Modularity and quasi-modularity of Feynman integrals

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Abstract: For the counter-terms of QFT, multiple polylogarithms suffice up to 7 loops. At 8 loops, a modular form of weight 3 and level 7 intervenes, associated with a K3 surface. By contrast, modular forms appear much earlier in evaluations of massive Feynman integrals. At two loops, an elliptic curve produces periods of a modular form of weight 3 and level 6. At three loops, a K3 surface leads to weight 4. I describe roles played by modular forms and their symmetric squares, up to 6 loops. Moreover, relations are found up to 22 loops between Feynman periods and critical values of L-functions whose local factors are given by Kloosterman sums over finite fields.

1. **Quasi-modularity** up to **3 loops** and in **topological string** theory
2. **Magnetic moment** of the electron and **black-hole** areas at **4 loops**
3. **Automorphic forms** up to **6 loops**
4. **Critical L-values** up to **22 loops**

1. Quasi-modularity up to 3 loops and in topological string theory.
Ramanujan's notebooks record this convenient evaluation

$$\zeta(3) = \frac{7\pi^3}{180} - 2\sum_{n>0} \frac{1}{n^3} \frac{q^n}{1-q^n}, \quad q = \exp(-2\pi) < \frac{1}{\mathbf{353}}$$

which I have recently improved to obtain $\zeta(3)$ in terms of $\sqrt{15}\pi^3$ and **Lambert series** with positive $q \leq \exp(-12\pi/\sqrt{\mathbf{15}}) < 1/\mathbf{16879}$.
His value of $\zeta(3)$ comes from a family of **quasi-modular** transformations of

$$T_s(\tau) = \frac{1}{2}\zeta(s) + \sum_{n>0} \frac{1}{n^s} \frac{q^n}{1-q^n}, \quad q = \exp(2\pi \mathrm{i}\tau) = \exp(-2\pi y)$$

with $\Im\tau = \Re y > 0$. This converges well at large y. To access $q \to 1^-$, as $y \to 0^+$, we use a **Fricke involution** $\tau \to -1/\tau$, when s is **odd**, for which he obtained

$$T_{2k+1}\left(\frac{-1}{\tau}\right) = \frac{\mathrm{i}}{\pi} \sum_{j=0}^{k+1} \frac{\zeta(2j)\zeta(2k+2-2j)}{\tau^{2j-1}} + \frac{T_{2k+1}(\tau)}{\tau^{2k}}$$

which evaluates $\zeta(3)$ when we set $k = 1$ and $\tau = \mathrm{i}$.

The **Schwinger** parametrization of the equal-mass **3-loop banana** integral is

$$J(t) = \int_0^\infty \int_0^\infty \int_0^\infty \left(\frac{1}{(1+a+b+c)(1+1/a+1/b+1/c) - t} \right) \frac{\mathrm{d}a}{a} \frac{\mathrm{d}b}{b} \frac{\mathrm{d}c}{c}$$

whose denominator defines a **K3 surface**. It is given by the **Bessel moment**

$$J(t) = 8 \int_0^\infty I_0(\sqrt{t}x) K_0^4(x) x \, \mathrm{d}x$$

for $t < 16$. **Stefano Laporta** and I discovered the **on-shell** evaluation

$$J(\mathbf{1}) = \frac{\pi^3}{2} \left(1 - \frac{1}{\sqrt{5}} \right) \left(1 + 2 \sum_{n>0} \exp(-\sqrt{\mathbf{15}} \pi n^2) \right)^4 = \frac{\Gamma\left(\frac{1}{15}\right) \Gamma\left(\frac{2}{15}\right) \Gamma\left(\frac{4}{15}\right) \Gamma\left(\frac{8}{15}\right)}{30\sqrt{5}}.$$

Jon Borwein and I developed the **Taylor series** $J(t) = \sum_{n \geq 0} C_n t^n$, with coefficients $C_0 = 7\zeta(3)$, $C_1 = C_0 - 6$ and for $n > 1$ the **recursion**

$$C_n = \left(1 - \frac{1}{2n} \right) \left(5 - \frac{5}{n} + \frac{2}{n^2} \right) C_{n-1} - 4 \left(1 - \frac{1}{n} \right)^3 C_{n-2}.$$

This recurrence relation determines the inhomogeneous **Picard-Fuchs** equation

$$t^2(t-4)(t-16)J''' + 6t(t^2-15t+32)J'' + (7t^2-68t+64)J' + (t-4)J + \mathbf{24} = 0$$

with **singularities** for $t \in \{0, 4, 16, \infty\}$. The **Feynman** integral is **regular** at $t = 4$. A **modular** parametrization of this equation was given by **Helena Verrill** in 1996 using the **Dedekind eta** function

$$\eta(\tau) = q^{1/24}\prod_{n=1}^{\infty}(1-q^n) = \sum_{n=-\infty}^{\infty}(-1)^n q^{(6n+1)^2/24} = \frac{\eta(-1/\tau)}{\sqrt{-\mathrm{i}\tau}}, \text{ with } q = \exp(2\pi\mathrm{i}\tau).$$

This was exploited by Spencer **Bloch**, Matt **Kerr** and Pierre **Vanhove**. Let

$$t = -64\left(\frac{\eta_2\eta_6}{\eta_1\eta_3}\right)^6, \quad \psi = \frac{\eta_1^2\eta_3^2}{\eta_2\eta_6}, \quad h = \frac{\eta_2^{16}}{\eta_1^8} - 9\frac{\eta_6^{16}}{\eta_3^8}$$

where $\eta_n = \eta(n\tau)$, ψ^2 is the **square** of a elliptic integral, obtained by **Geoff Joyce** in 1973 as a solution to the **homogeneous** equation, and h is a weight-4 **cuspform**. Then the **inhomogeneous** differential equation for the Feynman integral becomes

$$\left(q\frac{\mathrm{d}}{\mathrm{d}q}\right)^3\frac{J(t)}{\psi^2} = 24h = \mathbf{24}\sum_{n=1}^{\infty}\frac{n^3(q^n - 8q^{3n} + q^{5n})}{1-q^{6n}}.$$

This is easily integrated to give **Lambert series** in

$$\frac{J(t)}{\psi^2} = H(\tau) = 24T_3(\tau) - 3T_3(2\tau) - 8T_3(3\tau) + T_3(6\tau)$$

which converges well for $|t| \leq 8$. Note that at $t = 0$ we obtain the correct value of

$$J(0) = \tfrac{1}{2}(24 - 3 - 8 + 1)\zeta(3) = \mathbf{7}\zeta(3).$$

The **Fricke** involution $\tau \to -1/(6\tau)$ gives $t \to 64/t$ and **Ramanujan** delivers

$$48\tau^2 H\left(\frac{-1}{6\tau}\right) = 16(3T_3(\tau) - 6T_3(2\tau) - T_3(3\tau) + 2T_3(6\tau)) + 4\log^3(q)$$

for fast evaluation of $J(t)$ with $|t| \geq 8$. Then, at large $|t|$, we obtain

$$-tJ(t) = \mathbf{16}\zeta(3) + 4\log^3(-t)(1 + \mathcal{O}(1/t))$$

whose **asymptotic** constant $\mathbf{16}\zeta(3)$ is determined by $J(0) = \mathbf{7}\zeta(3)$.
Thus, Ramanujan's notebooks, from more than 100 years ago, provide us with a **connection** between expansions at **large and small momenta**. Later, I shall describe a method to relate expansions at **strong and weak couplings**.

Lambert series with a character at 2 loops.

The 2-loop **sunrise diagram** produces Lambert series with $s = 2$. So how can it **avoid** a divergent asymptotic series at large momenta?

For the sunrise integral $I(w^2) = 4\int_0^\infty I_0(wx)K_0^3(x)x\,dx$, the **modular** parameterization

$$w = 3\frac{\eta_2^2\eta_3^4}{\eta_1^4\eta_6^2}\,, \quad f = \frac{\eta_1^6\eta_6}{\eta_2^3\eta_3^2}\,, \quad g = \frac{\eta_3^9}{\eta_1^3} + \frac{\eta_6^9}{\eta_2^3}$$

neatly reduces the problem to solving

$$-\left(q\frac{\mathrm{d}}{\mathrm{d}q}\right)^2\frac{I(w^2)}{f} = 6g = 6\sum_{n=1}^{\infty}\frac{n^2(q^n - q^{5n})}{1-q^{6n}}$$

and the integration produces, inter alia, a **Lambert series with a character**

$$\mathcal{L}_2(\chi_{6,5};\tau) = \sum_{n=1}^{\infty}\frac{\chi_{6,5}(n)}{n^2}\left(\frac{q^n}{1-q^n}\right)$$

with $\chi_{6,5}(n) = \pm 1$ for $n = \pm 1 \bmod 6$ and $\chi_{6,5}(n) = 0$ otherwise. Then quasi-modularity allows one to expand about any of the singular points $w \in \{0, 1, 3, \infty\}$ corresponding to **cusps** of modular forms on the congruence subgroup $\Gamma_0(6)$, with **level 6**.

Two types of divisor sums with a character:

$$\begin{aligned}\mathcal{L}_s(\chi_r;\tau) &= \sum_{n>0} \frac{\chi_r(n)}{n^s}\frac{q^n}{1-q^n} \\ &= \sum_{n>0} \frac{\sigma'_{s,\chi_r}(n)\,q^n}{n^s}, \quad \sigma'_{s,\chi_r}(n) = \sum_{d|n} \chi_r(n/d)d^s, \\ \widetilde{\mathcal{L}}_s(\chi_r;\tau) &= \sum_{n>0} \frac{\sigma_{s,\chi_r}(n)\,q^n}{n^s}, \quad \sigma_{s,\chi_r}(n) = \sum_{d|n} \chi_r(d)d^s.\end{aligned}$$

If χ_r is a primitive **quadratic** character with fundamental discriminant D, then we have quasi-modularity for **odd** s when $D > 1$, and for **even** s, when $D < 0$. With conductor $N = |D|$, the modularity gap is given by **Bernoulli** polynomials in

$$\sqrt{D}\left(2\mathcal{L}_s(\chi_D;\tau) + \sum_{n>0}\frac{\chi_D(n)}{n^s}\right) + 2(-\tau)^{s-1}\widetilde{\mathcal{L}}_s\left(\chi_D;\frac{-1}{N\tau}\right) =$$

$$\frac{(2\pi\mathrm{i})^s}{(s+1)!}\sum_{j=0}^{\lfloor s/2\rfloor}\tau^{2j-1}\binom{s+1}{2j}B_{2j}(0)\sum_{m=1}^{N-1}\chi_D(m)B_{s+1-2j}(m/N).$$

For a fermionic **spectral trace** in **topological string** theory, compactified on a **Calabi-Yau** manifold with $\mathbb{P}^2$ geometry, **Veronica Fantini** and **Claudia Rella** obtained q-**Pochhammer** symbols in

$$\mathrm{Tr}(\rho_{\mathbb{P}^2}) = \frac{1}{3\sqrt{\tau}} \exp\left(-\frac{\pi \mathrm{i}}{12}\tau + \frac{\pi \mathrm{i}}{36}\tau^{-1} + \frac{\pi \mathrm{i}}{4}\right) \frac{(q^2;q^3)_\infty^2}{(q;q^3)_\infty} \frac{(\omega_3;\tilde{q})_\infty}{(\omega_3^{-1};\tilde{q})_\infty^2},$$

$$(xq^\alpha;q)_\infty := \prod_{n=0}^{\infty} \left(1 - xq^{\alpha+n}\right), \quad q = \exp(2\pi \mathrm{i}\tau), \quad \tilde{q} = \exp(-2\pi \mathrm{i}/(3\tau)), \quad \omega_3 = \exp(2\pi \mathrm{i}/3)$$

which involve the **odd** character $\chi_{3,2}(n) = \pm 1$ for $n = \pm 1 \bmod 3$. The behaviour of

$$\log\left[\mathrm{Tr}(\rho_{\mathbb{P}^2})\right] = -\frac{1}{2}\log(3^{\frac{5}{2}}\tau) - \frac{\pi \mathrm{i}}{4} + \frac{3}{2}\left(\widetilde{\mathcal{L}}_1(\chi_{3,2};\tau) - \sqrt{-3}\mathcal{L}_1(\chi_{3,2};-\tfrac{1}{3\tau})\right)$$

at strong or weak **coupling** $y = -\mathrm{i}\tau$ requires **Borel resummation** of an infinite number of perturbative terms, with exponentially suppressed **tails** in

$$\begin{pmatrix} \mathcal{L}_1(\chi_{3,2};\tau) \\ \widetilde{\mathcal{L}}_1(\chi_{3,2};\tau) \end{pmatrix} = \begin{pmatrix} \mathcal{S}_\mp\left[\mathcal{L}_1^{\mathrm{Pert}}\right](\chi_{3,2};\tau) \\ \mathcal{S}_\mp\left[\widetilde{\mathcal{L}}_1^{\mathrm{Pert}}\right](\chi_{3,2};\tau) \end{pmatrix} \pm \mathrm{i} \begin{pmatrix} 0 & \frac{1}{\sqrt{3}} \\ \sqrt{3} & 0 \end{pmatrix} \begin{pmatrix} \mathcal{L}_1(\chi_{3,2};-\frac{1}{3\tau}) \\ \widetilde{\mathcal{L}}_1(\chi_{3,2};-\frac{1}{3\tau}) \end{pmatrix}$$

that **resolve** the ambiguity of **directional** Borel resummation, indicated by $S_\mp$.

Spectral trace of local $\mathbb{P}^{m,n}$. Here we need to consider characters with **conductor** $N = m + n + 1$. With $(m, n) = (2, 1)$, we encounter the **odd** character $\chi_{4,3}(n) = \pm 1$ for $n = \pm 1 \bmod 4$ and $\chi_{4,3}(n) = 0$ otherwise. The logarithm of the spectral trace involves the combination

$$\widetilde{\mathcal{L}}_1(\chi_{4,3}; \tau) - \sqrt{-4}\mathcal{L}_1(\chi_{4,3}; -\tfrac{1}{4\tau})$$

and hence directional Borel resummation is required at strong and weak coupling. With $(m, n) = (3, 1)$, or $(m, n) = (2, 2)$, there are **three** non-principal characters:

$n \bmod 5$	1	2	3	4
$\chi_{5,2}(n)$	1	i	$-$i	-1
$\chi_{5,3}(n)$	1	$-$i	i	-1
$\chi_{5,4}(n)$	1	-1	-1	1

The **odd** complex characters give contributions that **require** Borel resummation. The **even** real character gives a contribution

$$\widetilde{\mathcal{L}}_1(\chi_{5,4}; \tau) - \sqrt{5}\mathcal{L}_1(\chi_{5,4}, -\tfrac{1}{5\tau})$$

with perturbative expansions that **terminate**, at both large and small coupling.

2. Magnetic moment of the electron and black-hole areas at 4 loops

With $N = 6$, Bessel functions the moments

$$M_{m,n}(z) = \int_0^\infty I_0(xz)[I_0(x)]^m[K_0(x)]^{5-m}x^{2n+1}\mathrm{d}x$$

were intensively studied because of their relevance to **four-loop** corrections to the **magnetic moment** of the **electron**, where **Stefano Laporta** found contributions from $M_{0,0}(\mathbf{1})$ and $M_{0,1}(\mathbf{1})$. **Jon Borwein** and I derived the **Calabi-Yau** equation

$$\sum_{k=0}^{3}(-z^2)^{1-k}P_k(\theta+2-k)M_{m,0}(z) = 5!\delta_{m,0}, \quad \theta = z\frac{\mathrm{d}}{\mathrm{d}z},$$

$$P_0(t) = t^4,\ P_1(t) = 35t^4 + 42t^2 + 3,\ P_2(t) = 259t^4 + 104t^2,\ P_3(t) = 225(t^2-1)^2$$

for $m = 0, 1, 2$, with an **inhomogeneous** constant in the **Feynman** case $m = 0$. Philip **Candelas**, Xenia **de la Ossa**, Mohamed **Elmi** and Duco **van Straten** encountered this Calabi-Yau equation in their construction of **black holes** in 4 spacetime dimensions by compactifying a 10-dimensional **supergravity** theory on a Calabi-Yau threefold with complex structure. Remarkably, their constructions involve the **special values** $z = \sqrt{-\mathbf{7}}$ and $z = \sqrt{\mathbf{17}} - 4$, where Bessel moments determine **areas** of black holes.

Modular forms of level 6 at $z = 1$: I discovered and **Yajun Zhou** proved that

$$\mathcal{M} = \frac{2}{\pi^2}\begin{bmatrix} 4M_{0,0}(1) & \frac{36}{5}\left(M_{0,0}(1)+M_{0,1}(1)\right) \\ \frac{5}{3}M_{1,0}(1) & 3\left(M_{1,0}(1)+M_{1,1}(1)\right) \end{bmatrix} = \begin{bmatrix} -\Omega_2 & \widehat{\Omega}_2 \\ -\Omega_1 & \widehat{\Omega}_1 \end{bmatrix}$$

is a matrix with elements determined by **Eichler integrals**

$$\frac{\Omega_s}{(2\pi)^s} = \int_{1/\sqrt{3}}^{\infty} f_{4,6}\left(\frac{1+\mathrm{i}y}{2}\right) y^{s-1} dy, \quad \frac{\widehat{\Omega}_s}{(2\pi)^s} = \int_{1/\sqrt{3}}^{\infty} \widehat{f}_{4,6}\left(\frac{1+\mathrm{i}y}{2}\right) y^{s-1} dy,$$

of **modular forms** of weight 4 and **level 6**, given by **eta quotients** with $\eta_n := \eta(n\tau)$:

$$f_{4,6}(\tau) = (\eta_1\eta_2\eta_3\eta_6)^2, \quad \frac{\widehat{f}_{4,6}}{f_{4,6}} = \frac{1}{32}\left(w+\frac{3}{w}\right)^4 - \frac{9}{16}\left(w+\frac{3}{w}\right)^2, \quad w = 3\frac{\eta_2^2\eta_3^4}{\eta_1^4\eta_6^2}.$$

The 4-loop **periods** $\Omega_{1,2}$, from $f_{4,6}$, and the **quasi-periods** $\widehat{\Omega}_{1,2}$, from the **weakly** holomorphic modular form $\widehat{f}_{4,6}$, have a rational **intersection number**, namely $\Omega_1\widehat{\Omega}_2 - \Omega_2\widehat{\Omega}_1 = \det\mathcal{M} = \frac{1}{12}$. David **Roberts** and I conjectured explicit forms of such **quadratic relations** for **all loops**. Javier **Fresán**, Claude **Sabbah** and Jeng-Daw **Yu** then proved our formulas up to **20 loops**, as did Yajun **Zhou**.

Modular forms of level 14 at $z = \sqrt{-7}$: Here the weight-4 **cuspform**

$$f_{4,14}(\tau) = \frac{(\eta_2\eta_7)^6}{(\eta_1\eta_{14})^2} - 4(\eta_1\eta_2\eta_7\eta_{14})^2 + \frac{(\eta_1\eta_{14})^6}{(\eta_2\eta_7)^2}$$

has **periods** given by **critical values** of the L-function

$$L(f_{4,14}, s) = \frac{(2\pi)^s}{\Gamma(s)} \int_0^\infty f_{4,14}(\mathrm{i}y) y^{s-1} \mathrm{d}y$$

which are easily evaluated by the **Bessel moments**

$$M_{2,0}(\sqrt{-7}) = \tfrac{1}{2} L(f_{4,14}, 2), \quad M_{1,0}(\sqrt{-7}) = L(f_{4,14}, 3) = \frac{\pi^2}{7} L(f_{4,14}, 1),$$

with the **black-hole area** determined by their **ratio** $M_{2,0}(\sqrt{-7})/M_{1,0}(\sqrt{-7})$.
I determined the corresponding **quasi-periods**, using an Eichler integral of

$$\widehat{f}_{4,14} = (253 + 645h + 1446h^2 + 2064h^3 + 1024h^4)(\eta_1\eta_2\eta_7\eta_{14})^2, \quad h = \left(\frac{\eta_2\eta_{14}}{\eta_1\eta_7}\right)^3.$$

Modular forms of level 34 at $z = \sqrt{\mathbf{17}} - 4$: Bessel moments determine the **area** of the event horizon of a **black hole**, with **charges** specified by (k, ℓ), constructed by **Candelas** at al. I found that the area is given by

$$A = \mathbf{34}\pi \left(\frac{k^2}{v} + \ell^2 v \right), \quad v = 4\pi \frac{M_{2,0}(\sqrt{\mathbf{17}} - 4)}{M_{1,0}(\sqrt{\mathbf{17}} - 4)}.$$

The **Bessel moments** also determine the **periods** of a **cuspform** $f_{4,34}$ whose **complex** Fourier coefficients are **Gaussian** integers. The **critical values** of its L-function are

$$\begin{aligned} L(f_{4,34}, 1) &= \left(\frac{7 - 11z + (11 + 7z)\mathrm{i}}{\pi^2} \right) M_{1,0}(z) \\ L(f_{4,34}, 2) &= 4 \left(\frac{5 - 3z + (3 + 5z)\mathrm{i}}{17} \right) M_{2,0}(z) \\ L(f_{4,34}, 3) &= \left(\frac{13 - z + (1 + 13z)\mathrm{i}}{17} \right) M_{1,0}(z) \end{aligned}$$

with $z = \sqrt{\mathbf{17}} - 4$. I identified the corresponding **quasi-periods** as Bessel moments. **Kilian Bönisch**, with assistance from **Don Zagier**, identified a **weakly** holomorphic modular form $\widehat{f}_{4,34}$ whose Eichler integrals also give those quasi-periods.

3. Automorphic up to 6 loops

With $N = a + b$ **Bessel** functions and $c \geq 0$, I define **moments**

$$M(a, b, c) := \int_0^\infty I_0^a(t) K_0^b(t) t^c \mathrm{d}t$$

that converge for $b > a \geq 0$. For $b = a = N/2$, we have convergence for $b > c+1$. The L-loop on-shell **banana** diagram in $D = 2$ dimensions gives

$$2^L M(1, L+1, 1) = \int_0^\infty \ldots \int_0^\infty \frac{\prod_{k=1}^L \mathrm{d}x_k/x_k}{(1 + \sum_{i=1}^L x_i)(1 + \sum_{j=1}^L 1/x_j) - 1}$$

as an integral over Schwinger parameters. $M(2, L, 1)$ is obtained by cutting an internal line. To obtain $M(1, L+1, 3)$ and $M(2, L, 3)$, we differentiate w.r.t. an external momentum, before taking the **on-shell** limit.

Kloosterman sums over finite fields

For $a \in \mathbf{F}_q$, with $q = p^k$, we define **Kloosterman sums**

$$K(a) := \sum_{x \in \mathbf{F}_q^*} \exp\left(\frac{2\pi \mathrm{i}}{p} \mathrm{Trace}\left(x + \frac{a}{x}\right)\right)$$

with a trace of Frobenius in $\mathbf{F}_q$ over $\mathbf{F}_p$. We obtain **integers**

$$c_N(q) := -\frac{1+S_N(q)}{q^2}$$

$$S_N(q) := \sum_{a\in\mathbf{F}_q^*}\sum_{k=0}^{N}[g(a)]^k[h(a)]^{N-k}$$

with $K(a) = -g(a) - h(a)$ and $g(a)h(a) = q$. Then

$$Z_N(p,T) = \exp\left(-\sum_{k>0}\frac{c_N(p^k)}{k}T^k\right)$$

is a **polynomial** in T. For $N < 8$, the appropriate L-series is

$$L_N(s) = \prod_p \frac{1}{Z_N(p,p^{-s})}$$

with a modification at $N = 8$:

$$L_8(s) = \prod_p \frac{Z_4(p,p^{2-s})}{Z_8(p,p^{-s})}\,.$$

Symmetric square at 5 loops. With $N = 7$ Bessel functions, the **local** factors of

$$L_7(s) = \prod_p \frac{1}{Z_7(p, p^{-s})}$$

are given, for prime p with $\gcd(p, 105) = 1$, by the **cubic**

$$Z_7(p, T) = \left(1 - \left(\frac{p}{105}\right) p^2 T\right) \left(1 + \left(\frac{p}{105}\right) (2p^2 - |\lambda_p|^2) T + p^4 T^2\right)$$

where $(\frac{p}{105}) = \pm 1$ is a **Kronecker** symbol and λ_p is a Hecke eigenvalue of a weight-3 newform with level 525, `https://www.lmfdb.org/ModularForm/GL2/Q/holomorphic/525/3/c/a/`. For the primes of **bad** reduction, I obtained

$$Z_7(3, T) = 1 - 10T + 3^4 T^2,\ Z_7(5, T) = 1 - 5^4 T^2,\ Z_7(7, T) = 1 + 70T + 7^4 T^2.$$

Then **Anton Mellit** suggested a **functional equation**

$$\Lambda_7(s) := \left(\frac{105}{\pi^3}\right)^{s/2} \Gamma\left(\frac{s-1}{2}\right) \Gamma\left(\frac{s}{2}\right) \Gamma\left(\frac{s+1}{2}\right) L_7(s) = \Lambda_7(5 - s)$$

that was validated at high precision and gave us the empirical result

$$24M(2, 5, 1) = 5\pi^2 L_7(2).$$

Modular form and subtleties at 6 loops

With $N = 8$ Bessel functions, the L-series comes from the **modular** form

$$f_{6,6} := \left(\frac{\eta_2^3\eta_3^3}{\eta_1\eta_6}\right)^3 + \left(\frac{\eta_1^3\eta_6^3}{\eta_2\eta_3}\right)^3$$

with weight 6 and level 6. I discovered and checked to 1000 digits that

$$M(4,4,1) = L_8(3), \quad 4M(3,5,1) = 9L_8(4), \quad 4M(2,6,1) = 27L_8(5).$$

Moreover, $2^6 M(1,7,1) = 72\pi^2 L_8(4)$ determines the **6-loop banana** integral.

There are **two subtleties**. First, Kloosterman moments at $N = 8$ do **not** deliver the local factors directly. $L_8(s) = \prod_p Z_4(p, p^{2-s})/Z_8(p, p^{-s})$ removes factors from $N = 4$. Secondly, there is an infinite family of **sum rules**, with

$$a(n) := \left(\frac{2}{\pi}\right)^4 \int_0^\infty \left(\pi^2 I_0^2(t) - K_0^2(t)\right) I_0(t) K_0^5(t)\, (2t)^{2n-1} \mathrm{d}t$$

delivering the **integers** of `http://oeis.org/A262961` as proved by **Yajun Zhou**.

4. Critical L-values up to 22 loops

Let $\Omega_{a,b}$ be the **determinant** of the $r \times r$ matrix with $M(a,b,1)$ at top left, size $r = \lceil (a+b)/4 - 1 \rceil$, powers of t^2 increasing to the right and powers of $I_0^2(t)$ increasing downwards. Thus $\Omega_{1,23}$ is a 5×5 determinant with the **22-loop banana** integral $M(1,23,1)$ at **top left** and $M(9,15,9)$ at bottom right. With $N = 4r + 4$ Bessel functions, **David Roberts** and I discovered that

$$
\begin{aligned}
L_8(4) &= \frac{2^2\,\Omega_{1,7}}{3^2\pi^2} \equiv \frac{4}{9\pi^2}\int_0^\infty I_0(t)K_0^7(t)t\mathrm{d}t \\
L_{12}(6) &= \frac{2^6\,\Omega_{1,11}}{3^4\times 5\pi^6} \\
L_{16}(8) &= \frac{2^{14}\,\Omega_{1,15}}{3^7\times 5^2\times 7\pi^{12}} \\
L_{20}(10) &= \frac{2^{22}\times 11\times \mathbf{131}\,\Omega_{1,19}}{3^{11}\times 5^6\times 7^3\pi^{20}} \quad \text{to 44 digits} \\
L_{24}(12) &= \frac{2^{29}\times \mathbf{12558877}\,\Omega_{1,23}}{3^{19}\times 5^9\times 7^3\times 11\pi^{30}} \quad \text{to 19 digits}
\end{aligned}
$$

With a **cut** of a line in the diagram at top left of the matrix, we found

$$
\begin{aligned}
L_8(5) &= \frac{2^2\,\Omega_{2,6}}{3^3} \equiv \frac{4}{27}\int_0^\infty I_0^2(t)K_0^6(t)t\mathrm{d}t \\
L_{12}(7) &= \frac{2^5\times 11\,\Omega_{2,10}}{3^6\times 5^2\pi^2} \\
L_{16}(9) &= \frac{2^{14}\times 13\,\Omega_{2,14}}{3^9\times 5^3\times 7^2\pi^6} \\
L_{20}(11) &= \frac{2^{19}\times 17\times 19\times \mathbf{23}\,\Omega_{2,18}}{3^{13}\times 5^7\times 7^3\pi^{12}} \\
L_{24}(13) &= \frac{2^{27}\times 17\times 19^2\times 23^2\times \mathbf{46681}\,\Omega_{2,22}}{3^{23}\times 5^{12}\times 7^4\times 11^2\pi^{20}}
\end{aligned}
$$

In the cases with $N = 4r + 2$, we obtained

$$
\begin{aligned}
L_6(2) &= \frac{2\,\Omega_{1,5}}{\pi^2} \equiv \frac{2}{\pi^2}\int_0^\infty I_0(t)K_0^5(t)t\mathrm{d}t \\
L_6(3) &= \frac{2\,\Omega_{2,4}}{3} \equiv \frac{2}{3}\int_0^\infty I_0^2(t)K_0^4(t)t\mathrm{d}t \\
L_{10}(4) &= \frac{2^7\,\Omega_{1,9}}{3^2\pi^6} \\
L_{10}(5) &= \frac{2^4\,\Omega_{2,8}}{3\times 5\pi^2} \\
L_{14}(7) &= \frac{2^{10}\times 11\times 13\,\Omega_{2,12}}{3^6\times 5^2\times 7\pi^6} \\
L_{18}(8) &= \frac{2^{21}\times 17\times \mathbf{19}\,\Omega_{1,17}}{3^5\times 5^4\times 7\pi^{20}} \\
L_{18}(9) &= \frac{2^{12}\times 13\times 17\times \mathbf{41}\,\Omega_{2,16}}{3^8\times 5^3\times 7^2\pi^{12}} \\
L_{22}(11) &= \frac{2^{23}\times 17\times 19\times \mathbf{11621}\,\Omega_{2,20}}{3^{14}\times 5^7\times 7^3\pi^{20}} \\
L_{14}(6) &= L_{22}(10) = \mathbf{0} \quad \text{(central values with } \mathbf{odd} \text{ signature)}
\end{aligned}
$$

For cases with odd N, we obtained

$$
\begin{aligned}
L_5(2) &= \frac{2^2\,\Omega_{2,3}}{3} \equiv \frac{4}{3}\int_0^\infty I_0^2(t)K_0^3(t)t\mathrm{d}t \\
L_7(2) &= \frac{2^3\times 3\,\Omega_{2,5}}{5\pi^2} \equiv \frac{24}{5\pi^2}\int_0^\infty I_0^2(t)K_0^5(t)t\mathrm{d}t \\
L_9(4) &= \frac{2^6\,\Omega_{2,7}}{3\times 5\pi^2} \\
L_{11}(4) &= \frac{2^8\times 5\,\Omega_{2,9}}{3\times 7\pi^6} \\
L_{13}(6) &= \frac{2^7\times \mathbf{149}\,\Omega_{2,11}}{3^3\times 5\times 7\pi^6} \\
L_{15}(6) &= \frac{2^8\times 7\times \mathbf{53}\,\Omega_{2,13}}{3^2\times 5\pi^{12}} && \text{to 43 digits} \\
L_{17}(8) &= \frac{2^{15}\times \mathbf{29}\,\Omega_{2,15}}{3^5\times 5^2\times 7\pi^{12}} && \text{to 23 digits} \\
L_{19}(8) &= \frac{2^{14}\times \mathbf{1093}\times \mathbf{13171}\,\Omega_{2,17}}{3^4\times 5^4\times 7\times 11\pi^{20}} && \text{to 14 digits}
\end{aligned}
$$

Comments

1. Up to 6 loops, with $N = L + 2 \leq 8$ Bessel functions, the results follow from Fourier expansions of **modular forms** or their **symmetric squares**.

2. Thereafter, we determined the **conductor**, **signature** and **functional relation** of the L-function **empricially**.

3. Then the procedure `lfuncreate` of `Pari/GP` defines the L-function.

4. To check the functional equation numerically, one may use `lfuncheckfeq`.

5. For **odd** numbers of Bessel functions the conductor grows quickly, with 17 loops at the boundary of computability.

6. For **even** numbers, we reached **22 loops**, finding the 8-digit prime **12558877** in a relation between a critical L-value and a determinant of periods. This appears to be the **largest prime** ever encountered in the **Langlands** programme.

Appendix: a note on stratons (1966–1974) based on an scholarly article by **Liu Jinyan**, *Chinese Physicists' Construction of the Straton Model in Social Context*, Chinese Annals of History of Science and Technology **2** (2018) 85–122, doi:10.3724/SP.J.1461.2018.01085

1. I learnt of this work on stratons in **1966** from **Dick Dalitz** (1925–2006) at **Oxford** and from reports by **Abdus Salam** (1926–1996) of a conference in **Beijing**, resulting from the interest of **Mao Zedong** in work by the Japanese physicist **Shoichi Sakata** (1911-1970) on the idea that hadrons are composite.

2. I was updated by conversation with **Yang Chen-Ning** (1922–2025) at Stanford, shortly after his meeting with **Zhou Enlai**, in July **1971**.

3. **Lee Tsung-Dao** (1926–2024) met with Premier Zhou in October **1972**.

4. *Acta Physica Sinica* resumed publication and translations became available in **1974**, showing work on **Bethe-Salpeter** equations for composite hadrons.

This historical episode illustrates the enduring power of science to engage enquiring minds, irrespective of nationality or ideology.