perp) = \frac{V_{p+1}}{k_\perp^2}$$

- One gets

$$h_{\mu\nu}(k) = 2T_p \frac{V_{p+1}}{k_\perp^2}(-A, A \dots A, B \dots B)$$

$$\phi(k) = -\frac{a}{\sqrt{D-2}} T_p \frac{V_{p+1}}{k_\perp^2} \qquad C_{01\dots p} = \mp 2\mu_p \frac{V_{p+1}}{k_\perp^2}$$

- The previous expressions can be obtained from the boundary state by computing in the NS sector($c = \frac{1}{2}$ for NS and $c = 0$ for R)

$$J^{\mu\nu} = {}_a\langle 0|_{\tilde{a}}0|\psi^\nu_{1/2}\psi^\mu_{1/2}|D|B\rangle = -\frac{T_p V_{p+1} S^{\mu\nu}}{2k_\perp^2}$$

$$D = \frac{\alpha'}{4\pi}\int \frac{d^2z}{|z|^2} z^{L_0-c}\bar{z}^{\tilde{L}_0-c}$$

- One must project over the various massless fields.
- For the dilaton one gets

$$\epsilon_\phi^{\mu\nu} = \frac{1}{\sqrt{D-2}}\left(\eta^{\mu\nu} - k^\mu \ell^\nu - k^\nu \ell^\mu\right)$$

$$\longrightarrow \phi(k) = \epsilon_\phi^{\mu\nu} J_{\mu\nu} = T_p \frac{V_{p+1}}{k_\perp^2} \frac{D-2p-4}{2\sqrt{D-2}}$$

that agrees with the expression obtained from the classical solution for $D = 10$.

- For the graviton one gets

$$h_{\mu\nu}(k) = \frac{2T_p V_{p+1}}{k_\perp^2}\left(-\frac{1}{4}S_{\mu\nu} - \frac{D-2p-4}{4(D-2)}\eta_{\mu\nu}\right)$$

$$= \frac{2T_p V_{p+1}}{k_\perp^2}\left(-\frac{D-3-p}{2(D-2)}\eta_{\alpha\beta} + \delta_{ij}\frac{p+1}{2(D-2)}\right)$$

that agrees with the expression obtained from the classical solution.

- For the R we get

$$C_{01\ldots p} = \mp 2\mu_p \frac{V_{p+1}}{k_\perp^2}$$

  again in agreement with the classical solution.
- For a review of the boundary state and its uses see [Liccardo and DV, hep-th/9912161 and hep-th/9912275]

Elastic scattering of tachyon on a D0 brane

- The elastic scattering of a tachyon on a D0-brane is given by

$$\int \frac{d^2z_1 d^2z_2}{dV_{MB}} {}_a\langle 0|_{\tilde{a}}\langle 0|\langle p=0| : e^{ik_1 X(z_1,\bar{z}_1)} :: e^{ik_2 X(z_2,\bar{z}_2)} : |B\rangle$$

where

$$|B\rangle = \frac{T_p}{2} \prod_{i=1}^{D-1} \delta(\hat{q}^i - y^i) \prod_{n=1}^{\infty} e^{a_n^\dagger S \tilde{a}_n^\dagger} |0\rangle_a |0\rangle_{\tilde{a}} |p=0\rangle$$

- After inverting the z with the bar one gets the following amplitude

$$\int \frac{d^2z_1 d^2z_2}{dV_{MB}} (z_1-\bar{z}_1)^{\frac{\alpha'}{2}k_1 S k_1} (z_2-\bar{z}_2)^{\frac{\alpha'}{2}k_2 S k_2} [(z_1-z_2)(\bar{z}_1-\bar{z}_2)]^{\frac{\alpha'}{2}k_1 k_2}$$

$$\times [(z_1-\bar{z}_1)(\bar{z}_1-z_2)]^{\frac{\alpha'}{2}k_1 S k_2} \quad ; \quad dV_{Mb} = \frac{dz_1 dz_2 d\bar{z}_1}{(z_1-\bar{z}_1)(z_1-z_2)(\bar{z}_1-z_2)}$$

- Choosing $z_1 = i$ and $z_2 = iy$ we get

$$\int_0^1 dy(1-y^2)y^{\frac{\alpha'}{2}k_1 Sk_2}(1-y)^{\alpha' k_1 k_2}(1+y)^{\alpha' k_1 Sk_2}$$

It is convenient to change variable as $y = \frac{1-\sqrt{x}}{1+\sqrt{x}}$ to get

$$\int_0^1 dx(1+\sqrt{x})^{2-\alpha' k_{2\perp}^2}(1-\sqrt{x})^{2-\alpha' k_{2\perp}^2} x^{\frac{\alpha'}{2}k_1 k_2}$$

$$= \int_0^1 dx(1-x)^{2-\alpha' k_{2\perp}^2} x^{\frac{\alpha'}{2}k_1 k_2} = \frac{\Gamma(-\frac{\alpha' t}{4}-1)\Gamma(-\alpha' s-1)}{\Gamma(-\frac{\alpha' t}{4}-2-\alpha' s)}$$

where

$$t = -(k_1+k_2)^2 \;\; ; \;\; \alpha' s = \alpha'(k_2^0)^2 = \alpha' k_{2\perp}^2 - 4$$

- Closed string poles exchanged in the t-channel: $2 + \frac{\alpha'}{2}t = N$
- Open string poles in the s-channel: $1 + \alpha' s = M$.
- N, M are non-negative integers.

Moving D0-branes

- For describing the scattering of D0-branes we need to boost the boundary state.
- In this way we obtain

$$|V_2, y_2\rangle_{\rm B} = \frac{N_2 T_0}{2}\sqrt{1-V_2^2}\,\delta(\hat{q}^1 - V_2\hat{q}^0)\left[\prod_{i=2}^{D-1}\delta(\hat{q}^i - y_2^i)\right]$$
$$\times\left[\prod_{k=2}^{D-1} e^{\sum_{n=1}^{\infty} a^\dagger_{n,k}\,\tilde{a}^\dagger_{n,k}}\right] e^{-\sum_{n=1}^{\infty} a^\dagger_n\cdot M(V_2)\cdot\tilde{a}^\dagger_n}|0\rangle_a|0\rangle_{\tilde{a}}|p=0\rangle$$

and

$${}_{\rm B}\langle V_1, y_1| = \frac{N_1 T_0}{2}\langle p=0|_a\langle 0|_{\tilde{a}}\langle 0|\sqrt{1-V_1^2}\,\delta(q^1 - V_1 q^0)\left[\prod_{i=2}^{D-1}\delta(\hat{q}^i - y_1^i\right.$$
$$\times\left[\prod_{k=1}^{D-1} e^{\sum_{n=1}^{\infty} a_{n,k}\,\tilde{a}_{n,k}}\right] e^{-\sum_{n=1}^{\infty} a_n\cdot M(V_1)\cdot\tilde{a}_n}$$

- $V_{1,2}$ are the four-velocities of the two branes.

- We assume that they are moving along direction 1 and they are located at a distance $y^{1,2}$ in the other directions.
- It is convenient to use the rapidities given by

$$V_{1,2} = \tanh v_{1,2}$$

- The matrices $M(V)$ appearing in the boundary states are

$$M(V) = \begin{pmatrix} \frac{1+V^2}{1-V^2} & \frac{2V}{1-V^2} \\ \frac{2V}{1-V^2} & \frac{1+V^2}{1-V^2} \end{pmatrix} = \begin{pmatrix} \cosh(2v) & \sinh(2v) \\ \sinh(2v) & \cosh(2v) \end{pmatrix}$$

  with $V = V_{1,2}$ and $v = v_{1,2}$.
- In the following we want to compute the five-point amplitude involving the elastic scattering of two moving D0-branes with the production of a graviton.

5-point amplitude with 4 D0- branes and one graviton

- The 5-point amplitude is given by

$$A_5 = \frac{\alpha'}{4}\frac{\kappa_D}{2\pi}\int\frac{d^2w}{|w|^2}\int d^2z\langle V_1, y_1|W^{\mu\nu}(z,\bar{z})w^{L_0-1}\bar{w}^{\bar{L}_0-1}|V_2, y_2\rangle$$

  where $W^{\mu\nu}$ is the vertex operator of the graviton

$$\epsilon_\mu\tilde{\epsilon}_\nu W^{\mu\nu}(z,\bar{z}) = \frac{2i}{\sqrt{\alpha'}}\epsilon_\mu\partial_z X^\mu\frac{2i}{\sqrt{\alpha'}}\tilde{\epsilon}_\nu\partial_{\bar{z}}X^\nu e^{ik\cdot X}, \quad k^2 = \epsilon\cdot k = \tilde{\epsilon}\cdot k = 0$$

  and the integral is performed in the interval $0 \leq |w| \leq 1$ and $|w| \leq |z| \leq 1$.
- The calculation is performed by exponentiating the terms containing the graviton polarisations.
- It is rather involved and at the end we get the following result.

$$A_5 = \frac{\kappa_D}{2\pi}\frac{\alpha' N_1 N_2 T_0^2}{8\pi|\sinh(v_{12})|}\int \frac{d^{D-2}q}{(2\pi)^{D-2}} e^{iq\cdot b}\int_0^1 \frac{d^2w}{|w|^4}\int_{|w|} d^2z\,|z|^{\frac{\alpha'}{2}(r^2-q^2)}$$

$$\times |w|^{\frac{\alpha'}{2}q^2} e^{\frac{\sqrt{2\alpha'}}{2}q(\frac{\epsilon}{z}+\frac{\tilde\epsilon}{\bar z})}\prod_{n=1}^{\infty}(1-2|w|^{2n}\cosh(2v_{21})+|w|^{4n})^{-1}$$

$$\times (1-|w|^{2n})^{4-D+2\alpha'\sinh^2(nv_{12})(k_0^2-k_1^2)}$$

$$\times \prod_{n=0}^{\infty}\Big[(1-|z|^2|w|^{2n})^{-2\tilde A_{v_1+nv_{12}}A_{v_1+nv_{12}}-\tilde A\cdot A}$$

$$\times (1-|\frac{w}{z}|^2|w|^{2n})^{-2\tilde A_{v_2+nv_{21}}A_{v_2+nv_{21}}-\tilde A\cdot A}\Big]$$

$$\times \prod_{n=1}^{\infty} e^{\sqrt{\frac{\alpha'}{2}}\frac{4|w|^{2n}}{1-|w|^{2n}}\cosh(nv_{12})\left[\frac{1}{z}(\epsilon_{nv_1}k_{nv_2}-\epsilon_{nv_2}k_{nv_1})+\frac{1}{\bar z}(\tilde\epsilon_{nv_1}k_{nv_2}-\tilde\epsilon_{nv_2}k_{nv_1})\right]}.$$

$$\times \prod_{n=1}^{\infty} e^{\sqrt{\frac{\alpha'}{2}}\frac{4|w|^{2n}}{1-|w|^{2n}}\cosh(nv_{12})\left[\frac{1}{z}(\epsilon_{nv_1}k_{nv_2}-\epsilon_{nv_2}k_{nv_1})+\frac{1}{\bar z}(\tilde\epsilon_{nv_1}k_{nv_2}-\tilde\epsilon_{nv_2}k_{nv_1})\right]}$$

where $A = \frac{\sqrt{2\alpha'}}{2}k + \epsilon\partial_z$, $\tilde A = \frac{\sqrt{2\alpha'}}{2}k + \tilde\epsilon\partial_{\bar z}z$ and $X_v = X^0\cosh v - X^1\sinh v$; r=q+k.

- In order to perform the field theory limit we look at the partition function:

$$\frac{1}{|w|^2}\prod_{n=1}^{\infty}\frac{1}{(1-2|w|^{2n}\cosh(2v_{12})+|w|^{4n})(1-|w|^{2n})^{D-4}}$$
$$\sim\frac{1}{|w|^2}+(D-4+2\cosh(2v_{12}))+\mathcal{O}(|w|^2).$$

- The first term corresponds to tachyon exchange, while the second term correspond to massless states exchange
- Naively one would think that only massless states contribute in the field theory limit.
- It turns out that also the tachyon exchange gives a contribution in the field theory limit.
- Higher mass states are irrelevant in the field theory limit.

- The contribution of the massless states is given by

$$A_5^{\text{massless}} = \frac{\kappa_D}{2\pi}\frac{\pi N_1 N_2 T_0^2}{8|\sinh(v_{12})|}(D-4+2\cosh(2v_{12}))\int \frac{\mathrm{d}^{D-2}q}{(2\pi)^{D-2}} e^{iq\cdot b}$$
$$\times\left\{\frac{8(q\cdot\epsilon)(q\cdot\tilde{\epsilon})}{q^2 r^2} - 4(\epsilon\cdot\tilde{\epsilon})\left(\frac{1}{q^2}+\frac{1}{r^2}\right)\right.$$
$$-\frac{4}{k_{v_1}^2 q^2}\left(k_{v_1}\epsilon_{v_1}(q\cdot\tilde{\epsilon}) + k_{v_1}\tilde{\epsilon}_{v_1}(q\cdot\epsilon) - (\tilde{\epsilon}_{v_1}\epsilon_{v_1})(k\cdot q)\right)$$
$$+\frac{4}{k_{v_2}^2 r^2}\left(k_{v_2}\epsilon_{v_2}(q\cdot\tilde{\epsilon}) + k_{v_2}\tilde{\epsilon}_{v_2}(q\cdot\epsilon) - (\tilde{\epsilon}_{v_2}\epsilon_{v_2})(k\cdot q)\right)$$
$$\left.+(\epsilon\cdot\tilde{\epsilon})\left(\frac{r^2}{k_{v_1}^2 q^2}+\frac{q^2}{k_{v_2}^2 r^2}\right)\right\} \qquad ; \qquad r = k+q$$

- That of the tachyon is given by

$$
\begin{aligned}
A_5^{\text{tachyonic}} &= \frac{\kappa_D}{2\pi}\frac{\pi N_1 N_2 T_0^2}{8|\sinh(v_{12})|}\int\frac{\mathrm{d}^{D-2}q}{(2\pi)^{D-2}}e^{iq\cdot b}\\
&\times\left\{\frac{32k_{v_1}^2k_{v_2}^2}{q^2r^2}\left(\frac{\epsilon_{v_1}}{k_{v_1}}-\frac{\epsilon_{v_2}}{k_{v_2}}\right)\left(\frac{\tilde\epsilon_{v_1}}{k_{v_1}}-\frac{\tilde\epsilon_{v_2}}{k_{v_2}}\right)\right.\\
&+\frac{32\cosh(v_{12})k_{v_1}k_{v_2}}{q^2r^2}\left[\left(\frac{\epsilon_{v_1}}{k_{v_1}}-\frac{\epsilon_{v_2}}{k_{v_2}}\right)(q\cdot\tilde\epsilon)+\left(\frac{\tilde\epsilon_{v_1}}{k_{v_1}}-\frac{\tilde\epsilon_{v_2}}{k_{v_2}}\right)(q\cdot\epsilon)\right]\\
&-\frac{16\cosh(v_{12})k_{v_1}k_{v_2}}{q^2}\left[\left(\frac{\epsilon_{v_1}}{k_{v_1}}-\frac{\epsilon_{v_2}}{k_{v_2}}\right)\frac{\tilde\epsilon_{v_1}}{k_{v_1}}+\left(\frac{\tilde\epsilon_{v_1}}{k_{v_1}}-\frac{\tilde\epsilon_{v_2}}{k_{v_2}}\right)\frac{\epsilon_{v_1}}{k_{v_1}}\right]\\
&+\frac{16\cosh(v_{12})k_{v_1}k_{v_2}}{r^2}\left[\left(\frac{\epsilon_{v_1}}{k_{v_1}}-\frac{\epsilon_{v_2}}{k_{v_2}}\right)\frac{\tilde\epsilon_{v_2}}{k_{v_2}}+\left(\frac{\tilde\epsilon_{v_1}}{k_{v_1}}-\frac{\tilde\epsilon_{v_2}}{k_{v_2}}\right)\frac{\epsilon_{v_2}}{k_{v_2}}\right]\\
&+16(\epsilon\cdot\tilde\epsilon)\left(\frac{k_{v_1}^2+k_{v_2}^2}{q^2r^2}+\frac{\cosh(v_{12})(\cosh(v_{12})k_{v_1}-k_{v_2})}{k_{v_1}q^2}\right.\\
&\left.\left.+\frac{\cosh(v_{12})(\cosh(v_{12})k_{v_2}-k_{v_2})}{k_{v_2}r^2}\right)\right\}
\end{aligned}
$$

Both contributions are gauge invariant.

- We now introduce the velocities of the two incoming D-particles participating in the scattering,

$$u_1^\mu = (\cosh v_1, \sinh v_1, 0_\perp), \quad u_2^\mu = (\cosh v_2, \sinh v_2, 0_\perp)$$

- and define the relative boost factor

$$\gamma = -u_1 \cdot u_2 = \cosh(v_{12})$$

- With these conventions, we have

$$k_{v_1} = -u_1 \cdot k, \quad k_{v_2} = -u_2 \cdot k, \quad \epsilon_{v_1} = -u_1 \cdot \epsilon, \quad \epsilon_{v_2} = -\epsilon \cdot u_2$$

- It is also convenient to define the following two vectors

$$P_{12}^\mu = (k \cdot u_1) u_2^\mu - (k \cdot u_2) u_1^\mu, \quad Q_{12}^\mu = 2q^\mu - \frac{r^2}{u_1 \cdot k} u_1^\mu + \frac{q^2}{u_2 \cdot k} u_2^\mu$$

- They are gauge invariant

$$k_\mu P_{12}^\mu = k_\mu Q_{12}^\mu = 0$$

- In terms of these quantities, the massless contribution can be written in a more compact form:

$$A_5^{\text{massless}} = \frac{\kappa_D^3 N_1 N_2 m_0^2}{4\sqrt{\gamma^2-1}}\left(\frac{D-6}{4}+\gamma^2\right)\int \frac{\mathrm{d}^{D-2}q}{(2\pi)^{D-2}} e^{iq\cdot b} \times \left\{ \frac{2(\epsilon\cdot Q_{12})(\tilde{\epsilon}\cdot Q_{12})}{q^2 r^2} + (\epsilon\cdot\tilde{\epsilon})\left(\frac{r^2}{k_{v_1}^2 q^2}+\frac{q^2}{k_{v_2}^2 r^2}-\frac{4}{q^2}-\frac{4}{r^2}\right)\right.$$

where we have used the relation $\cosh(2v_{12}) = 2\gamma^2 - 1$.

- The tachyonic contribution becomes

$$A_5^{\text{tachyonic}} = \frac{\kappa_D^3 N_1 N_2 m_0^2}{16\sqrt{\gamma^2-1}} \int \frac{\mathrm{d}^{D-2}q}{(2\pi)^{D-2}} e^{iq\cdot b}$$
$$\times \left\{ \frac{16[2(\epsilon\cdot P_{12})(\tilde{\epsilon}\cdot P_{12}) - \gamma(\epsilon\cdot P_{12})(\tilde{\epsilon}\cdot Q_{12}) - \gamma(\tilde{\epsilon}\cdot P_{12})(\epsilon\cdot Q_{12})]}{q^2 r^2} \right.$$
$$\left. + 16(\epsilon\cdot\tilde{\epsilon})\left(\frac{k_{v_1}^2 + k_{v_2}^2}{q^2 r^2} + \frac{\gamma(\gamma k_{v_1} - k_{v_2})}{k_{v_1} q^2} + \frac{\gamma(\gamma k_{v_2} - k_{v_1})}{k_{v_2} r^2} \right) \right\}.$$

- Alltogether we get

$$A_5 = \frac{\kappa_D^3 N_1 N_2 m_0^2}{4\sqrt{\gamma^2-1}} \int \frac{\mathrm{d}^{D-2}q}{(2\pi)^{D-2}} e^{iq\cdot b}$$
$$\times \left\{ \frac{4[2(\epsilon\cdot P_{12})(\tilde{\epsilon}\cdot P_{12}) - \gamma(\epsilon\cdot P_{12})(\tilde{\epsilon}\cdot Q_{12}) - \gamma(\tilde{\epsilon}\cdot P_{12})(\epsilon\cdot Q_{12})]}{q^2 r^2} \right.$$
$$+ \left(\frac{D-6}{4} + \gamma^2\right) \left[\frac{2(\epsilon\cdot Q_{12})(\tilde{\epsilon}\cdot Q_{12})}{q^2 r^2} + (\epsilon\cdot\tilde{\epsilon}) \left(\frac{r^2}{k_{v_1}^2 q^2} + \frac{q^2}{k_{v_2}^2 r^2} - \frac{4}{q^2} \right.\right.$$
$$\left.\left. - \frac{4}{r^2} \right)\right] + 4(\epsilon\cdot\tilde{\epsilon}) \left(\frac{k_{v_1}^2 + k_{v_2}^2}{q^2 r^2} + \frac{\gamma(\gamma k_{v_1} - k_{v_2})}{k_{v_1} q^2} + \frac{\gamma(\gamma k_{v_2} - k_{v_1})}{k_{v_2} r^2} \right) \Bigg\}$$

- We have used the following relations:

$$T_0^2 = 8\pi G_N m_0^2 = \kappa_D^2 m_0^2$$

- They follow from the following relations

$$m_0 = \frac{1}{\sqrt{\alpha'} g_s}, \quad T_0 = \frac{\sqrt{\pi}}{2^{\frac{D-10}{4}}} (2\pi\sqrt{\alpha'})^{\frac{D-4}{2}}; \quad G_D = 2^{-\frac{D-4}{2}} (2\pi\sqrt{\alpha'})^{D-4} \alpha' g_s^2$$

- The graviton amplitude is given by

$$A_5^{\text{graviton}} = \epsilon_\mu \tilde{\epsilon}_\nu \frac{\kappa_D^3 N_1 N_2 m_0^2}{2\sqrt{\gamma^2-1}} \int \frac{\mathrm{d}^{D-2}q}{(2\pi)^{D-2}} e^{iq\cdot b} \times \left\{ \frac{4P_{12}^{(\mu}(P_{12}^{\nu)} - \gamma Q_{12}^{\nu)})}{q^2 r^2} + \left(\frac{D-6}{4} + \gamma^2\right) \frac{Q_{12}^\mu Q_{12}^\nu}{q^2 r^2} \right\}$$

- To be compared with the same amplitude for 4 perturbative scalars in Einstein gravity

$$A_5^{\text{graviton}} = \epsilon_\mu \tilde{\epsilon}_\nu \frac{m_1 m_2 \kappa_D^3}{2\sqrt{\gamma^2-1}} \int \frac{d^D q}{(2\pi)^{D-2}} e^{ib\cdot q} \times \left[\frac{4P_{12}^{(\mu}(P_{12}^{\nu)} - \gamma Q_{12}^{\nu)})}{q^2 r^2} + \left(\gamma^2 - \frac{1}{D-2}\right) \left(\frac{Q_{12}^\mu Q_{12}^\nu}{q^2 r^2} - \frac{P_{12}^\mu P_{12}^\nu}{k_{v_1}^2 k_{v_2}^2} \right) \right]$$

where $A^{(\mu} B^{\nu)} = \frac{1}{2}(A^\mu B^\nu + A^\nu B^\mu)$.

- By saturating with $\epsilon^\mu \tilde{\epsilon}^\nu \longrightarrow \frac{1}{\sqrt{D-2}}\left(\eta^{\mu\nu} - \frac{k^\mu \bar{k}^\nu + k^\nu \bar{k}^\mu}{k\cdot\bar{k}}\right)$ one gets the dilaton amplitude

$$A_5^{\text{dilaton}} = \frac{\kappa_D^3 N_1 N_2 m_0^2}{4\sqrt{\gamma^2-1}} \frac{D-4}{\sqrt{D-2}} \int \frac{\mathrm{d}^{D-2}q}{(2\pi)^{D-2}} e^{iq\cdot b}$$
$$\times \left\{ 4\left(\frac{k_{v_1}^2 + k_{v_2}^2}{q^2 r^2} + \frac{\gamma(\gamma k_{v_1} - k_{v_2})}{k_{v_1} q^2} + \frac{\gamma(\gamma k_{v_2} - k_{v_1})}{k_{v_2} r^2} \right) \right.$$
$$\left. + \left(\frac{D-6}{4} + \gamma^2\right)\left(\frac{r^2}{q^2 k_{v_1}^2} + \frac{q^2}{r^2 k_{v_2}^2} - \frac{4}{q^2} - \frac{4}{r^2} \right) \right\}.$$

- Graviton amplitude in the supersymmetric case

$$A^{\text{graviton}} = \epsilon_\mu \tilde{\epsilon}_\nu \frac{\kappa_D^3 N_1 N_2 m_0^2}{2\sqrt{\gamma^2-1}} \cdot \int \frac{d^{D-2}q}{(2\pi)^{D-2}} e^{iq\cdot b}$$
$$\times \left\{ \frac{4 P_{12}^{(\mu}(P_{12}^{\nu)} - (\gamma-1)Q_{12}^{\nu)})}{q^2 r^2} + \left(\frac{D-10}{4} + (\gamma-1)^2\right) \frac{Q_{12}^\mu Q_{12}^\nu}{q^2 r^2} \right\}$$

It vanishes for $v_1 = v_2, \gamma = 1$ in $D = 10$: BPS and stable.

Elastic scattering

- If we neglect the graviton vertex operator in the previous expression we get the elastic amplitude with four D0-branes

$$A_4 = \frac{\alpha' N_1 N_2 T_0^2}{8\pi |\sinh(v_{12})|} \int \frac{d^{D-2}q}{(2\pi)^{D-2}} e^{iq\cdot b} \int_0^1 \frac{d^2 w}{|w|^4} |w|^{\frac{\alpha'}{2} q^2}$$
$$\times \prod_{n=1}^{\infty} (1 - 2|w|^{2n} \cosh(2v_{21}) + |w|^{4n})^{-1} (1 - |w|^{2n})^{4-D}$$

[Bachas, 1996 ; Billo', Cangemi, Di Vecchia, 1997]

- In the field theory limit we expand the terms in the second line

$$1 + (D - 4 + 2cosh2v_{12})|w|^2 + \mathcal{O}(|w|^4)$$

where the first term is the tachyon contribution, while the second term is the contribution of the massless states.

- In the field theory limit ($\alpha' \to 0$) only the second term contributes and we get

$$A_4 = \frac{\alpha' N_1 N_2 T_0^2}{8\pi |\sinh(v_{12})|}(D-4+2cosh2v_{12})|$$
$$\times \int \frac{d^{D-2}q}{(2\pi)^{D-2}} e^{iq\cdot b}\left[(2\pi)\int_0^1 d|w||w|^{\frac{\alpha'}{2}q^2-1} = \frac{4\pi}{\alpha' q^2}\right]$$

- Using $T_0^2 = m_0^2\kappa_D^2$; $\sinh v_{12} = \sqrt{\gamma^2-1}$; $cosh2v_{12} = 2\gamma^2-1$ we get

$$A_4 = \frac{8\pi N_1 N_2 m_0^2 G_D}{\sqrt{\gamma^2-1}}\left(\frac{D-6}{4}+\gamma^2\right)\int \frac{d^{D-2}q}{(2\pi)^{D-2}} e^{iq\cdot b}\frac{1}{q^2}$$
$$= \frac{2N_1 N_2 G_D m_0^2 \Gamma(\frac{D-4}{2})(\pi b^2)^{\frac{4-D}{2}}}{\sqrt{\gamma^2-1}}\left(\frac{D-6}{4}+\gamma^2\right)$$

to be compared with the eikonal with four perturbative scalars

$$\frac{2m_1 m_2 G_D \Gamma(\frac{D-4}{2})(\pi b^2)^{\frac{4-D}{2}}}{\sqrt{\gamma^2-1}}(\gamma^2 - \frac{1}{D-2})$$

Field-theory calculation with world-line formalism

- The action consists of two terms.
- The first is the super-gravity action consisting of the metric, the dilaton and the RR vector field given in the Einstein frame by

$$S_{\text{grav}} = \frac{1}{2\kappa_D^2}\int d^D x\sqrt{-G}\left[R(G) - \frac{4}{D-2}G^{\mu\nu}(\partial_\mu\Phi)(\partial_\nu\Phi) - \frac{1}{4}e^{-2a\Phi}G^{\mu\rho}G^{\nu\sigma}F_{\mu\nu}F_{\rho\sigma}\right] \tag{1}$$

  where $a = -\frac{D-4}{D-2}$ and $F_{\mu\nu} - \partial_\mu C_\nu - \partial_\nu C_\mu$.
- The second one is the action of the two world-lines coupled to supergravity

$$S_{\text{D0}}^{(i)} = -m_i\int d\tau\, e^{a\Phi(X_i)}\sqrt{-G_{\mu\nu}(X_i)\dot{X}_i^\mu\dot{X}_i^\nu} + m_i\int d\tau\,\dot{X}_i^\mu C_\mu(X_i).$$

- It is convenient to rewrite it in the Polyakov form as

$$S_{\text{D0}}^{(i)} = \frac{m_i}{2}\int d\tau\left[G_{\mu\nu}(X_i)\dot{X}_i^\mu\dot{X}_i^\nu - e^{2a\Phi(X_i)}\right] + m_i\int d\tau\,\dot{X}_i^\mu C_\mu(X_i)$$

- The pure gravity contribution has been already computed and it is given by

$$\frac{im_1 m_2 \kappa_D^3}{2} \int \frac{d^D q}{(2\pi)^{D-2}} e^{ib\cdot q} \delta(u_1 \cdot q)\delta(u_2 \cdot r)$$
$$\times \left[\frac{4 P_{12}^{(\mu}(P_{12}^{\nu)} - \gamma Q_{12}^{\nu)})}{q^2 r^2} + \left(\gamma^2 - \frac{1}{D-2}\right)\left(\frac{Q_{12}^{\mu} Q_{12}^{\nu}}{q^2 r^2} - \frac{P_{12}^{\mu} P_{12}^{\nu}}{k_{v_1}^2 k_{v_2}^2}\right) \right]$$

  where $A^{(\mu} B^{\nu)} = \frac{1}{2}(A^{\mu} B^{\nu} + A^{\nu} B^{\mu})$
  [Mogull, Plefka and Steinhoff, 2020].
- Adding the diagrams with the contribution of the dilaton, we reproduce the field theory result obtained from the bosonic string or from the NS-NS and R-R sectors of superstring, except for the terms localised at zero impact parameter.
- A reason of this mismatch at $b = 0$ can be connected with the fact that, when the two D0-branes are coincident, the field theory becomes non-abelian.

Outlook

- Solve the problem of mismatch at $b = 0$.
- Compute the 5-point amplitude at one loop in $N = 11$ supergravity with four gravitons having polarisations in the 11th direction.