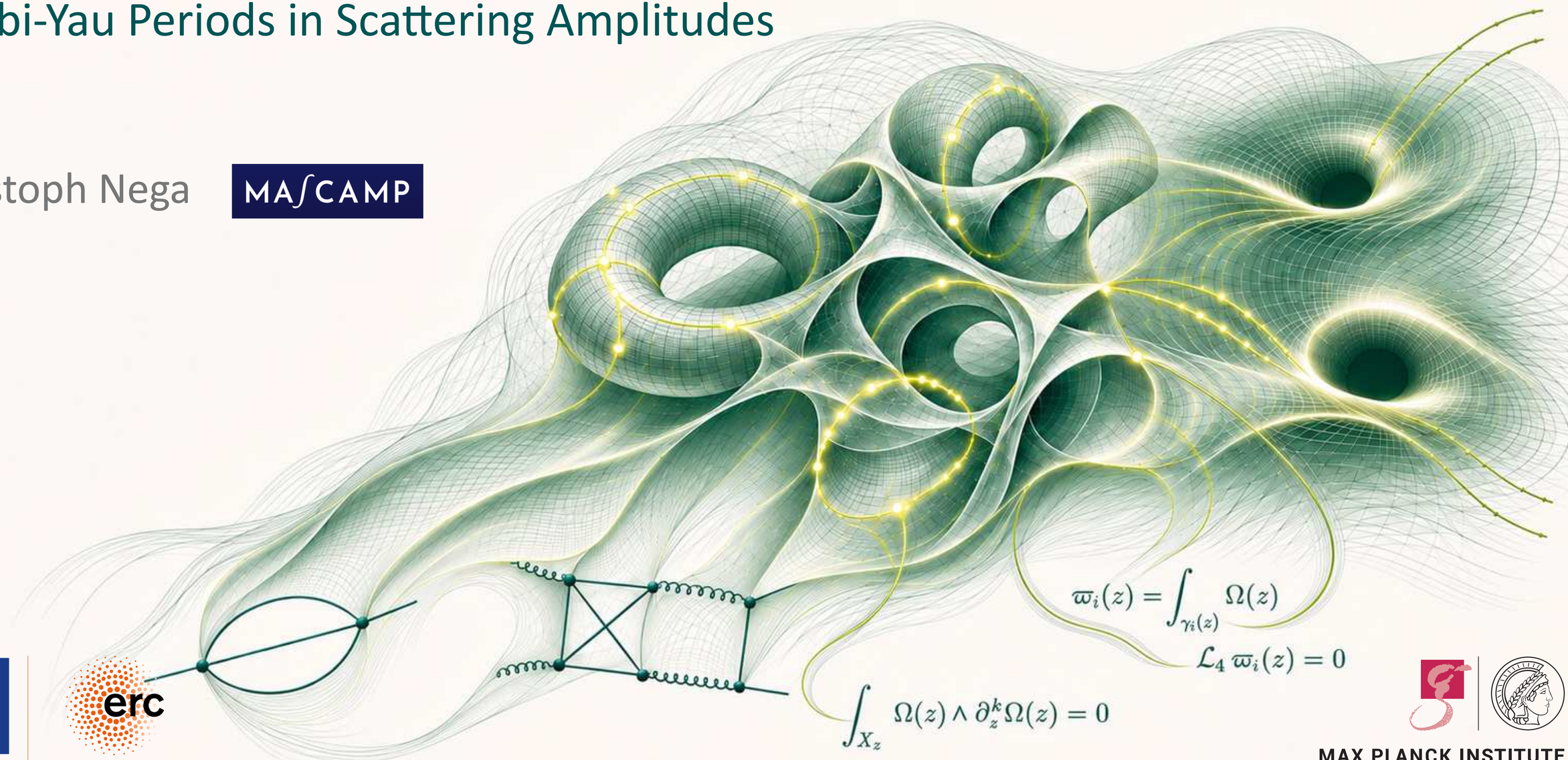
From Particle Scattering to Black Hole Physics

Calabi-Yau Periods in Scattering Amplitudes

Christoph Nega

MA∫CAMP



Funded by the European Union

European Research Council
Established by the European Commission

Recent Progress in Amplitudes

MAX PLANCK INSTITUTE
FOR GRAVITATIONAL PHYSICS
(ALBERT EINSTEIN INSTITUTE)

Collaborations

Joint work with:

Federico Buccioni, Matthias Driesse, Claude Duhr, Felix Forner, Federico Gasparotto, Lennard Görges, Gustav Jakobsen, Albrecht Klemm, Sara Maggio, Andreas von Manteuffel, Cesare C. Mella, Gustav Mogull, Jan Plefka, Benjamin Sauer, Lorenzo Tancredi, Johann Usovitsch, Fabian J. Wagner, and Stefan Weinzierl

Collaborations:

"Integrand Analysis, Leading Singularities and Canonical Bases beyond Polylogarithms" **[1]**,
"Aspects of canonical differential equations for Calabi-Yau geometries and beyond" **[2]**,
"On a procedure to derive ϵ-factorised differential equations beyond polylogarithms" **[3]**,

"QED self-energies up to four loops" **[4]** (appearing soon),
"On the photon self-energy to three loops in QED" **[5]**,
"On the electron self-energy to three loops in QED" **[6]**,

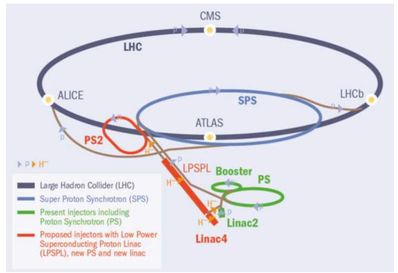
"Analytic two-loop corrections to top-quark pair production at hadron colliders" **[7],**

"Conservative Black Hole Scattering at Fifth Post-Minkowskian and Second Self-Force Order" **[8]**,
"Emergence of Calabi–Yau manifolds in high-precision black-hole scattering" **[9]**

Motivation

- **Feynman integrals** are the backbone of many **perturbative computations**. While they have traditionally been used in **quantum field theory**, they are now also relevant in **classical gravitational problems**.
- **Current** and **future experiments** produce **highly precise data**. Thus, **high-precision predictions** are **essential** to compare this data with theory. This typically means **higher-loop calculations**.

Particle accelerators



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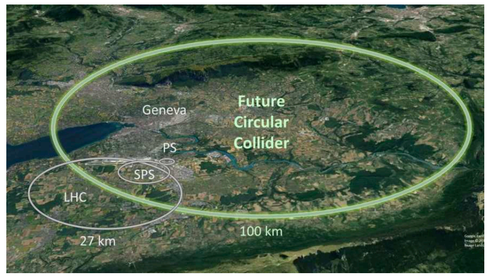

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Gravitational wave detectors



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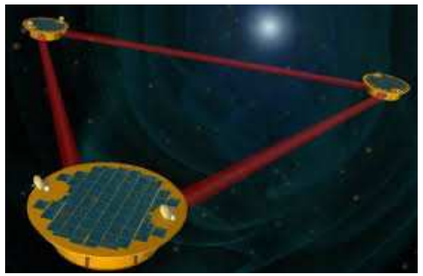

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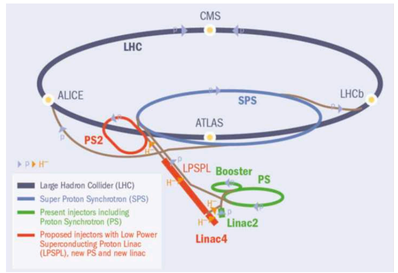

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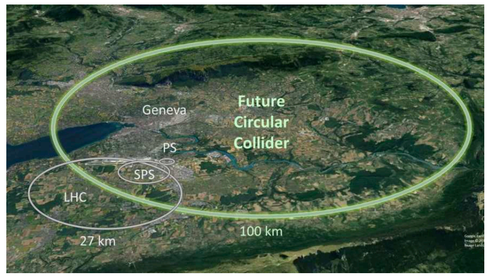

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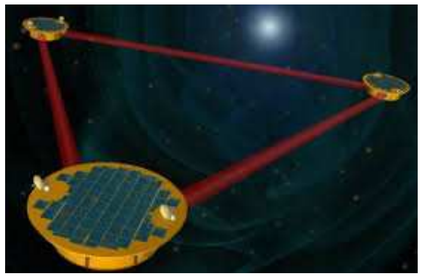

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- In **higher-loop calculations** involving **massive particles**, we have observed **new classes of functions** beyond multiple polylogarithms. Their **analytic structure** is governed by **non-trivial geometries**, particularly **Calabi-Yau manifolds**.

higher dimensions, growth in complexity, additional differential forms



sphere | torus | genus-two | K3 | Calabi-Yau

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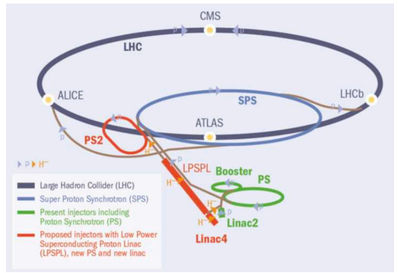

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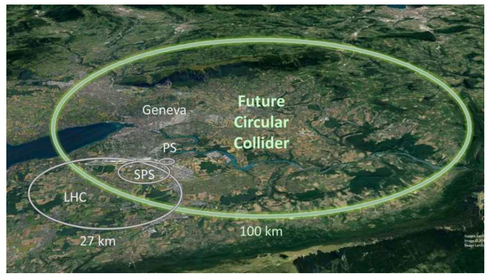

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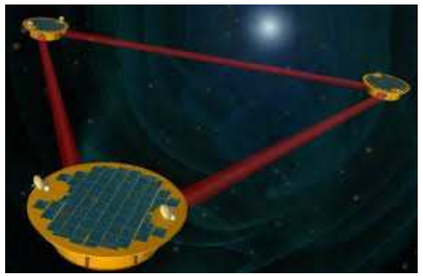

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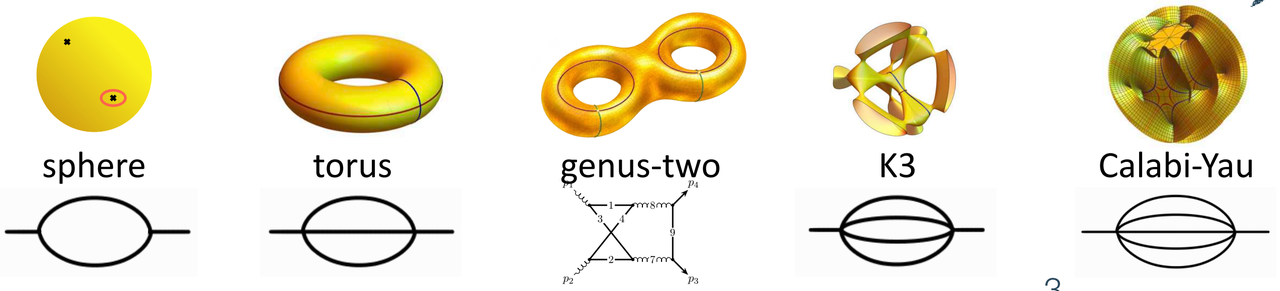

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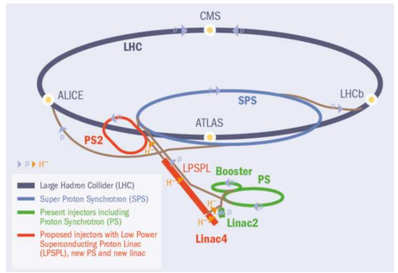

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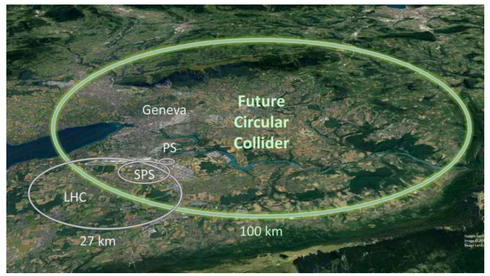

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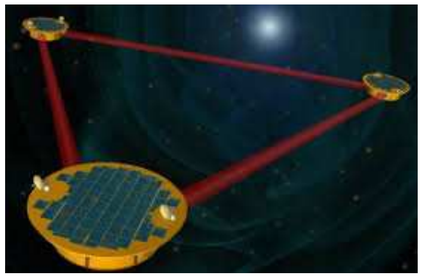

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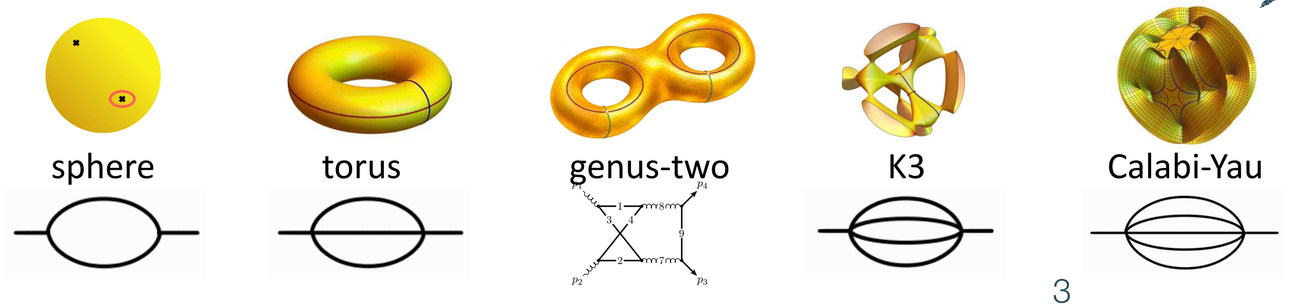

Today, I will argue that we now have good analytic control over function spaces beyond polylogarithms that are associated with these geometries.

What does *analytic* mean?

- **Thomas Gehrmann at HP2 conference:**

What is an analytical result?

Expressed as formula (FORM include file or Mathematica notebook)

- Not really sufficient…

Proposal of a workable definition: an expression which has

- No hidden zeros (also means: can be simplified systematically)
- Prescription to perform a numerical evaluation for all valid input parameters (systematically improvable, e.g. as convergent series)

[Ettore Remiddi]

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[Ettore Remiddi]

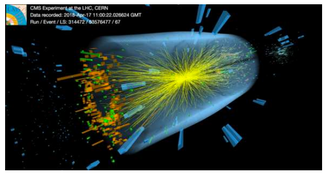
→ In this sense, **our analytic treatment** will yield:

- Simple and compact expressions.
- Strategies for fast numerical evaluations.

Why *canonical* Integrals?

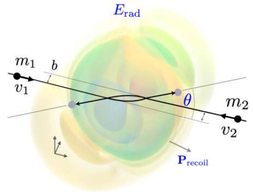
- Many **state-of-the-art scattering amplitude computations** use **canonical integrals**. These integrals make the **analytic properties manifest**.

Particle Physics



standard approach

Gravity



worldline approach

Special analytic properties

poles of $\mathscr{A}$

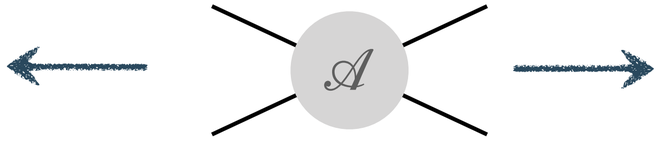
$\sim \frac{1}{s}$

single particles go on-shell

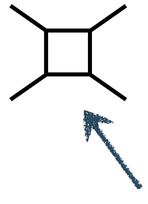
branch cuts of $\mathscr{A}$

$\sim \{\sqrt{s}, \log(s)\}$

multi-particle threshold



Decomposition through scalar Feynman integrals

$\mathscr{A} \sim T^{\mu\nu}$ $R(\underline{s})$ 

tensor structure rational scalar integrals

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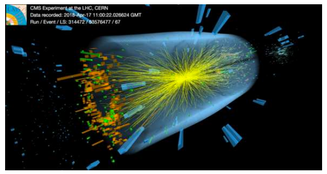

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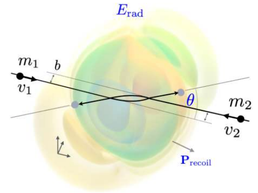

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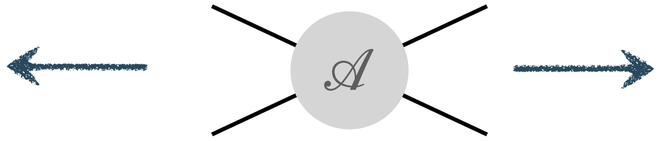

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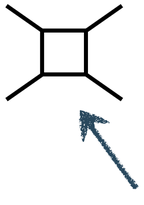

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- Also, **canonical integrals** simplify the **differential equations** and allow for **pure & uniform transcendental** solutions.

$$\frac{\partial}{\partial z_i}\underline{I} = B(\underline{z};\epsilon)\underline{I} \quad \xrightarrow{\textbf{canonical integrals}} \quad \frac{\partial}{\partial z_i}\underline{J} = \epsilon A(\underline{z})\underline{J}$$

[Kotikov, Gehrmann and Remiddi] [Henn]

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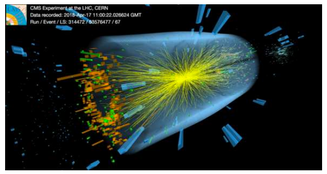

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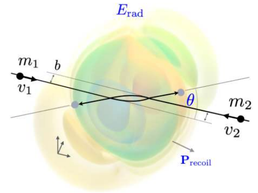

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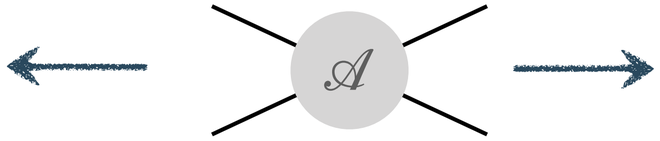

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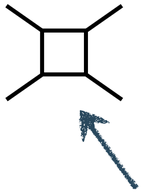

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canonical integrals →

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canonical: [2]

- ϵ-**factorized**
- **single poles**
- **independent cohomology classes**

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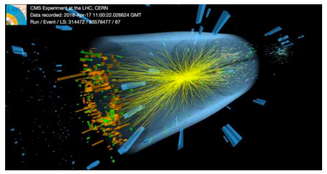

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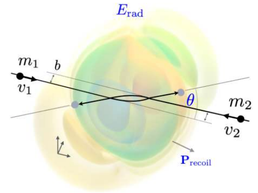

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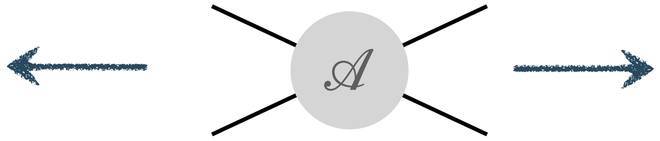

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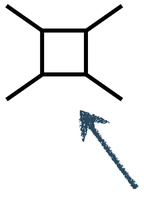

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advantages:

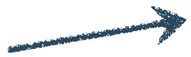
- Formal solution: **pure** & **uniform transcendental iterated integrals** [Chen]
- **Series expansion** for numerics
- **Numerical codes** [Czakon and Tancredi]

How to find them?

- For **polylogarithmic integrals**, it is well-known how to find canonical integrals. One approach is given by **studying the integrand** of candidate integrals.

[Arkani-Hamd et al.]

$$\mathscr{A} \sim T^{\mu\nu} \quad R(\underline{s}) \quad \int \mathrm{d}\log\alpha_r \wedge \ldots \wedge \mathrm{d}\log\alpha_1$$



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$$f(x) = \frac{1}{x}\log(x-1)$$

$$\frac{\mathrm{d}f}{\mathrm{d}x} = -\frac{1}{x}f(x) + \frac{1}{x(x-1)}$$

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(not unipotent)

normalization by leading singularity

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Integrals with **normalized leading singularities** give rise to **canonical integrals**.

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- We have developed a **general method** [1-3] to **compute canonical integrals beyond the polylogarithmic case,** which is divided into the following steps:

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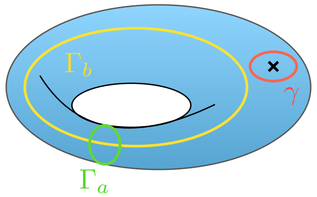
1 Find a **start basis** that reflects the underlying geometry (**mixed Hodge structure**)



1st-kind (no pole)	**2nd-kind** (double pole)	**3rd-kind** (single pole)
$\frac{\mathrm{d}X}{Y}$	$\partial\frac{\mathrm{d}X}{Y}$	$\frac{X\mathrm{d}X}{Y}$

The **derivative basis** is the reason why our approach terminates!

Generalization beyond Polylogs

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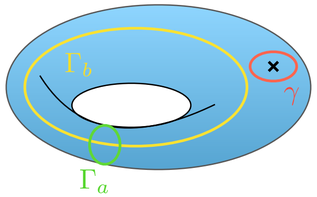

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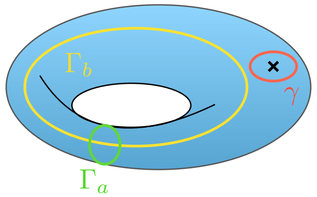

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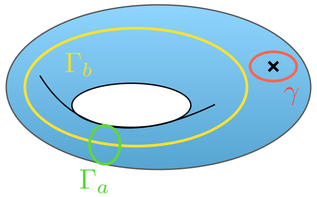

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- In our **recent paper** [1], we have given these **three steps** an **interpretation** in terms of **integrand analysis**.

Leading Singularities beyond Polylogs

- To argue that our procedure [2-3] indeed produces integrals with normalized leading singularities satisfying canonical ϵ-factorized differential equations, we first have to **generalize** the **concept of Leading Singularities**:

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Polylog case:

$$f(x) = \frac{1}{x}\log(x-1) = \int^{x} \mathrm{d}y \frac{1}{x(y-1)}$$

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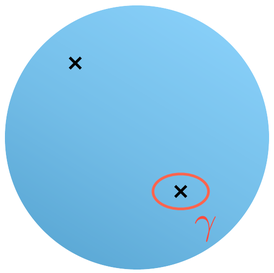

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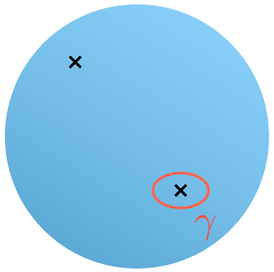

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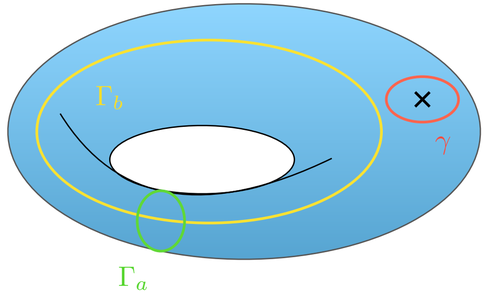

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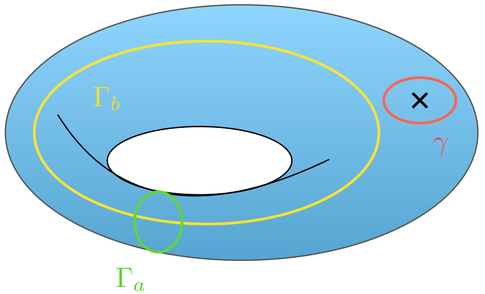

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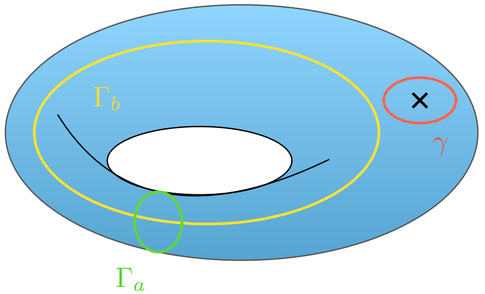

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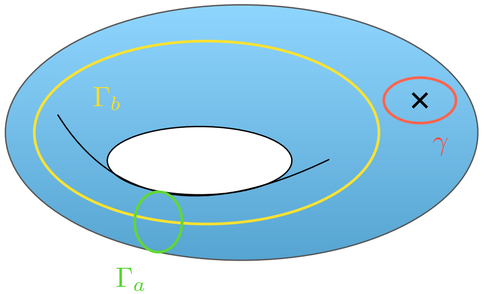

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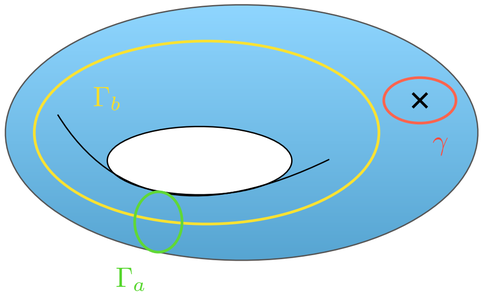

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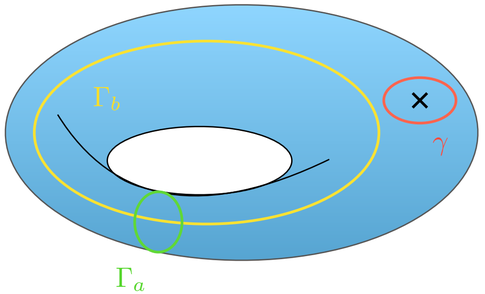

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Beyond the **polylogarithmic case**, **leading singularities** are themselves **transcendental functions**. Some integrals, see I_3, can have **multiple different leading singularities**.

Unit Leading Singularities

- Second, we have to generalize the notion of **unit leading singularities**:

> *A basis of integrals has unit leading singularities if, locally, all integrals have maximal transcendental weight, all corresponding integrands have at most single poles, and all leading singularities associated with the holomorphic cycle(s) and with the cycles encircling the single poles are constant to order* ϵ^0.

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Producing integrals with **unit leading singularities** is similar to multiplying by the **inverse semi-simple** part of the **period matrix/Wronskian**.

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- If the integrals are extended to ϵ-dimensions, we have to analyze the **second integral** because of the **weight drop** to **one order higher in** ϵ.

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- For the **third integral**, I_3, we have to **remove additional leading singularities**.

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Additional LS must be **removed,** requiring **new transcendental functions,** which we collectively call **G-functions.**

Integrand Analysis Beyond Polylogarithms: Summary

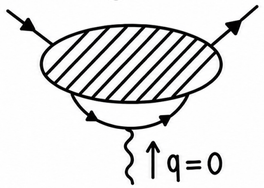
Integrand Analysis, Leading Singularities and Canonical Bases beyond Polylogarithms [1]

- In summary, we have established the following connections:
  - We have identified the **periods** as **leading singularities** for $\epsilon = 0$ that have to be normalized throught $(\mathrm{W}^{\mathrm{SS}})^{-1}$.
    - → Works for **arbitrary geometries** (mathematically: **Mixed Hodge Structure**)
  - Choosing the **second-kind forms as derivatives** guarantees that our **procedure terminates** after finitely many steps.
  - **Higher poles** of differential forms produce a **weight drop,** which requires studying the integrals to higher orders in ϵ.
    - → For **polylogarithmic integrals**, forms with **higher poles are not independent**, and a basis containing **only single pole**s exists. Therefore, additional steps are not needed.
  - The **clean-up step normalizes** leading singularities **at higher orders** in ϵ. In other words, **G-functions** are just **LS** of our integrals, either for $\epsilon = 0$ or at higher orders in ϵ.
  - In [1], we have also shown the **equivalence** between **normalizing LS** at the **integrand level** and via the **differential equations**.
  - **Canonical bases** constructed via **our procedure** should **define/be related to** the correct **pure multiple polylogarithms** on the **considered geometries**.

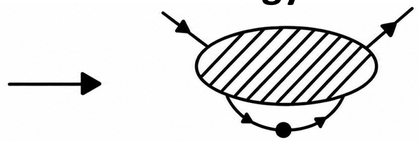
QED Self-Energies

- An important place where **Calabi-Yau manifolds** play a significant role is in **self-energies in QED**.

g-2



self-energy with dot



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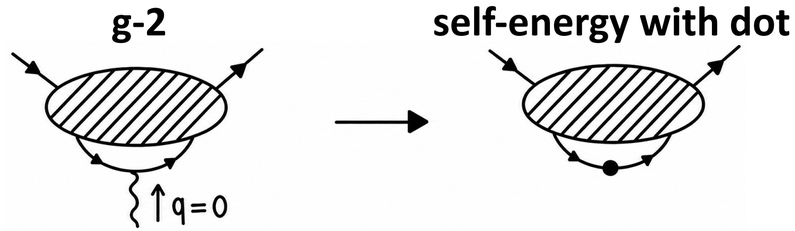

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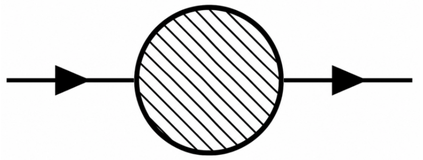

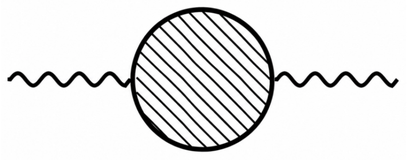

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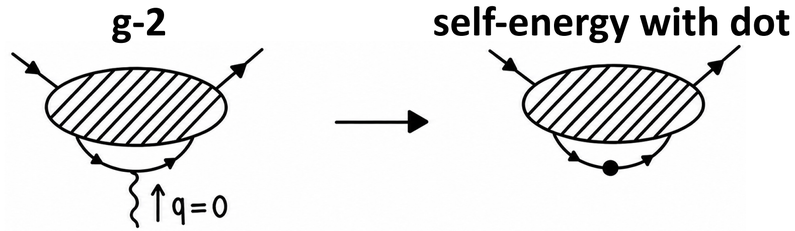

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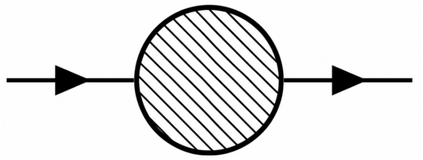

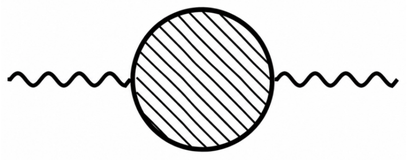

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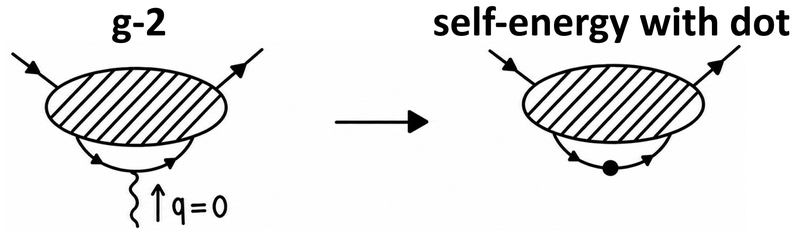

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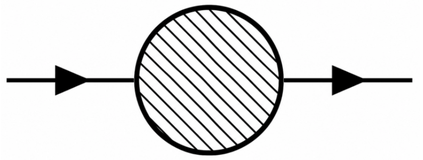

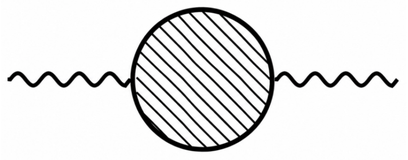

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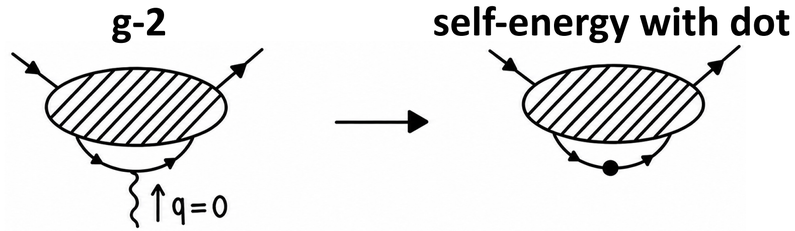

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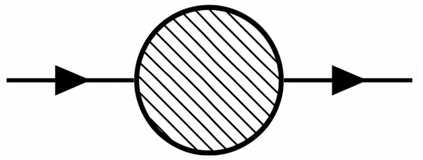

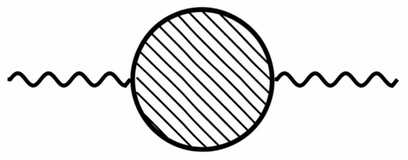

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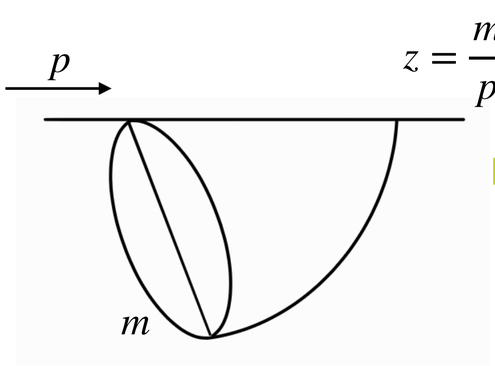

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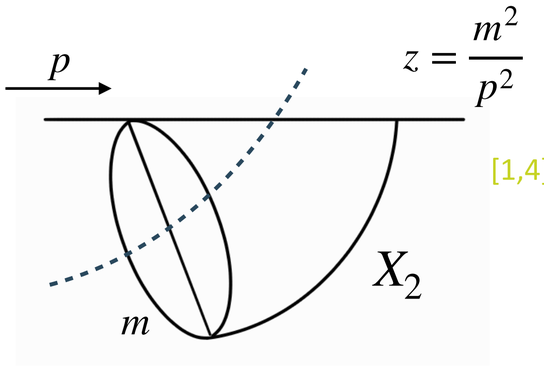

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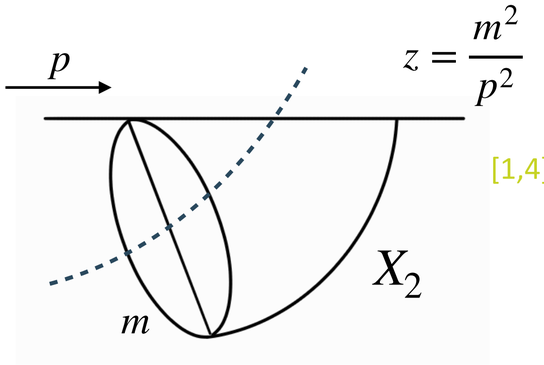

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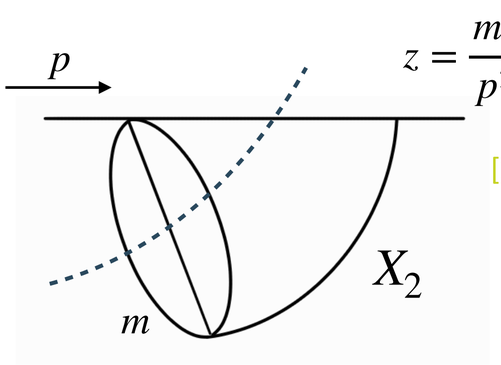

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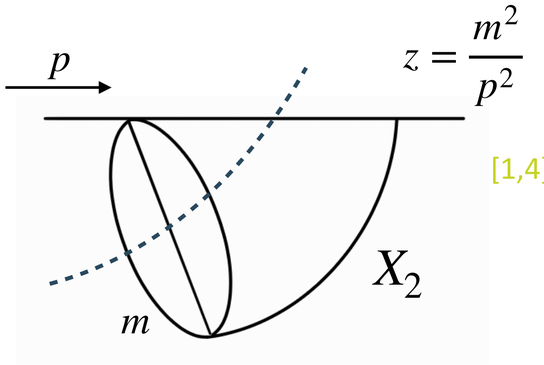

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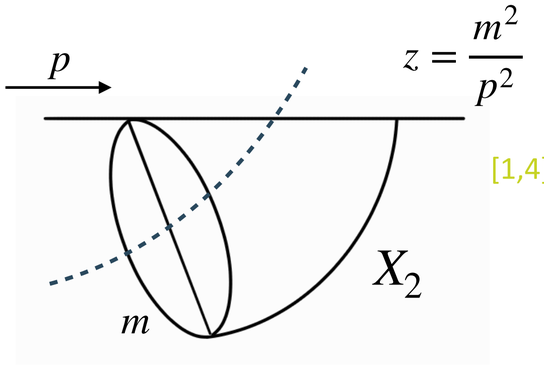

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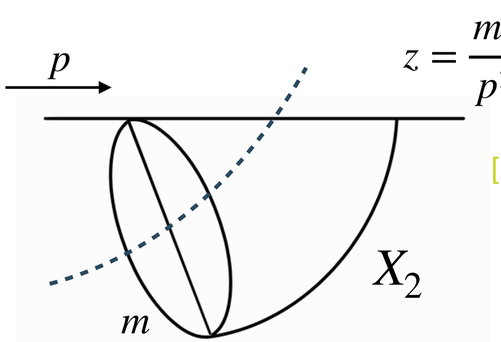

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Can we provide **more information** about these **new functions?**

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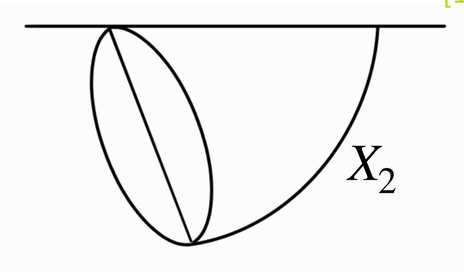

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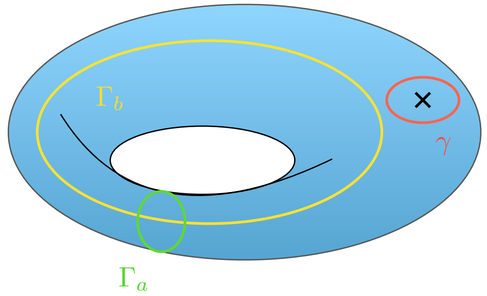

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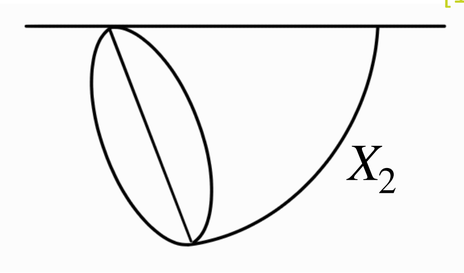

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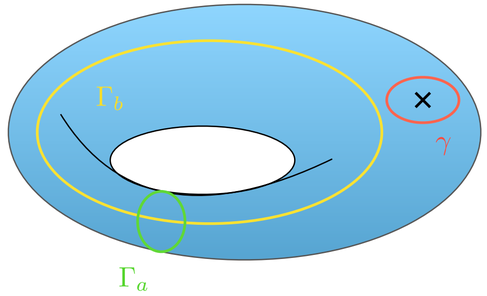

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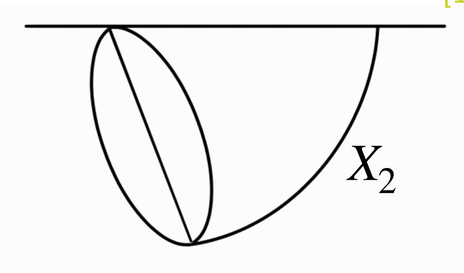

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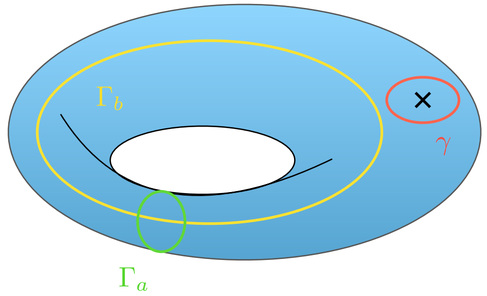

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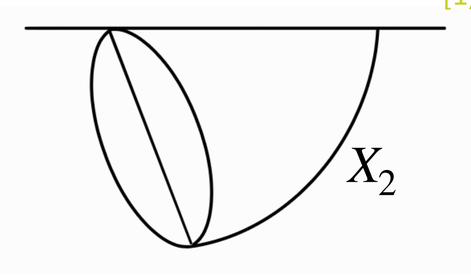

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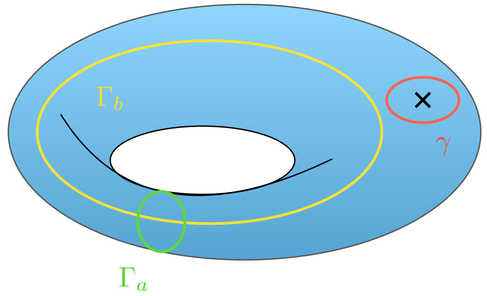

→ $\mathrm{LS}_1 = \dfrac{z}{\sqrt{1-4z}}$ fixes the **normalization** of the integral

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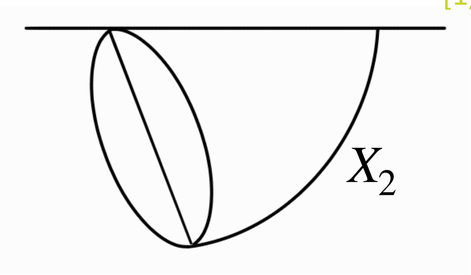

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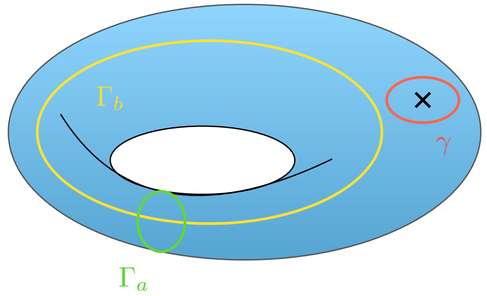
$$\longrightarrow \mathrm{LS}_1 = \frac{z}{\sqrt{1-4z}}$$ fixes the **normalization** of the integral



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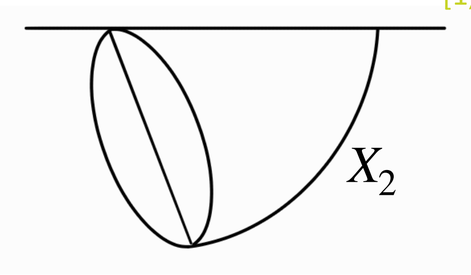
The Eyeball Graph @ 3L

- Using a **Baikov representation** of the eyeball integral, we find:

[1,4]



$$I \sim z \int \mathrm{d}X_1 \mathrm{d}X_2 \frac{1}{1-X_2} \frac{1}{\sqrt{X_1}\sqrt{4-X_1}\sqrt{Q(X_1,X_2)}\sqrt{Q(X_2 z, z)}}$$

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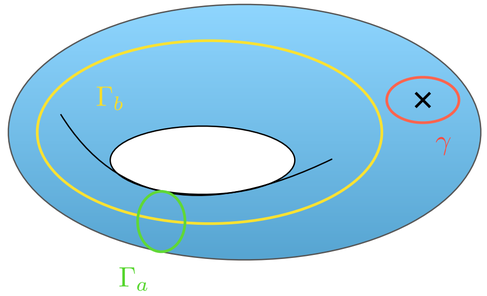

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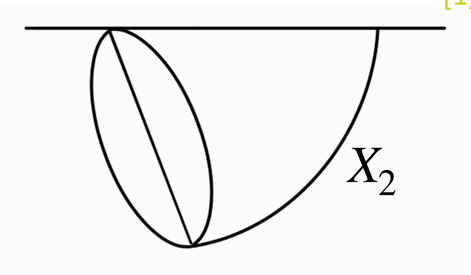

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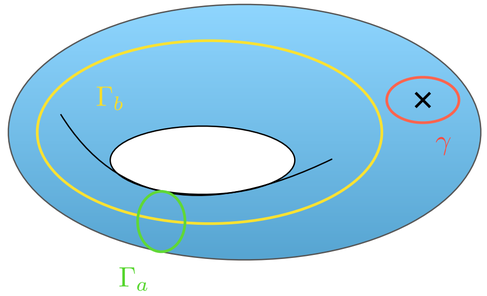

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→ **Additional leading singularity** connected to a geometric obstruction

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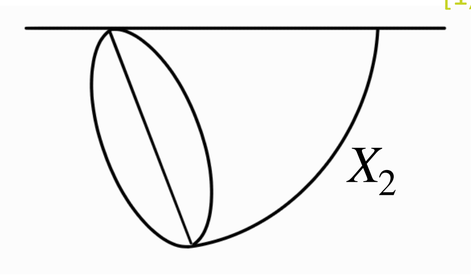

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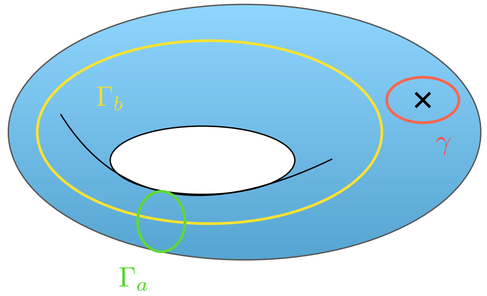

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Can we say even more about this function? Is there a **new transcendental function** in the **canonicalization process**, or not?

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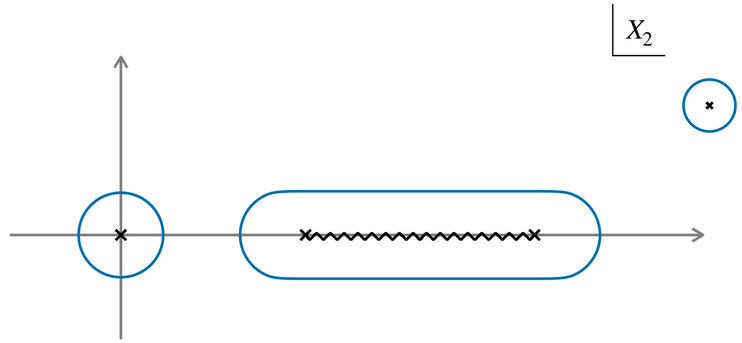

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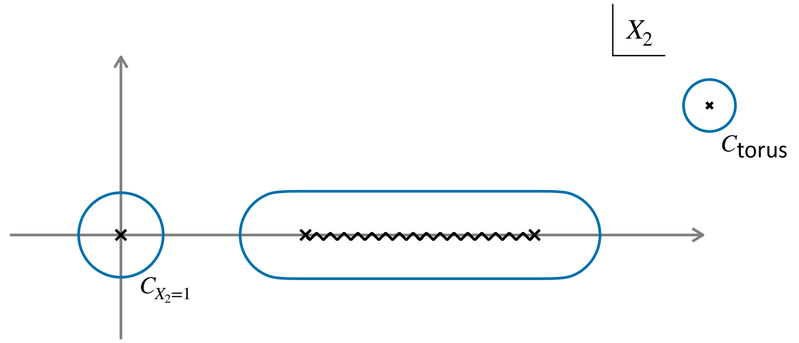

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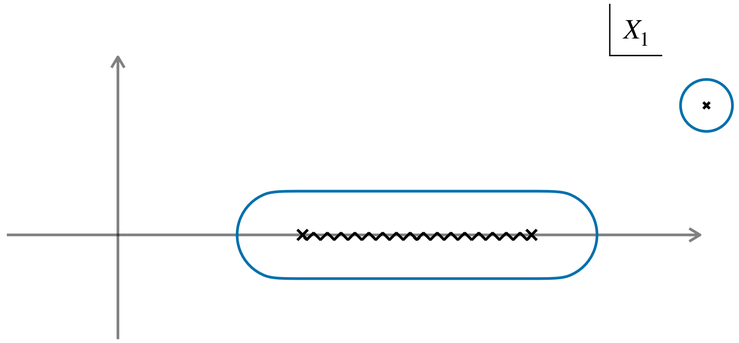

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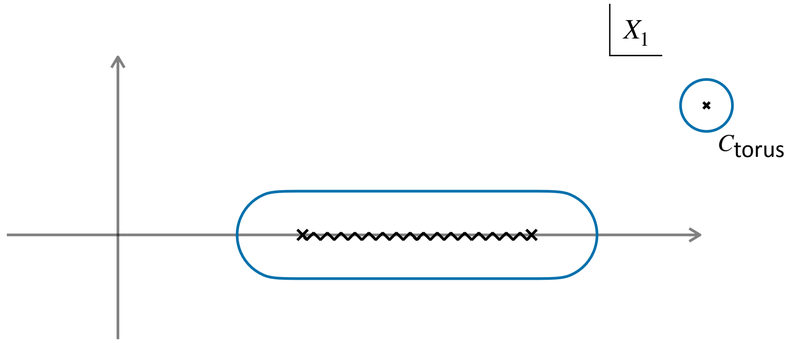

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$$= z(z+8z^2+64z^3+576z^4+5876z^5+66176z^6+798848z^7+\mathcal{O}(z^8))$$

$$=: G(z)$$

By evaluating the series expansion to sufficiently high orders, we obtain the following **inhomogeneous differential equation**:

$$\underbrace{\left(2(1-4z)z\frac{\partial}{\partial z}-2(1-2z)\right)}_{\text{homogeneous solution } \frac{z}{\sqrt{1-4z}}}\,G(z)=\underbrace{z\,\varpi_0^{\text{K3}}-3z^2\,\partial_z\varpi_0^{\text{K3}}}_{\text{Not a total derivative!}}$$

→ **New transcendental function**, necessary for canonical form!

$$J=\left(\frac{z}{\sqrt{1-4z}}\right)^{-1}\left\{I-G(z)\cdot\frac{I_{\text{3L Ban}}}{\varpi_0^{\text{K3}}}\right\}$$

The Eyeball Graph @ 3L

- We can derive a **differential equation** directly from the **Baikov representation**:

Torus-Period ⟷ **Hard branch** in **expansion by regions** [1,4]

"We first expand in z and then perform the integration."

$$z\oint_{C_{\text{torus}}} \mathrm{d}X_1\mathrm{d}X_2\,\frac{1}{1-X_2}\,\frac{1}{\sqrt{X_1}\sqrt{4-X_1}\sqrt{Q(X_1,X_2)}\sqrt{Q(X_2 z,z)}}$$

$$= z\oint_{C_{\text{torus}}} \mathrm{d}X_1\mathrm{d}X_2\,\frac{1}{1-X_2}\,\frac{1+(1+X_2)z+(1+4X_2+X_2^2)z^2+\mathcal{O}(z^3)}{\sqrt{X_1}\sqrt{4-X_1}\sqrt{Q(X_1,X_2)}}$$

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QED Self-Energies @ 3L

- At **three loops**, we found the following:

Electron

$$\Pi_{e,\mathrm{bare}}(p,m) = \frac{i}{\not{p} - m + \hat{\Sigma}(p,m)}$$ [6]





@3L 32 Sectors, 51 MI

Photon

$$D^{\mu\nu}_{\gamma,\,\mathrm{bare}}(p,m) = -\frac{i}{p^2}\left[\frac{1}{1-\Pi(p,m)}\left(g^{\mu\nu} - \frac{p^\mu p^\nu}{p^2}\right) + \xi\frac{p^\mu p^\nu}{p^2}\right]$$ [5]





@3L 21 Sectors, 36 MI

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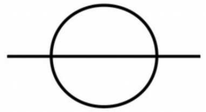
Elliptic curve + Residue



Periods: ϖ_0

Photon

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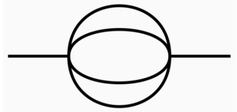
@3L 21 Sectors, 36 MI

K3 surface



Periods: ϖ_0, G_1, G_2, G_3 third-kind

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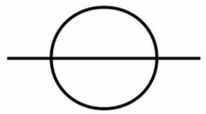
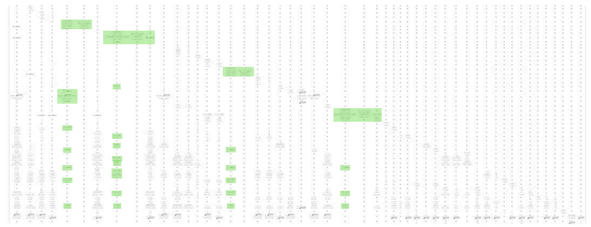


Periods: ϖ_0

↓

Canonical diff. eq.



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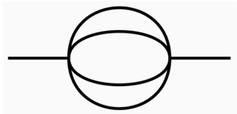
K3 surface



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↓

Canonical diff. eq.



K3

Function Space @ 3L

- In total, we found the following **integration kernels** in the differential equations. [5,6]

Electron

$$\left\{\frac{1}{x+3},\frac{1}{x+1},\frac{1}{x},\frac{1}{x-1},\frac{1}{x-2},\frac{1}{x-9},\frac{1}{\sqrt{(3+x)(1-x)}},\frac{1}{\sqrt{(1-x)(9-x)}},\frac{1}{\sqrt{(1-x)(9-x)}}\frac{1}{x}\right\}$$

16=9+7

$$\left\{\frac{1}{x(x-1)(x-9)\varpi_0(x)^2},\varpi_0(x),\frac{\varpi_0(x)}{x-1},\frac{(x-3)\varpi_0(x)}{\sqrt{(1-x)(9-x)}},\right.$$
$$\left.\frac{(x+3)^4\varpi_0(x)^2}{x(x-1)(x-9)},\frac{(x+3)(x-1)\varpi_0(x)^2}{x(x-9)},\frac{\varpi_0(x)^2}{(x-1)(x-9)}\right\}$$

Photon

$$\left\{\frac{1}{x},\quad\frac{1}{x-1},\quad\frac{1}{x-4},\quad\frac{1}{r_1(x)r_2(x)},\quad\frac{1}{16-x}\right\}\qquad r_1(x)=\sqrt{x}\,,\quad r_2(x)=\sqrt{4-x}\,,\quad r_3(x)=\sqrt{16-x}$$

24=5+19

$$\left\{\frac{1}{r_2(x)r_3(x)x\varpi_0(x)},\frac{G_2(x)}{r_2(x)r_3(x)x\varpi_0(x)},\frac{G_2(x)^2}{r_2(x)r_3(x)x\varpi_0(x)},\frac{G_2(x)^3}{r_2(x)r_3(x)x\varpi_0(x)},\frac{G_3(x)}{r_2(x)r_3(x)x\varpi_0(x)},\frac{G_2(x)G_3(x)}{r_2(x)r_3(x)x\varpi_0(x)},\right.$$
$$\frac{G_3(x)}{r_1(x)r_2(x)},\frac{G_3(x)}{x},\frac{G_3(x)}{4-x},\frac{G_3(x)}{16-x},\varpi_0(x),\frac{\varpi_0(x)}{x},\frac{\varpi_0(x)}{4-x},\frac{r_1(x)r_2(x)\varpi_0(x)}{x},\frac{r_1(x)\varpi_0(x)}{r_2(x)},\frac{r_1(x)\varpi_0(x)}{r_2(x)(4-x)},$$
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Photon

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Photon

24=5+19

15=4+11

$$\left\{\frac{1}{x}, \frac{1}{x-1}, \frac{1}{x-4}, \frac{1}{r_1(x)r_2(x)}, \cancel{\frac{1}{16-x}}\right\} \qquad r_1(x)=\sqrt{x}\,, \quad r_2(x)=\sqrt{4-x}\,, \quad r_3(x)=\sqrt{16-x}$$

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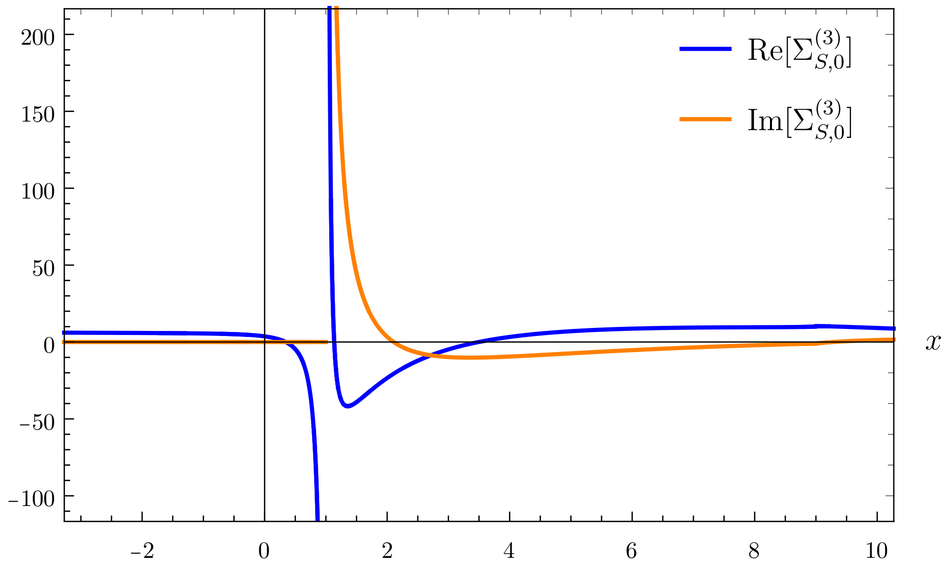
Bootstrap?

Numerics of QED Self-Energies

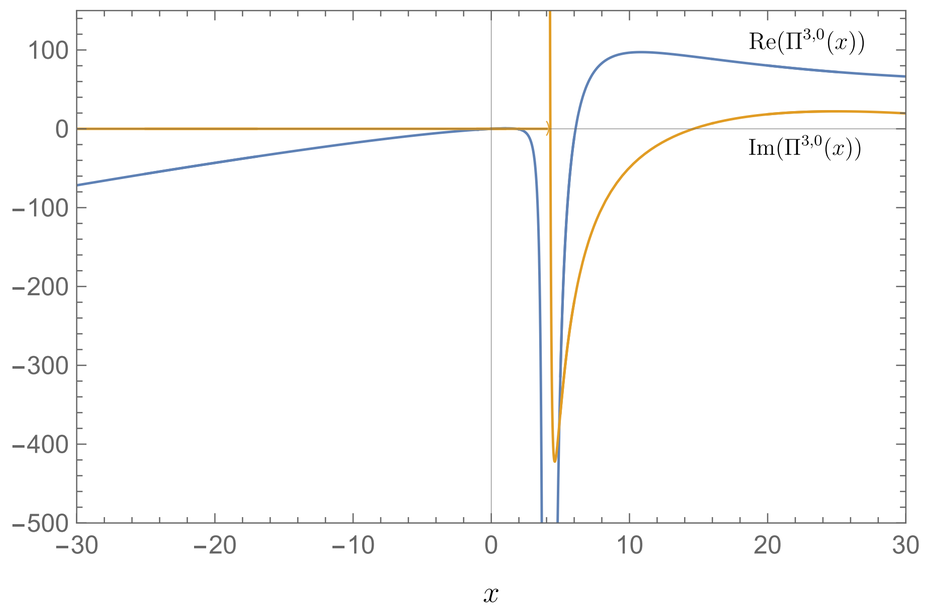
[5,6]

- Using **series expansions** of the canonical integrals, we can cover the **whole parameter space**.
  - For the transcendental functions ϖ_0^2, G_i, we can use their **local series expansions** obtained from their differential equations.

Electron



Photon



- We see that although elliptic/K3 integrals are involved in this problem, we can **derive and evaluate** the electron and photon **self-energies up to 3L** in QED.

Two-Loop Corrections to Top-Pair Production

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  - Number of **diagrams:**

Process	tree level	one loop	two loops
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  - **14 integral families** (5 planar, 9 non-planar) requiring **417 master integrals**.
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HARD

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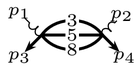
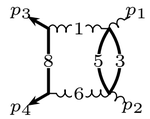
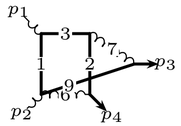
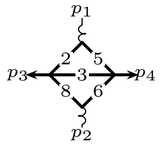
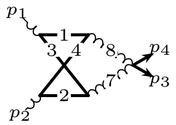
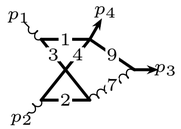
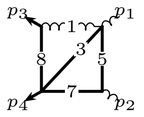
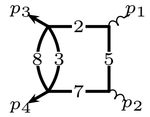
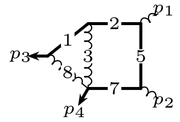
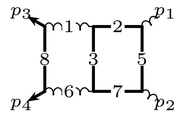
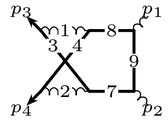
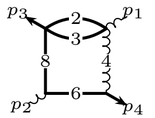
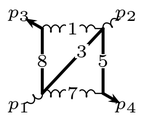
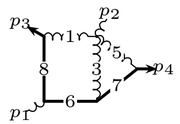
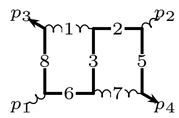
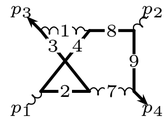
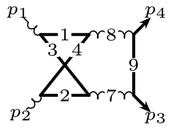
E_1 elliptic	 pB148 2 (0)	E_2 elliptic	 pB181 3 (0)	E_3 elliptic	 nB359 5 (0)
E_4 elliptic	 pE182 3 (0)	E_5 elliptic	 nB207 2 (0)	E_6 elliptic	 nB335 4 (2)
E_7 elliptic	 pB213 4 (0)		 pB214 3 (0)		 pC215 3 (0)		 pB247 5 (0)		 nC463$^{\times 12}$ 4 (0)
E_8 elliptic	 pE174 3 (0)		 pE213$^{\times 12}$ 5 (0)		 pD245$^{\times 12}$ 3 (0)		 pE247$^{\times 12}$ 4 (1)		 nI463 6 (1)
genus two	 nB463 5 (0)								

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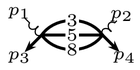
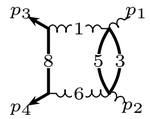
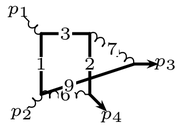
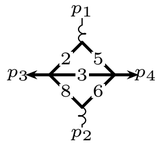
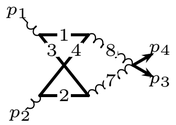
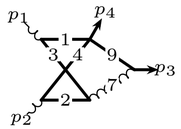
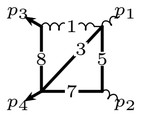
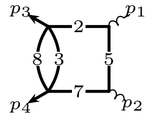
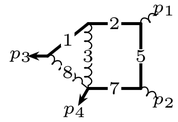
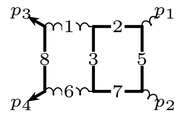
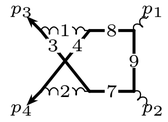
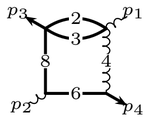
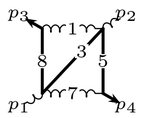
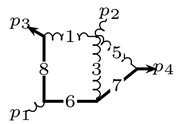
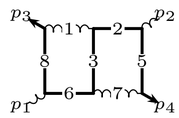
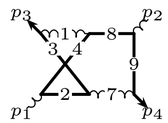
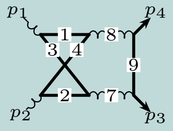
E_1 elliptic	 pB148 2 (0)	E_2 elliptic	 pB181 3 (0)	E_3 elliptic	 nB359 5 (0)
E_4 elliptic	 pE182 3 (0)	E_5 elliptic	 nB207 2 (0)	E_6 elliptic	 nB335 4 (2)
E_7 elliptic	 pB213 4 (0)		 pB214 3 (0)	 pC215 3 (0)	 pB247 5 (0)	 nC463$^{\times 12}$ 4 (0)
E_8 elliptic	 pE174 3 (0)	 pE213$^{\times 12}$ 5 (0)	 pD245$^{\times 12}$ 3 (0)	 pE247$^{\times 12}$ 4 (1)	 nI463 6 (1)
genus two	 nB463 5 (0)				

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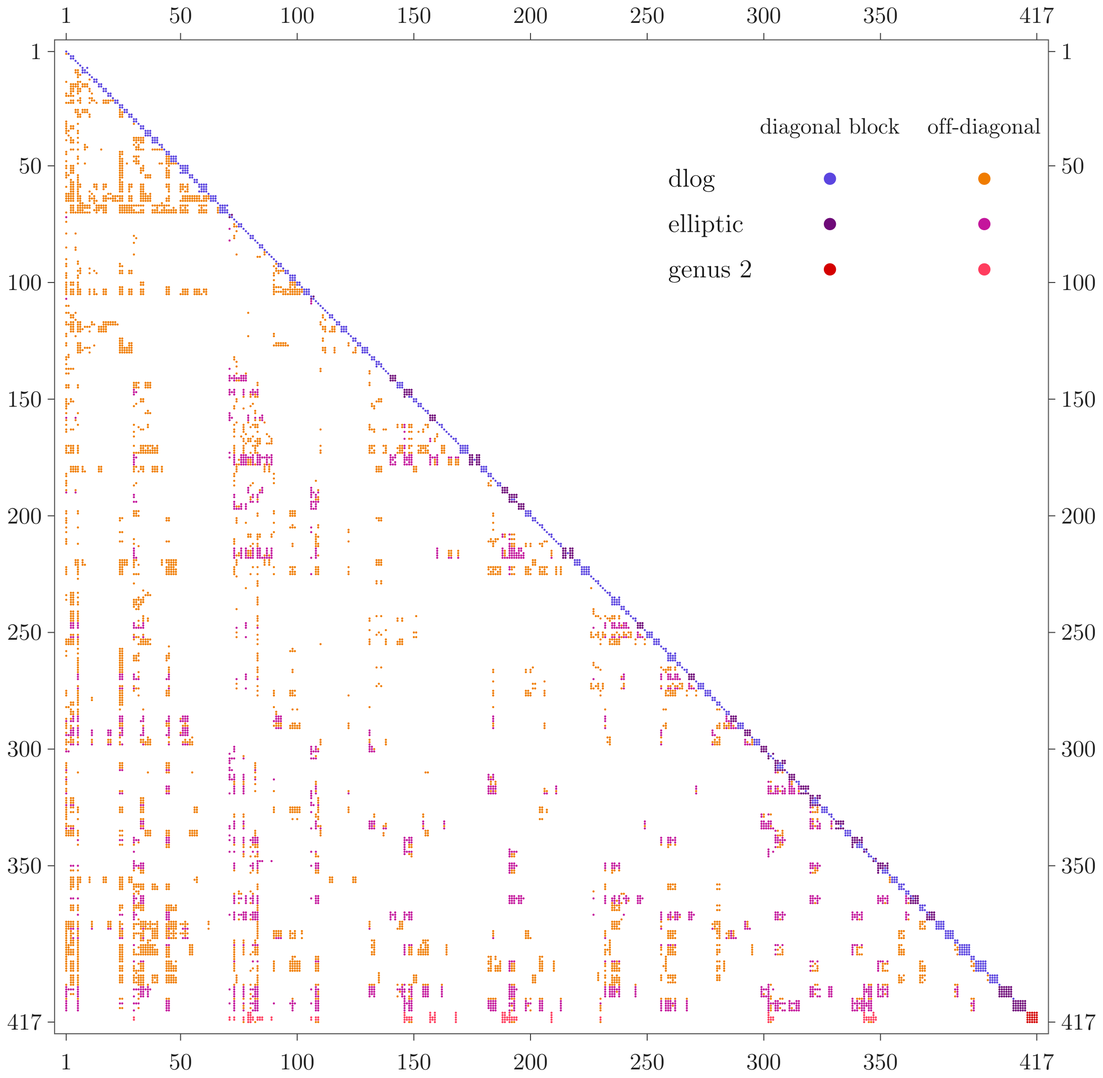

Using **our method**, we found a **fully canonical basis** that also **includes** the **genus-two curve**.

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  - There is an **interesting dropping pattern** of **integration kernels** when comparing the **differential-equation level** with the **amplitude level**.

Differential equations			**Amplitudes**	
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x_1, x_2: **blow-up variables** (disentangle singularities)

$(x_1, x_2) = (0,0)$: **threshold**, **boundary values** computed with **AMFlow**

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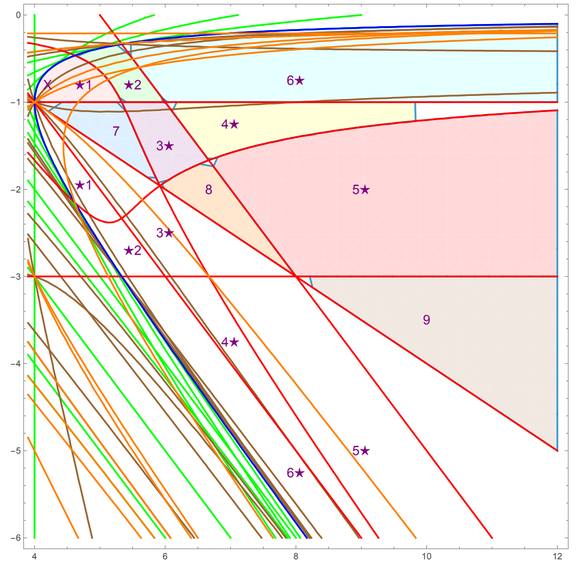

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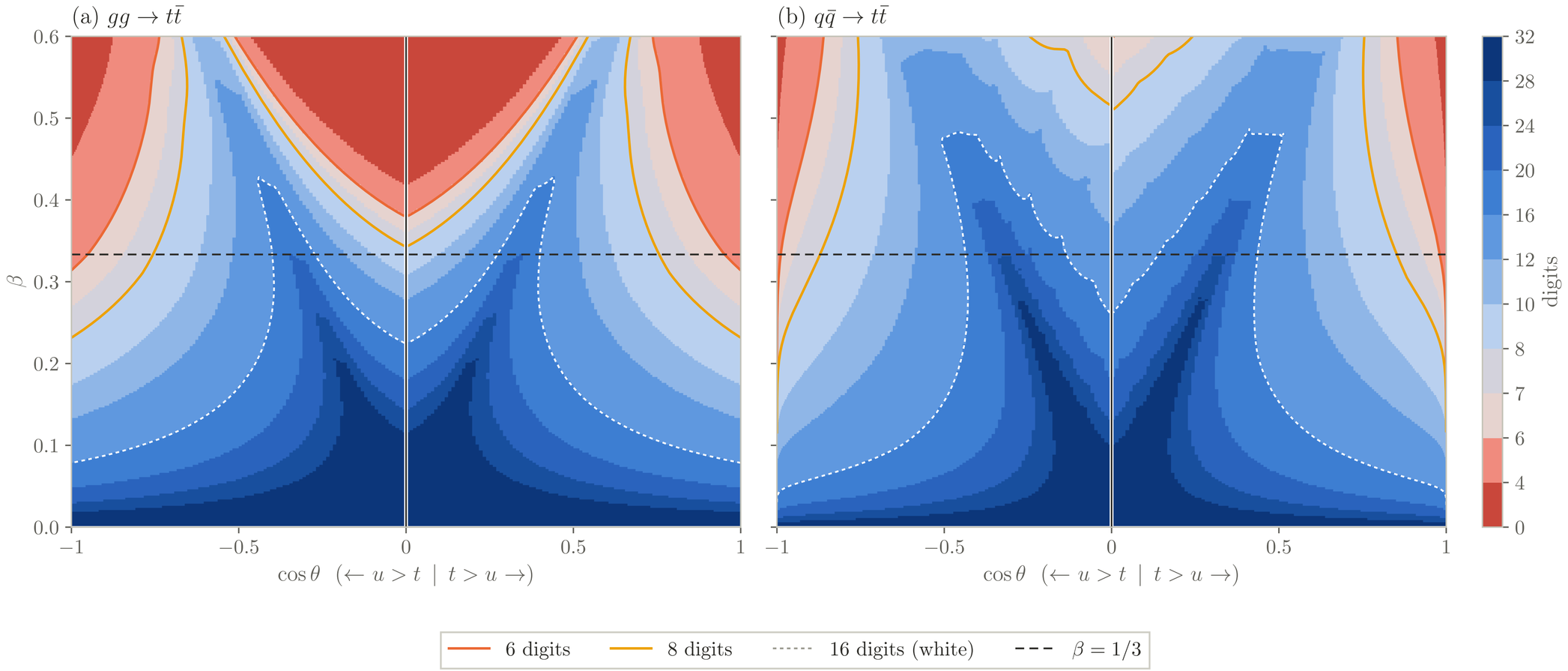
angle: $t = m_t^2 - \frac{s}{2}(1 - \beta \cos\theta)$

velocity: $\beta = \sqrt{1 - \frac{4m_t^2}{s}}$

Precision:



Numerical Results

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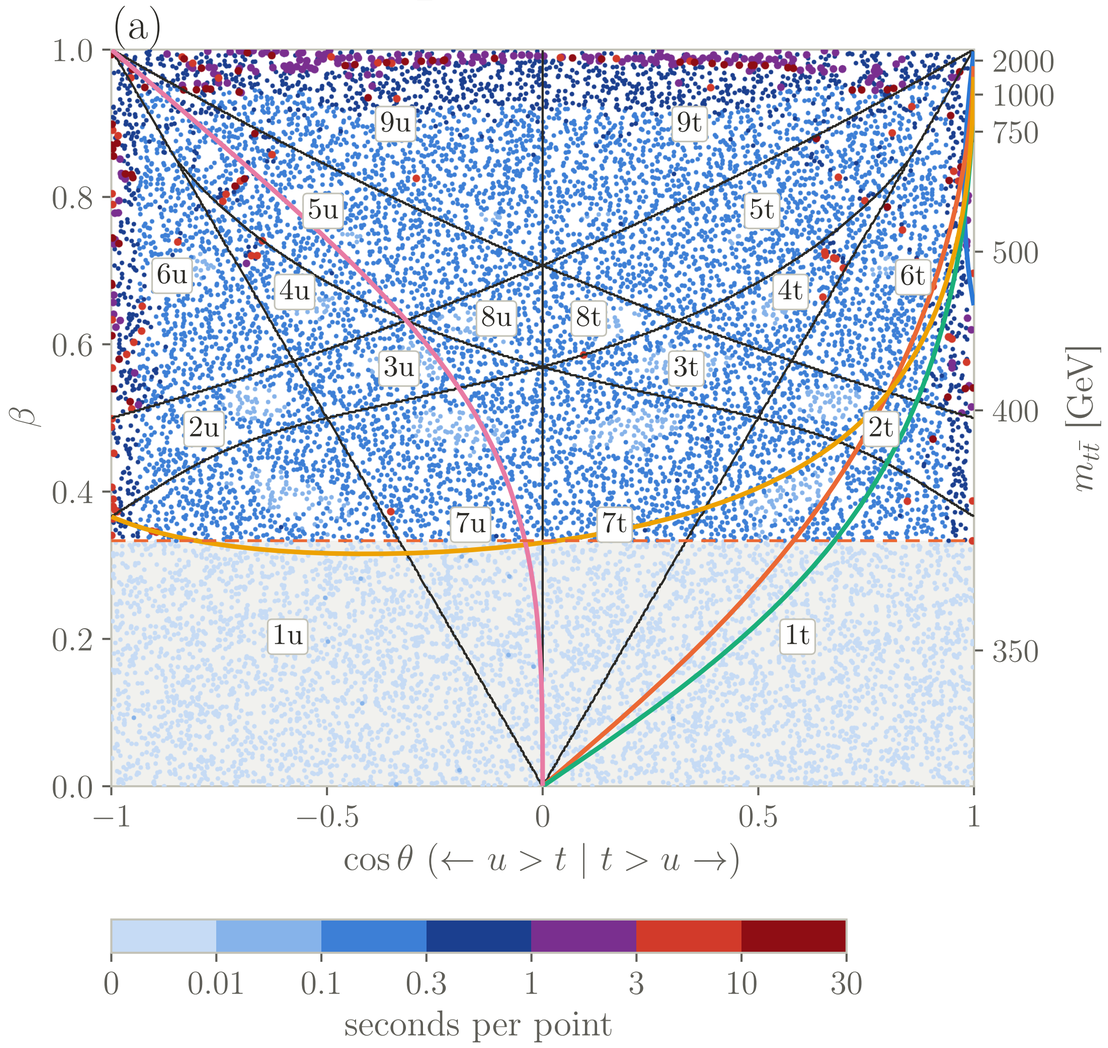
Evaluation time:

angle: $t = m_t^2 - \frac{s}{2}(1 - \beta \cos\theta)$ velocity: $\beta = \sqrt{1 - \frac{4m_t^2}{s}}$



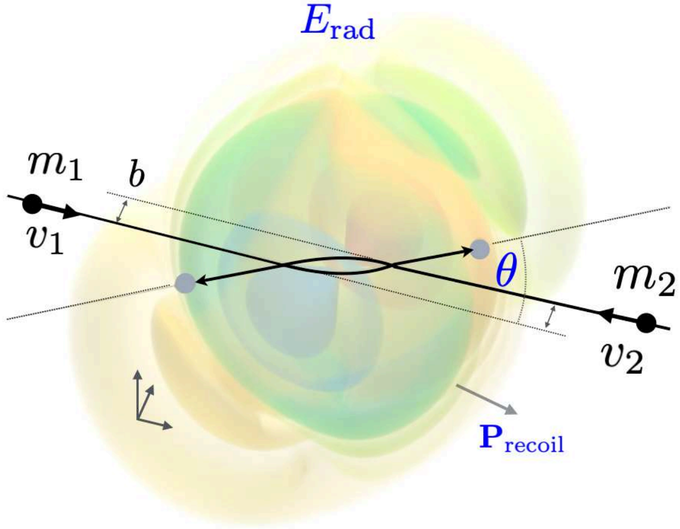
series: ~ 7ms
numerical code: ~ 0.39s

Black Hole Scattering

- Through the **Worldline QFT** approach, we can model the scattering of **black holes**.

$$S = -\sum_{i=1}^{2} m_i \int \mathrm{d}\tau \left[\frac{1}{2} g_{\mu\nu} \dot{x}_i^\mu \dot{x}_i^\nu\right] - \frac{1}{16\pi G} \int \mathrm{d}^4x \sqrt{-g} R(g) + S_{\text{g.f.}}$$

[Mogull, Plefka, Steinhoff]



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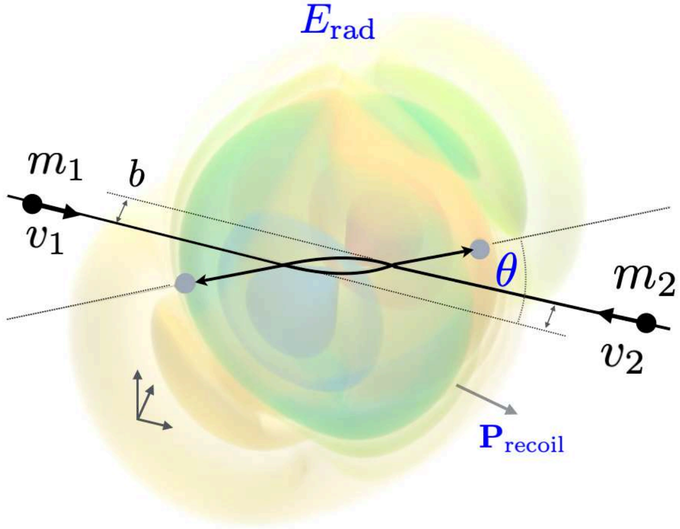

- **Post-Minkoskian** approximation: We expand in Newton's constant G.

deflection

$$x_i^\mu = b_i^\mu + v_i^\mu \tau + z_i^\mu$$

deflection angle θ ← → radiated energy E_{rad}

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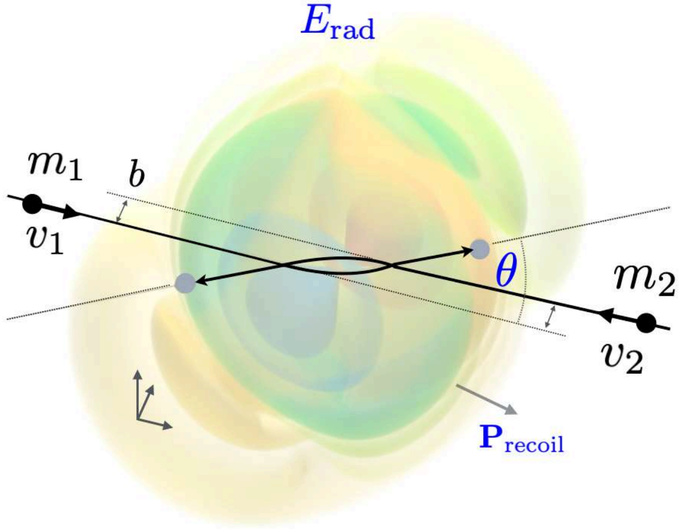

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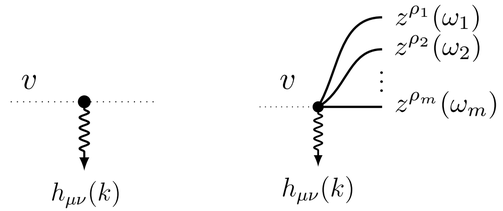
- The advantage of the **WQFT** approach is that **tree-level one-point functions solve the classical e.o.m.**
- Due to the **WQFT Feynman rules tree-level** functions still contain **"loop"-integrals**.

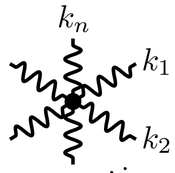
world line vertices



only energy conservation

$$\sim \delta\Big(v \cdot k + \sum_i \omega_i\Big)$$

background field vertices



full four-momentum conservation

$$\sim \delta^{(4)}\Big(\sum_i k_i\Big)$$

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retarded worldline propagators (in-in)
retarded graviton propagators (in-in)

physical result

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We discuss today:
5PM 1SF dissipative [9]
5PM 2SF conservative [8]

Calabi-Yaus @ 5PM

- In the black hole scattering problem at 5PM, we find **two distinct K3 surfaces and two Calabi-Yau three-folds**, with **one of each** appearing at **1SF** and **2SF**. These are connected to the following **graphs** and **differential operators**.

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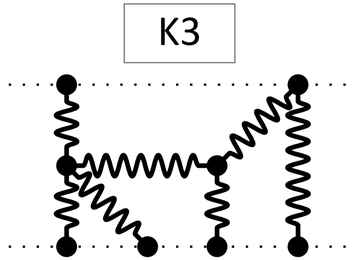

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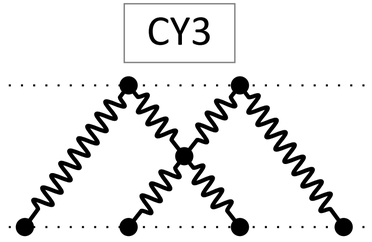

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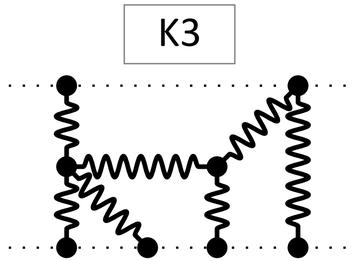

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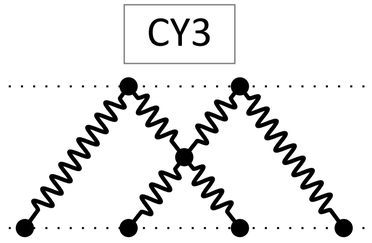

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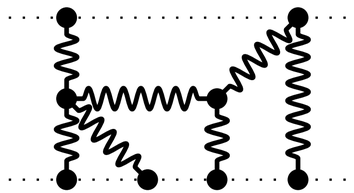

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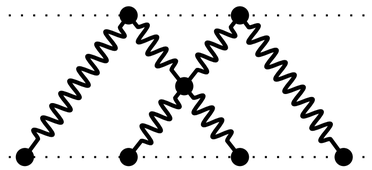

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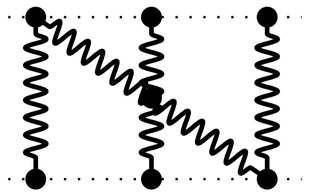

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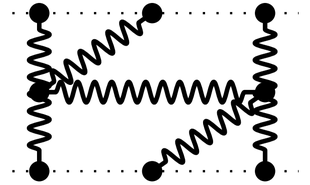

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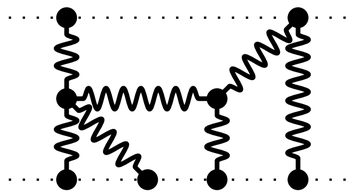

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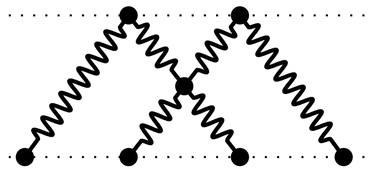

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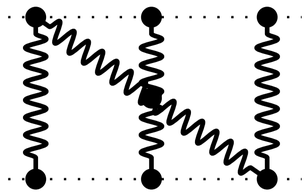

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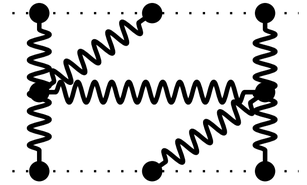

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complicated, many third-kinds on max cut, K3 7x7, CY3 5x5

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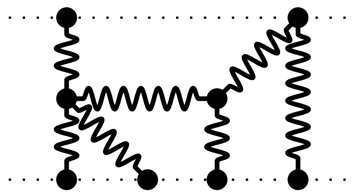

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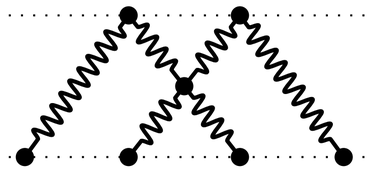

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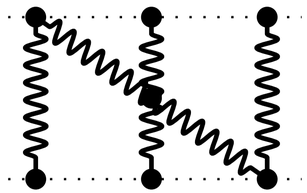

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Apery operator [Apery, Zagier]

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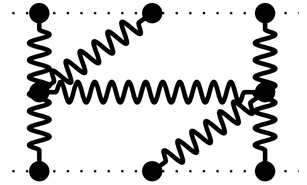
- It seems that there is **another new K3 surface** in the **dissipative sector**.

CY3



$$\mathcal{L}_2^{(4)} = \theta^4 - 2^{30}z^3(\theta+\tfrac{1}{2})^4 - 2^4 z(192\theta^4 + 128\theta^3 + 112\theta^2 + 48\theta + 7) + 2^{14}z^2(192\theta^4 + 256\theta^3 + 208\theta^2 + 64\theta + 7)$$

again Hadamard product [AESZ, van Straten]

complicated, many third-kinds on max cut, K3 7x7, CY3 5x5

Calabi-Yaus @ 5PM

- At **5PM**, we found **two interesting families containing** the **Calabi-Yau integrals**. At **1SF**, this family contained **232 masters**, and at **2SF**, **321 masters**. We managed to bring **both systems** into **canonical form**.

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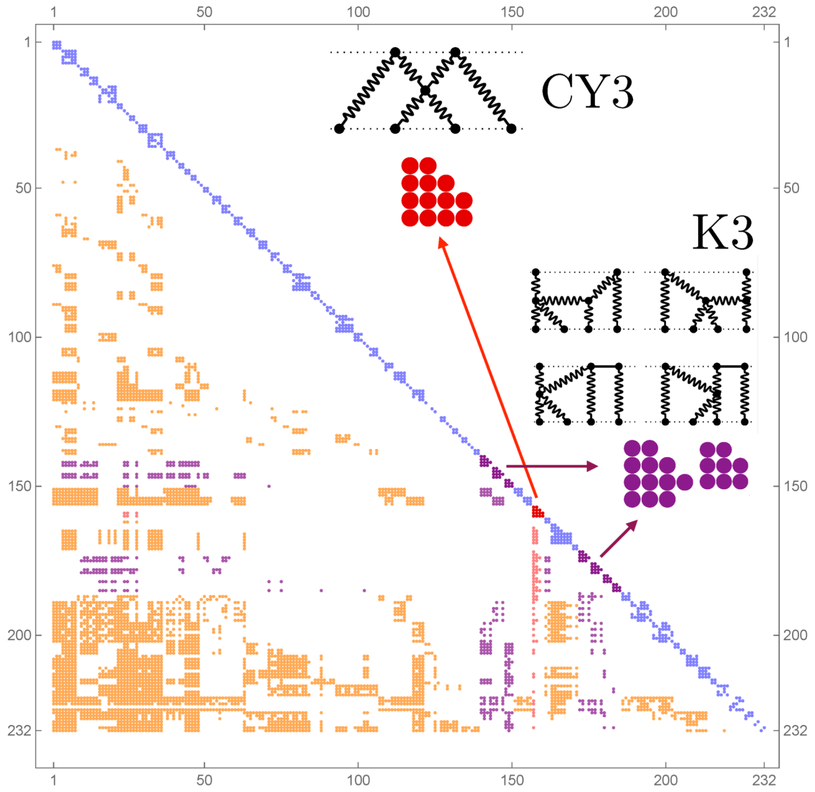

2SF

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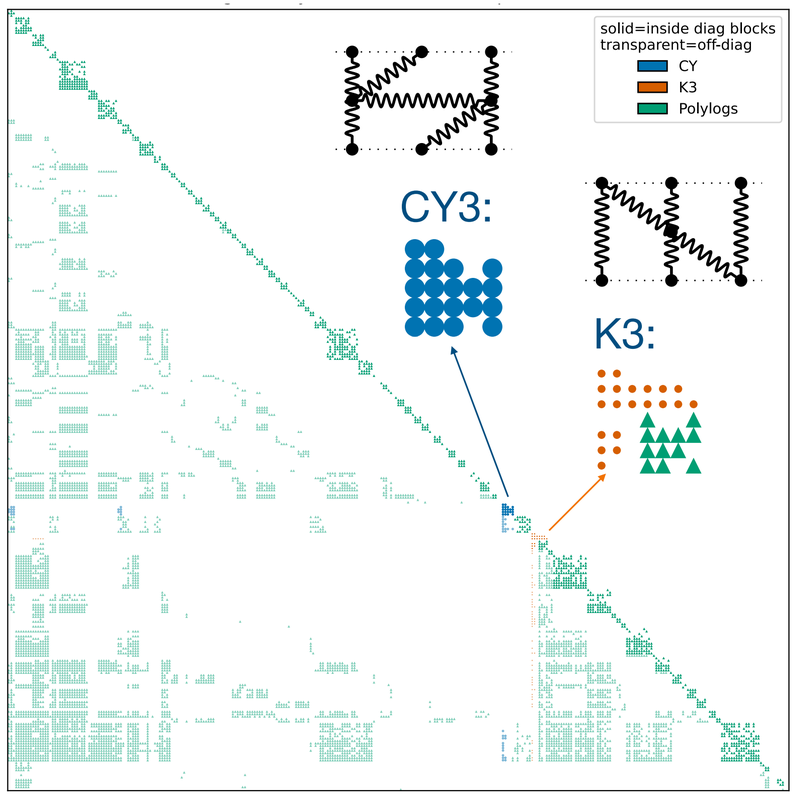

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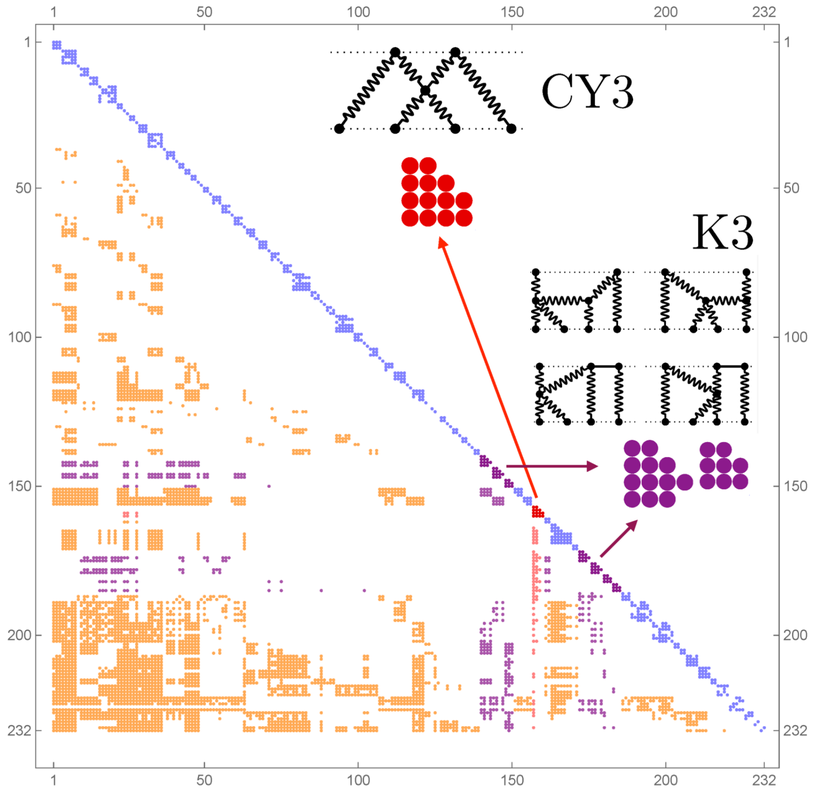

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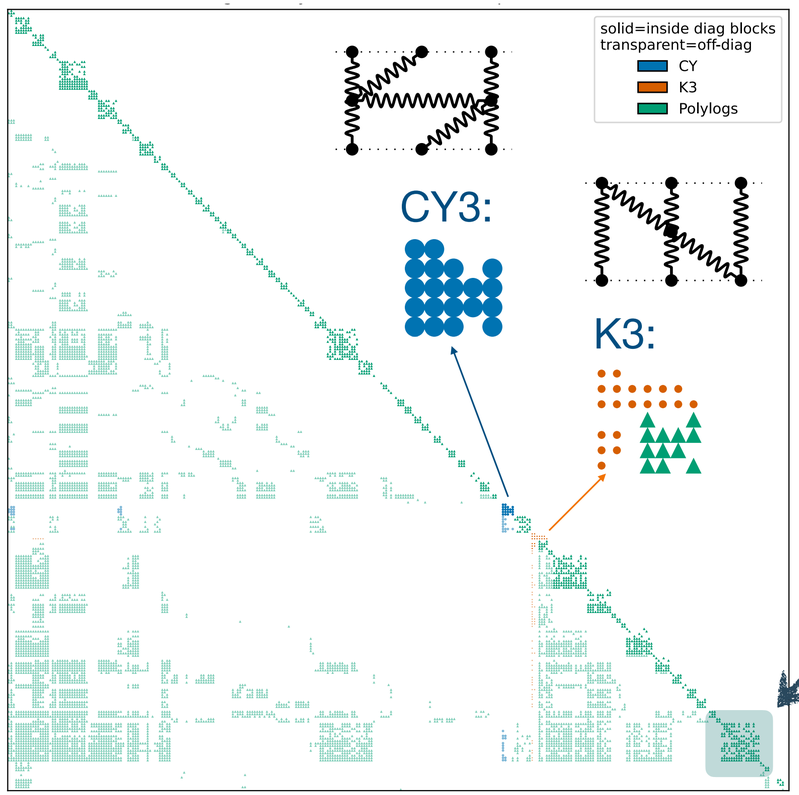

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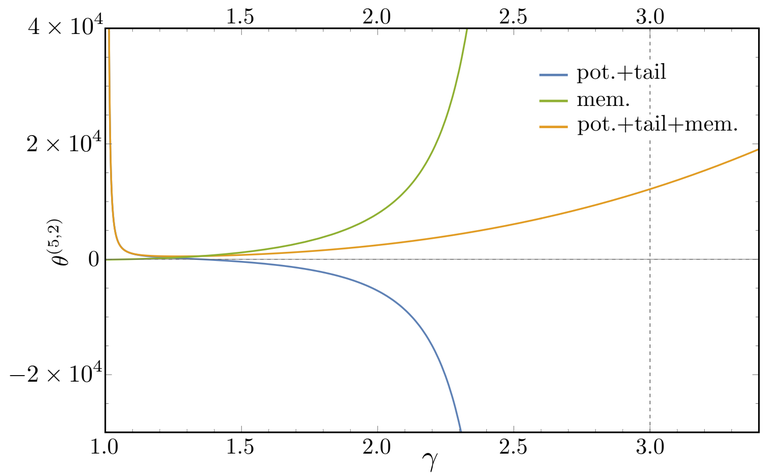

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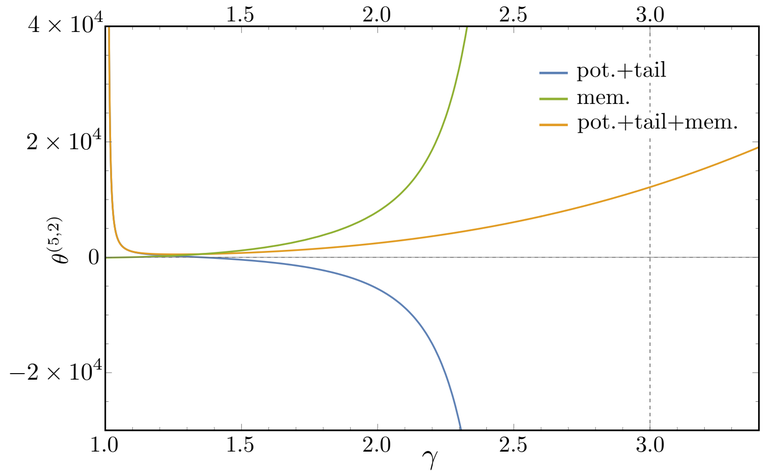

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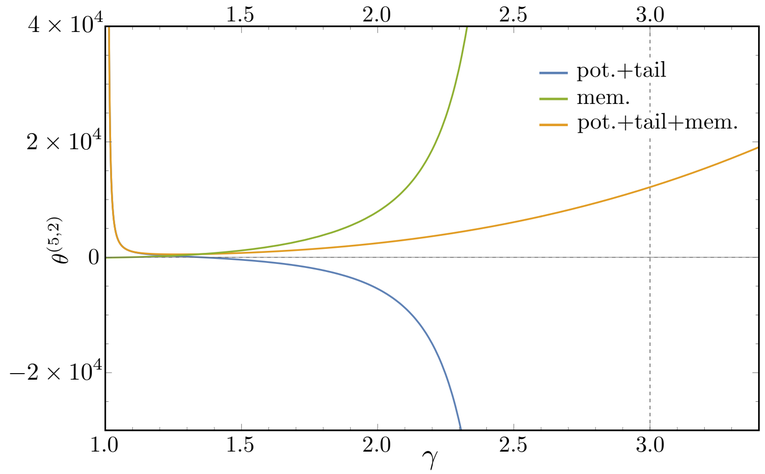

- At **1SF dissipative**, the **Calabi-Yau three-fold periods** survive in the final **result** for the **radiated energy**. [9]
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Summary

- We have discussed how to do **analytic Feynman integral computations** involving **non-polylogarithmic functions**.
- One crucial part of such computations is calculating **canonical integrals beyond polylogs**.
- The **underlying geometry** tells us what the right integrals are.
- **New transcendental functions** were necessary in our examples. They are related to **periods** of the associated **geometry**, **third-kind functions**, and **iterated integrals** thereof.
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Don't be shy when amplitudes contain functions beyond polylogarithms.
We now have all the techniques and tools to deal with them.

Thank you for your attention

