Hypergeometric function representation of Feynman integral

费曼积分的超几何函数表示

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Recent Progress in Amplitudes

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Research Background: The Feynman Loop Calculations

Feynman Loop Diagrams: A Core Tool in Theoretical Physics

Core Mathematical Tools

Feynman loop diagrams are the core mathematical tools for calculating the interaction processes of elementary particles in quantum field theory.

Experimentally Driven Precision

With the improvement of experimental measurement precision, the requirements for the precision of theoretical predictions have become stringent.

High-Precision Calculation Needs

To obtain high-precision theoretical predictions, it is essential to perform accurate analytic or numerical calculations of multi-loop Feynman diagrams.

Key Research Directions

Therefore, developing efficient and accurate methods for Feynman loop calculations is a key issue in the field of theoretical physics.



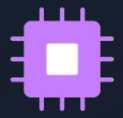
Quantum Field Theory

Bottlenecks of Traditional Methods: Exponential Growth of Computational Complexity



Difficulties in Analytic Calculation

Traditional parametrization methods lead to extremely complex multiple integrals when dealing with multi-loop diagrams, making it difficult to obtain analytic solutions.



Inefficiency in Numerical Calculation

Although numerical calculation methods are universal, they are slow and inefficient when dealing with multi-loop and multi-scale problems, and have high requirements for CPU and memory.



Slow Research Progress

The development of analytic calculation methods for multi-loop Feynman integrals has been slow over the past four decades. But recently, there are many developments.

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Core Thought:
Hypergeometric Function Representation



New Path: Hypergeometric Function Representation

Core Idea: Transform complex Feynman integrals into analytic expressions represented by hypergeometric functions through a series of mathematical transformations.

Hypergeometric Function

univariate Gauss function $${}_2F_1(a, b; c \mid x) = \Sigma\, (a)_n\, (b)_n / [n!\, (c)_n] \cdot x^n\,, \quad |x| < 1$$

First Appell function $$F_1(a; b_1, b_2; c \mid x, y) = \Sigma\, (a)_{m+n}\, (b_1)_m\, (b_2)_n / [m!\, n!\, (c)_{m+n}] \cdot x^m\, y^n$$

Lauricella function $$F_B(a_1, a_2, a_3; b_1, b_2, b_3; c \mid x_1, x_2, x_3)\,, \quad |x_1|+|x_2|+|x_3| < 1$$

Pochhammer symbol $$(a)_n = \Gamma(a + n) / \Gamma(a)$$

Advantage

Core Idea: Transform complex Feynman integrals into analytic expressions represented by hypergeometric functions through a series of mathematical transformations.

Analyticity

Within convergence intervals, precise analytic solutions can be obtained, which are concise and elegant in form and easy to verify.

Efficiency

Computational programs based on analytic expressions are much faster than traditional numerical algorithms.

Universality

The method demonstrates strong adaptability and can handle various complex Feynman diagrams from one loop to multi-loop.

This method provides a new approach for the analytical calculation of Feynman diagrams.

03

Theoretical Discovery:

Analytical Solutions from one/two Loops to Multi-Loops

Phase 1: Basic Series Solutions for one/two Loops (2018-2019)

Research Objects

one-loop B0, C0 functions, two-loop vacuum diagrams, and sunset diagrams.

Main Achievements

- Derived the hypergeometric function representations of these Feynman integrals.
- Obtained precise analytic series solutions within the convergence interval around the **zero point**.
- Derived the homogeneous linear partial differential equations satisfied by these solutions.

Published in

Nuclear Physics B (2018, 2019)

Phase 2: Analytic Continuation and Hybrid Algorithm (2020-2023)

Core Challenge

How to analytically continue the solution from the neighborhood of zero to other regions? This is the key connecting local solutions to global solutions.

Innovative Method

Proposed a algorithm of **Mellin-Barnes representation and Miller transformation**, breaking through traditional continuation limits and achieving efficient computation.

Key Achievements

Established the **GKZ PDEs** on the ordinary manifold; solved the fundamental solution systems in the neighborhoods of **zero and infinity**, covering the core parameter space.

Published in: Nuclear Physics B (2020), European Physical Journal C (2023)

Phase 3: Introduction of Grassmannian Manifold (2022)

Core Challenge

How to solve the fundamental solution systems in the neighborhoods of all singularities?

Innovative Breakthrough

Introduced the geometric theory of Grassmannian manifold into the Feynman integral representation.

Key Achievements

Constructed GKZ hypergeometric systems on Grassmannian manifolds. Enabled fundamental solution systems in the neighborhoods of all singularities.

Published in: Physical Review D (2022)

Phase 4: Global Analytic Continuation (2025)

Core Goal

How to efficiently extend the analytic solution to the entire domain of definition?

Innovative Method

Proposed a method to complete analytic continuation by extending the **three types of Gauss relations** of hypergeometric functions.

Key Achievements

obtained the Laurent series in the neighborhood of spacetime dimension D=4, which is crucial for physical applications.

Published in

Physical Review D (2025)

Through the clever application of the Gauss relations of hypergeometric functions, we achieved the leap from local solutions to global solutions.

Polylogarithms and Hypergeometric Functions

Basic Relations

Logarithm $\mathrm{Li}_1(x) = -\ln(1-x) = x \cdot {}_2F_1(1, 1; 2; x)$

Dilogarithm $\mathrm{Li}_2(x) = x \cdot {}_3F_2(1, 1, 1; 2, 2; x)$

Gauss Relations

Gauss inverse Relation (Mellin–Barnes contour)

$x \to 1/x$: connects the two branches

(real identity: $x < 0$)

Gauss–Kummer Relation

$x \to 1-x$, $x \to x/(x-1)$: continuation to the whole domain

Dilogarithm Identities

$$\mathrm{Li}_2(x) + \mathrm{Li}_2(1/x) = -\pi^2/6 - \tfrac{1}{2}\ln^2(-x)$$

$$\mathrm{Li}_2(x) + \mathrm{Li}_2(1-x) = \pi^2/6 - \ln x \cdot \ln(1-x)$$

$$\mathrm{Li}_2(x) + \mathrm{Li}_2(x/(x-1)) = -\tfrac{1}{2}\ln^2(1-x)$$

The hypergeometric representation is a larger language than polylogarithms:

$$\mathrm{Li}_n(x) = x \cdot {}_{n+1}F_n(1, \ldots, 1; 2, \ldots, 2; x).$$

Present: Breakthrough in Multi-Loop Problems

Core Verification

Is the Hypergeometric Function Method applicable to multi-loop Feynman diagrams?

Research Objects

Three-loop and four-loop vacuum diagrams.

Key Achievements

Derived their GKZ hypergeometric equations and fundamental solution systems on the ordinary manifold.

Published in

Journal of High Energy Physics

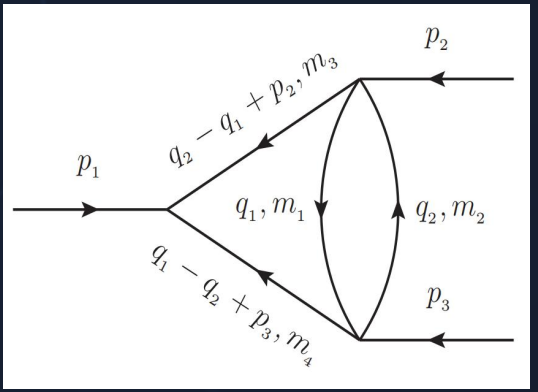
(2023, 2025)

This indicates the applicability of the hypergeometric function representation in the multi-loop calculations.

Present: two-Loop triangle diagram

Research Objects

The massive two-loop triangle diagram with 4 propagators.



Main Achievements

- Feynman integral of the 2-loop massive triangle diagram is embedded in Grassmannian $G_{5,8}$
- Through geometric configurations, the hypergeometric function solutions are attributed to ten types.
- The hypergeometric functions can be transformed into each other by the various Gauss relations.

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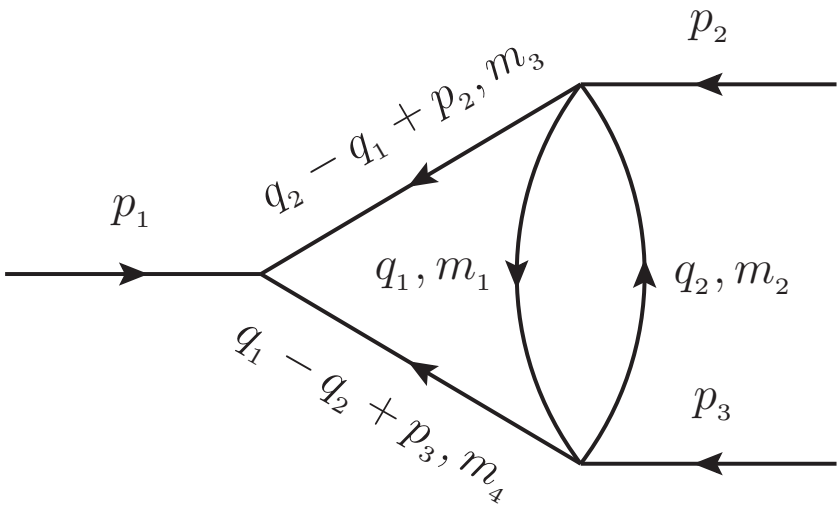
Two-Loop Triangle Diagram: An example



Feynman diagram



Parameterization

$$A_{DU}(\{p_i^2\},\{m_j^2\}) = \frac{\left(\Lambda^2_{\rm RE}\right)^{4-D}\Lambda^2\exp\{i\pi(1-\frac{D}{2})\}\Gamma(4-D)}{(4\pi)^D(-i)^{4-D}}$$
$$\times\int_S \omega_5(t)\Big\{t_1t_2+t_1t_3+t_1t_4+t_2t_3+t_2t_4\Big\}^{4-3D/2}$$
$$\times\delta(\sum_{j=1}^4 t_jm_j^2+t_5\Lambda^2)\Big[(t_1+t_2)t_3t_4p_1^2+t_1t_2t_3p_2^2+t_1t_2t_4p_3^2$$
$$+t_5[t_1t_2+t_1t_3+t_1t_4+t_2t_3+t_2t_4]\Lambda^2\Big]^{D-4}, \qquad (1)$$

where

$$\omega_k(t)=\sum_{i=1}^k(-1)^{i-1}t_i\,dt_1\wedge\cdots\wedge dt_{i-1}\wedge dt_{i+1}\wedge\cdots\wedge dt_k \qquad (2)$$

is the volume element of $(k-1)$-dimensional projective space.

Matroid of Grassmannian $G_{5,8}$ embedded

The matroid $\boldsymbol{\xi}_{DU}$ of size 5×8

$$\boldsymbol{\xi}_{DU} = \left(\begin{array}{c|c} I_5 & \boldsymbol{Z}_{DU} \end{array} \right) \tag{3}$$

with the exponent vector $\boldsymbol{\beta}_{DU} = (0, 0, 0, 0, 0, -D, D-4, -1) \in \mathbf{C}^8$, and

$$\boldsymbol{Z}_{DU} = \begin{pmatrix} 1 & p_1^2 & m_1^2 \\ 1 & p_2^2 & m_2^2 \\ 1 & p_3^2 & m_3^2 \\ 1 & p_4^2 & m_4^2 \\ 0 & \Lambda^2 & \Lambda^2 \end{pmatrix} . \tag{4}$$

The expression for one connected component in the definition domain

The Feynman integral can be written as a linear combination involving 14 functions of split coordinates

$$A_{DU}(\{p_i^2\},\{m_j^2\}) = \sum_{\alpha=\mathrm{I}}^{\mathrm{XIV}} C_\alpha \Phi^\alpha_{\{12345\}}(\{p_i^2\},\{m_j^2\})\ . \tag{5}$$

Here every fundamental solution Φ_α takes the form of a product between a power factor of split coordinates and the corresponding generalized hypergeometric function of dimensionless variables.

The expression for one connected component in the definition domain

Besides the hypergeometric power series expansions of the Feynman integral in some neighborhood of the origin, we should know how to

- analytically continue the resulting expansions to the entire domain (inverse relations);
- characterize the analytic properties of the Feynman integral near its singular locus (Kummer relations);
- reduce at some special regular points (adjacent relations).

The expression for $\Phi^{\mathrm{I}}_{\{12345\}}$

The function is

$$\Phi^{\mathrm{I}}_{\{12345\}}(\{p_i^2\},\{m_j^2\}) = P^{\mathrm{I}}_{\{12345\}}(\boldsymbol{\beta},\boldsymbol{\xi}) H_{\mathrm{I}}\left(\begin{matrix} (a_1,a_2,a_3,a_4,a_5,a_6) \\ c \end{matrix} \middle| \boldsymbol{x} \right) , \quad (6)$$

where

$$\boldsymbol{x} = \left(\frac{m_1^2}{m_2^2}, \frac{m_1^2}{m_2^2}, \frac{m_1^2}{m_3^2}, \frac{p_2^2 m_1^2}{p_1^2 m_3^2}, \frac{m_1^2}{m_4^2}, \frac{p_3^2 m_1^2}{p_1^2 m_4^2}, 0, \frac{m_1^2}{p_1^2}\right) ,$$
$$P^{\mathrm{I}}_{\{12345\}}(\boldsymbol{\beta},\boldsymbol{\xi}) = (p_1^2)^{-a_6} (m_1^2)^{c-1} (m_2^2)^{-a_1} (m_3^2)^{-a_2} (m_4^2)^{-a_3} (\Lambda^2)^{-a_4} . \quad (7)$$

with $a_1 = a_2 = a_3 = a_4 = 1$, $a_5 = D$, $a_6 = 4 - D$, and $c = 4$.

The expression for $\Phi^{\mathrm{I}}_{\{12345\}}$

Since $x_1 = x_2$, we can apply the Bailey transformation to reduce the double sum over $x_{1,2}$ to a single sum. However, the resulting expression cannot be written as a function on the Grassmannian, which makes the subsequent computation inconvenient.

The definitions of those generalized hypergeometric functions will be given below.

The expression for $\Phi^{\rm II}_{\{12345\}}$

The function is

$$\Phi^{\rm II}_{\{12345\}}(\{p_i^2\},\{m_j^2\}) = P^{\rm II}_{\{12345\}}(\boldsymbol{\beta},\boldsymbol{\xi}) H_{\rm II}\left(\begin{matrix} (a_1,a_2,a_3,a_4),b \\ (c_1,c_2) \end{matrix} \middle| \boldsymbol{x} \right) , \qquad (8)$$

where

$$\boldsymbol{x} = (\frac{m_1^2}{m_2^2}, 1, \frac{m_2^2}{m_3^2}, \frac{p_2^2 m_2^2}{p_1^2 m_3^2}, \frac{m_2^2}{m_4^2}, \frac{p_3^2 m_2^2}{p_1^2 m_4^2}, 0, \frac{m_2^2}{p_1^2}) \ , \ ,$$
$$P^{\rm II}_{\{12345\}}(\boldsymbol{\beta},\boldsymbol{\xi}) = (p_1^2)^{c_2-b-1}(m_2^2)^{c_1-1}(m_3^2)^{-a_1}(m_4^2)^{-a_2}(\Lambda^2)^{-a_3} \ , \qquad (9)$$

with $a_1 = a_2 = a_3 = 1$, $a_4 = c_2 = D$, and $b = c_1 = 3$.

The expression for $\Phi^{\mathrm{II}}_{\{12345\}}$

Since $x_2 = 1$, we can use the Gauss summation formula to reduce the number of summations in the above expression by one. However, the resulting expression cannot be written as a function on the Grassmannian, which makes the subsequent computation inconvenient.

The expression for $\Phi^{\mathrm{III}}_{\{12345\}}$

The function is

$$\Phi^{\mathrm{III}}_{\{12345\}}(\{p_i^2\},\{m_j^2\}) = P^{\mathrm{III}}_{\{12345\}}(\boldsymbol{\beta},\boldsymbol{\xi}) H_{\mathrm{III}}\left(\begin{matrix} (a_1,a_2,a_3,a_4),b \\ (c_1,c_2) \end{matrix} \middle| \boldsymbol{x} \right) , \quad (10)$$

where

$$\boldsymbol{x} = (\frac{p_2^2 m_1^2}{p_1^2 m_3^2}, 1, \frac{p_2^2 m_2^2}{p_1^2 m_3^2}, \frac{p_1^2}{p_2^2}, \frac{p_1^2 m_3^2}{p_2^2 m_4^2}, \frac{p_3^2 m_3^2}{p_2^2 m_4^2}, 0, \frac{m_3^2}{p_2^2}) ,$$
$$P^{\mathrm{III}}_{\{12345\}}(\boldsymbol{\beta},\boldsymbol{\xi}) = (p_1^2)^{c_1-a_1-1}(p_2^2)^{-b}(m_3^2)^{c_2-1}(m_4^2)^{-a_2}(\Lambda^2)^{-a_3} \quad (11)$$

with $a_1 = a_2 = a_3 = 1$, $a_4 = c_1 = D$, and $b = c_2 = 2$.

The expression for $\Phi^{\mathrm{IV}}_{\{12345\}}$

The function is

$$\Phi^{\mathrm{IV}}_{\{12345\}}(\{p_i^2\},\{m_j^2\}) = P^{\mathrm{IV}}_{\{12345\}}(\boldsymbol{\beta},\boldsymbol{\xi}) H_{\mathrm{IV}}\left(\begin{matrix} (a_1,a_2,a_3,a_4),b \\ (c_1,c_2) \end{matrix} \middle| \boldsymbol{x} \right) , \quad (12)$$

where

$$\boldsymbol{x} = (\frac{p_3^2 m_1^2}{p_1^2 m_4^2}, 1, \frac{p_3^2 m_2^2}{p_1^2 m_4^2}, \frac{p_1^2}{p_2^2}, \frac{p_3^2 m_3^2}{p_2^2 m_4^2}, \frac{p_1^2}{p_3^2}, 0, \frac{m_4^2}{p_3^2}) ,$$
$$P^{\mathrm{IV}}_{\{12345\}}(\boldsymbol{\beta},\boldsymbol{\xi}) = (p_1^2)^{c_1-a_1-1}(p_2^2)^{-a_2}(p_3^2)^{-b}(m_4^2)^{c_2-1}(\Lambda^2)^{-a_3} \quad (13)$$

with $a_1 = a_2 = a_3 = 1$, $a_4 = c_1 = D$, and $b = c_2 = 1$.

The expression for $\Phi^{\rm V}_{\{12345\}}$

The function is

$$\Phi^{\rm V}_{\{12345\}}(\{p_i^2\},\{m_j^2\}) = P^{\rm V}_{\{12345\}}(\boldsymbol{\beta},\boldsymbol{\xi}) H_{\rm V}\left(\begin{matrix} (a_1,a_2,a_3,a_4,a_5),b \\ c \end{matrix} \middle| \boldsymbol{x} \right) , \quad (14)$$

where

$$\boldsymbol{x} = (\frac{m_1^2}{p_1^2}, 1, \frac{m_2^2}{p_1^2}, \frac{p_1^2}{p_2^2}, \frac{m_3^2}{p_2^2}, \frac{p_1^2}{p_3^2}, \frac{m_4^2}{p_3^2}, 0) ,$$
$$P^{\rm V}_{\{12345\}}(\boldsymbol{\beta},\boldsymbol{\xi}) = (p_1^2)^{c-a_1-1}(p_2^2)^{-a_2}(p_3^2)^{-a_3}(\Lambda^2)^{-a_4-b} \quad (15)$$

with $a_1 = a_2 = a_3 = a_4 = 1$, $a_5 = c = D$, and $b = 0$.

The expression for $\Phi^{\rm VI}_{\{12345\}}$

The function is

$$\Phi^{\rm VI}_{\{12345\}}(\{p_i^2\},\{m_j^2\}) = P^{\rm VI}_{\{12345\}}(\boldsymbol{\beta},\boldsymbol{\xi}) H_{\rm VI}\left(\begin{matrix} (a_1,a_2,a_3,a_4,a_5),b \\ c \end{matrix} \middle| \boldsymbol{x} \right) , \quad (16)$$

where

$$\boldsymbol{x} = (1, \frac{m_1^2}{m_2^2}, \frac{m_2^2}{m_3^2}, \frac{p_2^2 m_2^2}{p_1^2 m_3^2}, \frac{m_2^2}{m_4^2}, \frac{p_3^2 m_2^2}{p_1^2 m_4^2}, 0, \frac{m_2^2}{p_1^2}) ,$$
$$P^{\rm VI}_{\{12345\}}(\boldsymbol{\beta},\boldsymbol{\xi}) = (p_1^2)^{-a_5} (m_2^2)^{c-1} (m_3^2)^{-a_2} (m_4^2)^{-a_3} (\Lambda^2)^{-a_4} \quad (17)$$

with $a_1 = a_2 = a_3 = a_4 = 1$, $a_5 = 4 - D$, $b = D - 1$, and $c = 3$.

The expression for $\Phi^{\mathrm{VII}}_{\{12345\}}$

The function is

$$\Phi^{\mathrm{VII}}_{\{12345\}}(\{p_i^2\},\{m_j^2\}) = P^{\mathrm{VII}}_{\{12345\}}(\boldsymbol{\beta},\boldsymbol{\xi}) H_{\mathrm{VII}}\left(\begin{matrix} (a_1,a_2,a_3),(b_1,b_2) \\ (c_1,c_2) \end{matrix} \middle| \boldsymbol{x} \right) , \quad (18)$$

where

$$\boldsymbol{x} = (1, \frac{p_2^2 m_1^2}{p_1^2 m_3^2}, \frac{p_2^2 m_2^2}{p_1^2 m_3^2}, \frac{p_1^2}{p_2^2}, \frac{p_1^2 m_3^2}{p_2^2 m_4^2}, \frac{p_3^2 m_3^2}{p_2^2 m_4^2}, 0, \frac{m_3^2}{p_2^2}) ,$$
$$P^{\mathrm{VII}}_{\{12345\}}(\boldsymbol{\beta},\boldsymbol{\xi}) = (p_1^2)^{c_2-1}(p_2^2)^{-b_2}(m_3^2)^{c_1-1}(m_4^2)^{-a_2}(\Lambda^2)^{-a_3} \quad (19)$$

with $a_1 = a_2 = a_3 = 1$, $b_1 = c_2 = D-1$, and $b_2 = c_1 = 2$.

The expression for $\Phi^{\rm VIII}_{\{12345\}}$

The function is

$$\Phi^{\rm VIII}_{\{12345\}}(\{p_i^2\},\{m_j^2\}) = P^{\rm VIII}_{\{12345\}}(\boldsymbol{\beta},\boldsymbol{\xi}) H_{\rm VIII}\left(\begin{matrix} (a_1,a_2,a_3),(b_1,b_2) \\ (c_1,c_2) \end{matrix} \middle| \boldsymbol{x} \right) , \quad (20)$$

where

$$\boldsymbol{x} = (1, \frac{p_3^2 m_1^2}{p_1^2 m_4^2}, \frac{p_3^2 m_2^2}{p_1^2 m_4^2}, \frac{p_1^2}{p_2^2}, \frac{p_3^2 m_3^2}{p_2^2 m_4^2}, \frac{p_1^2}{p_3^2}, 0, \frac{m_4^2}{p_3^2}) ,$$
$$P^{\rm VIII}_{\{12345\}}(\boldsymbol{\beta},\boldsymbol{\xi}) = (p_1^2)^{c_1-1}(p_2^2)^{-a_2}(p_3^2)^{-b_2}(m_4^2)^{c_2-1}(\Lambda^2)^{-a_3} \quad (21)$$

with $a_1 = a_2 = a_3 = 1$, $b_1 = c_1 = D-1$, and $b_2 = c_2 = 1$.

The expression for $\Phi^{\rm IX}_{\{12345\}}$

The function is

$$\Phi^{\rm IX}_{\{12345\}}(\{p_i^2\},\{m_j^2\}) = P^{\rm IX}_{\{12345\}}(\boldsymbol{\beta},\boldsymbol{\xi}) H_{\rm IV}\left(\begin{matrix} (a_1,a_2,a_3,a_4),b \\ (c_1,c_2) \end{matrix} \middle| \boldsymbol{x} \right) , \quad (22)$$

where

$$\begin{aligned} \boldsymbol{x} &= (0, \frac{m_4^2}{p_3^2}, \frac{p_1^2}{p_3^2}, \frac{m_3^2}{p_2^2}, \frac{p_1^2}{p_2^2}, \frac{m_2^2}{p_1^2}, \frac{m_1^2}{p_1^2}, 1) , \\ P^{\rm IX}_{\{12345\}}(\boldsymbol{\beta},\boldsymbol{\xi}) &= (p_1^2)^{-b} (p_2^2)^{-a_2} (p_3^2)^{-a_1} (\Lambda^2)^{c_1-a_4-1} \end{aligned} \quad (23)$$

with $a_1 = a_2 = a_3 = a_4 = c_1 = 1$, and $b_1 = c_2 = 2 - D$.

The expression for $\Phi^{\rm X}_{\{12345\}}$

The function is

$$\Phi^{\rm X}_{\{12345\}}(\{p_i^2\},\{m_j^2\}) = P^{\rm X}_{\{12345\}}(\boldsymbol{\beta},\boldsymbol{\xi}) H_{\rm IX}\left(\begin{matrix} (a_1,a_2,a_3,a_4,a_5),b \\ c \end{matrix} \middle| \boldsymbol{x} \right) , \quad (24)$$

where

$$\boldsymbol{x} = (\frac{p_1^2}{p_2^2},\frac{m_1^2}{m_3^2},\frac{p_1^2}{p_2^2},\frac{m_2^2}{m_3^2},\frac{m_3^2}{m_4^2},\frac{p_3^2 m_3^2}{p_2^2 m_4^2},0,\frac{m_3^2}{p_2^2}) ,$$
$$P^{\rm X}_{\{12345\}}(\boldsymbol{\beta},\boldsymbol{\xi}) = (p_1^2)^{-a_5}(m_3^2)^{c-1}(m_4^2)^{-a_3}(\Lambda^2)^{-a_4} \quad (25)$$

with $a_1 = a_2 = a_3 = a_4 = 1$, $a_5 = 4 - D$, $b = D - 2$, and $c = 2$.

The expression for $\Phi^{\rm XI}_{\{12345\}}$

The function is

$$\Phi^{\rm XI}_{\{12345\}}(\{p_i^2\},\{m_j^2\}) = P^{\rm XI}_{\{12345\}}(\boldsymbol{\beta},\boldsymbol{\xi}) H_{\rm VII}\left(\begin{matrix} (a_1,a_2,a_3),(b_1,b_2) \\ (c_1,c_2) \end{matrix} \middle| \boldsymbol{x} \right) , \quad (26)$$

where

$$\boldsymbol{x} = \left(\frac{m_4^2}{p_3^2}, 0, \frac{p_2^2}{p_3^2}, \frac{p_3^2 m_3^2}{p_2^2 m_4^2}, \frac{p_3^2 m_2^2}{p_2^2 m_4^2}, \frac{p_1^2}{p_2^2}, \frac{p_3^2 m_1^2}{p_2^2 m_4^2}, \frac{p_1^2}{p_2^2}\right) ,$$

$$P^{\rm XI}_{\{12345\}}(\boldsymbol{\beta},\boldsymbol{\xi}) = (p_2^2)^{-b_2} (p_3^2)^{c_2-1} (m_4^2)^{-b_1} (\Lambda^2)^{-a_1} \quad (27)$$

with $a_1 = a_2 = a_3 = 1$, $b_1 = c_2 = 0$, and $b_2 = c_1 = 3 - D$.

The expression for $\Phi^{\mathrm{XII}}_{\{12345\}}$

The function is

$$\Phi^{\mathrm{XII}}_{\{12345\}}(\{p_i^2\},\{m_j^2\}) = P^{\mathrm{XII}}_{\{12345\}}(\boldsymbol{\beta},\boldsymbol{\xi}) H_{\mathrm{III}}\left(\begin{matrix} (a_1,a_2,a_3,a_4),b \\ (c_1,c_2) \end{matrix} \middle| \boldsymbol{x} \right) , \quad (28)$$

where

$$\boldsymbol{x} = (0, \frac{m_4^2}{p_3^2}, \frac{p_2^2}{p_3^2}, (\frac{m_3^2}{p_2^2}, \frac{m_2^2}{p_2^2}, \frac{p_1^2}{p_2^2}, \frac{m_1^2}{p_2^2}, \frac{p_1^2}{p_2^2}) ,$$
$$P^{\mathrm{XII}}_{\{12345\}}(\boldsymbol{\beta},\boldsymbol{\xi}) = (p_2^2)^{-b}(p_3^2)^{-a_1}(\Lambda^2)^{c_1-a_4-1} \quad (29)$$

with $a_1 = a_2 = a_3 = a_4 = c_1 = 1$, $b = c_2 = 3 - D$.

The expression for $\Phi^{\mathrm{XIII}}_{\{12345\}}$

The function is

$$\Phi^{\mathrm{XIII}}_{\{12345\}}(\{p_i^2\},\{m_j^2\}) = P^{\mathrm{XIII}}_{\{12345\}}(\boldsymbol{\beta},\boldsymbol{\xi})\, H_{\mathrm{VI}}\left(\begin{matrix} (a_1,a_2,a_3,a_4,a_5),b \\ c \end{matrix} \middle| \boldsymbol{x} \right) , \quad (30)$$

where

$$\boldsymbol{x} = \left(\frac{m_4^2}{p_3^2}, 0, \frac{m_3^2}{m_4^2}, \frac{p_2^2}{p_3^2}, \frac{m_2^2}{m_4^2}, \frac{p_1^2}{p_3^2}, \frac{m_1^2}{m_4^2}, \frac{p_1^2}{p_3^2}\right) ,$$
$$P^{\mathrm{XIII}}_{\{12345\}}(\boldsymbol{\beta},\boldsymbol{\xi}) = (p_3^2)^{-a_5}(m_4^2)^{-b}(\Lambda^2)^{-a_1} \quad (31)$$

with $a_1 = a_2 = a_3 = a_4 = 1$, $a_5 = c = 4 - D$, and $b = 0$.

The expression for $\Phi^{\mathrm{XIV}}_{\{12345\}}$

The function is

$$\Phi^{\mathrm{XIV}}_{\{12345\}}(\{p_i^2\},\{m_j^2\}) = P^{\mathrm{XIV}}_{\{12345\}}(\boldsymbol{\beta},\boldsymbol{\xi}) H_{\mathrm{II}}\left(\begin{matrix} (a_1,a_2,a_3,a_4),b \\ (c_1,c_2) \end{matrix} \middle| \boldsymbol{x} \right) , \quad (32)$$

where

$$\boldsymbol{x} = (0, \frac{m_4^2}{p_3^2}, \frac{m_3^2}{p_3^2}, \frac{p_2^2}{p_3^2}, \frac{m_2^2}{p_3^2}, \frac{p_1^2}{p_3^2}, \frac{m_1^2}{p_3^2}, \frac{p_1^2}{p_3^2}) ,$$
$$P^{\mathrm{XIV}}_{\{12345\}}(\boldsymbol{\beta},\boldsymbol{\xi}) = (p_3^2)^{-b} (\Lambda^2)^{c_2-a_4-1} \quad (33)$$

with $a_1 = a_2 = a_3 = a_4 = c_2 = 1$, and $b = c_1 = 4 - D$.

To determine the coefficients

To get uniquely the combination coefficients C_α in the formula

$$A_{DU}(\{p_i^2\},\{m_j^2\}) = \sum_{\alpha=\mathrm{I}}^{\mathrm{XIV}} C_\alpha \Phi^\alpha_{\{12345\}}(\{p_i^2\},\{m_j^2\}) \ , \tag{34}$$

we need to know the analytic expressions of the Feynman integral and the 14 fundamental solutions at some regular points.

The Feynman Integral at $p_3 = 0$, $m_1 = m_2 = m_3 = m_4 = 0$

Using the Gegenbaur polynomials, we have

- $p_3 = 0$, $m_1 = m_2 = m_3 = m_4 = 0$

$$A_{DU}(\{p_i^2\},\{m_j^2\}) = \frac{(-1)^{D-3}2^{2D-1}(\frac{D}{2}-1)\Gamma^5(\frac{D}{2}-1)}{(4\pi)^{2D}}\left(\frac{\Lambda^2_{\rm RE}}{p_1^2}\right)^{4-D} \times\left[\frac{1}{4-D}-\frac{1}{6-2D}\right]\frac{\Gamma(4-D)}{\Gamma(\frac{3}{2}D-4)} \tag{35}$$

The Feynman Integral at $p_2 = 0$, $m_1 = m_2 = m_3 = m_4 = 0$

- $p_2 = 0$, $m_1 = m_2 = m_3 = m_4 = 0$

$$A_{DU}(\{p_i^2\},\{m_j^2\}) = (-1)^{3-D}\frac{2^{2D-1}(\frac{D}{2}-1)\Gamma^5(\frac{D}{2}-1)}{(4\pi)^{2D}}\left(\frac{\Lambda^2_{\rm RE}}{p_1^2}\right)^{4-D} \times\left\{\frac{1}{4-D}-\frac{1}{6-2D}\right\}\frac{\Gamma(4-D)}{\Gamma(\frac{3}{2}D-4)} \tag{36}$$

The Feynman Integral at $p_1 = 0$, $m_1 = m_2 = m_3 = m_4 = 0$

- $p_1 = 0$, $m_1 = m_2 = 0$, and $m_3 = m_4 = m_{\rm IR} \to 0$, we have

$$A_{DU}(\{p_i^2\},\{m_j^2\}) = -\frac{2^{D-1}(\frac{D}{2}-1)\Gamma^3(\frac{D}{2}-1)}{(4\pi)^{3D/2}}\left(\frac{\Lambda^2_{\rm RE}}{p_2^2}\right)^{4-D} \times\left\{\frac{\Gamma(\frac{D}{2}-2)\Gamma(4-D)}{\Gamma(\frac{3}{2}D-4)} + \left(\frac{m^2_{\rm IR}}{p_2^2}\right)^{D/2-2}\frac{\Gamma^2(2-\frac{D}{2})}{\Gamma(D-2)}\right\} \quad (37)$$

The Feynman Integral at $p_1 = p_2 = p_3 = 0$, $m_2 = m_3 = m_4 = 0$, $m_1 \neq 0$

- $p_1 = p_2 = p_3 = 0$, $m_2 = 0$, $m_3 = m_4 = m_{\rm IR} \to 0$, and $m_1 \neq 0$, we have

$$A_{DU}(\{p_i^2\},\{m_j^2\}) = -\frac{\Gamma(\frac{D}{2}-1)\Gamma(3-\frac{D}{2})}{(4\pi)^D\Gamma(\frac{D}{2})}\left(\frac{\Lambda_{\rm RE}^2}{m_1^2}\right)^{4-D} \times\left\{\Gamma(4-D)\Gamma(\frac{D}{2}-2)+\frac{2\Gamma(2-\frac{D}{2})}{4-D}\left(\frac{m_{\rm IR}^2}{m_1^2}\right)^{D/2-2}\right\} \quad (38)$$

The Feynman Integral at $p_1 = p_2 = p_3 = 0$, $m_1 = m_3 = m_4 = 0$, $m_2 \neq 0$

- $p_1 = p_2 = p_3 = 0$, $m_1 = 0$, $m_3 = m_4 = m_{\rm IR} \to 0$, and $m_2 \neq 0$, we have

$$A_{DU}(\{p_i^2\},\{m_j^2\}) = -\frac{\Gamma(\frac{D}{2}-1)\Gamma(3-\frac{D}{2})}{(4\pi)^D\Gamma(\frac{D}{2})}\left(\frac{\Lambda_{\rm RE}^2}{m_2^2}\right)^{4-D} \times\left\{\Gamma(4-D)\Gamma(\frac{D}{2}-2)+\frac{2\Gamma(2-\frac{D}{2})}{4-D}\left(\frac{m_{\rm IR}^2}{m_2^2}\right)^{D/2-2}\right\} \quad (39)$$

The Feynman Integral at $p_1 = p_2 = p_3 = 0$, $m_1 = m_2 = m_4 = 0$, $m_3 \neq 0$

- $p_1 = p_2 = p_3 = 0$, $m_1 = m_2 = m_4 = 0$, $m_3 \neq 0$,

$$A_{DU}(\{p_i^2\},\{m_j^2\}) = -\frac{2^{D-1}\Gamma^3(\frac{D}{2}-1)\Gamma(3-\frac{D}{2})\Gamma(4-D)}{(4\pi)^{3D/2}}\Big(\frac{\Lambda^2_{\rm RE}}{m_3^2}\Big)^{4-D}$$
$$\times\Big[-\frac{1}{3-D}+\frac{2}{4-D}\Big] \qquad (40)$$

The Feynman Integral at $p_1 = p_2 = p_3 = 0$, $m_1 = m_2 = m_3 = 0$, $m_4 \neq 0$

- $p_1 = p_2 = p_3 = 0$, $m_1 = m_2 = m_3 = 0$, $m_4 \neq 0$,

$$A_{DU}(\{p_i^2\},\{m_j^2\}) = -\frac{2^{D-1}\Gamma^3(\frac{D}{2}-1)\Gamma(3-\frac{D}{2})\Gamma(4-D)}{(4\pi)^{3D/2}}\Big(\frac{\Lambda^2_{\rm RE}}{m_4^2}\Big)^{4-D}$$
$$\times\Big[-\frac{1}{3-D}+\frac{2}{4-D}\Big] \qquad (41)$$

The Feynman Integral at $p_3 = 0,\ m_2 = m_3 = m_4 = 0,\ p_1^2 = p_2^2 = m_1^2 \neq 0$

- $p_3 = 0,\ m_2 = m_3 = m_4 = 0,\ p_1^2 = p_2^2 = m_1^2 \neq 0,$

$$A_{DU}(\{p_i^2\},\{m_j^2\}) = \frac{2^{2D-2}(\frac{D}{2}-1)\Gamma^4(\frac{D}{2}-1)}{(4\pi)^{2D}}\left(\frac{\Lambda^2_{\rm RE}}{m_1^2}\right)^{4-D}$$
$$\times\left\{\left[1-(-1)^{2-D/2}\right]\frac{\Gamma(2-\frac{D}{2})\Gamma(3-D)}{\Gamma(D-3)\Gamma(\frac{3}{2}D-4)}\right.$$
$$\left.+(-1)^{3-D/2}\frac{\Gamma(2-\frac{D}{2})\Gamma(3-D)}{\Gamma(\frac{D}{2}-1)}\right\} \qquad (42)$$

The Feynman Integral at $p_3 = 0,\ m_1 = m_3 = m_4 = 0,\ p_1^2 = p_2^2 = m_2^2 \neq 0$

- $p_3 = 0,\ m_1 = m_3 = m_4 = 0,\ p_1^2 = p_2^2 = m_2^2 \neq 0,$

$$A_{DU}(\{p_i^2\},\{m_j^2\}) = \frac{2^{2D-2}(\frac{D}{2}-1)\Gamma^4(\frac{D}{2}-1)}{(4\pi)^{2D}}\left(\frac{\Lambda^2_{\rm RE}}{m_2^2}\right)^{4-D}$$
$$\times\left\{\left[1-(-1)^{2-D/2}\right]\frac{\Gamma(2-\frac{D}{2})\Gamma(3-D)}{\Gamma(D-3)\Gamma(\frac{3}{2}D-4)}\right.$$
$$\left.+(-1)^{3-D/2}\frac{\Gamma(2-\frac{D}{2})\Gamma(3-D)}{\Gamma(\frac{D}{2}-1)}\right\} \qquad (43)$$

The Feynman Integral at $p_3 = 0,\ m_1 = m_2 = m_4 = 0,\ p_1^2 = p_2^2 = m_3^2 \neq 0$

- $p_3 = 0,\ m_1 = m_2 = m_4 = 0,\ p_1^2 = p_2^2 = m_3^2 \neq 0,$

$$A_{DU}(\{p_i^2\},\{m_j^2\}) = -\frac{2^{2D-2}\Gamma^3(\frac{D}{2}-1)\Gamma(3-\frac{D}{2})\Gamma(4-D)}{(4\pi)^{2D}}\left(\frac{\Lambda^2_{\rm RE}}{m_3^2}\right)^{4-D}$$
$$\times\Big[-\frac{1}{3-D}+\frac{2}{4-D}\Big]\frac{\Gamma(\frac{D}{2})\Gamma(2D-7)}{\Gamma(D-3)\Gamma(\frac{3}{2}D-4)} \qquad (44)$$

The Feynman Integral at $p_2 = 0,\ m_2 = m_3 = m_4 = 0,\ p_1^2 = p_3^2 = m_1^2 \neq 0$

- $p_2 = 0,\ m_2 = m_3 = m_4 = 0,\ p_1^2 = p_3^2 = m_1^2 \neq 0,$

$$A_{DU}(\{p_i^2\},\{m_j^2\}) = \frac{2^{2D-2}(\frac{D}{2}-1)\Gamma^4(\frac{D}{2}-1)}{(4\pi)^{2D}}\left(\frac{\Lambda^2_{\rm RE}}{m_1^2}\right)^{4-D}$$
$$\times\left\{\left[1-(-1)^{2-D/2}\right]\frac{\Gamma(2-\frac{D}{2})\Gamma(3-D)}{\Gamma(D-3)\Gamma(\frac{3}{2}D-4)}\right.$$
$$\left.+(-1)^{3-D/2}\frac{\Gamma(2-\frac{D}{2})\Gamma(3-D)}{\Gamma(\frac{D}{2}-1)}\right\} \qquad (45)$$

The Feynman Integral at $p_2 = 0,\ m_1 = m_3 = m_4 = 0,\ p_1^2 = p_3^2 = m_2^2 \neq 0$

- $p_2 = 0,\ m_1 = m_3 = m_4 = 0,\ p_1^2 = p_3^2 = m_2^2 \neq 0,$

$$A_{DU}(\{p_i^2\},\{m_j^2\}) = \frac{2^{2D-2}(\frac{D}{2}-1)\Gamma^4(\frac{D}{2}-1)}{(4\pi)^{2D}}\left(\frac{\Lambda^2_{\rm RE}}{m_2^2}\right)^{4-D}$$
$$\times\left\{\left[1-(-1)^{2-D/2}\right]\frac{\Gamma(2-\frac{D}{2})\Gamma(3-D)}{\Gamma(D-3)\Gamma(\frac{3}{2}D-4)}\right.$$
$$\left.+(-1)^{3-D/2}\frac{\Gamma(2-\frac{D}{2})\Gamma(3-D)}{\Gamma(\frac{D}{2}-1)}\right\} \qquad (46)$$

The Feynman Integral at $p_2 = 0,\ m_1 = m_2 = m_3 = 0,\ p_1^2 = p_3^2 = m_4^2 \neq 0$

- $p_2 = 0,\ m_1 = m_2 = m_3 = 0,\ p_1^2 = p_3^2 = m_4^2 \neq 0,$

$$A_{DU}(\{p_i^2\},\{m_j^2\}) = -\frac{2^{2D-2}\Gamma^3(\frac{D}{2}-1)\Gamma(3-\frac{D}{2})\Gamma(4-D)}{(4\pi)^{2D}}\left(\frac{\Lambda^2_{\rm RE}}{m_4^2}\right)^{4-D}$$
$$\times\Big[-\frac{1}{3-D}+\frac{2}{4-D}\Big]\frac{\Gamma(\frac{D}{2})\Gamma(2D-7)}{\Gamma(D-3)\Gamma(\frac{3}{2}D-4)} \qquad (47)$$

The Feynman Integral at $p_1 = 0,\ m_2 = m_3 = m_4 = 0,\ p_2^2 = p_3^2 = m_1^2 \neq 0$

- $p_1 = 0,\ m_2 = 0,\ m_3 = m_4 = m_{\rm IR} \to 0,\ p_2^2 = p_3^2 = m_1^2 \neq 0,$

$$A_{DU}(\{p_i^2\},\{m_j^2\}) = -\frac{2^{D-1}\Gamma(\frac{D}{2}-1)\Gamma(\frac{D}{2}-2)}{(4\pi)^{3D/2}}\left(\frac{\Lambda^2_{\rm RE}}{m_1^2}\right)^{4-D}$$
$$\times\Big[\Gamma(3-\frac{D}{2})\Gamma(4-D)\frac{\Gamma(\frac{D}{2})\Gamma(2D-7)}{\Gamma(D-3)\Gamma(\frac{3}{2}D-4)}$$
$$-\frac{\Gamma(3-\frac{D}{2})}{\Gamma(\frac{D}{2}-1)}\left(\frac{m^2_{\rm IR}}{m_1^2}\right)^{D/2-2}\Gamma(2-\frac{D}{2})\frac{\Gamma(\frac{D}{2})\Gamma(D-3)}{\Gamma(\frac{D}{2}-1)\Gamma(D-2)}\Big] \quad (48)$$

The Feynman Integral at $p_1 = 0,\ m_1 = m_3 = m_4 = 0,\ p_2^2 = p_3^2 = m_2^2 \neq 0$

- $p_1 = 0,\ m_1 = 0,\ m_3 = m_4 = m_{\rm IR} \to 0,\ p_2^2 = p_3^2 = m_2^2 \neq 0,$

$$A_{DU}(\{p_i^2\},\{m_j^2\}) = -\frac{2^{D-1}\Gamma(\frac{D}{2}-1)\Gamma(\frac{D}{2}-2)}{(4\pi)^{3D/2}}\left(\frac{\Lambda^2_{\rm RE}}{m_2^2}\right)^{4-D}$$
$$\times\Big[\Gamma(3-\frac{D}{2})\Gamma(4-D)\frac{\Gamma(\frac{D}{2})\Gamma(2D-7)}{\Gamma(D-3)\Gamma(\frac{3}{2}D-4)}$$
$$-\frac{\Gamma(3-\frac{D}{2})}{\Gamma(\frac{D}{2}-1)}\left(\frac{m^2_{\rm IR}}{m_2^2}\right)^{D/2-2}\Gamma(2-\frac{D}{2})\frac{\Gamma(\frac{D}{2})\Gamma(D-3)}{\Gamma(\frac{D}{2}-1)\Gamma(D-2)}\Big] \quad (49)$$

The Feynman Integral at $p_1 = 0$, $m_1 = m_2 = m_4 = 0$, $p_2^2 = p_3^2 = m_3^2 \neq 0$

- $p_1 = 0$, $m_1 = m_2 = m_4 = 0$, $p_2^2 = p_3^2 = m_3^2 \neq 0$,

$$A_{DU}(\{p_i^2\},\{m_j^2\}) = -\frac{2^{2D-2}(\frac{D}{2}-1)\Gamma^4(\frac{D}{2}-1)}{(4\pi)^{2D}}\left(\frac{\Lambda^2_{\rm RE}}{m_3^2}\right)^{4-D} \times\Big[-\frac{\Gamma(3-\frac{D}{2})\Gamma(3-D)\Gamma(2D-6)}{\Gamma(D-3)\Gamma(\frac{3}{2}D-3)} +\frac{\Gamma(2-\frac{D}{2})\Gamma(4-D)\Gamma(2D-6)}{\Gamma(D-2)\Gamma(\frac{3}{2}D-4)}\Big] \tag{50}$$

The Feynman Integral at $p_1 = 0,\ m_1 = m_2 = m_3 = 0,\ p_2^2 = p_3^2 = m_4^2 \neq 0$

- $p_1 = 0,\ m_1 = m_2 = m_3 = 0,\ p_2^2 = p_3^2 = m_4^2 \neq 0,$

$$A_{DU}(\{p_i^2\},\{m_j^2\}) = -\frac{2^{2D-2}(\frac{D}{2}-1)\Gamma^4(\frac{D}{2}-1)}{(4\pi)^{2D}}\left(\frac{\Lambda_{\rm RE}^2}{m_4^2}\right)^{4-D} \times\Big[-\frac{\Gamma(3-\frac{D}{2})\Gamma(3-D)\Gamma(2D-6)}{\Gamma(D-3)\Gamma(\frac{3}{2}D-3)} +\frac{\Gamma(2-\frac{D}{2})\Gamma(4-D)\Gamma(2D-6)}{\Gamma(D-2)\Gamma(\frac{3}{2}D-4)}\Big] \tag{51}$$

The values of fundamental solutions on those points are presented in the following tables. Those results are derived through the inverse Gauss relations and the Gauss-Kummer relations of the relevant hypergeometric functions. Those relations are all satisfy the nilpotent property of order two.

The values of fundamental solutions on some regular points

We should check if the chosen points are essential singularities of the 14 hypergeometric functions.

#	I	II	III	IV	V	VI	VII	VIII	IX	X	XI	XII	XIII	XIV
1	$p_1=0, p_2=p_3$ $m_1=m_2=m_3=m_4=0$	0	0	0	0	$\frac{(D-2)(D-3)(p_2-p_1)^{D-4}}{2^D}$	0	0	0	$\frac{(D-2)(D-3)(p_2-p_1)^{D-4}}{2^D}$	0	0	$\frac{(D-2)(D-3)(p_2-p_1)^{D-4}}{2^D}$	0	0
2	$p_1=0, p_2=p_3$ $m_1=m_2=m_3=m_4=0$	0	0	0	0	$\frac{(D-2)(D-3)(p_2-p_1)^{D-4}}{2^D}$	0	0	0	$\frac{(D-2)(D-3)(p_2-p_1)^{D-4}}{2^D}$	0	0	$\frac{0}{p_1^2+p_2^2}$	0	$\frac{(D-2)(D-3)(p_2-p_1)^{D-4}}{2^D}$
3	$p_1=0, p_2=p_3$ $m_1=m_2=m_3=m_4=0$	0	0	0	0	0	0	0	0	0	0	0	$\frac{(D-3)(p_2-p_1)^{D-4}}{2^D}$	0	$\frac{(D-3)(p_2-p_1)^{D-4}}{2^D}$
4	$p_1=p_2=p_3=0$ $m_1\neq 0,\ m_2=m_3=m_4=0$	$\frac{4(-1)^{D-4}(m_1^2)^{D-4}}{(D-3)(D-2)(D-1)2^D}$	$\frac{(-1)^{D-4}(m_1^2)^{D-4}}{2^D}$	$\frac{(-1)^{D-4}(m_1^2)^{D-4}}{2^D}$	$\frac{\cos(\pi D)(m_1^2)^{D-4}}{2^D}$	$\frac{(-1)^{D-4}(m_1^2)^{D-4}}{2^D}$	0	0	0	0	0	0	0	0	0
5	$p_1=p_2=p_3=0$ $m_2\neq 0,\ m_1=m_3=m_4=0$	0	0	0	0	0	$\frac{2(-1)^D(m_2^2)^{D-4}}{2^D(D-2)(D-3)}$	$\frac{(-1)^D(m_2^2)^{D-4}}{(1-D)2^D}$	$\frac{(-1)^D(m_2^2)^{D-4}}{2^D}$	$\frac{\cos(\pi D)(m_2^2)^{D-4}}{2^D}$	0	0	0	0	0
6	$p_1=p_2=p_3=0$ $m_3\neq 0,\ m_1=m_2=m_4=0$	0	0	0	0	0	0	0	0	0	$\frac{(-1)^{D-4}(m_3^2)^{D-4}}{2^D(D-3)}$	$\frac{(-1)^{D-4}(m_3^2)^{D-4}}{2^D}$	$\frac{(m_3^2)^{D-4}}{2^D}$	0	0
7	$p_1=p_2=p_3=0$ $m_4\neq 0,\ m_1=m_2=m_3=0$	0	0	0	0	0	0	0	0	0	0	0	0	$\frac{(m_4^2)^{D-4}}{2^D}$	$\frac{(-1)^{D-4}(m_4^2)^{D-4}}{2^D}$
8	$p_1=0, p_2=p_3$ $m_1\neq 0,\ m_2=m_3=m_4=0$	$\frac{16(p_2)^{D-4}}{(D-1)(D-2)2^D}$	$\frac{0}{m_1^2+p_2^2}$	$\frac{0}{m_1^2+p_2^2}$	$\frac{(D-2)(p_2)^{D-4}}{2^D}$ $m_1^2+p_2^2$	$\frac{(D-3)(p_2)^{D-4}}{2^D}$	$\frac{2(p_2)^{D-4}}{2^D(D-2)}$	$\frac{(D-3)(p_2)^{D-4}}{2^D}$ $m_1^2+p_2^2$	$\frac{(D-3)(p_2)^{D-4}}{2^D}$ $m_1^2+p_2^2$	$\frac{(D-3)(p_2)^{D-4}}{2^D}$ $m_1^2+p_2^2$	$\frac{(p_2)^{D-4}}{2^D}$ $m_1^2+p_2^2$	$\frac{(D-3)(p_2)^{D-4}}{2^D}$ $m_1^2+p_2^2$	$\frac{(D-3)(p_2)^{D-4}}{2^D}$ $m_1^2+p_2^2$	$\frac{0}{m_1^2+p_2^2}$	$\frac{0}{m_1^2+p_2^2}$
9	$p_1=0, p_2=p_3$ $m_2\neq 0,\ m_1=m_3=m_4=0$	0	$\frac{(D-3)(p_2)^{D-4}}{2^D}$ $m_2^2+p_2^2$	$\frac{(D-3)(p_2)^{D-4}}{2^D}$ $m_2^2+p_2^2$	$\frac{(D-3)(p_2)^{D-4}}{2^D}$ $m_2^2+p_2^2$	$\frac{(D-3)(p_2)^{D-4}}{2^D}$	0 $m_2^2+p_2^2$	0 $m_2^2+p_2^2$	$\frac{(D-3)(p_2)^{D-4}}{2^D}$ $m_2^2+p_2^2$	$\frac{(D-3)(p_2)^{D-4}}{2^D}$ $m_2^2+p_2^2$	$\frac{(p_2)^{D-4}}{2^D}$ $m_2^2+p_2^2$	$\frac{(D-3)(p_2)^{D-4}}{2^D}$ $m_2^2+p_2^2$	$\frac{(D-3)(p_2)^{D-4}}{2^D}$ $m_2^2+p_2^2$	0 $m_2^2+p_2^2$	0 $m_2^2+p_2^2$

The values of fundamental solutions on some regular points

#	I	II	III	IV	V	VI	VII	VIII	IX	X	XI	XII	XIII	XIV
10	$p_1=0, p_2=p_3$ $m_1\neq 0, m_2=m_3=m_4=0$	0	0	$\frac{(2-3)(p_1)^{2-4}}{s^2}$ $m_2^2=p_2^2$	$\frac{(2-3)(p_1)^{2-4}}{s^2}$ $m_2^2=p_2^2$	$-\frac{(2-3)(p_1)^{2-4}}{s^2}$	0	$\frac{(2-3)(p_1)^{2-4}}{s^2}$ $m_2^2=p_2^2$	$\frac{(2-3)(p_1)^{2-4}}{s^2}$ $m_2^2=p_2^2$	$\frac{(2-3)(p_1)^{2-4}}{s^2}$ $m_2^2=p_2^2$	$-\frac{(p_1)^{2-2}}{s^2}$ $m_2^2=p_2^2$	$\frac{(2-3)(p_1)^{2-4}}{s^2}$ $m_2^2=p_2^2$	$\frac{(2-3)(p_1)^{2-4}}{s^2}$ $m_2^2=p_2^2$	0 $m_2^2=p_2^2$	0 $m_2^2=p_2^2$
11	$p_1=0, p_2=p_3$ $m_1\neq 0, m_2=m_3=m_4=0$	0	0	0	$\frac{(2s^2-8(2+7)(p_1)^{2-4}}{8s^2}$ $m_2^2=p_2^2$	$\frac{(2s^2-8(2+7)(p_1)^{2-4}}{8s^2}$	0	0	$\frac{(2s^2-8(2+7)(p_1)^{2-4}}{8s^2}$ $p_1^2=p_2^2$ $p_2^2=m_2^2$	$\frac{(2s^2-8(2+7)(p_1)^{2-4}}{8s^2}$ $p_1^2=p_2^2$ $p_2^2=m_2^2$	0	$\frac{(2s^2-8(2+7)\times(p_1)^{2-4}}{4.8^2m_2^2}$ $m_2^2=p_2^2$	$\frac{(2s^2-8(2+7)(p_1)^{2-4}}{4.8^2m_2^2}$ $m_2^2=p_2^2$	$\frac{(-1)^{1-4}(m_1)^{2-4}}{8s^2}$ $m_2^2=p_2^2$	$\frac{(-1)^{1-4}(m_1)^{2-4}}{8s^2}$ $m_2^2=p_2^2$
12	$p_1=0, p_2=p_3$ $m_1\neq 0, m_2=m_3=m_4=0$	$\frac{18(p_1)^{2-4}}{(2-1)(2-2)8^2}$ $m_2^2=p_2^2$	$\frac{(2-3)(p_1)^{2-4}}{s^2}$ $m_2^2=p_2^2$	$\frac{2(2-3)(p_1)^{2-4}}{s^2}$ $m_2^2=p_2^2$	0 $m_2^2=p_2^2$	$\frac{(2s^2-8(2+7)(p_1)^{2-4}}{8s^2}$	$\frac{(p_1)^{2-4}}{s^2(2-2)}$ $m_2^2=p_2^2$	$-\frac{(2-3)(p_1)^{2-4}}{s^2}$ $m_2^2=p_2^2$	$\frac{(2-3)(p_1)^{2-4}}{s^2}$ $m_2^2=p_2^2$	$\frac{(2-3)(p_1)^{2-4}}{s^2}$ $m_2^2=p_2^2$	0 $m_2^2=p_2^2$	0 $m_2^2=p_2^2$	0 $m_2^2=p_2^2$	$\frac{(2-3)(p_1)^{2-4}}{s^2}$ $m_2^2=p_2^2$	$\frac{(2-3)(p_1)^{2-4}}{s^2}$ $m_2^2=p_2^2$
13	$p_1=0, p_2=p_3$ $m_1\neq 0, m_2=m_3=m_4=0$	0	$\frac{(2-3)(p_1)^{2-4}}{s^2}$ $m_2^2=p_2^2$	$\frac{(2-3)(p_1)^{2-4}}{s^2}$ $m_2^2=p_2^2$	$\frac{(2-3)(p_1)^{2-4}}{s^2}$ $m_2^2=p_2^2$	$-\frac{(2-3)(p_1)^{2-4}}{s^2}$	0 $m_2^2=p_2^2$	$\frac{(2-3)(2^2-2-1)(p_1)^{2-4}}{s^2(2-1)}$ $m_2^2=p_2^2$	0 $m_2^2=p_2^2$	$\frac{(2-3)(p_1)^{2-4}}{s^2}$ $m_2^2=p_2^2$	0 $m_2^2=p_2^2$	0 $m_2^2=p_2^2$	0 $m_2^2=p_2^2$	$\frac{(2-3)(p_1)^{2-4}}{s^2}$ $m_2^2=p_2^2$	$\frac{(2-3)(p_1)^{2-4}}{s^2}$ $m_2^2=p_2^2$
14	$p_1=0, p_2=p_3$ $m_1\neq 0, m_2=m_3=m_4=0$	0	0	$\frac{(2s^2-8(2+7)(p_1)^{2-4}}{8s^2}$ $m_2^2=p_2^2$	$\frac{(2s^2-8(2+7)(p_1)^{2-4}}{8s^2}$ $m_2^2=p_2^2$	$\frac{(2s^2-8(2+7)(p_1)^{2-4}}{8s^2}$	0	$\frac{(2s^2-8(2+7)(p_1)^{2-4}}{8s^2}$ $m_2^2=p_2^2$	$\frac{(2s^2-8(2+7)(p_1)^{2-4}}{8s^2}$ $m_2^2=p_2^2$	$\frac{(-1)^{2-4}2(m_1)^{2-4}}{8s^2(2-2)}+\frac{(2s^2-8(2+7)(p_1)^{2-4}}{8s^2(2-3)}$ $m_2^2=p_2^2$	$\frac{(-1)^{2-4}(m_1)^{2-4}}{8s^2p_1^2}+\frac{(2s^2-8(2+7)(p_1)^{2-4}}{8s^2}$ $m_2^2=p_2^2$	$\frac{(-1)^{2-4}(m_1)^{2-4}}{8s^2p_1^2}$ $m_2^2=p_2^2$	$\frac{(2s^2-8(2+7)(p_1)^{2-4}}{8s^2}$ $m_2^2=p_2^2$	$\frac{(2s^2-8(2+7)(p_1)^{2-4}}{8s^2}$ $m_2^2=p_2^2$
15	$p_1=0, p_2=p_3$ $m_1\neq 0, m_2=m_3=m_4=0$	0	0	0	$\frac{(2-3)(p_1)^{2-4}}{s^2}$ $m_2^2=p_2^2$	$-\frac{(2-3)(p_1)^{2-4}}{s^2}$	0	0	$\frac{(2-3)(p_1)^{2-4}}{s^2}$ $m_2^2=p_2^2$	$\frac{(2-3)(p_1)^{2-4}}{s^2}$ $m_2^2=p_2^2$	0	0 $m_2^2=p_2^2$	0 $m_2^2=p_2^2$	$\frac{(2-3)(p_1)^{2-4}}{s^2}$ $m_2^2=p_2^2$	$\frac{(2-3)(p_1)^{2-4}}{s^2}$ $m_2^2=p_2^2$
16	$p_1=0, p_2=p_3$ $m_1\neq 0, m_2=m_3=m_4=0$	$-\frac{12(2(0-5))(p_1)^{2-4}}{(2-1)(2-2)(2-3)8^2}$ $+\frac{2(2(0-5))(p_1)^{2-4}}{3(-1)^{1-4}(m_1)^{2-4}}$ $+\frac{3(-1)^{2-4}(m_1)^{2-4}}{2(2-1)(2-2)(2-3)(p_1)^2s^2}$ $m_2^2=p_2^2$	$\frac{2(2(0-5))(p_1)^{2-4}}{s^2}$ $+\frac{(-1)^{2-4}(m_1)^{2-4}}{4(p_1)^2s^2}$ $m_2^2=p_2^2$	$\frac{(-1)^{2-4}(m_1)^{2-4}}{4(p_1)^2s^2}$ $m_2^2=p_2^2$	$\frac{(-1)^{2-4}(m_1)^{2-4}}{4(p_1)^2s^2}$ $m_2^2=p_2^2$	0 $m_2^2=p_2^2$	0 $m_2^2=p_2^2$	0 $m_2^2=p_2^2$	0 $m_2^2=p_2^2$	$-\frac{(2(0-5))(p_1)^{2-4}}{4.8^2(0-3)}$ $m_2^2=p_2^2$	$\frac{(p_1)^{2-4}}{2.8^2}$ $m_2^2=p_2^2$	0	$\frac{(2(0-5))(p_1)^{2-4}}{4.8^2}$ $m_2^2=p_2^2$	0
17	$p_1=0, p_2=p_3$ $m_1\neq 0, m_2=m_3=m_4=0$	0	0	0	0	0	$\frac{(2(0-5))(p_1)^{2-4}}{2^2(2-2)(2-3)}$ $-\frac{(-1)^{1-4}(m_1)^{2-4}}{2.8^2(p_1)^2(2-2)(0-3)}$ $m_2^2=p_2^2$	$\frac{(-1)^{2-4}(m_1)^{2-4}}{4.8^2(p_1)^2}$ $m_2^2=p_2^2$	$\frac{(p_1)^{2-4}}{8.8^2}$ $+\frac{(-1)^{2-4}(m_1)^{2-4}}{4.8^2(p_1)^2}$	$\frac{(m_1)^{2-4}}{4.8^2(p_1)^2}$	$-\frac{(2(0-5))(p_1)^{2-4}}{4.8^2(0-3)}$	$\frac{(p_1)^{2-4}}{2.8^2}$	0	$\frac{(2(0-5))(p_1)^{2-4}}{4.8^2}$ $m_2^2=p_2^2$	0
18	$p_1=0, p_2=p_3$ $m_1\neq 0, m_2=m_3=m_4=0$	0	0	0	0	0	0	0	0	0	$\frac{2(p_1)^{2-4}}{3^2(0-3)}$ $m_2^2=p_2^2$	0 $m_2^2=p_2^2$	$\frac{(p_1)^{2-4}}{2^2}$ $m_2^2=p_2^2$	$\frac{(p_1)^{2-4}}{2^2}$ $m_2^2=p_2^2$	0
19	$p_1=0, p_2=p_3$ $m_1\neq 0, m_2=m_3=m_4=0$	0	0	0	0	0	0	0	0	0	$-\frac{(p_1)^{2-4}}{2^2}$ $m_2^2=p_2^2$ $p_1^2=p_2^2$	0	$\frac{(p_1)^{2-4}}{2^2}$ $m_2^2=p_2^2$	$\frac{(-1)^{1-4}(m_1)^{2-4}}{2^2}$

The definitions of the generalized hypergeometric functions

The definition of the generalized hypergeometric function is determined by the geometric configuration of the corresponding exponent matrix. The indices of columns a, b, c, d, e, f, g, h belong to the set $\{1, \cdots, 8\}$ and are all distinct from one another. The geometric configurations of the exponent matrices of hypergeometric functions are divided into the following types.

The definition of H_{I}

- The configuration I: any 2×2 minors of the 5×3 matrix composed by the columns with the indices a, b, and c are zero, and any 5×5 minors of the 5×6 matrix composed by the columns with the indices a (or b, or c), d, e, f, g, and h are nonzero. The geometric configuration corresponds to the following generalized hypergeometric function

$$H_{\mathrm{I}}\left(\begin{array}{c} \boldsymbol{a} \\ c \end{array} \middle| \boldsymbol{x} \right) = \sum_{\boldsymbol{n} \in \boldsymbol{Z}_{+}^{8}} \frac{(a_1)_{|n|_{12}} (a_2)_{|n|_{34}} (a_3)_{|n|_{56}} (a_4)_{|n|_{78}} (a_5)_{|n|_{1357}} (a_6)_{|n|_{2468}}}{\left[\prod_{i=1}^{8} n_i! \right] (c)_{|n|}} \boldsymbol{x}^{\boldsymbol{n}} \quad (52)$$

here the parameter vector $\boldsymbol{a} \in \boldsymbol{C}^6$ in the nominator, and the parameter $c \in \boldsymbol{C}$ in the denominator, $|n| = |n|_{1\cdots 8} = \sum_{i=1}^{8} n_i$, $|n|_{i_1 i_2 \cdots} = n_{i_1} + n_{i_2} + \cdots$, respectively. In addition, $(a)_n$ is the Pochhammer notation, and the abbreviation $\boldsymbol{x}^{\boldsymbol{n}} = \prod^{8} x_i^{n_i}$ for multi-variable notation.

The definition of H_{II}

- The geometric configuration II: any 2×2 minors of the 5×2 matrix composed by the columns with the indices a and b are zero, any 3×3 minors of the 5×3 matrix composed by the columns with the indices a (or b), c, and d are zero, and the determinant of the 5×5 matrix composed by the columns with the indices c, e, f, g, and h is zero. The generalized hypergeometric function is

$$H_{\mathrm{II}}\left(\begin{matrix} \boldsymbol{a}, b \\ \boldsymbol{c} \end{matrix} \middle| \boldsymbol{x} \right) = \sum_{\boldsymbol{n} \in \boldsymbol{Z}_+^8} \frac{(a_1)_{|n|_{34}} (a_2)_{|n|_{56}} (a_3)_{|n|_{78}} (a_4)_{|n|_{2357}} (b)_{-n_1+|n|_{2\cdots 8}}}{\left[\prod_{i=1}^{8} n_i! \right] (c_1)_{-n_1+|n|_{3\cdots 8}} (c_2)_{-n_1+|n|_{2357}}} \times (-1)^{|n|_{1357}} \boldsymbol{x}^{\boldsymbol{n}} , \tag{53}$$

where the parameter vector $\boldsymbol{a} \in \boldsymbol{C}^4$, the parameter $b \in \boldsymbol{C}$ in the nominator, and the parameter vector $\boldsymbol{c} \in \boldsymbol{C}^2$ in the denominator, respectively.

The definition of H_{III}

- The geometric configuration III: any 2×2 minors of the 5×2 matrix composed by the columns with the indices a and b are zero, any 4×4 minors of the 5×4 matrix composed by the columns with the indices a (or b), c, d, and e are zero, and any 4×4 minors of the 5×4 matrix composed by the columns with the indices c, f, g, and h are zero, respectively. The generalized hypergeometric function is

$$H_{\mathrm{III}}\left(\begin{matrix} \boldsymbol{a}, b \\ \boldsymbol{c} \end{matrix} \middle| \boldsymbol{x} \right) = \sum_{\boldsymbol{n} \in \boldsymbol{Z}_+^8} \frac{(a_1)_{|n|_{23}} (a_2)_{|n|_{56}} (a_3)_{|n|_{78}} (a_4)_{|n|_{2457}} (b)_{-|n|_{13}+|n|_{4\cdots 8}}}{\left[\prod_{i=1}^{8} n_i! \right] (c_1)_{-n_1+|n|_{2457}} (c_2)_{-|n|_{13}+|n|_{5678}}} \times (-1)^{|n|_{157}} \boldsymbol{x}^{\boldsymbol{n}} , \tag{54}$$

where the parameter vector $\boldsymbol{a} \in \boldsymbol{C}^4$, the parameter $b \in \boldsymbol{C}$ in the nominator, and the parameter vector $\boldsymbol{c} \in \boldsymbol{C}^2$ in the denominator, respectively.

The definition of $H_{\rm IV}$

- The geometric configuration IV: any 2×2 minors of the 5×2 matrix composed by the columns with the indices a and b are zero, any 3×3 minors of the 5×3 matrix composed by the columns with the indices c, d, and e are zero, and the determinant of the 5×5 matrix composed by the columns with the indices a (or b), c, f, g, and h is zero. The function is

$$H_{\rm IV}\left(\begin{array}{c} \boldsymbol{a}, b \\ \boldsymbol{c} \end{array} \middle| \boldsymbol{x} \right) = \sum_{\boldsymbol{n} \in \boldsymbol{Z}_+^8} \frac{(a_1)_{|n|_{23}} (a_2)_{|n|_{45}} (a_3)_{|n|_{78}} (a_4)_{|n|_{2467}} (b)_{-|n|_{135}+|n|_{678}}}{\left[\prod_{i=1}^{8} n_i! \right] (c_1)_{-n_1+|n|_{2467}} (c_2)_{-|n|_{135}+|n|_{78}}} \times (-1)^{|n|_{17}} \boldsymbol{x}^{\boldsymbol{n}} , \tag{55}$$

where the parameter vector $\boldsymbol{a} \in \boldsymbol{C}^4$, the parameter $b \in \boldsymbol{C}$ in the nominator, and the parameter vector $\boldsymbol{c} \in \boldsymbol{C}^2$ in the denominator, respectively.

The definition of $H_{\rm V}$

- The geometric configuration V: any 2×2 minors of the 5×2 matrix composed by the columns with the indices a and b are zero, any 2×2 minors of the 5×2 matrix composed by the columns with the indices c and d are zero, and any 5×5 minors of the 5×6 matrix composed by the columns with the indices a (or b), c (or d), e, f, g, and h are nonzero, respectively. The function is

$$H_{\rm V}\left(\begin{array}{c} \boldsymbol{a}, b \\ c \end{array} \middle| \boldsymbol{x}\right) = \sum_{\boldsymbol{n}\in \boldsymbol{Z}_+^8} \frac{(a_1)_{|n|_{23}} (a_2)_{|n|_{45}} (a_3)_{|n|_{67}} (a_4)_{|n|_{1357}} (a_5)_{|n|_{2468}} (b)_{-|n|_{1357}+n_8}}{\left[\prod_{i=1}^{8} n_i!\right] (c)_{-n_1+|n|_{2468}}} \times (-1)^{|n|_{357}} \boldsymbol{x}^{\boldsymbol{n}} , \tag{56}$$

where the parameter vector $\boldsymbol{a} \in \boldsymbol{C}^5$, the parameter $b \in \boldsymbol{C}$ in the nominator, and the parameter $\boldsymbol{c} \in \boldsymbol{C}$ in the denominator, respectively.

The definition of $H_{\rm VI}$

- The geometric configuration VI: any 2×2 minors of the 5×2 matrix composed by the columns with the indices a and b are zero, any 3×3 minors of the 5×3 matrix composed by the columns with the indices a (or b), c and d are zero, and the determinant of the 5×5 matrix composed by the columns with the indices a (or b), e, f, g, and h is zero, respectively. The function is

$$H_{\rm VI}\left(\begin{array}{c} \boldsymbol{a}, b \\ c \end{array} \middle| \boldsymbol{x} \right) = \sum_{\boldsymbol{n} \in \boldsymbol{Z}_+^8} \frac{(a_1)_{|n|_{12}} (a_2)_{|n|_{34}} (a_3)_{|n|_{56}} (a_4)_{|n|_{78}} (a_5)_{|n|_{1468}} (b)_{-|n|_{12}+|n|_{357}}}{\left[\prod_{i=1}^{8} n_i! \right] (c)_{-n_2+|n|_{3\cdots 8}}} \times (-1)^{n_1} \boldsymbol{x}^{\boldsymbol{n}} , \tag{57}$$

where the parameter vector $\boldsymbol{a} \in \boldsymbol{C}^5$ in the nominator, and the parameter $c \in \boldsymbol{C}$ in the denominator, respectively.

The definition of H_{VII}

- The geometric configuration VII: any 3×3 minors of the 5×3 matrix composed by the columns with the indices a, b, and c are zero, any 3×3 minors of the 5×3 matrix composed by the columns with the indices c, d, and e are zero, and any 4×4 minors of the 5×4 matrix composed by the columns with the indices d, f, g, and h are zero, respectively. The function is

$$H_{\mathrm{VII}}\left(\begin{array}{c} \boldsymbol{a}, \boldsymbol{b} \\ \boldsymbol{c} \end{array} \middle| \boldsymbol{x} \right) = \sum_{\boldsymbol{n} \in \boldsymbol{Z}_+^8} \frac{(a_1)_{|n|_{12}} (a_2)_{|n|_{56}} (a_3)_{|n|_{78}} (b_1)_{-|n|_{12}+|n|_{457}} (b_2)_{-|n|_{23}+|n|_{4\cdots 8}}}{\left[\prod_{i=1}^{8} n_i! \right] (c_1)_{-|n|_{123}+|n|_{457}} (c_2)_{-|n|_{23}+|n|_{5678}}} \times (-1)^{|n|_{2357}} \boldsymbol{x}^{\boldsymbol{n}} , \tag{58}$$

where the parameter vectors $\boldsymbol{a} \in \boldsymbol{C}^3$, $\boldsymbol{b} \in \boldsymbol{C}^2$ in the nominator, and the parameter $\boldsymbol{c} \in \boldsymbol{C}^2$ in the denominator, respectively.

The definition of H_{VIII}

- The geometric configuration VIII: any 3×3 minors of the 5×3 matrix composed by the columns with the indices a, b, and c are zero, any 3×3 minors of the 5×3 matrix composed by the columns with the indices d, e, and f are zero, and any 4×4 minors of the 5×4 matrix composed by the columns with the indices c, d, g, and h are zero, respectively. The function is

$$H_{\mathrm{VIII}}\left(\begin{matrix} \boldsymbol{a}, \boldsymbol{b} \\ \boldsymbol{c} \end{matrix} \middle| \boldsymbol{x} \right) = \sum_{\boldsymbol{n} \in \boldsymbol{Z}_+^8} \frac{(a_1)_{|n|_{12}} (a_2)_{|n|_{45}} (a_3)_{|n|_{78}} (b_1)_{-|n|_{12}+|n|_{467}} (b_2)_{-|n|_{235}+|n|_{678}}}{\left[\prod_{i=1}^{8} n_i! \right] (c_1)_{-|n|_{123}+|n|_{467}} (c_2)_{-|n|_{235}+|n|_{78}}} \times (-1)^{|n|_{237}} \boldsymbol{x}^{\boldsymbol{n}} , \tag{59}$$

where the parameter vectors $\boldsymbol{a} \in \boldsymbol{C}^3$, $\boldsymbol{b} \in \boldsymbol{C}^2$ in the nominator, and the parameter $\boldsymbol{c} \in \boldsymbol{C}^2$ in the denominator, respectively.

The definition of H_{IX}

- The geometric configuration IX: any 2×2 minors of the 5×2 matrix composed by the columns with the indices d, and e are zero, any 4×4 minors of the 5×4 matrix composed by the columns with the indices a, b, c, and d (or e) are zero, and any 4×4 minors of the 5×4 matrix composed by the columns with the indices d (or e), f, g, and h are zero, respectively. The function is

$$H_{\mathrm{IX}}\left(\begin{array}{c} \boldsymbol{a}, b \\ c \end{array} \middle| \boldsymbol{x}\right) = \sum_{\boldsymbol{n} \in \boldsymbol{Z}_+^8} \frac{(a_1)_{|n|_{12}} (a_2)_{|n|_{34}} (a_3)_{|n|_{56}} (a_4)_{|n|_{78}} (a_5)_{|n|_{1368}} (b)_{-|n|_{1234}+|n|_{57}}}{\left[\prod_{i=1}^{8} n_i!\right] (c)_{-|n|_{24}+|n|_{5678}}} \times (-1)^{|n|_{13}} \boldsymbol{x}^{\boldsymbol{n}}, \tag{60}$$

where the parameter vector $\boldsymbol{a} \in \boldsymbol{C}^5$, the parameter $b \in \boldsymbol{C}$ in the nominator, and the parameter $c \in \boldsymbol{C}$ in the denominator, respectively.

The definition of H_{X}

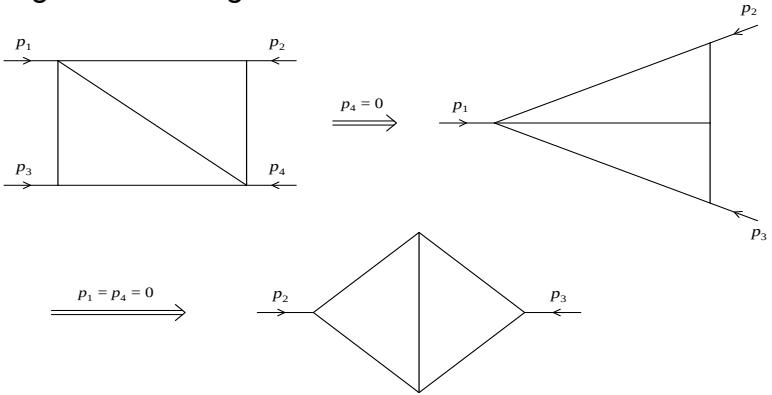
- The geometric configuration X: any 3×3 minors of the 5×3 matrix composed by the columns with the indices a, b and c are zero, any 3×3 minors of the 5×3 matrix composed by the columns with the indices b, d, and e are zero, and the determinant of the 4×4 matrix composed by the columns with the indices b, f, g, and h is zero. The function is

$$H_{\mathrm{X}}\left(\begin{matrix} \boldsymbol{a}, \boldsymbol{b} \\ c \end{matrix} \middle| \boldsymbol{x} \right) = \sum_{\boldsymbol{n} \in \boldsymbol{Z}_{+}^{8}} \frac{(a_1)_{|n|_{12}} (a_2)_{|n|_{34}} (a_3)_{|n|_{56}} (a_4)_{|n|_{78}} (b_1)_{|n|_{168} - |n|_{34}} (b_2)_{-|n|_{12} + |n|_{357}}}{\left[\prod_{i=1}^{8} n_i! \right] (c)_{-|n|_{24} + |n|_{5678}}} \times (-1)^{|n|_{13}} \boldsymbol{x}^{\boldsymbol{n}} , \tag{61}$$

where the parameter vector $\boldsymbol{a} \in \boldsymbol{C}^4$, the parameter $\boldsymbol{b} \in \boldsymbol{C}^2$ in the nominator, and the parameter vector $c \in \boldsymbol{C}$ in the denominator, respectively.

The 2-loop Feynman diagram

Since there are six Lorentz invariants constructed from the external momenta, this integral can be regarded as a function on the Grassmannian $G_{7,10}$.



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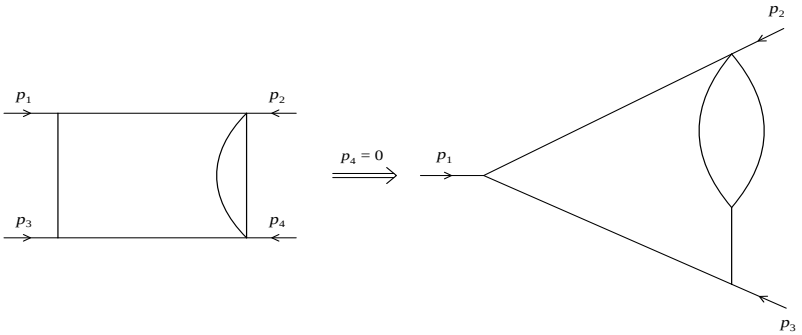

Thanks for your attention!

Thank you to the organizers for inviting us to this workshop!



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河北省计算物理基础学科研究中心
河北大学物理科学与技术学院

THANKS