Hadronic Vacuum Polarization at Three Loops in ChPT

Shanghai Tech, 9-12 Nov. 2026

Based on [2510.12885] [2603.15252]

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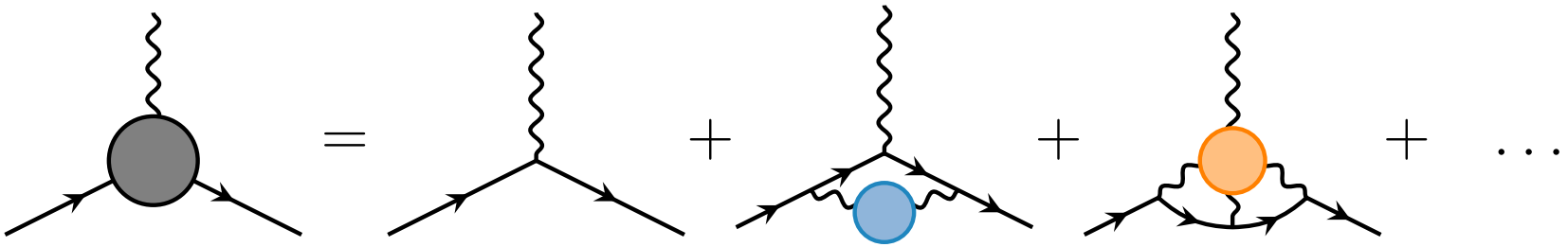
A linguistic aside

河 ≈ Sjö*

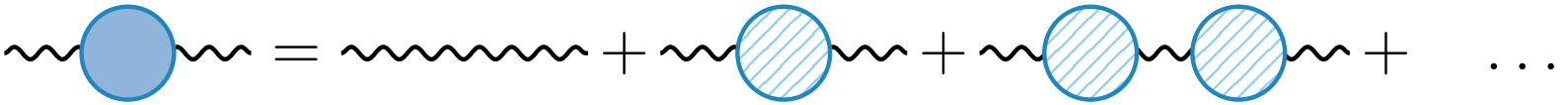
Introduction

HVP recap

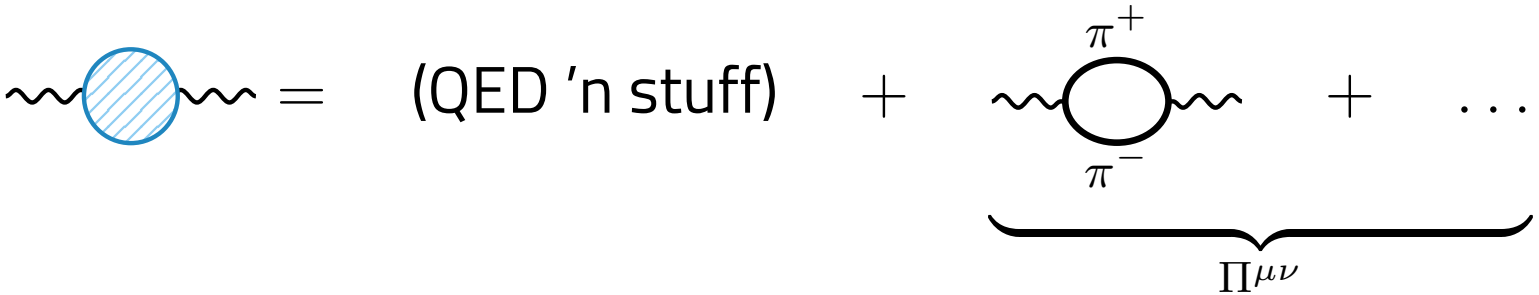
- Fermion-photon coupling (g-factor, etc.)



- ...from which the *vacuum polarization* 



- ...from which the *hadronic* vacuum polarization $\Pi^{\mu\nu}$



The challenge

PHYSICAL REVIEW D **102,** 094511 (2020)

Application of effective field theory to finite-volume effects in a_μ^{HVP}

Christopher Aubin,[1] Thomas Blum,[2] Maarten Golterman,[3] and Santiago Peris[4]

appear already at two loops, which is the minimum number. While it is unlikely that the full N^{3}LO analysis will ever be worked out in practice, it is important to establish the validity of the EFT framework for the study of a_μ^{HVP} in order to be assured that even the NNLO analysis of finite-volume effects carried out in Ref. [10] has a solid EFT basis. We believe that our discussion in this paper illustrates why indeed this is the case. We expect the same separation

Arthur C. Clarke's 1st law

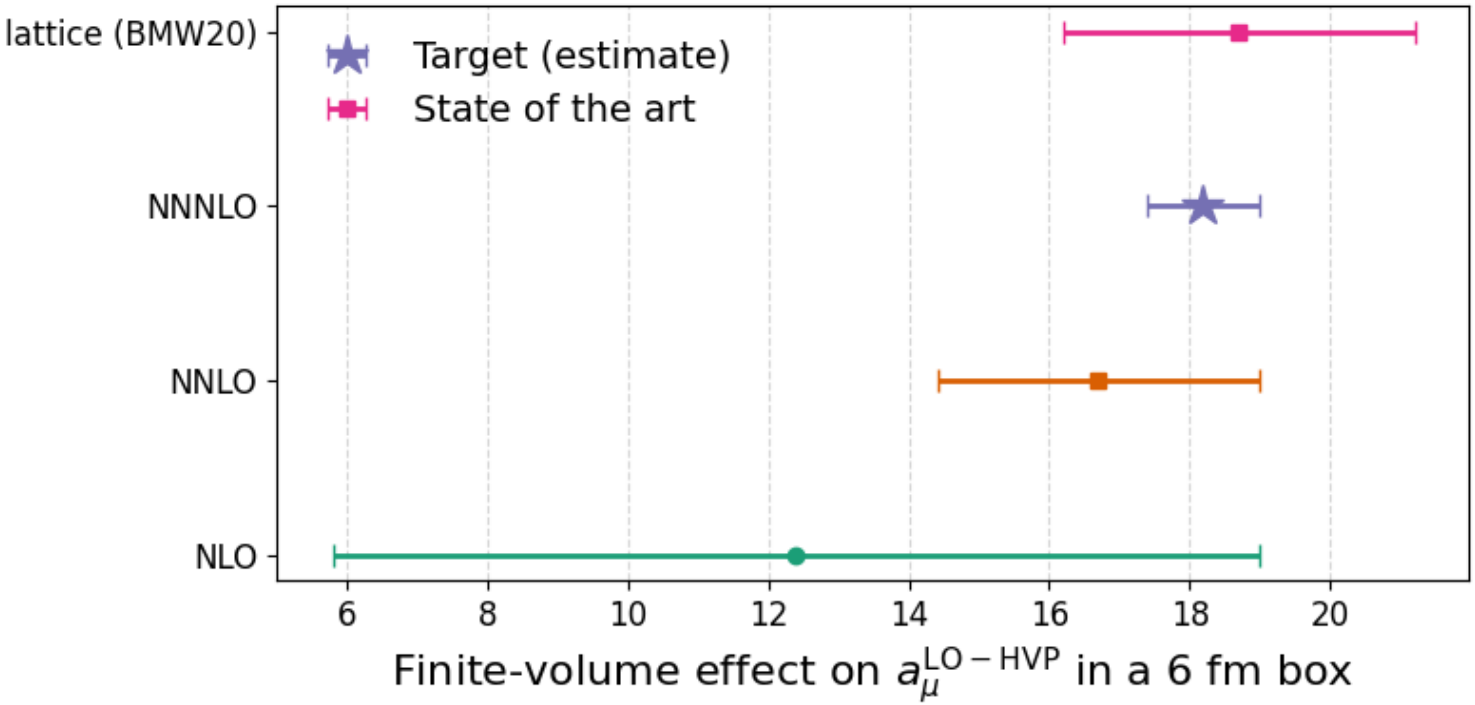
"When a distinguished [...] scientist [...] states that something is impossible, he is very probably wrong."

Lattice QCD vs. ChPT

	Lattice QCD	ChPT
Physics	QCD + add-ons	QCD + add-ons
Concept	First principles	Effective field theory
Perturbative?	No	Yes
Metric	Euclidean	Minkowski or Euclidean
Volume	Finite	Infinite (easier) or finite (harder)
Spacetime	Discrete	Continuous (easier) or discrete (harder)
High energy?	Discretization effects	Not valid
Low energy?	Finite-volume effects	No problems
Requirements	Collaborations & supercomputers	A laptop, a blackboard & some grit

The ultimate goal

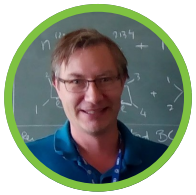
Assuming a geometric progression of FVE errors...



The collaboration

originator  **Kálmán Szábo** (Wuppertal/Jülich)

showrunner  **Mattias Sjö** (Wuppertal)

loop wizard  **Pierre Vanhove** (IPhT Saclay)

connection to reality   **Alessandro Lupo** (CERN)
Laurent Lellouch (CPT Marseille)

finite-volume reinforcements    **Antonin Portelli** (Edinburgh)
Johann Usovitsch (CERN)
Giuseppe de Anglelis (Padova)

The setup

The ChPT Lagrangian

- 2 flavors (pions only), non-dynamic photon field
- Vertices with $2n$ pions and up to 2 photons, drawn from

$$\mathcal{L}_{\text{ChPT}} = \mathcal{L}_{\text{LO}} + \mathcal{L}_{\text{NLO}} + \mathcal{L}_{\text{NNLO}} + \mathcal{L}_{\text{N}^3\text{LO}} + \cdots$$

- $\mathcal{L}_{\text{N}^3\text{LO}}$ → 475* $\mathcal{O}(p^8)$ counterterms
  Bijnens, (Herman-Trued)sson & Wang [1810.06384]
- $\mathcal{L}_{\text{NNLO}}$ → 57* $\mathcal{O}(p^6)$ counterterms
  Bijnens, Colangelo & Ecker [hep-ph/9902437], [hep-ph/9907333]
- $\mathcal{L}_{\text{NLO}}$ → 7* $\mathcal{O}(p^4)$ counterterms
  Gasser & Leutwyler (1984)
- $\mathcal{L}_{\text{LO}}$ → Basic $\mathcal{O}(p^2)$ vertex = NLSM but with masses, charges, ...

Power counting (the simple way)

$$\text{N}^k\text{LO diagram:} \qquad k = (\#\text{ loops}) + \sum_{\text{vertices}} (\#\text{ N's})$$

*These are 2-flavor numbers! It's 11,94,1254 for 3 flavors and 12,115,1862 for $\geq$ 4 flavors

A representative Lagrangian term

$$c_i \sim \bar{c}_i + \gamma_i^{(1)}\Lambda + \gamma_i^{(2)}\Lambda^2$$

$$\mathcal{L}_{\text{NNLO}} \supset i c_{35}\,\mathrm{Tr}\left(f_+^{\mu\nu} u_\mu \chi_+ u_\nu\right)$$

$$f_\pm^{\mu\nu} = u F^{\mu\nu} u^\dagger \pm u^\dagger F^{\mu\nu} u$$

$$F^{\mu\nu} = Q(\partial^\mu A^\nu - \partial^\nu A^\mu)$$

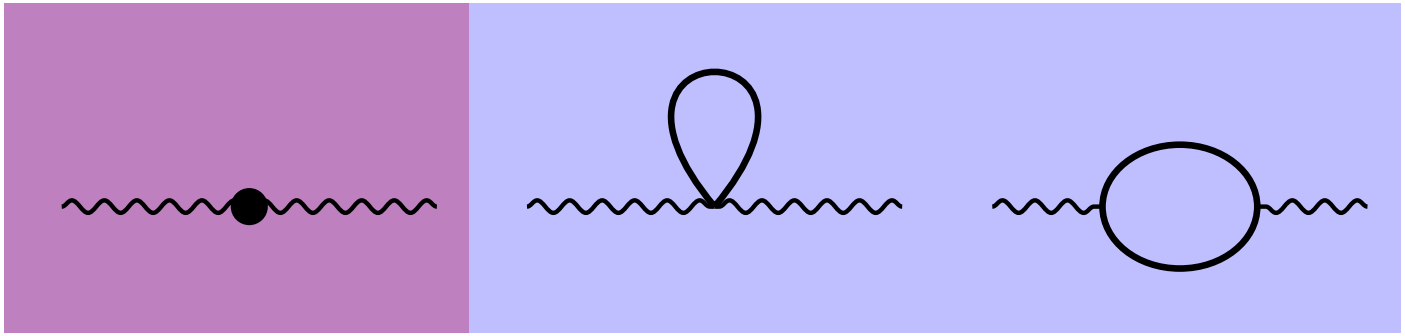
$$u_\mu = i\left[u^\dagger(\partial_\mu - iQA_\mu)u - u(\partial_\mu - iQA_\mu)u^\dagger\right]$$

$$u = \exp\left[\frac{i}{2F_{\pi,0}}\begin{pmatrix} +\pi^0/\sqrt{2} & \pi^+ \\ \pi^- & -\pi^0/\sqrt{2}\end{pmatrix}\right]$$

$$\chi_\pm = u^\dagger \chi u^\dagger \pm u \chi^\dagger u$$

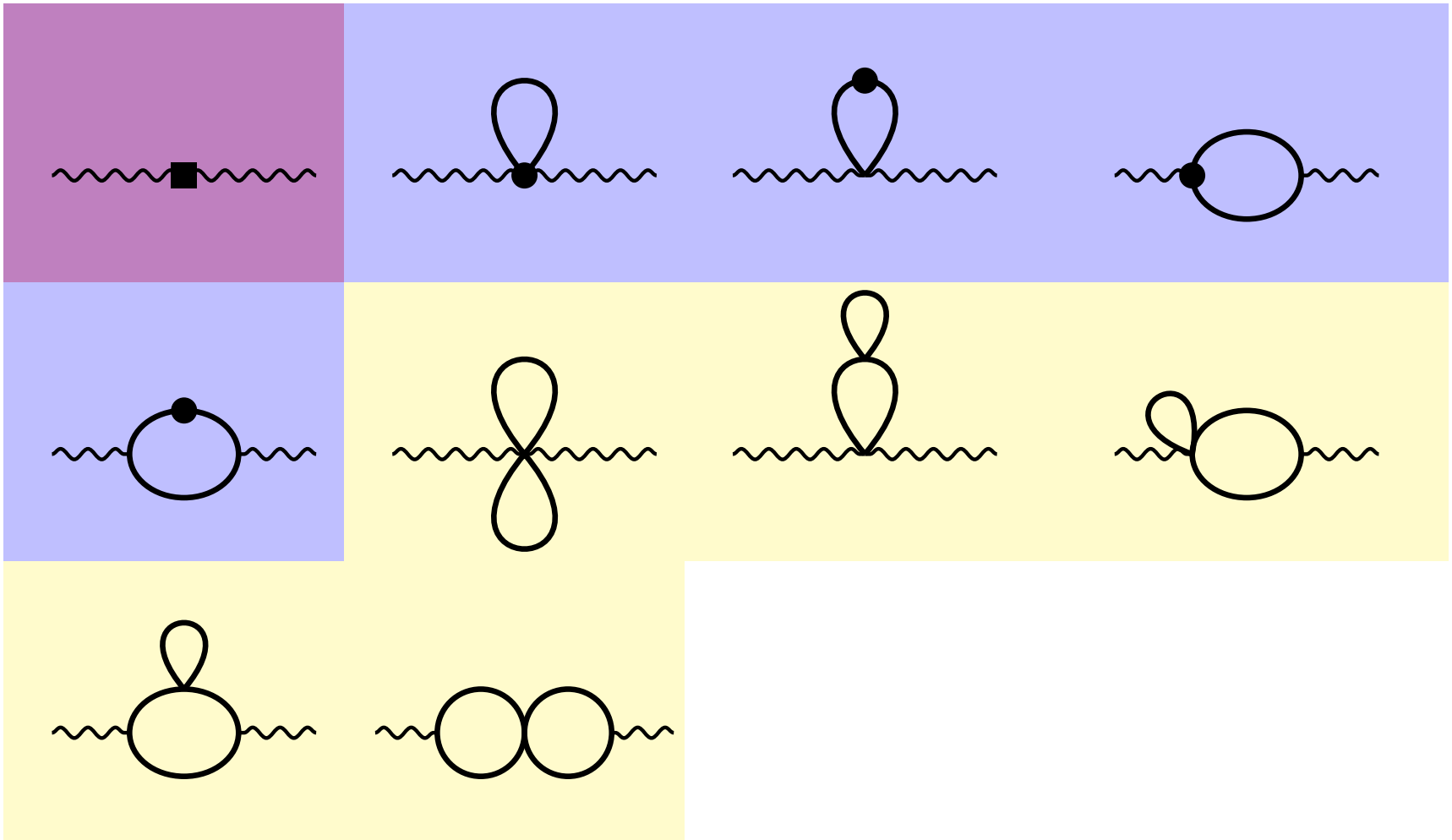
$$\chi \sim M_{\pi,0}^2$$

NLO diagrams



● = NLO (unmarked) = LO

NNLO diagrams



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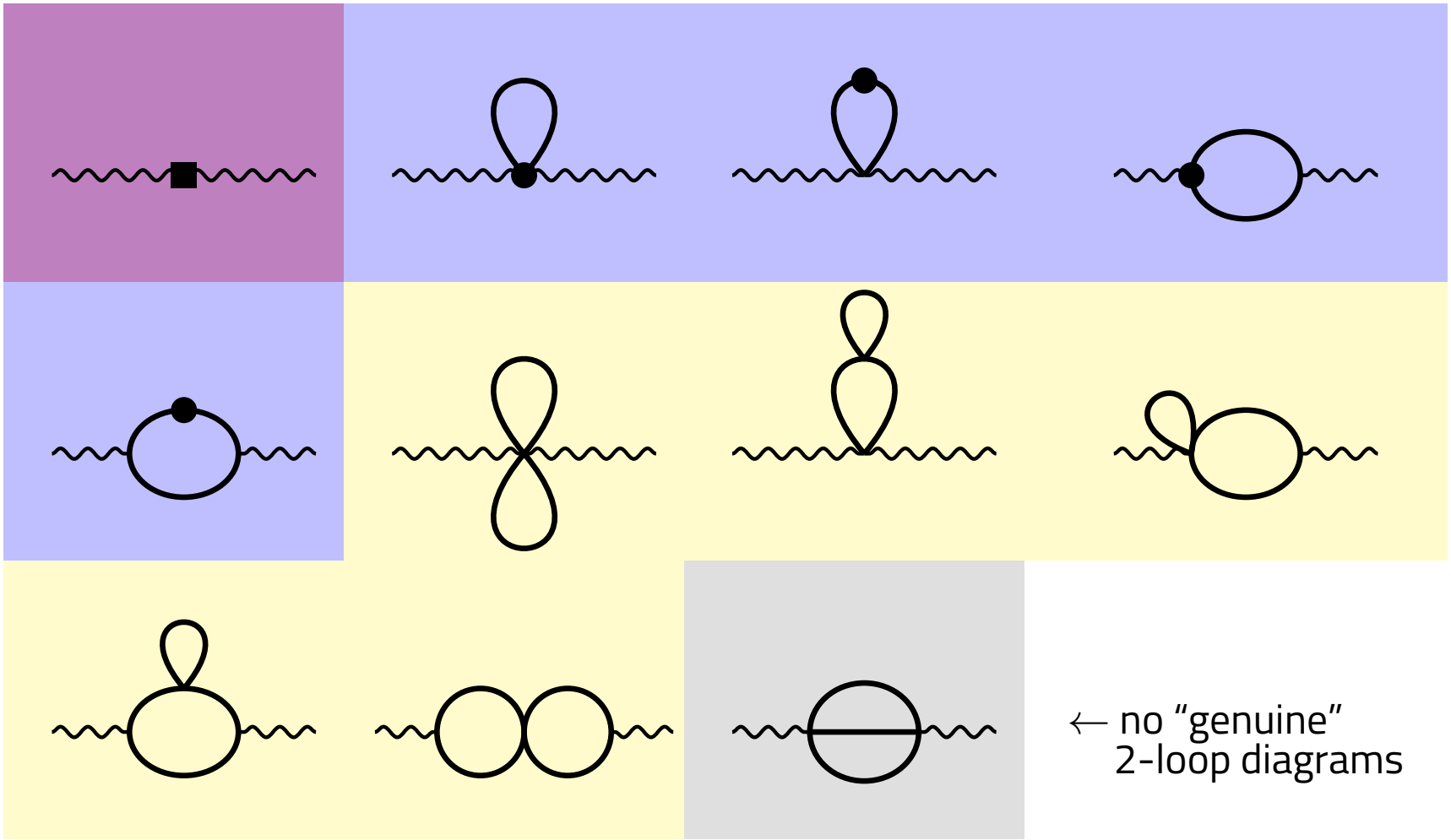
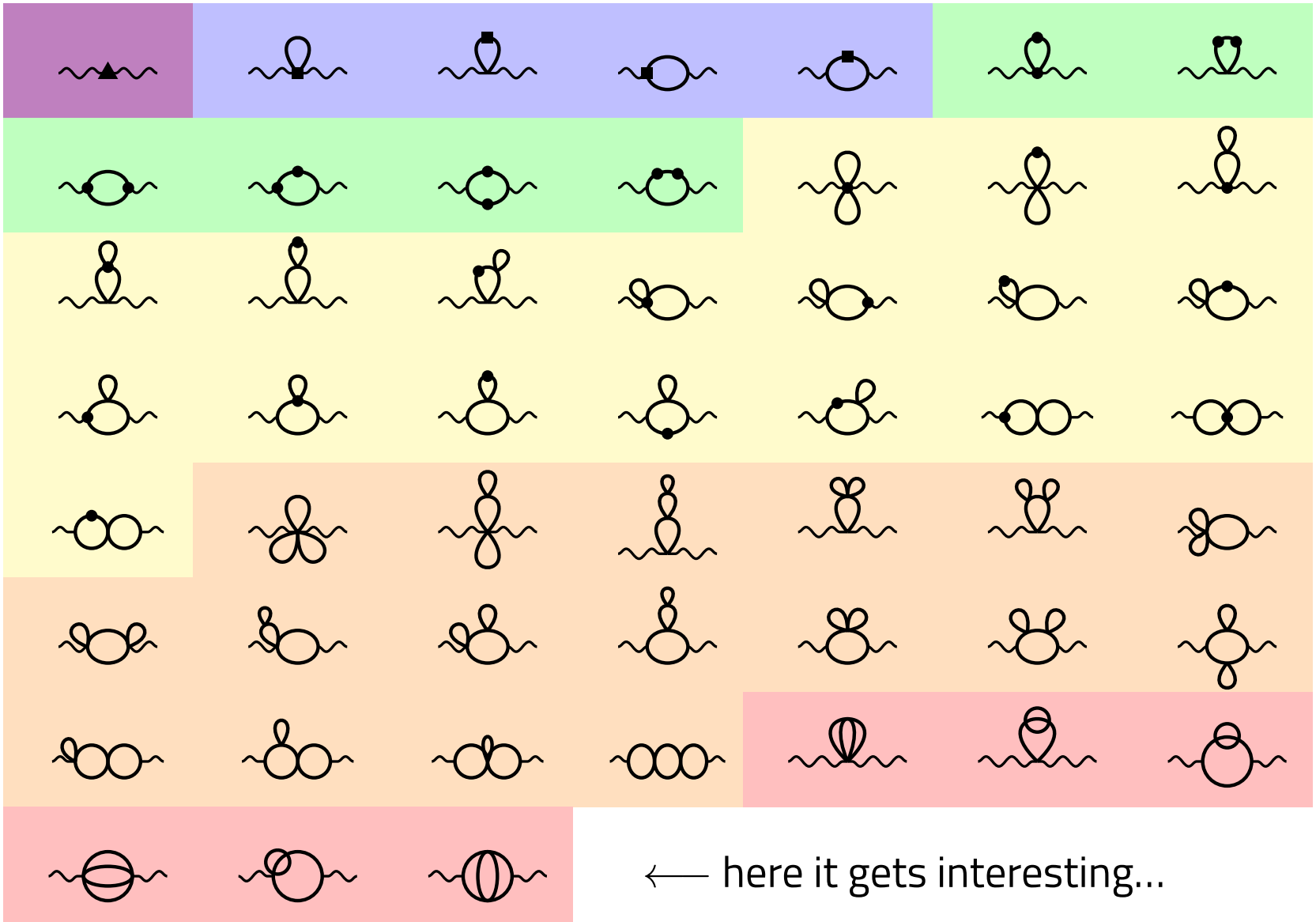
NNLO diagrams



■ = NNLO ● = NLO (unmarked) = LO

N^3LO diagrams



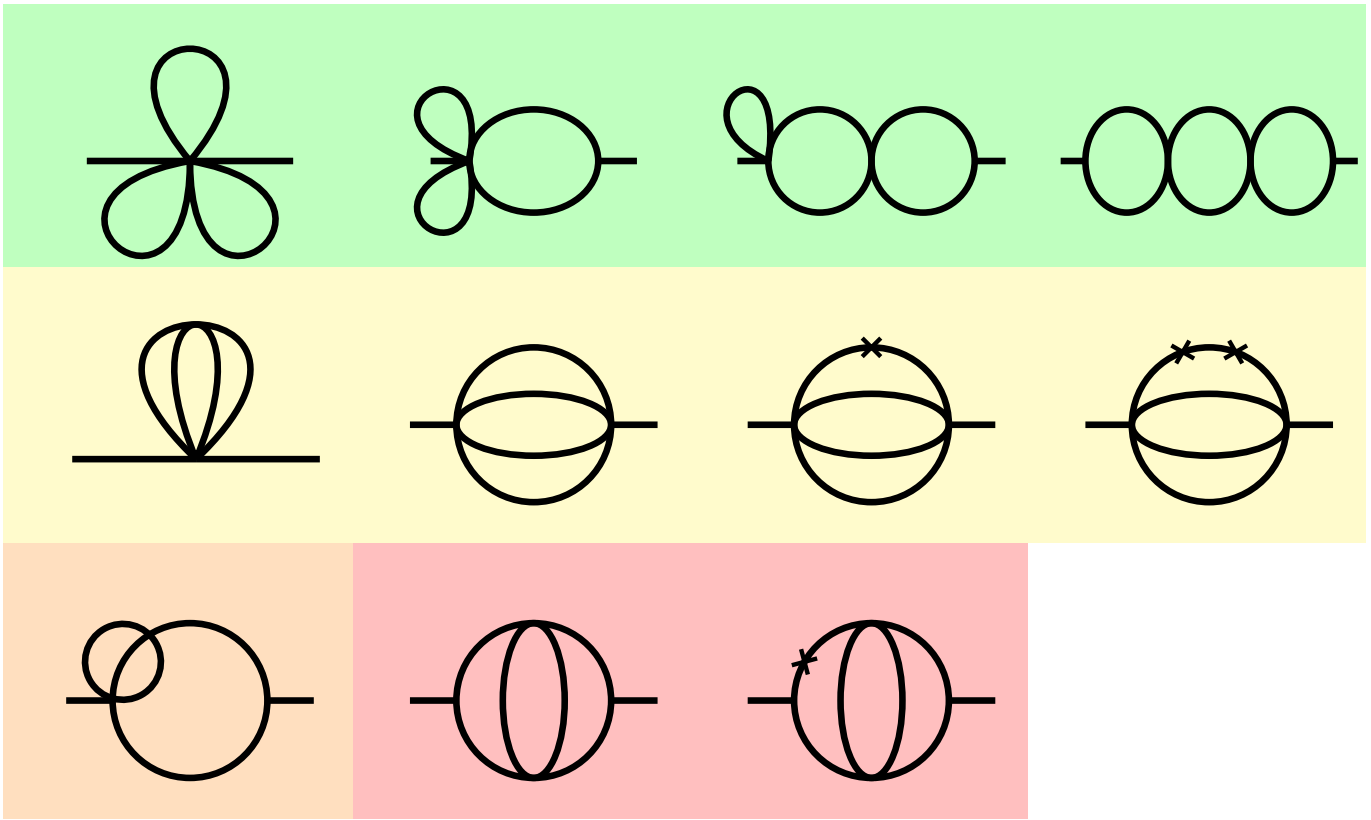
⟵ here it gets interesting...

Procedure

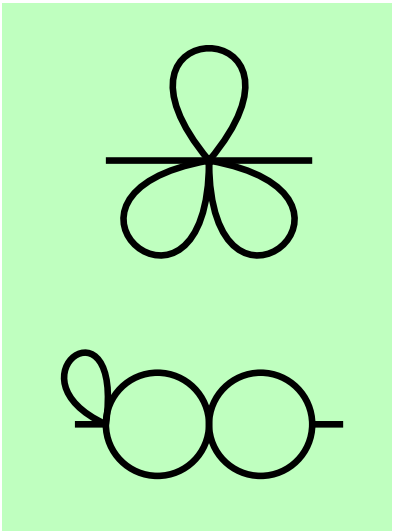
- ChPT Feynman rules, diagram-building, renormalization, convention-wrangling, bookkeeping, …
  using ChPTlib Sjö, `github.com/mssjo/ChPTlib`
  written in FORM Vermaseren [math-ph/0010025]
- Master integral reduction in LiteRed2 Roman N. Lee [1212.2685]
- MIs rendered finite using Tarasov's dimensional shift
  Tarasov [hep-th/9609018]
- Verification: Ward identity, Nambu–Goldstone manifold symmetry, finiteness, …
- All pipelined and automatized `github.com/mssjo/HVP-3loop`

The loop integrals

The 11 master integrals

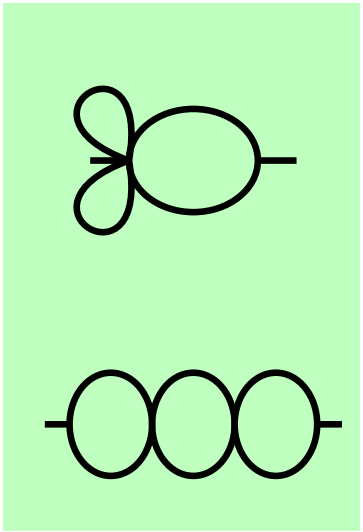


Factorizable MIs



$= \mathbb{I}_{\bigcirc}^{3}$

$= \mathbb{I}_{\bigcirc}\mathbb{I}_{\bigcirc\!\!-\!\!\bigcirc}^{2}(t)$

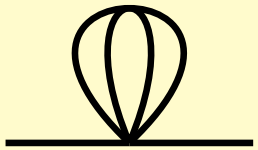


$= \mathbb{I}_{\bigcirc}^{2}\mathbb{I}_{\bigcirc\!\!-\!\!\bigcirc}(t)$

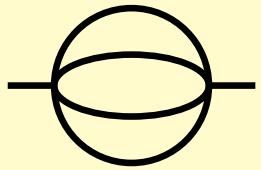
$= \mathbb{I}_{\bigcirc\!\!-\!\!\bigcirc}^{3}(t)$

- Divergent (even in 2D) but simple
- Bubble integral required up to $\mathcal{O}(\epsilon^2)$
- Just a pile of (poly)logarithms of $t = q^2/M_\pi^2$

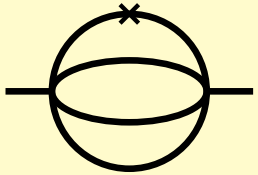
The 3-loop banana *et al.*



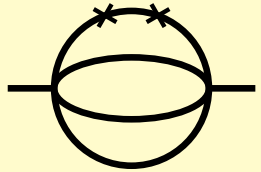
$= \mathbb{I}_{\Theta}(0) = 7\zeta(3)$



$\equiv \mathbb{I}_{\Theta}(t)$



$\sim \mathbb{I}'_{\Theta}(t)$

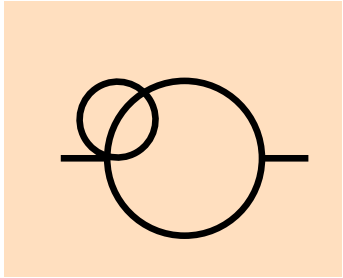


$\sim \mathbb{I}''_{\Theta}(t)$

- $\mathbb{I}_{\Theta}(t)$ in 2D satisfies an inhomogeneous Picard–Fuchs differential equation
- Related to the regulator periods of a family of K3 surfaces
- Known in terms of elliptic polylogs or Eisenstein series

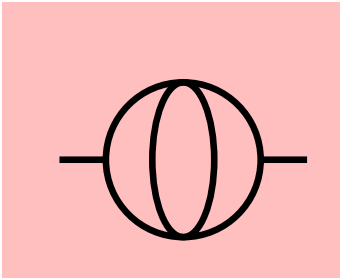
  Bloch, Kerr & Vanhove [1406.2664]
- The linchpin of our amplitude

The master integral that wasn't

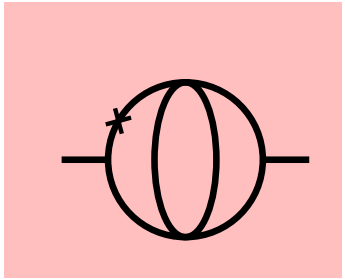


- Less well-studied, no explicit solution known before
- **Vanishes** under dimensional shift: leaves
  $[\mathbb{I}_{\ominus}(t), \mathbb{I}'_{\ominus}(t), \mathbb{I}''_{\ominus}(t), \mathbb{I}_{\bigcirc}(t)]$ ← lin. comb. with t-dependent coeffs
- Interesting test case:
  - *numerically* much harder than $\mathbb{I}_{\ominus}(t)$
  - *analytically* equivalent to it

The cat's eye integral



$\equiv \mathbb{I}_{\text{⦶}}(t)$



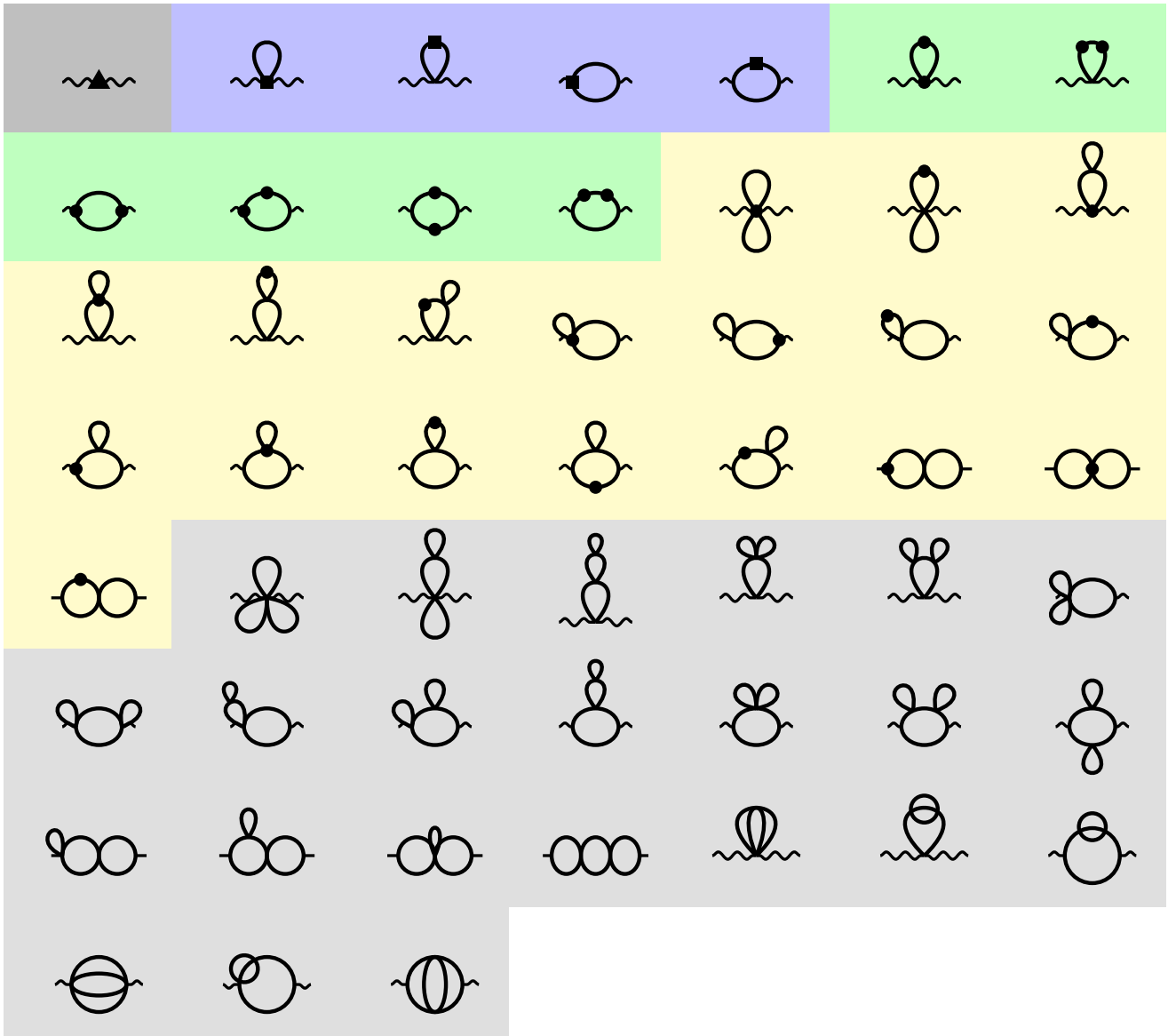
$\sim \mathbb{I}'_{\text{⦶}}(t)$

- Very difficult
- Seemingly breaks renormalization:

$$\mathbb{I}^{(d=4-2\epsilon)}_{\text{⊜}} \sim \overbrace{\frac{1}{\epsilon^3} + \frac{[\mathbb{I}_{\text{○}}]}{\epsilon^2} + \frac{[\mathbb{I}_{\text{○}}]}{\epsilon}}^{\text{managed by counterterms}} + \overbrace{\bar{\mathbb{I}}_{\text{⊜}}}^{\text{finite}}$$

$$\mathbb{I}^{(d=4-2\epsilon)}_{\text{⦶}} \sim \frac{1}{\epsilon^3} + \overbrace{\frac{[\mathbb{I}_{\text{○}}, \bar{\mathbb{I}}_{\text{⊜}}, \bar{\mathbb{I}}_{\text{⦶}}]}{\epsilon^2} + \frac{[\mathbb{I}_{\text{○}}, \bar{\mathbb{I}}_{\text{⊜}}, \bar{\mathbb{I}}_{\text{⦶}}]}{\epsilon}}^{\textbf{no counterterm!}} + \overbrace{\bar{\mathbb{I}}_{\text{⦶}}}^{\text{finite}}$$

The cat's eye integral



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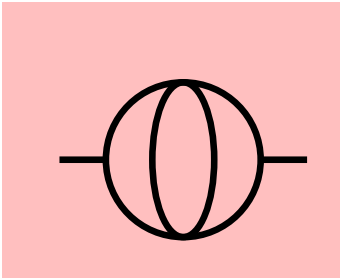

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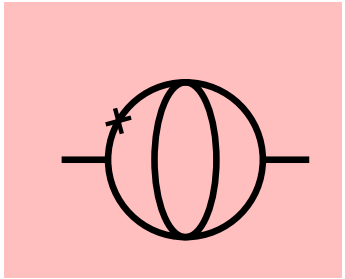

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Schouten relations

Gram determinant — Schouten identity

$$\mathbb{G}(v_1, \ldots, v_n)_{ij} = v_i \cdot v_j \quad \Rightarrow \quad |\mathbb{G}(u, v, w)| = 0 \quad \text{in } d = 2$$

- Put $\mathbb{G}(\ell_1, \ell_2, \ell_3)$ as numerator in suitable integral
  $\Rightarrow$ new relation $[\mathbb{I}_{\bigcirc}, \bar{\mathbb{I}}_{\ominus}, \bar{\mathbb{I}}_{\oslash}] = 0$ at $d = 2$
  $\Rightarrow$ renormalization restored
- Effectively eliminates master integral $\mathbb{I}'_{\oslash}(t)$ in 2D
  $\Rightarrow$ non-IBP relation!
- Novel here, but known in 2-loop sunset integrals with 3 independent masses Remiddi & Tancredi [1311.3342]

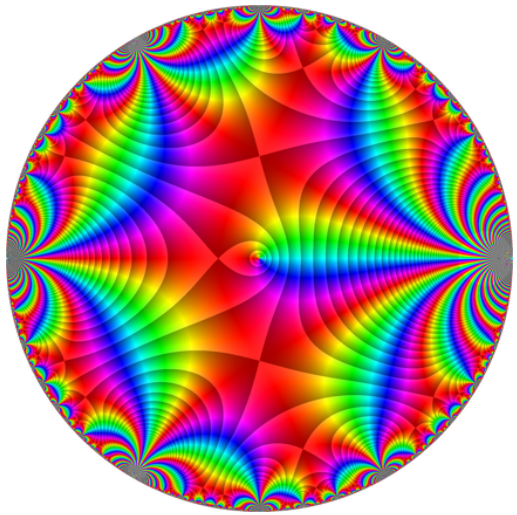
Integral evaluation

The elliptic wonderland

Take a nome q, $|q| < 1$ such that

$$t = -\left[\frac{\eta(q)\eta(q^3)}{\eta(q^2)\eta(q^6)}\right]^6,$$

using the Dirichlet η function

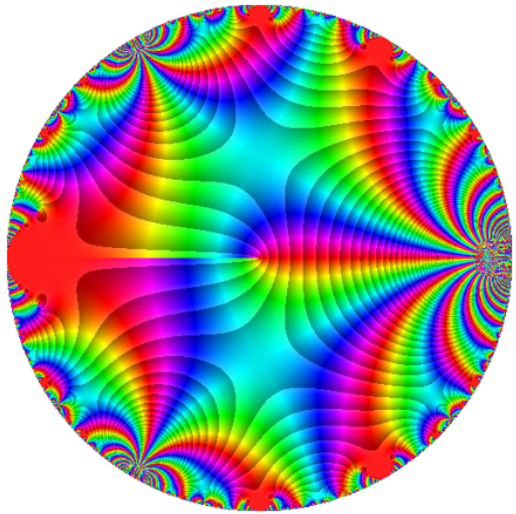


Then

$$\mathbb{I}_{\ominus}(t) = \frac{\left[\eta(q^2)\eta(q^6)\right]^4}{\left[\eta(q)\eta(q^3)\right]^2}$$

$$\times\left(3(\log q)^3 - 16\zeta(3) + \sum_{n=1}^{\infty} \frac{\psi_{\ominus}(n)}{n^3}\frac{q^n}{1-q^n}\right)$$

$\psi_{\ominus} = \{1, -15, -8, -15, 1, 120\}$ repeated



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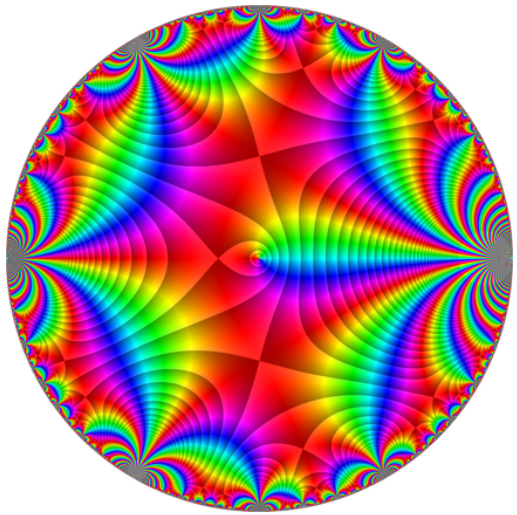

Then

$$\mathbb{I}'_{\ominus}(t) = \frac{\partial}{\partial t}\frac{\left[\eta(q^2)\eta(q^6)\right]^4}{\left[\eta(q)\eta(q^3)\right]^2}$$

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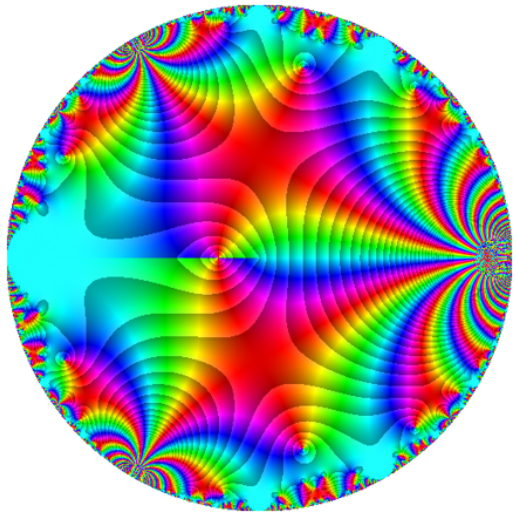

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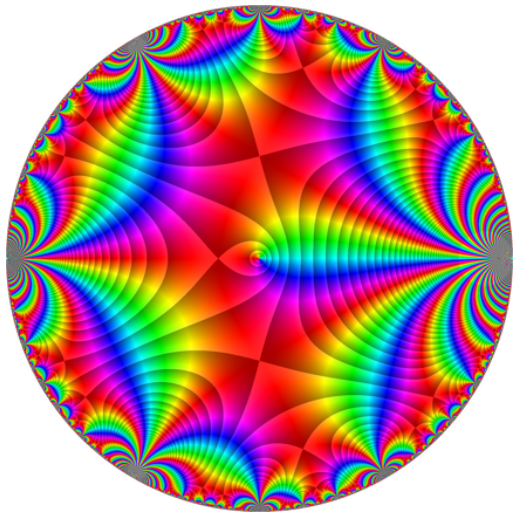

Then

$$\mathbb{I}''_{\ominus}(t) = \frac{\partial^2}{\partial^2 t}\frac{\left[\eta(q^2)\eta(q^6)\right]^4}{\left[\eta(q)\eta(q^3)\right]^2}$$

$$\times\left(3(\log q)^3 - 16\zeta(3) + \sum_{n=1}^{\infty}\frac{\psi_{\ominus}(n)}{n^3}\frac{q^n}{1-q^n}\right)$$

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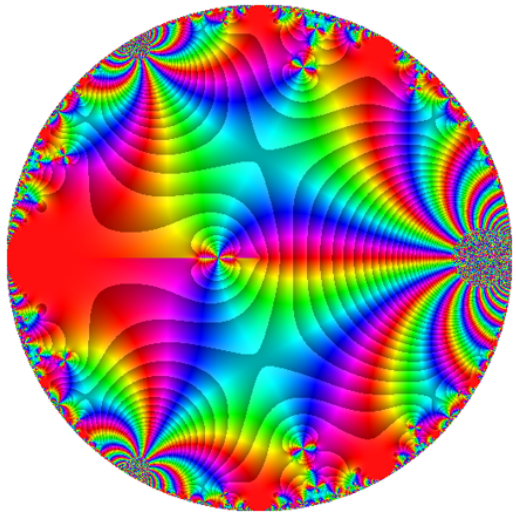


The inverse is known, they said...

$$\rho_F(z) = {}_2F_1\left(\begin{matrix}1/12, 5/12\\ 1\end{matrix}\middle|\, \frac{1728(z-9)(z-1)^3z^2}{(z-3)^3(z^3-9z^2+3z-3)^3}\right)$$

$$\varpi_c(z) = \frac{2\pi\rho_F(z)}{\sqrt[4]{z-3}\rho_4(z)} \longrightarrow \rho_4(z) = \sqrt[4]{z^3-9z^2+3z-3}$$

$$q = \exp\left[-2\pi\frac{\varpi_c(t_\ominus)}{\varpi_r(t_\ominus)}\right] \longrightarrow t = \frac{-64t_\ominus}{(t_\ominus-9)(t_\ominus-1)}$$

$$\varpi_r(z) = \frac{12\sqrt{3}}{z-9}\varpi_c\left(\frac{9z-9}{z-9}\right)$$

Bloch & Vanhove [1309.5865]

Quoting Donald E. Knuth:

"Beware of bugs in the above code; I have only proved it correct, not tried it."

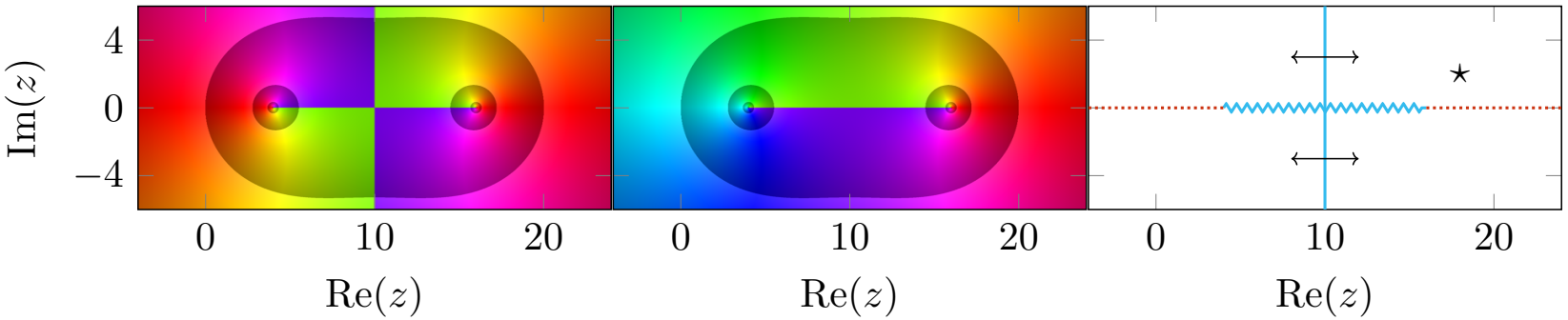
Choose your sheet carefully

$$t_{\ominus} = \frac{5t - 32 + 4\rho_2(t)}{t} \longrightarrow \rho_2(z) = \sqrt{(z-4)(z-16)}$$

Wrong $\rho_2(z)$ (principal sheet)

Right $\rho_2(z)$

Branch cut layout



...very carefully

$$\rho_4(z) = \sqrt[4]{z^3 - 9z^2 + 3z - 3}$$

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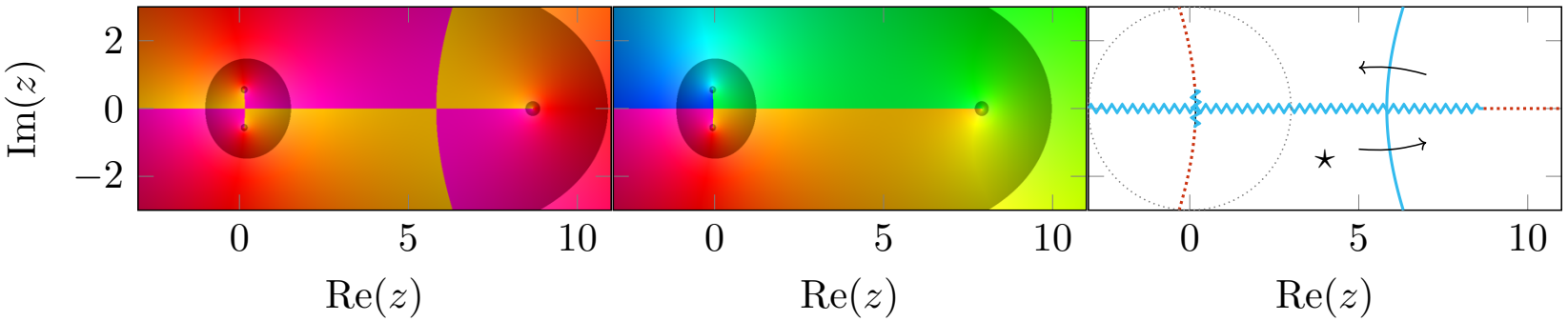
Right $\rho_4(z)$

Branch cut layout

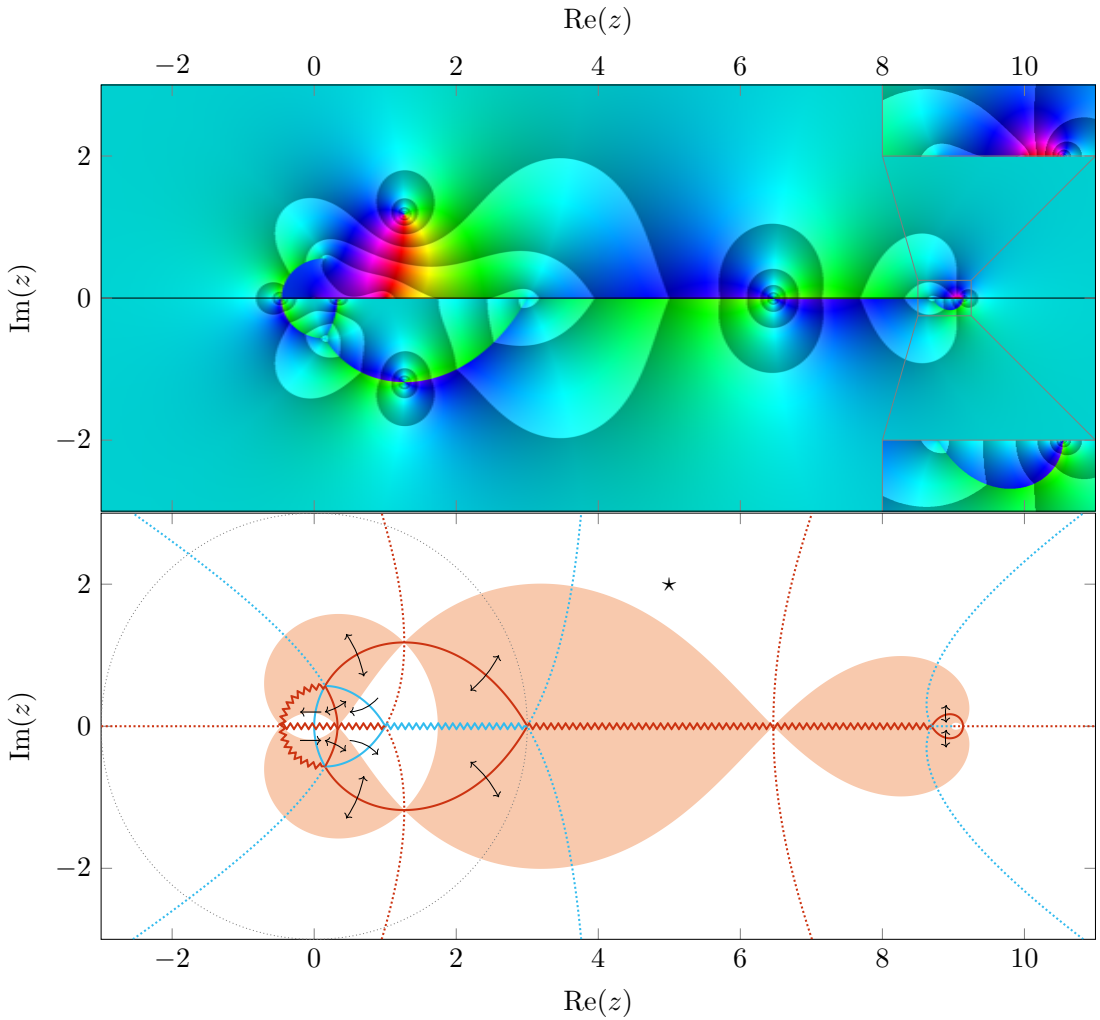


...very, very carefully

$$\rho_F(z) = {}_2F_1\left(\begin{matrix}1/12, 5/12\\ 1\end{matrix}\middle|\, \frac{1728(z-9)(z-1)^3z^2}{(z-3)^3(z^3-9z^2+3z-3)^3}\right)$$

Right $\rho_F(z) - \rho_F(1)$

Wrong $\rho_F(z) - \rho_F(1)$



On to the cat's eye

Using $t = \frac{4}{1-\beta^2}$,

$$\beta^2\left[\frac{\mathrm{d}^2}{\mathrm{d}\beta^2} - \frac{2\beta^2}{\beta^2-1}\right]\bar{\mathbb{I}}_{\textcircled{\scriptsize 0}}(\beta) = S(\beta)\,, \qquad S(\beta) \sim [\mathbb{I}_{\ominus}, \mathbb{I}_{\bigcirc}]$$

Homogeneous solution:

$$g_1(\beta) = \beta^2 - 1 \qquad g_2(\beta) = \frac{\beta^2-1}{4}\log\left(\frac{\beta+1}{\beta-1}\right) - \frac{\beta}{2}$$

Full Wrońskian solution:

$$\bar{\mathbb{I}}_{\textcircled{\scriptsize 0}}(\beta) = g_1(\beta)\left[c_1 - \int_{\xi_1}^{\beta} S(\xi)g_2(\xi)\frac{\mathrm{d}\xi}{\xi^2}\right] + g_2(\beta)\left[c_2 + \int_{\xi_2}^{\beta} S(\xi)g_1(\xi)\frac{\mathrm{d}\xi}{\xi^2}\right]$$

Must **integrate** $\mathbb{I}_{\ominus}$!

Divide and conquer

- Let $S(\xi) = S_N(\xi) + S_A(\xi)$:
  - $\int S_A(\xi) g_n(\xi) \mathrm{d}\xi/\xi^2$ analytically solvable
    $\Rightarrow$ just insert primitive function
  - Numerically integrate $\int_\beta^\infty S_N(\xi) g_n(\xi) \mathrm{d}\xi/\xi^2$
- Adjust subdivision so that numerical integral converges
- Fix $c_{1,2}$ from known $\bar{\mathbb{I}}_{\text{⌀}}(\beta = \infty) = \bar{\mathbb{I}}_{\text{⌀}}(t = 0)$
- Divert through complex plane when above threshold
- Do integrals involving $\mathbb{I}_{\ominus}$ in its native q-space (or rather over $\tau = \frac{\log q}{2\pi i}$)
- Cache fixed part of integration contour to save on computation

The cat's eye, vanquished

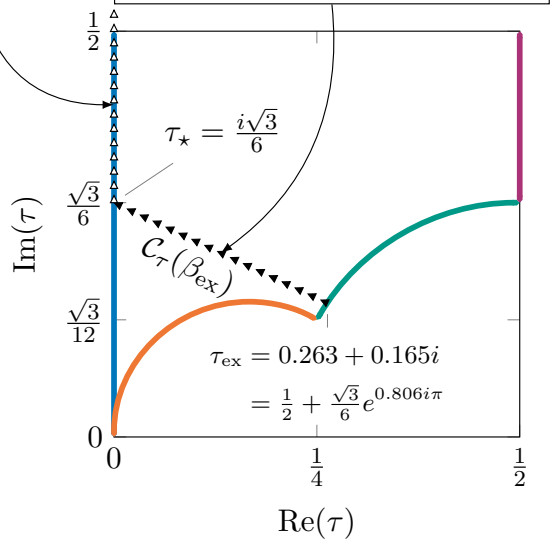
$$\mathcal{I}^{(1)}_{\rm rat}(\beta_{\rm ex}) = 21.034 - 2.937i \qquad \mathcal{I}_{\rm div}(\beta_{\rm ex}) = 47.842 - 8.179i$$
$$\mathcal{I}^{(2)}_{\rm rat}(\beta_{\rm ex}) = \ \ 2.394 - 1.602i$$

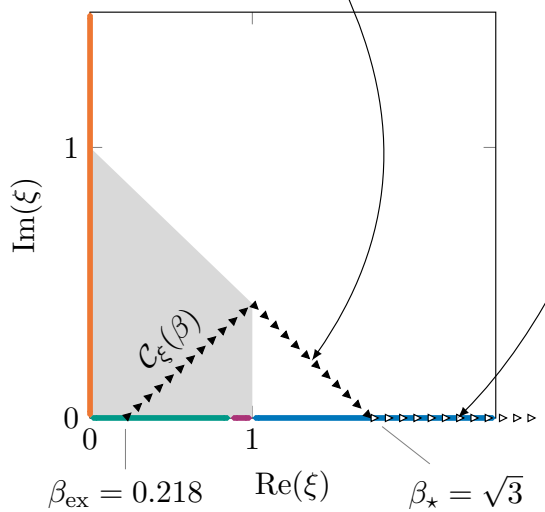
$$\mathcal{I}^{(1)}_{E}(\beta_\star) = (-6.020)_{<\chi} - (0.095)_{>\chi} = -6.115$$
$$\mathcal{I}^{(2)}_{E}(\beta_\star) = (\ \ 2.826)_{<\chi} + (0.001)_{>\chi} = +2.827$$

$$\mathcal{I}^{(1)}_{J}(\beta_\star) = (-0.897)_{<\chi} - (2\cdot 10^{-4}\ )_{>\chi} = -0.897$$
$$\mathcal{I}^{(2)}_{J}(\beta_\star) = (\ \ 0.008)_{<\chi} + (1\cdot 10^{-17})_{>\chi} = 0.008$$

$$\int_{C_\tau(\beta_{\rm ex})} h_1 \ldots {\rm d}\tau = 44.172 + 16.284i$$
$$\int_{C_\tau(\beta_{\rm ex})} h_2 \ldots {\rm d}\tau = \ \ 8.462 - 67.465i$$

$$\int_{\mathcal{C}_\xi(\beta_{\rm ex})} g_1 \ldots {\rm d}\xi = 70.061 - 21.617i$$
$$\int_{\mathcal{C}_\xi(\beta_{\rm ex})} g_2 \ldots {\rm d}\xi = \ \ 2.283 - 45.262i$$

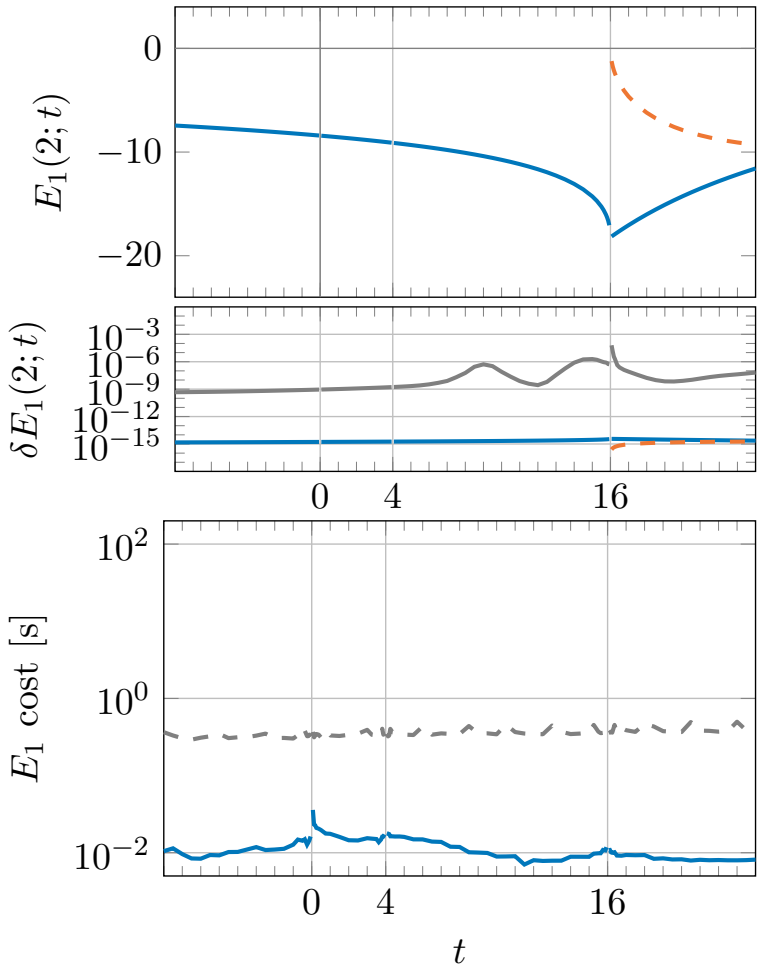


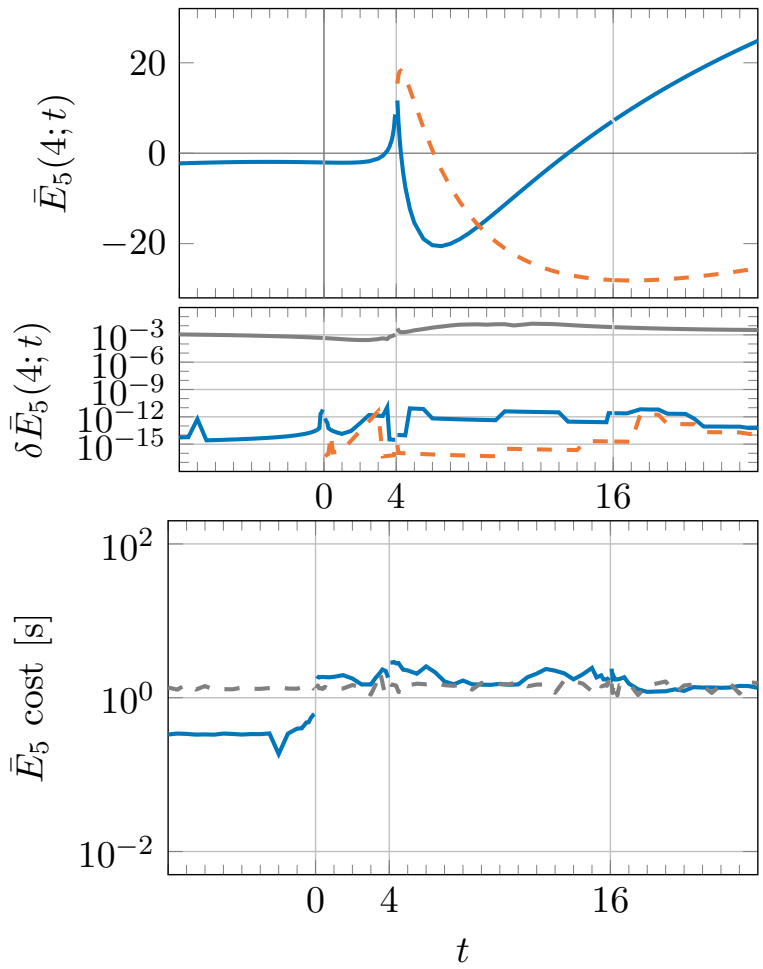


$$G_{15}(\beta_{\rm ex}) = \ -0.952$$
$$G_{25}(\beta_{\rm ex}) = \ -0.215 + \ 0.748i$$
$$B_5(\beta_{\rm ex}) = \ \ 12.853$$
$$\mathcal{I}[\beta_{\rm ex}; G_{15}(\beta_{\rm ex}), G_{25}(\beta_{\rm ex})] = -10.294 + 18.144i$$
$$\boldsymbol{\bar{E}_5(4, t_{\rm ex}) = 2.559 + 18.144i}$$

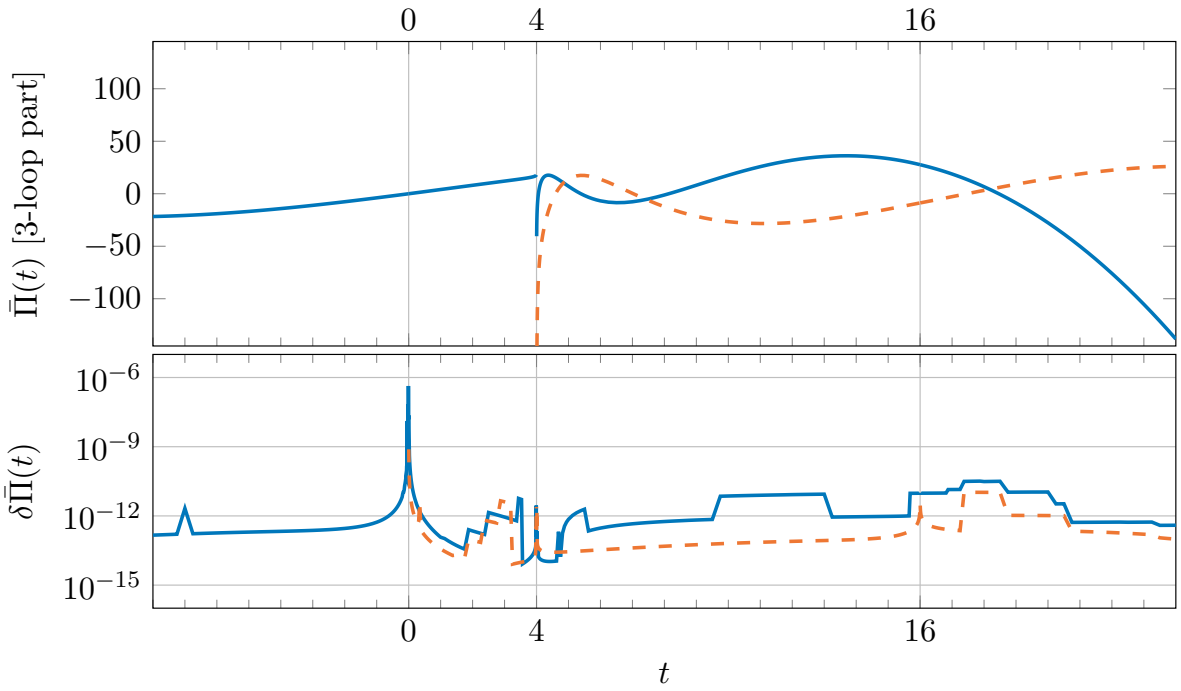
$$G_{16}(\beta_{\rm ex}) = \ \ 0$$
$$G_{26}(\beta_{\rm ex}) = \ \ 0.546$$
$$B_6(\beta_{\rm ex}) = -48.299 + 0.841i$$
$$\mathcal{I}[\beta_{\rm ex}; G_{16}(\beta_{\rm ex}), G_{26}(\beta_{\rm ex})] = \ \ 96.069 - 8.974i$$
$$\boldsymbol{\bar{E}_6(4, t_{\rm ex}) = 47.770 - 8.134i}$$

Benchmarking against pySecDec





The three-loop part



Evaluation of master integrals

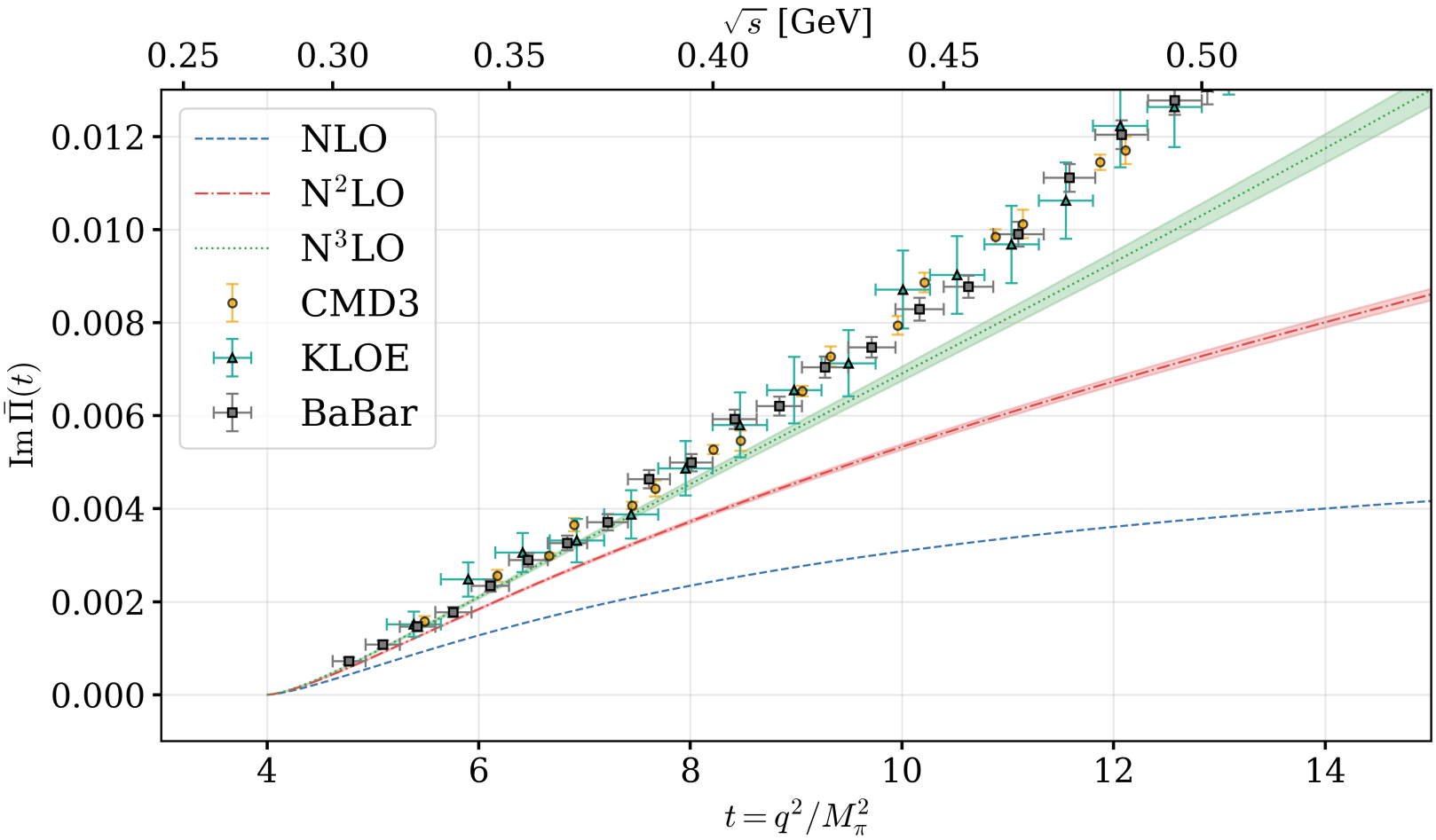
- Stable over entire complex plane
- 10^3–10^{12} times more precise than `pySecDec`
  Borowka et al. [1703.09692]
- 1–1000 times faster than `pySecDec`

The result in context

Phenomenological inputs

Parameter	**Source** Review: Bijnens & Ecker [1405 .6488]
l_1, l_2	$\pi\pi$ scattering [hep-ph/0103088]
l_4	π scalar radius [hep-ph/0103088]
l_6	π EM radius [hep-ph/9805389], $\pi^+ \to e^+\nu\gamma$ [0810.0760]
c_{50}	hadronic VA correlator [new]
c_{56}	$\frac{\partial}{\partial t}\Pi^{\text{NNLO}}(t)\big\|_{t=0}$ [new]
r_{V1}	π EM radius, VMD [hep-ph/9805389]
r_{V2}	π vector FF [hep-ph/9805389], lattice [2110.05493]
r_1	$\frac{\partial}{\partial t}\Pi^{\text{N}^3\text{LO}}(t)\big\|_{t=0}$ [new]
r_2	$\frac{\partial^2}{\partial t^2}\Pi^{\text{N}^3\text{LO}}(t)\big\|_{t=0}$ [new]

Convergence



Part of an in-depth study (led by Alessandro) coming soon



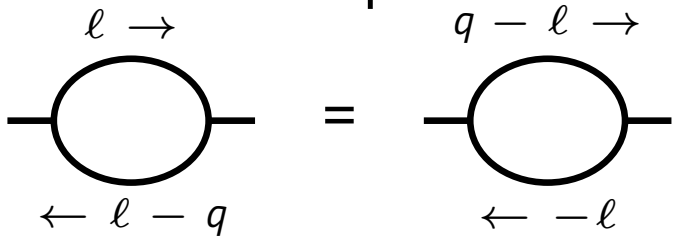
The finite volume

Our holy grail

- $\Pi(L) - \Pi(L = \infty)$ is a direct handle on a major source of lattice error
- Sadly, all loop tools break in the finite volume:
  - Loop *sums* — generally nastier than integrals
  - No Lorentz invariance
  - No dimreg
  - IBP invalidated
  - No parametric form
- FV HVP has been done at two-loops [2008.03809] (factorizes to 1×1 loop)
- FV two-loop sunset known [1311.3531]

Agnostic MI reduction

- Utilize freedom to reroute loop momenta



  - $\Rightarrow \ell \rightarrow q - \ell$ is a symmetry of the denominator
  - ...but antisymmetry of numerator factor $\ell \cdot q$ (up to constant term)
- Trivial at 1 loop, rich at 3 loops
  $\Rightarrow$ IBP-like system of relations (but weaker)
- Works for any remotely Feynman-integral-like object
  - Regulator-independent
  - Sum/integral agnostic
- **27** 3-loop MIs in our case (**7** with inifnite-volume IBP)

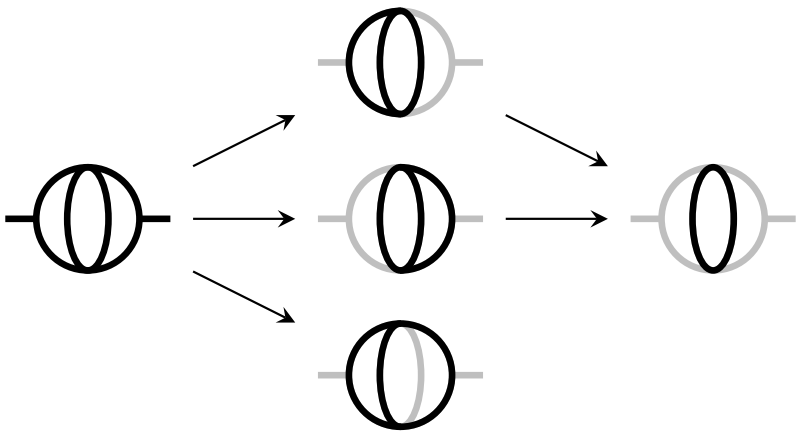
Finite-volume regularization

BPHZ subtraction

$$\bar{\mathbb{I}}(t) = \sum_{\Gamma} \left(\prod_{\gamma \in \Gamma} t_\gamma \right) \mathbb{I}(t)$$

$$t_\gamma J(q) = \sum_{n=0}^{\omega_\gamma} \frac{-1}{n!} \left. \frac{\partial^n J(q)}{\partial q_{\mu_1} \cdots \partial q_{\mu_n}} \right|_{q=0} q_{\mu_1} \cdots q_{\mu_n}$$

Γ-forest: disjoint or nested divergent subgraphs



Finite-volume evaluation

Monte Carlo summation (with Antonin Portelli)

- $\bar{\mathbb{I}}_{\ominus}$ doable with `vegas+` [2009.05112]
- Stability is an issue, $\bar{\mathbb{I}}_{\oplus}$ very hard
- Numerator-sensitive
  $\Rightarrow$ ideally needs IBP



Finite-volume evaluation

Poisson resummation(with Johann Usovitsch)

$$\frac{1}{L^{4n}} \sum_{\ell_i^\mu \in \mathbb{Z}^4} J(\{\ell_i\}) = \sum_{r_i^\mu \in \mathbb{Z}^4} \int_{\{\ell_i\}} J(\{\ell_i\}) e^{L \sum_i r_i \cdot \ell_i}$$

$$= \int_{\{r_i\}} \sum_{r_i^\mu \in \mathbb{Z}^4} \int_{\{\ell_i\}} J(\{\ell_i\}) \prod_i e^{iL\tau_i} \delta(\tau_i - r_i \cdot \ell_i)$$

- Restores integral form with winding numbers
- IBP reduction possible but impractical
- Evaluation with Bessel or theta functions
  - Limited success at 2 loops [1311.3531]
  - Challenging at 3 loops



Finite-volume evaluation

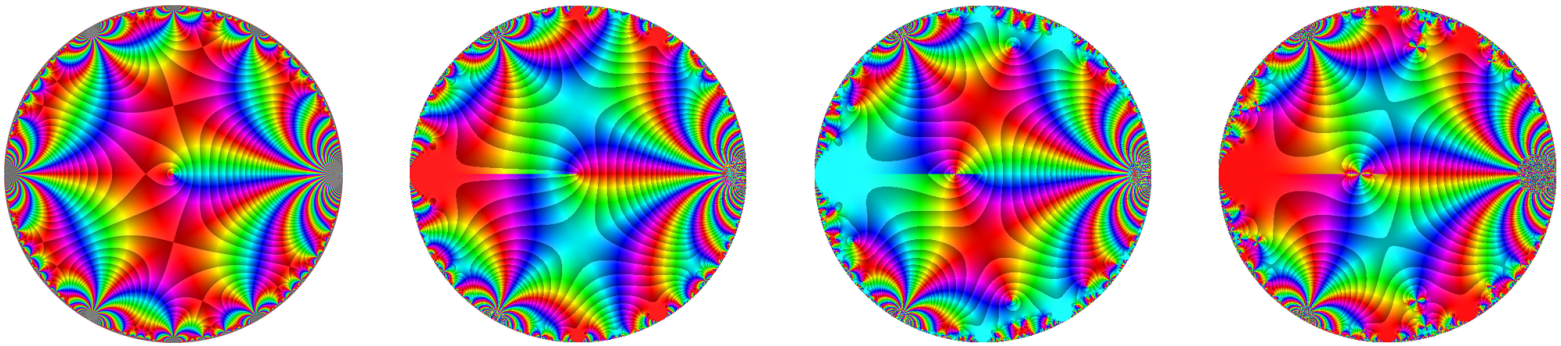
Monte Carlo summation (with Giuseppe de Angelis)

- Sum over discretized position space
- Guaranteed convergence, robust against numerators
- Scales with #vertices, not #loops
- ϕ^4 proof-of-concept successful, including cat's eye



Summary and Outlook

- Amplitude calculation published [2510.12885]
- Loop integral deep dive published [2603.15252]
- Phenomenological context completed soon
- Finite volume making progress—stay tuned



Lower orders

- Leading order Gasser & Leutwyler (1985)

$$\bar{\Pi}^{\mathrm{NLO}}(t) = +2B(t) + \tfrac{2}{3}$$

- Next-to-leading order Golowich & Kambor (1995)

$$\bar{\Pi}^{\mathrm{NNLO}}(t) = +tB(t)^2 - 4tB(t)l_6 - 8c_{56}t$$

Function of t only | LECs known | LECs TBD | LECs unknown

Dispersive+FVE | FVE only (?) | No Dispersive/FVE

Bubble function

$$B(t) = -\frac{1}{9} + \frac{4-t}{3t}\int_0^1 \log\left[1 - x(1-x)t\right]\,\mathrm{d}x$$

Three-loop order

- Next-to-next-to-next-to-leading order This work (2025)

$$
\begin{aligned}
\bar{\Pi}^{\mathrm{N^3LO}}(t) = \;& + \left[\mathbb{I}_{\bigcirc}, \bar{\mathbb{I}}_{\ominus}, \bar{\mathbb{I}}'_{\ominus}, \bar{\mathbb{I}}''_{\ominus}, \bar{\mathbb{I}}_{\oplus}, \bar{\mathbb{I}}'_{\oplus}\right] \\
& + tB(t)^2\left[t(l_2 - 2l_1 - 2l_6) + 2l_4\right] \\
& + 2tB(t)l_6(4l_4 - l_6 t) \\
& + 4tB(t)\left(r_{V1} + r_{V2}t\right) \\
& - 16l_4 c_{56} t \\
& + 16t(\tilde{c}_{332} - \tilde{c}_{333}) - 8t^2\tilde{c}_{459}
\end{aligned}
$$

Function of t only | LECs known | LECs TBD | LECs unknown

Dispersive+FVE | FVE only (?) | No Dispersive/FVE

Pion vector form factor NNLO LECs (known)

$$r_{V1} = -4(2c_{53} + 2c_{35} + 4c_6)\,, \qquad r_{V2} = 4(c_{53} - c_{51})$$