Dual conformal regularization and slightly off-shell amplitudes in N=4 SYM

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November 9, 2026, Recent Progress in Amplitudes

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[a] In collaboration with A. Belitsky, L. Bork, A. Onishchenko, and V. Smirnov, based on [2509.12056], [2510.16471], [2605.17057], and [2605.17056].

Outline

Motivation

$\mathcal{N} = 4$ SYM and dual conformal symmetry

Method of regions

DCI regularization

2- and 3-loop DCI pentagons

Conclusion and Outlook

Motivation

Motivation

Disclaimer

I am not an expert in $\mathcal{N} = 4$ SYM, I am rather an expert in multiloop calculations methods. So I will present a **new approach** to the evaluation of slightly off-shell integrals which is definitely very effective for $\mathcal{N} = 4$ SYM calculations, but may be useful for more realistic setup.

- Perturbative calculations provide input for all-loop conjectures (like Bern-Dixon-Smirnov ansatz).
- Planar $\mathcal{N} = 4$ SYM is a nice playground for elaborating the approaches which might be useful also for more realistic QFT.
- Method presented in this talk can be useful also for the more realistic QFTs.

$\mathcal{N} = 4$ SYM and dual conformal symmetry

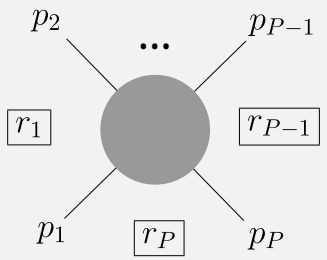
Slightly off-shell amplitudes in planar $\mathcal{N}=4$ SYM

- Amplitudes in $\mathcal{N}=4$ SYM exhibit **dual conformal invariance (DCI)**. However they contain infrared divergences, therefore some regularization is needed.
- Standard choice is the dimensional regularization $d=4-2\epsilon$. However, it breaks DCI, although in somewhat controlled way.
- Another option is to consider slightly off-shell "amplitudes". In particular, we can consider "amplitudes" with small off-shellness of external legs. Such a regularization preserves DCI.
- While in conventional dimensional regularization the collinear singularities are governed by Γ_{cusp}, in small-off-shellness asymptotics the leading logs are governed by Γ_{oct} known exactly in 't Hooft coupling.
- Within this approach, we want to calculate the small-off-shellness asymptotics of these amplitudes up to power corrections.

Dual conformal symmetry

P-point amplitude A_P in planar SYM

To formulate the dual conformal invariance, one introduces **dual variables** r_i, s.t. $p_i = r_i - r_{i-1}$.



- Then the A_P is invariant under conformal group acting on r_i and generated by

shifts: $r_i \to r_i + a$, rotations&dilat.: $r_i \to \Lambda r_i$, inversion: $r_i \to r_i / r_i^2$

- As a result, the amplitudes depend only on

DCI cross-ratios

$$\frac{r_{ij}^2 r_{kl}^2}{r_{il}^2 r_{kj}^2}\,, \qquad \text{where } r_{kl} = r_k - r_l = \textstyle\sum_{i=l+1}^{k} p_i .$$

$$\#(\text{off}) = (p-1)P - (p+1)p/2, \qquad \text{where } p = \min(P, d+1)$$

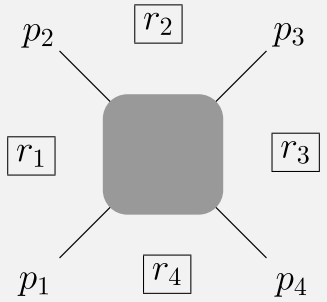
Simple exercise: check that these cross-ratios are DCI invariant.
Less simple exercise: Find their number for off-shell and on-shell cases.

Four-point amplitude A_4 [Caron-Huot and Coronado, 2022]

Simplest example: 4-point amplitude $A_{P=4} = A_4$

A_4 depends on $s = (p_1 + p_2)^2 = r_{24}^2$, $t = (p_2 + p_3)^2 = r_{31}^2$, and $m_i^2 = p_i^2 = r_{i,i-1}^2$. Due to DCI symmetry, these invariants appear only via two cross-ratios

$$u_1 = \frac{m_1^2 m_3^2}{st}, \quad u_2 = \frac{m_2^2 m_4^2}{st}$$



We have a nice all-loop conjecture for M_4:

The form of $\log M_4 = \log(A_4/A_4^{\text{tree}})$

$$\log M_4 = -\frac{1}{16}\Gamma_{oct}(g)\log^2(u_1 u_2) - \frac{g^2}{4}\log^2(u_1/u_2) - \frac{1}{2}D(g) + O(m^2)$$

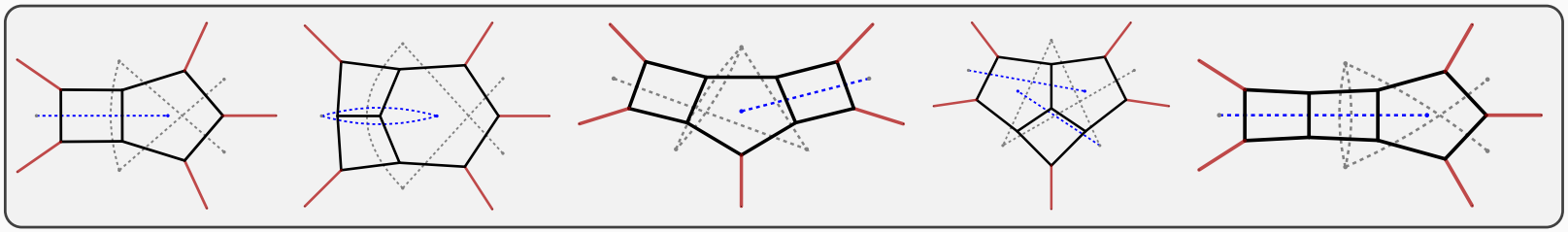
where

$$\Gamma_{oct}(g) = \frac{2}{\pi^2}\log\cosh(2\pi g) = 4g^2 - 16\zeta_2 g^4 + 256\zeta_4 g^6 + \dots$$

$$D(g) = \frac{1}{4}\log\frac{\sinh(4\pi g)}{4\pi g} = 4\zeta_2 g^2 - 32\zeta_4 g^4 + \frac{1024}{3}\zeta_6 g^6 + \dots$$

Five-point amplitude A_5

- Can we discover a similar all-loop conjecture for five-point amplitude A_5?
- We need to be able to calculate several first terms of perturbation theory. E.g., in two and three loops the following topologies contribute:



- Until the last year even the first diagram, slightly off-shell DCI pentabox integral was not calculated!

The form of $\log M_5 = \log(A_5/A_5^{\text{tree}})$

$$\log M_5 = -\frac{5}{4}\Gamma_{oct}(g)\log^2(m^2) + ???????$$

Method of regions

Method of regions

- Method of regions is a tool for calculating asymptotics of integrals as functions of parameters.
- In order to separate the contribution of each region it uses factor of the form $p(x)^\nu$, either existing or introduced, in the integrand.

Example: calculate asymptotics of complete elliptic integral

$$\mathrm{K}(1-a) = \int\limits_0^\infty \frac{dx}{\sqrt{(1+x^2)(a+x^2)}} \overset{a\to+0}{\longrightarrow} ?$$

1. Introduce factor $x^{2\epsilon}$ in the integrand.
2. Find regions: $x \sim a^0$, $x \sim a^{1/2}$.
3. For each region $x \sim a^q$ rescale $x \to a^q x$ and expand wrt a under $\int$.
4. Take the integrals, assuming $x^{2\epsilon}$ provides convergence.
5. Add up contributions, calculate limit $\epsilon \to 0$.

Method of regions (contd.)

In this oversimplified example we can write down a general term of expansion in each region. We have

$$C(x \sim a^0) = \sum_{n=0}^{\infty} \frac{a^n \Gamma\left(n - \epsilon + \frac{1}{2}\right) \Gamma\left(-n + \epsilon\right)}{2\Gamma\left(\frac{1}{2} - n\right) \Gamma(n+1)}$$

$$C(x \sim \sqrt{a}) = \sum_{n=0}^{\infty} \frac{a^{n+\epsilon} \Gamma\left(n + \epsilon + \frac{1}{2}\right) \Gamma\left(-n - \epsilon\right)}{2\Gamma\left(\frac{1}{2} - n\right) \Gamma(n+1)}$$

Note that highlighted parts turn into ∞ at $\epsilon = 0$.

Adding contributions and taking the limit $\epsilon \to 0$ term-wise, we obtain

Full asymptotic series

$$\mathrm{K}(1-a) \simeq \sum_n a^n \frac{\Gamma\left(n + \frac{1}{2}\right)^2}{\pi \Gamma(n+1)^2} \left[\psi(n+1) - \psi\left(n + \frac{1}{2}\right) - \frac{1}{2}\log a\right]$$

NB: in fact, this series has finite convergence radius ($= 1$).

Method of regions (contd.)

How can we find the regions for $a \to 0$ asymptotics of the integral

$$\int\limits_{\mathbb{R}^n_+} \prod_{i=1}^{n} dx_i \prod_k p_k(\boldsymbol{x}, a)^{\mu_k} ?$$

Newton polytope of the polynomial $p(x) = \prod_k p_k(x)$

If $p(\boldsymbol{x}) = \sum c_k \boldsymbol{x}^{\boldsymbol{n_k}}$ then $\mathrm{Newt}(p(\boldsymbol{x})) = \{\sum \alpha_k \boldsymbol{n_k} | \alpha_k \geqslant 0, \sum \alpha_k \leqslant 1\}$. In other words, $\mathrm{Newt}(p(\boldsymbol{x}))$ is a convex hull of exponents of its terms.

Lower faces of $\mathrm{Newt}(p(\boldsymbol{x}, a)) \Leftrightarrow$ regions

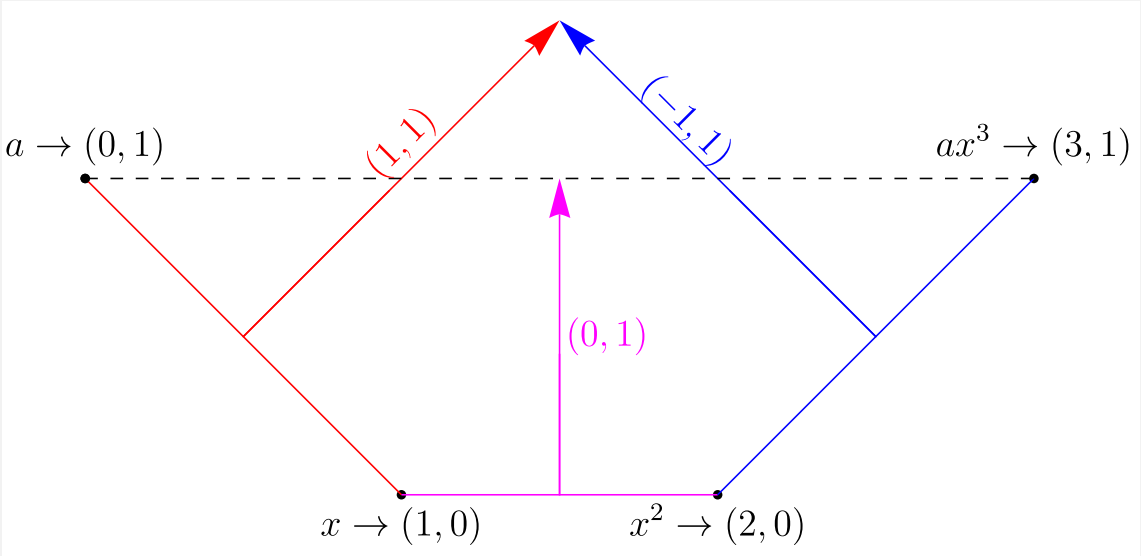
For the polynomial $p = \prod_k p_k$ with positive (or sufficiently generic) coefficients c_k each lower[a] face F of $\mathrm{Newt}(p(\boldsymbol{x}, a))$ corresponds to a separate region: let $\boldsymbol{N}_F = (\boldsymbol{q}, 1)$ be an inner normal vector to F, then the region is given by rescaling $\boldsymbol{x} \to a^{\boldsymbol{q}} \boldsymbol{x}$.

[a] down-up axis corresponds to the exponents of a.

Method of regions (contd.)

Example: calculate $a \to 0$ asymptotics of the integral

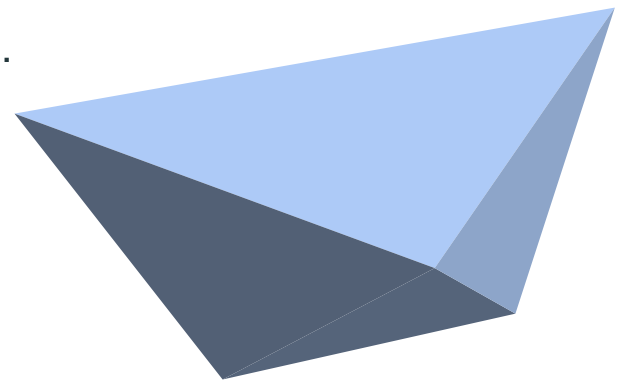
$$\int_0^\infty \frac{dx}{(a + x + x^2 + ax^3)^\nu} \overset{a \to +0}{\longrightarrow} ?$$



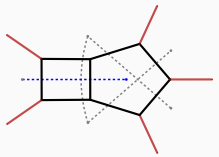
Region I: $x \to a^1 x$ Region II: $x \to a^0 x$ Region III: $x \to a^{-1} x$

Method of regions (contd.)

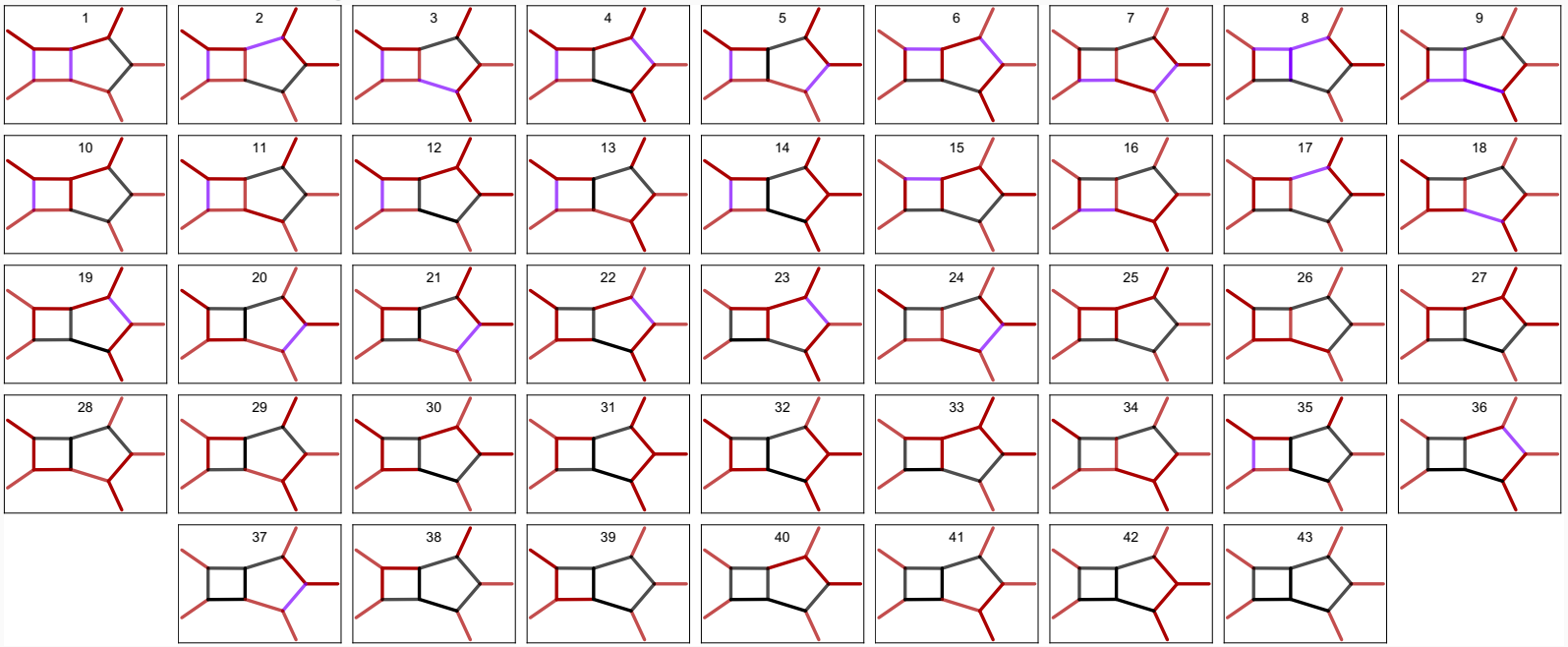
- In order to separate the regions, some kind of regularization is needed.
- In multiloop calculations the **dimensional regularization** is often sufficient.
- Sometimes, in order to separate the regions it is necessary to introduce also the analytic regularization.
- Finding the full set of regions can be automatized (with some reservations) with `asy` [Pak and Smirnov, 2011, Semenova et al., 2019], or `GetRegions` (`https://github.com/rnlg/Get-regions`).
- Both dimensional and analytic regularizations preserve the Lorentz invariance of the original scalar multiloop integral.



Method of regions for pentabox integral

In order to obtain M_5 at two loops, one had to calculate one missing master integral — slightly off-shell pentabox . The first idea was, of course, to try the MofR.

- There are 43 regions



where the color denotes scalings of Feynman parameters as m^0,m^{-2},m^{-4}.

M_5 at two loops

Method of regions in dimensional regularization

- "Only" 32 regions contribute in the leading order in m.
- The contributions of 9 regions was expressed via Γ-functions.
- The contributions of 10 regions was expressed via ${}_pF_q$-functions.
- The contributions of 7 regions was expressed via 2-fold MB.
- The contributions of 2 regions was expressed via 3-fold MB.
- The contributions of 2 regions was expressed via 4-fold FP.
- The contributions of 2 regions was expressed via 5-fold FP.

So, the calculation is very complicated.

Method of regions for pentabox integral (cont.)

In a week after I learned about the problem, the paper [arXiv:2508.14298] by Belitsky&Smirnov appeared where they followed the same approach and succeeded! But their result was quite complicated:

Belitsky&Smirnov result

$$PB = 3\log^4 m^2 - \left[\tfrac{5}{2}\log(s_{12}s_{23}s_{34}s_{51}) - 2\log(s_{45})\right]\log^3 m^2 + \boxed{1.2\text{Kb}}\log^2 m^2 + \boxed{32\text{Kb}}\log m^2 + \boxed{2.4\text{Mb}}$$

The coefficients were expressed via GPL up to weight 5
example: $G(1 - s_{51}/s_{34}, 1 - s_{51}/s_{34}, 1, 0, 1 - s_{51}/s_{34}|1)$.

This complicated form of the result is especially unpleasant given that the amplitude is expected to be expressed via logs only!

Method of regions for pentabox integral (cont.)

Maybe, there is a better way to calculate this integral?

Idea

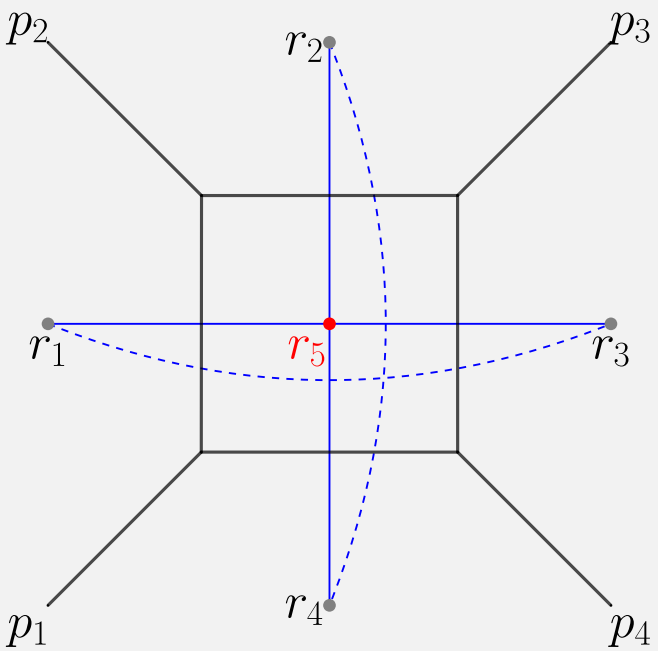
- Dimensional regularization breaks DCI, so the contribution of each region is not DCI. Only the sum of the contributions is DCI when we remove the regularization.
- Meanwhile, we always want the regularization to retain as much symmetry as possible. In particular, the dimensional regularization itself was introduced as conserving most of the theory symmetries.
- **Maybe, if we introduce the regularization which preserves DCI, the calculation will be easier?**

DCI regularization

DCI regularization

In perturbation theory the amplitudes of $\mathcal{N} = 4$ SYM are expressed via planar multiloop integrals which also inherit DCI.

Dual graphs



Dual graph (blue lines)

- draw a dual graph (vertices $\leftrightarrow$ faces, adjacent faces $\leftrightarrow$ edges).
- assign a dual variable to each vertex of dual graph, edge momentum = difference of adjacent dual variables.
- numerators are drawn as dashed lines.

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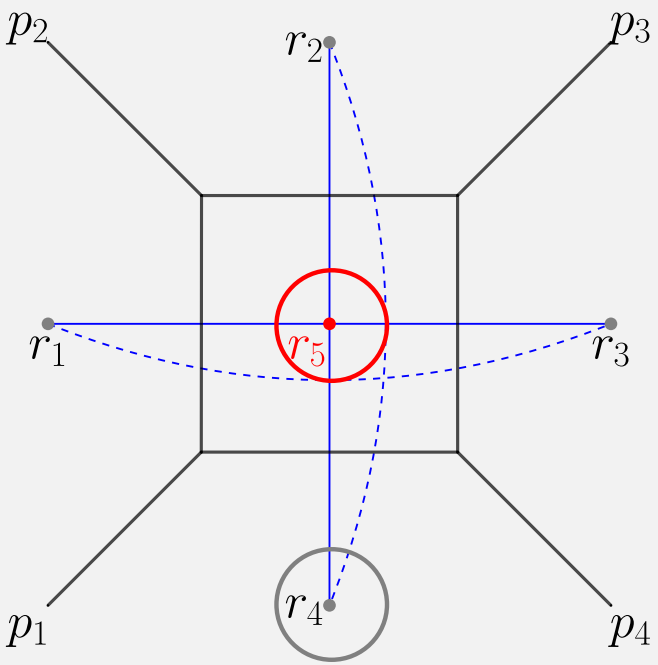

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- draw a dual graph (vertices $\leftrightarrow$ faces, adjacent faces $\leftrightarrow$ edges).
- assign a dual variable to each vertex of dual graph, edge momentum = difference of adjacent dual variables.
- numerators are drawn as dashed lines.

Sum of edge powers at vertex

- equals 0 at external vertex
- equals $d=4$ at internal vertex
- in dim.reg. $d=4-2\epsilon \implies$ analytic reg. is needed.

Pedagogical example: slightly off-shell one-loop box.

Generic regularization

$$B^{\text{reg}} = \int \frac{d^d r_5}{\pi^{d/2}} \frac{\left[r_{13}^2\right]^{1-\alpha_5} \left[r_{24}^2\right]^{1-\alpha_6} \left[r_{21}^2\right]^{-\alpha_7} \left[r_{32}^2\right]^{-\alpha_8}}{\left[r_{15}^2\right]^{1+\alpha_1} \left[r_{25}^2\right]^{1+\alpha_2} \left[r_{35}^2\right]^{1+\alpha_3} \left[r_{45}^2\right]^{1+\alpha_4}},$$

- Dimensional regularization: $\alpha = 0$
- DCI regularization:

$$\alpha_4 + \alpha_6 = 0,\ \alpha_1 + \alpha_5 + \alpha_7 = 0,\ \alpha_3 + \alpha_5 + \alpha_8 = 0,$$
$$\alpha_2 + \alpha_6 + \alpha_7 + \alpha_8 = 0,\ \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = -2\epsilon.$$

[1] `github.com/rnlg/Get-regions`

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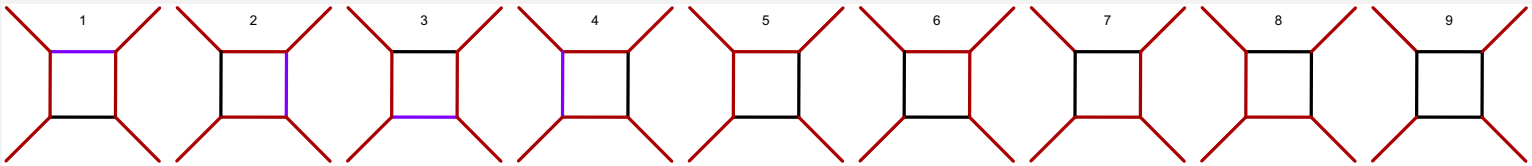
- Dimensional regularization: $\alpha = 0$
- DCI regularization:

$$\alpha_4 + \alpha_6 = 0,\ \alpha_1 + \alpha_5 + \alpha_7 = 0,\ \alpha_3 + \alpha_5 + \alpha_8 = 0,$$
$$\alpha_2 + \alpha_6 + \alpha_7 + \alpha_8 = 0,\ \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = -2\epsilon.$$

- With `GetRegions`[1] we find 9 regions:



[1] `github.com/rnlg/Get-regions`

Pedagogical example: slightly off-shell one-loop box.

#	dim reg.
1	$s^{\epsilon}\Gamma(1-\epsilon)\Gamma(\epsilon)^2\left(p_1^2p_2^2\right)^{-\epsilon}$
2	$t^{\epsilon}\Gamma(1-\epsilon)\Gamma(\epsilon)^2\left(p_2^2p_3^2\right)^{-\epsilon}$
3	$s^{\epsilon}\Gamma(1-\epsilon)\Gamma(\epsilon)^2\left(p_3^2p_4^2\right)^{-\epsilon}$
4	$t^{\epsilon}\Gamma(1-\epsilon)\Gamma(\epsilon)^2\left(p_4^2p_1^2\right)^{-\epsilon}$
5	$\Gamma(-\epsilon)^2\Gamma(\epsilon)\left(p_1^2\right)^{-\epsilon}/\Gamma(-2\epsilon)$
6	$\Gamma(-\epsilon)^2\Gamma(\epsilon)\left(p_2^2\right)^{-\epsilon}/\Gamma(-2\epsilon)$
7	$\Gamma(-\epsilon)^2\Gamma(\epsilon)\left(p_3^2\right)^{-\epsilon}/\Gamma(-2\epsilon)$
8	$\Gamma(-\epsilon)^2\Gamma(\epsilon)\left(p_4^2\right)^{-\epsilon}/\Gamma(-2\epsilon)$
9	B_9^{dim}

$$B_9^{\text{dim}} = \frac{st\pi\Gamma(-\epsilon)}{\sin(\pi\epsilon)\Gamma(-2\epsilon)}\left[2\pi\cot(\pi\epsilon)\left(\tfrac{s+t}{st}\right)^{\epsilon} - \tfrac{ts^{-1-\epsilon}}{1+\epsilon}\,{}_2F_1\left(1,1;2+\epsilon;-\tfrac{t}{s}\right)\right.$$
$$\left. - \tfrac{st^{-1-\epsilon}}{1+\epsilon}\,{}_2F_1\left(1,1;2+\epsilon;-\tfrac{s}{t}\right)\right]$$

Pedagogical example: slightly off-shell one-loop box.

#	dim reg.	dci.reg
1	$s^{\epsilon}\Gamma(1-\epsilon)\Gamma(\epsilon)^2\left(p_1^2p_2^2\right)^{-\epsilon}$	$\frac{\Gamma(1-\alpha_1-\epsilon,\alpha_1+\alpha_2+\epsilon,\alpha_1+\alpha_4+\epsilon)}{\Gamma(1+\alpha_1,1+\alpha_2,1+\alpha_4)}u_1^{\alpha_2+\alpha_3+\epsilon}$
2	$t^{\epsilon}\Gamma(1-\epsilon)\Gamma(\epsilon)^2\left(p_2^2p_3^2\right)^{-\epsilon}$	$\frac{\Gamma(1-\alpha_2-\epsilon,\alpha_1+\alpha_2+\epsilon,\alpha_2+\alpha_3+\epsilon)}{\Gamma(1+\alpha_1,1+\alpha_2,1+\alpha_3)}$
3	$s^{\epsilon}\Gamma(1-\epsilon)\Gamma(\epsilon)^2\left(p_3^2p_4^2\right)^{-\epsilon}$	$\frac{\Gamma(1-\alpha_3-\epsilon,\alpha_3+\alpha_4+\epsilon,\alpha_2+\alpha_3+\epsilon)}{\Gamma(1+\alpha_2,1+\alpha_3,1+\alpha_4)}u_2^{\alpha_1+\alpha_2+\epsilon}$
4	$t^{\epsilon}\Gamma(1-\epsilon)\Gamma(\epsilon)^2\left(p_4^2p_1^2\right)^{-\epsilon}$	$\frac{\Gamma(1-\alpha_4-\epsilon,\alpha_3+\alpha_4+\epsilon,\alpha_1+\alpha_4+\epsilon)}{\Gamma(1+\alpha_1,1+\alpha_3,1+\alpha_4)}$ $\times u_1^{\alpha_2+\alpha_3+\epsilon}u_2^{\alpha_1+\alpha_2+\epsilon}$
5	$\Gamma(-\epsilon)^2\Gamma(\epsilon)\left(p_1^2\right)^{-\epsilon}/\Gamma(-2\epsilon)$	0
6	$\Gamma(-\epsilon)^2\Gamma(\epsilon)\left(p_2^2\right)^{-\epsilon}/\Gamma(-2\epsilon)$	0
7	$\Gamma(-\epsilon)^2\Gamma(\epsilon)\left(p_3^2\right)^{-\epsilon}/\Gamma(-2\epsilon)$	0
8	$\Gamma(-\epsilon)^2\Gamma(\epsilon)\left(p_4^2\right)^{-\epsilon}/\Gamma(-2\epsilon)$	0
9	B_9^{dim}	0

$$B_9^{\text{dim}} = \frac{st\pi\Gamma(-\epsilon)}{\sin(\pi\epsilon)\Gamma(-2\epsilon)}\left[2\pi\cot(\pi\epsilon)\left(\tfrac{s+t}{st}\right)^{\epsilon} - \tfrac{ts^{-1-\epsilon}}{1+\epsilon}{}_2F_1\left(1,1;2+\epsilon;-\tfrac{t}{s}\right)\right.$$
$$\left. - \tfrac{st^{-1-\epsilon}}{1+\epsilon}{}_2F_1\left(1,1;2+\epsilon;-\tfrac{s}{t}\right)\right]$$

Pedagogical example: slightly off-shell one-loop box.

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4	$t^{\epsilon}\Gamma(1-\epsilon)\Gamma(\epsilon)^2\left(p_4^2p_1^2\right)^{-\epsilon}$	$\frac{\Gamma(1-\alpha_4-\epsilon,\alpha_3+\alpha_4+\epsilon,\alpha_1+\alpha_4+\epsilon)}{\Gamma(1+\alpha_1,1+\alpha_3,1+\alpha_4)}$ $\times u_1^{\alpha_2+\alpha_3+\epsilon}u_2^{\alpha_1+\alpha_2+\epsilon}$
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$$B_9^{\text{dim}}=\frac{st\pi\Gamma(-\epsilon)}{\sin(\pi\epsilon)\Gamma(-2\epsilon)}\left[2\pi\cot(\pi\epsilon)\left(\tfrac{s+t}{st}\right)^{\epsilon}-\tfrac{ts^{-1-\epsilon}}{1+\epsilon}{}_2F_1\left(1,1;2+\epsilon;-\tfrac{t}{s}\right)\right.$$
$$\left.-\tfrac{st^{-1-\epsilon}}{1+\epsilon}{}_2F_1\left(1,1;2+\epsilon;-\tfrac{s}{t}\right)\right]$$

All contributions in dci regularization are expressed in terms of Γ-functions, no ${}_2F_1$. Many contributions vanished.

2- and 3-loop DCI pentagons

Back to pentabox

The most general **DCI** regularization of pentabox integral reads

$$PB = PB(\epsilon, \alpha_1, \ldots, \alpha_6) = \int \frac{d^d r_6}{\pi^{d/2}} \frac{d^d r_7}{\pi^{d/2}}$$

$$\times \frac{r_{14}^{2\left(-\alpha_{2\bar{3}46}\right)} r_{25}^{2(1+\alpha_{45})} r_{31}^{2\left(\alpha_{12\bar{3}46}\right)} r_{42}^{2\left(1+\alpha_{2\bar{5}}\right)} r_{53}^{2\left(1-\alpha_{12\bar{3}4}\right)} r_{17}^{2}}{r_{56}^{2\left(1-\alpha_{12\bar{3}}\right)} r_{57}^{2(1+\alpha_5)} r_{61}^{2(1+\alpha_1)} r_{62}^{2(1+\alpha_2)} r_{72}^{2(1+\alpha_4)} r_{73}^{2(1+\alpha_6)} r_{74}^{2\left(1-\alpha_{\bar{3}456}\right)} r_{67}^{2(1-\alpha_3-2\epsilon)}}$$

Here $\boxed{\alpha_{ij\ldots k} = \alpha_i + \alpha_j + \ldots + \alpha_k}$ and $\boxed{\alpha_{\bar{i}} = -\alpha_i}$.

With this regularization we have

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Here $\boxed{\alpha_{ij\ldots k} = \alpha_i + \alpha_j + \ldots + \alpha_k}$ and $\boxed{\alpha_{\bar{i}} = -\alpha_i}$.

With this regularization we have

Success

- Zero regions expressed via n-fold FP.
- Zero regions expressed via n-fold MB.
- Zero regions expressed via ${}_pF_q$.
- **32 (all) regions either vanish or evaluate via Γ-functions!**

The calculation became almost elementary!.

Typical calculation of one region contribution

$$PB_{38}^{\text{dci}} = m^{2(\alpha_{\bar{3}45}-\epsilon)} \frac{\Gamma(3-\epsilon) s_1^{1+\alpha_{45}} s_2^{\alpha_{12\bar{3}46}} s_3^{1+\alpha_{2\bar{5}}} s_4^{1-\alpha_{12\bar{3}4}} s_5^{-\alpha_{2\bar{3}46}}}{\alpha_1!\,\alpha_2!\alpha_{\bar{1}\bar{2}3}!\alpha_4!\alpha_5!\alpha_{3\bar{4}\bar{5}\bar{6}}!,\alpha_6!(\alpha_{\bar{3}}-2\epsilon)!\Gamma\left(\alpha_{\bar{3}}-1-\epsilon\right)} \int\limits_{\mathbb{R}_+^8} dx_1 \dots dx_8$$

$$\times \frac{x_1^{\alpha_1} x_2^{\alpha_2} x_3^{-\alpha_{12\bar{3}}} x_4^{-\alpha_3-2\epsilon} x_5^{\alpha_4} x_6^{\alpha_5} x_7^{\alpha_6} x_8^{-\alpha_{\bar{3}456}} x_{1234}(s_2 x_7 + s_5 x_8)}{\left(x_{1234}(x_{78}+\hat{x}_{78}+s_3\hat{x}_{58}+s_4\hat{x}_{67}) + s_1\hat{x}_{23}x_{78} + s_2\hat{x}_{147} + s_3\hat{x}_{248} + s_4\hat{x}_{347} + s_5\hat{x}_{148}\right)^{3-\epsilon}}$$

Here $\boxed{x_{ij\dots k} = x_i + x_j + \dots + x_k}$ and $\boxed{\hat{x}_{ij\dots k} = x_i \cdot x_j \cdot \dots \cdot x_k}$.

Change of variables and integration

We make subsequent rescalings

$$x_k \to x_4 x_k \ (k \neq 4), \quad x_2 \to x_1 x_2, \quad x_8 \to x_6 x_8$$

and perform the integration in the order $x_4, x_1, x_3, x_5, x_6, x_7, x_2, x_8$ using one master formula

$$\int\limits_0^\infty dx \frac{x^{a-1}}{(A+Bx)^c} = \frac{\Gamma(a, c-a)}{\Gamma(c) A^{c-a} B^a}$$

Typical calculation of one region contribution

$$PB_{38}^{\text{dci}} = m^{2(\alpha_{\bar{3}45}-\epsilon)} \frac{\Gamma(3-\epsilon) s_1^{1+\alpha_{45}} s_2^{\alpha_{12\bar{3}46}} s_3^{1+\alpha_{2\bar{5}}} s_4^{1-\alpha_{12\bar{3}4}} s_5^{-\alpha_{2\bar{3}46}}}{\alpha_1!\,\alpha_2!\alpha_{\bar{1}\bar{2}3}!\alpha_4!\alpha_5!\alpha_{3\bar{4}\bar{5}\bar{6}}!,\alpha_6!(\alpha_{\bar{3}}-2\epsilon)!\Gamma\left(\alpha_{\bar{3}}-1-\epsilon\right)} \int\limits_{\mathbb{R}_+^8} dx_1 \dots dx_8$$

$$\times \frac{x_1^{\alpha_1} x_2^{\alpha_2} x_3^{-\alpha_{12\bar{3}}} x_4^{-\alpha_3-2\epsilon} x_5^{\alpha_4} x_6^{\alpha_5} x_7^{\alpha_6} x_8^{-\alpha_{\bar{3}456}} x_{1234}(s_2 x_7 + s_5 x_8)}{\left(x_{1234}(x_{78} + \hat{x}_{78} + s_3\hat{x}_{58} + s_4\hat{x}_{67}) + s_1\hat{x}_{23}x_{78} + s_2\hat{x}_{147} + s_3\hat{x}_{248} + s_4\hat{x}_{347} + s_5\hat{x}_{148}\right)^{3-\epsilon}}$$

Here $\boxed{x_{ij\dots k} = x_i + x_j + \dots + x_k}$ and $\boxed{\hat{x}_{ij\dots k} = x_i \cdot x_j \cdot \dots \cdot x_k}$.

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and perform the integration in the order $x_4, x_1, x_3, x_5, x_6, x_7, x_2, x_8$ using one master formula

$$\int\limits_0^\infty dx \frac{x^{a-1}}{(A+Bx)^c} = \frac{\Gamma(a, c-a)}{\Gamma(c) A^{c-a} B^a}$$

All regions except one (hard) were calculated in a similar way.

Calculation of the hard region,#43

$$PB_{43}^{\text{dci}} = m^{2(\alpha_{\bar{3}45}-\epsilon)} \frac{\Gamma(3-\epsilon) s_1^{1+\alpha_{45}} s_2^{\alpha_{12\bar{3}46}} s_3^{1+\alpha_{2\bar{5}}} s_4^{1-\alpha_{12\bar{3}4}} s_5^{-\alpha_{2\bar{3}46}}}{\alpha_1!\,\alpha_2!\alpha_{\bar{1}\bar{2}3}!\alpha_4!\alpha_5!\alpha_{3\bar{4}\bar{5}\bar{6}}!, \alpha_6!(\alpha_{\bar{3}}-2\epsilon)!\Gamma\left(\alpha_{\bar{3}}-1-\epsilon\right)}$$
$$\times \int_{\mathbb{R}_+^8} dx_1 \dots dx_8 (x_{1234}\left(s_2 x_7 + s_5 x_8 + 1\right) + s_1 \hat{x}_{23})/G_{43}^{3-\epsilon}$$

$$G_{43} = x_{1234}\left(s_1\hat{x}_{56} + s_3\hat{x}_{58} + s_4\hat{x}_{67} + x_{5678}\right) + s_1\left(\hat{x}_{246} + \hat{x}_{345} + \hat{x}_{23}x_{45678}\right)$$
$$+ s_2\hat{x}_{147} + s_5\hat{x}_{148} + s_3\hat{x}_{248} + s_4\hat{x}_{347} + x_4x_{123}$$

Problem

For this form we failed to find the appropriate variable change.

Solution (trick)

This contribution is

- Dual conformal invariant (thanks to DCI regularization)
- Does not depend on m.

Calculation of the hard region,#43

$$PB_{43}^{\text{dci}} = m^{2(\alpha_{\bar{3}45}-\epsilon)} \frac{\Gamma(3-\epsilon) s_1^{1+\alpha_{45}} s_2^{\alpha_{12\bar{3}46}} s_3^{1+\alpha_{2\bar{5}}} s_4^{1-\alpha_{12\bar{3}4}} s_5^{-\alpha_{2\bar{3}46}}}{\alpha_1!\,\alpha_2!\alpha_{\bar{1}\bar{2}3}!\alpha_4!\alpha_5!\alpha_{3\bar{4}\bar{5}\bar{6}}!, \alpha_6!(\alpha_{\bar{3}}-2\epsilon)!\Gamma\left(\alpha_{\bar{3}}-1-\epsilon\right)}$$
$$\times \int\limits_{\mathbb{R}_+^8} dx_1 \dots dx_8 (x_{1234}\left(s_2 x_7 + s_5 x_8 + 1\right) + s_1 \hat{x}_{23})/G_{43}^{3-\epsilon}$$

$$G_{43} = x_{1234}\left(s_1\hat{x}_{56} + s_3\hat{x}_{58} + s_4\hat{x}_{67} + x_{5678}\right) + s_1\left(\hat{x}_{246} + \hat{x}_{345} + \hat{x}_{23}x_{45678}\right)$$
$$+ s_2\hat{x}_{147} + s_5\hat{x}_{148} + s_3\hat{x}_{248} + s_4\hat{x}_{347} + x_4x_{123}$$

Problem

For this form we failed to find the appropriate variable change.

Solution (trick)

This contribution is constant!

Instead of searching for nontrivial variable change, we can use

$$PB_{43} = \lim_{s_4\to 0}\lim_{s_2\to 0}\lim_{s_5\to 0}\lim_{s_3\to 0}\lim_{s_1\to 0} PB_{43}$$

Calculation of the hard region,#43 (contd)

Taking the limits, we obtain a much simpler expression

$$PB_{43}^{\text{dci}} = \frac{\Gamma(3-\epsilon)}{\alpha_1!\,\alpha_2!\alpha_{\bar{1}\bar{2}3}!\alpha_4!\alpha_5!\alpha_{3\bar{4}\bar{5}\bar{6}}!,\alpha_6!(\alpha_{\bar{3}}-2\epsilon)!\Gamma\left(\alpha_{\bar{3}}-1-\epsilon\right)} \int\limits_{\mathbb{R}_+^8} dx_1\ldots dx_8$$

$$\times \frac{x_{24}x_{78}x_1^{\alpha_1}x_2^{\alpha_2}x_3^{-\alpha_{12\bar{3}}}x_5^{\alpha_4}x_6^{\alpha_5}x_7^{\alpha_6}x_8^{-\alpha_{\bar{3}456}}x_4^{-\alpha_3-2\epsilon}}{\Big(x_8\left(x_4x_{125}+\hat{x}_{25}\right)+x_7\left(x_4x_{136}+x_6x_{24}\right)+\left(x_{36}+1\right)\left(\hat{x}_{24}+\hat{x}_{25}+\hat{x}_{45}\right)\Big)^{3-\epsilon}}$$

Change of variables and integration

We make subsequent rescalings

$$x_k \to x_4x_k\ (k\neq 4), \quad x_{5,7} \to \tfrac{x_2}{1+x_2}x_{5,7}$$

and perform the integration in the order $x_4, x_1, x_3, x_6, x_8, x_2, x_7, x_5$ using the same master formula $\int_0^\infty dx \frac{x^{a-1}}{(A+Bx)^c} = \frac{\Gamma(a,c-a)}{\Gamma(c)A^{c-a}B^a}$.

Result

$$PB_{43}^{\text{dci}} = \frac{\alpha_{\bar{1}\bar{2}3\bar{4}}!\alpha_{2\bar{5}}!(\epsilon+\alpha_3)!\Gamma\left(-\alpha_{2\bar{3}46},\alpha_{12\bar{3}46},-\alpha_{12}-\epsilon,\alpha_{2\bar{3}}-\epsilon,\alpha_{\bar{3}45}-\epsilon\right)}{\alpha_2!\alpha_{\bar{1}\bar{2}3}!\alpha_{3\bar{4}\bar{5}\bar{6}}!\alpha_6!(-2\epsilon-\alpha_3)!(\alpha_{2\bar{3}4}-\epsilon)!(\alpha_{\bar{1}\bar{2}5}-\epsilon)!\Gamma\left(\alpha_1\right)}$$

Result

Adding up and calculating the limit $\epsilon, \alpha_i \to 0$, we obtain

Result for DCI pentabox integral

$$PB = \tfrac{1}{2} L_1 \left(L_3 L_2^2 + 2 L_3 L_4 L_2 + L_4 L_5^2 + 2 L_3 L_4 L_5\right) + \tfrac{1}{2}\big(4 L_1 L_2 + 4 L_1 L_5 + 4 L_3 L_4 + 2 L_1 L_3 + 2 L_1 L_4 - 2 L_4 L_2 - 2 L_3 L_5 - L_2^2 - L_5^2\big)\zeta_2 + (L_3 + L_4 - 2 L_1)\,\zeta_3 + 5\zeta_4,$$

where $L_i = \log v_i = \log \frac{m^2 s_i}{s_{i-2} s_{i+2}}$

This is to be compared with the result obtained in dimensional regularization:

Belitsky&Smirnov result

$$PB = 3 \log^4 m^2 - \left[\tfrac{5}{2} \log(s_{12} s_{23} s_{34} s_{51}) - 2 \log(s_{45})\right] \log^3 m^2 + \boxed{1.2\text{Kb}} \log^2 m^2 + \boxed{32\text{Kb}} \log m^2 + \boxed{2.4\text{Mb}}$$

The coefficients were expressed via GPL up to weight 5

example: $G(1 - s_{51}/s_{34}, 1 - s_{51}/s_{34}, 1, 0, 1 - s_{51}/s_{34}|1)$.

All-loop ansatz for $\log M_5$

Amplitude up to two loops

$$\log M_5 = -\left(g^2 - 4\zeta_2 g^4\right) \log v_1 \log v_2 - g^4 \zeta_2 \log \frac{v_1}{v_2} \log \frac{v_2}{v_3} - \zeta_2 g^2 + \frac{27}{4}\zeta_4 g^4 + O(g^6) + \text{cyclic permutations.}$$

All-loop conjecture

$$\log M_5 = -\tfrac{1}{4}\Gamma_{\text{oct}} \log v_1 \log v_2 - \tfrac{1}{8}\left(\Gamma_{\text{cusp}} - \Gamma_{\text{oct}}\right) \log \frac{v_1}{v_2} \log \frac{v_2}{v_3} + \text{cyclic permutations} + D_5(g),$$

where $D_5(g) = -5\zeta_2 g^2 + \frac{135}{4}\zeta_4 g^4 + \ldots$

All-loop ansatz for $\log M_5$

Amplitude up to two loops

$$\log M_5 = -\left(g^2 - 4\zeta_2 g^4\right) \log v_1 \log v_2 - g^4 \zeta_2 \log \frac{v_1}{v_2} \log \frac{v_2}{v_3}$$
$$- \zeta_2 g^2 + \frac{27}{4}\zeta_4 g^4 + O(g^6) + \text{cyclic permutations.}$$

All-loop conjecture

$$\log M_5 = -\tfrac{1}{4}\Gamma_{\text{oct}} \log v_1 \log v_2 - \tfrac{1}{8}\left(\Gamma_{\text{cusp}} - \Gamma_{\text{oct}}\right) \log \frac{v_1}{v_2} \log \frac{v_2}{v_3}$$
$$+ \text{cyclic permutations} + D_5(g),$$

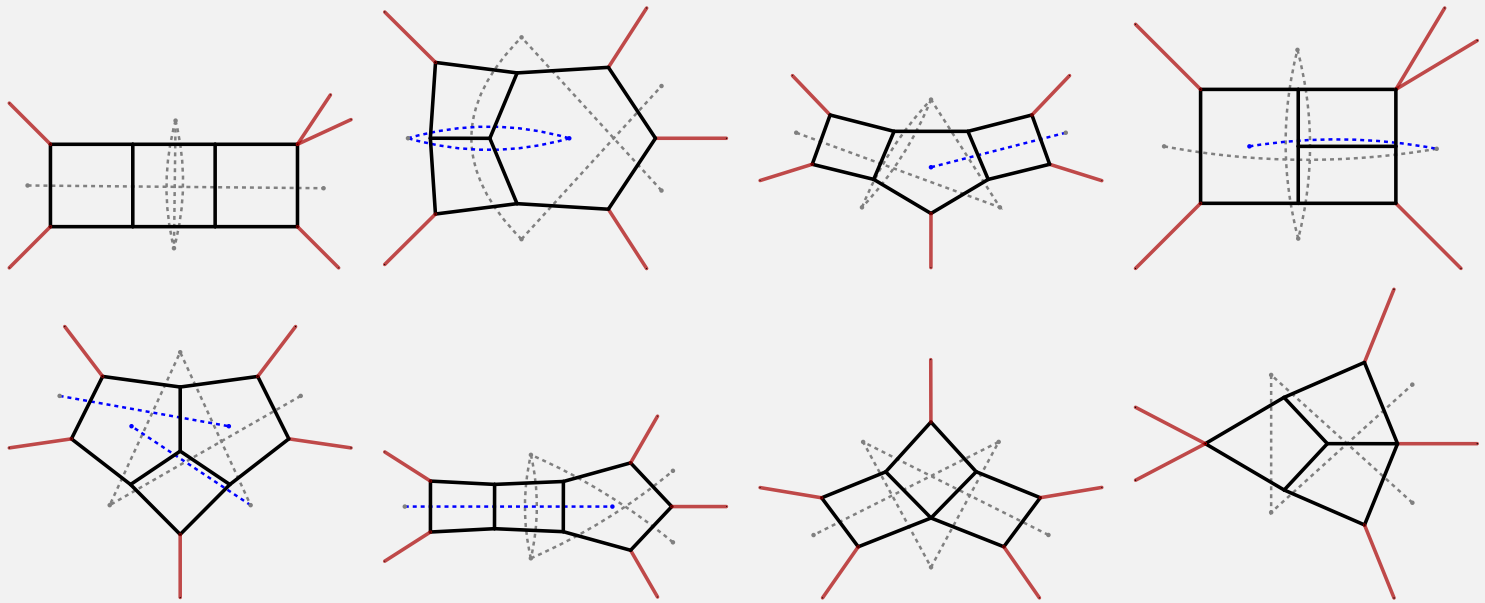
where $D_5(g) = -5\zeta_2 g^2 + \frac{135}{4}\zeta_4 g^4 + \ldots$

It is clear that our all-loop conjecture was not sufficiently supported[2]. And indeed, our recent three-loop calculations have shown that it is not correct.

[2] I made a bet with my coauthor that our conjecture is not correct in the finite terms.

Recent development: Three loops

How about 3 loops?



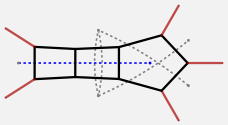
- [2605.17057]— calculation of masters
- [2605.17056]— construction of the amplitude

Calculation of master integrals

We used the same approach, but it was not sufficient. In addition to DCI regularization more tricks were needed:

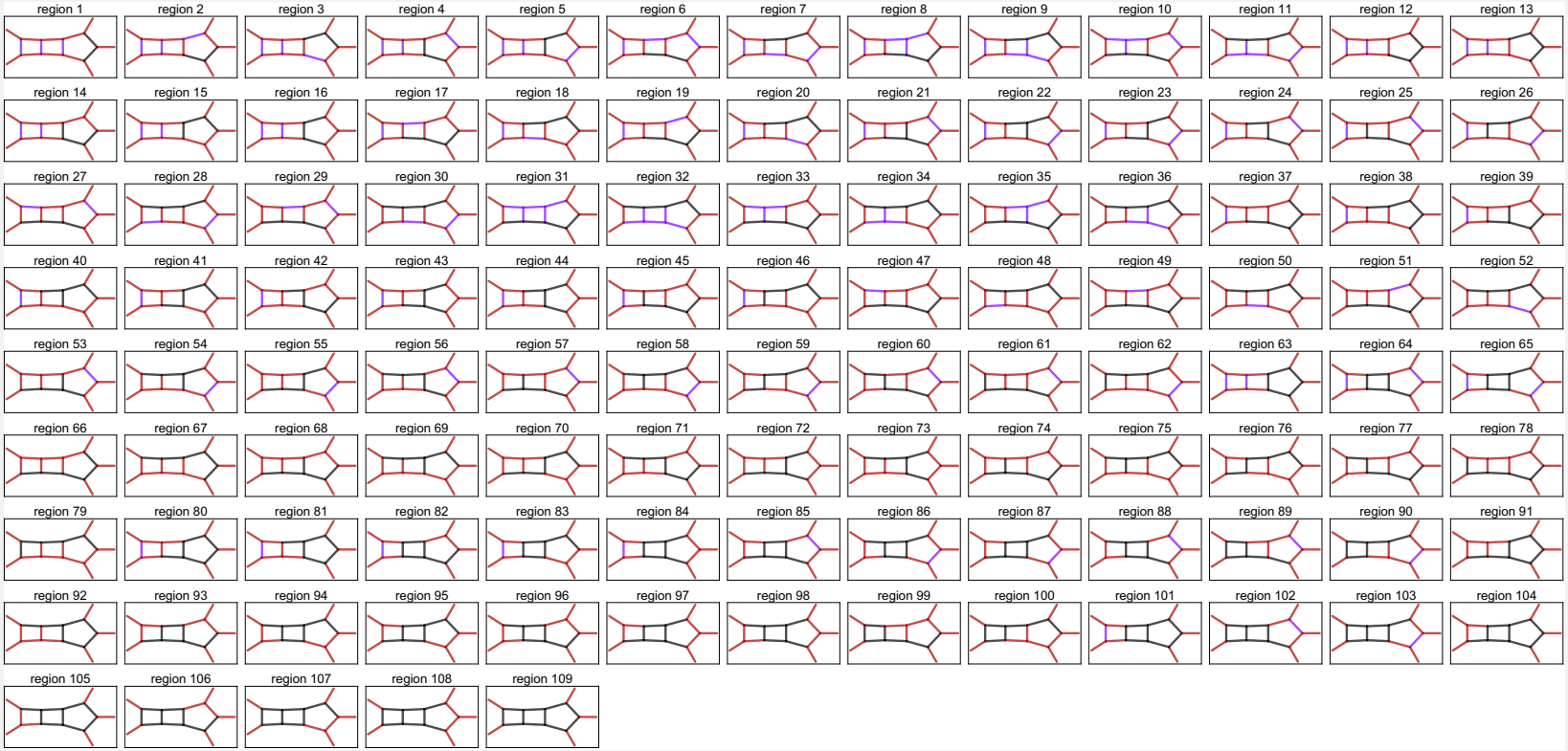
Additional techniques

- Pulling out dependence on DCI cross ratios.
- Successive asymptotics to simplify the integrands.
-  Magic identities.
-  Detecting finite integrals.
-  IBP reduction in parametric representation.
-  Using direct integration with `HyperInt`.

Example: $\mathcal{I}_6^{(3)} =$ 

Regions

There are 149 regions (color encodes scaling m^0,m^{-2},m^{-4})



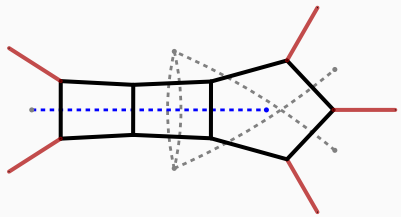
Problem

It was not possible to find appropriate variable change for almost all regions.

$\Longrightarrow$ **We need further simplifications.**

Pulling out dependence on DCI cross ratios

We would like to apply an approach, based on successive limits, similar to what we used for the hard region in pentabox. But first we need to pull out the dependence on the cross ratios (for the hard region there was no dependence).



Idea: use different masses to tag cross ratios

We use different masses, $m_i^2 = r_{i-2,i+2}^2$, and pull out the power of each mass from the integral for a contribution of each region. Then $(m_i^2)^\nu$ implies the factor $v_i^\nu = \left(\frac{m_i^2 s_i}{s_{i-2} s_{i+2}}\right)^\nu$. Thus, for each contribution we know its scaling with all five cross ratios, $(\mathcal{I}_3)_{\text{region } k} \propto \prod_l v_l^{\nu_{lk}}$. Then we use again

$$\frac{\left(\mathcal{I}_3^{(3)}\right)_{\text{region } k}}{\prod_l v_l^{\nu_{lk}}} = \lim_{s_4\to 0} \lim_{s_2\to 0} \lim_{s_5\to 0} \lim_{s_3\to 0} \lim_{s_1\to 0} \frac{\left(\mathcal{I}_3^{(3)}\right)_{\text{region } k}}{\prod_l v_l^{\nu_{lk}}}$$

Important note:

Already after this step we know that the dependence on the cross-rations is exclusively via logarithms (from expansion of $\prod_l v_l^{\nu_{lk}}$)! No nontrivial polylogs.

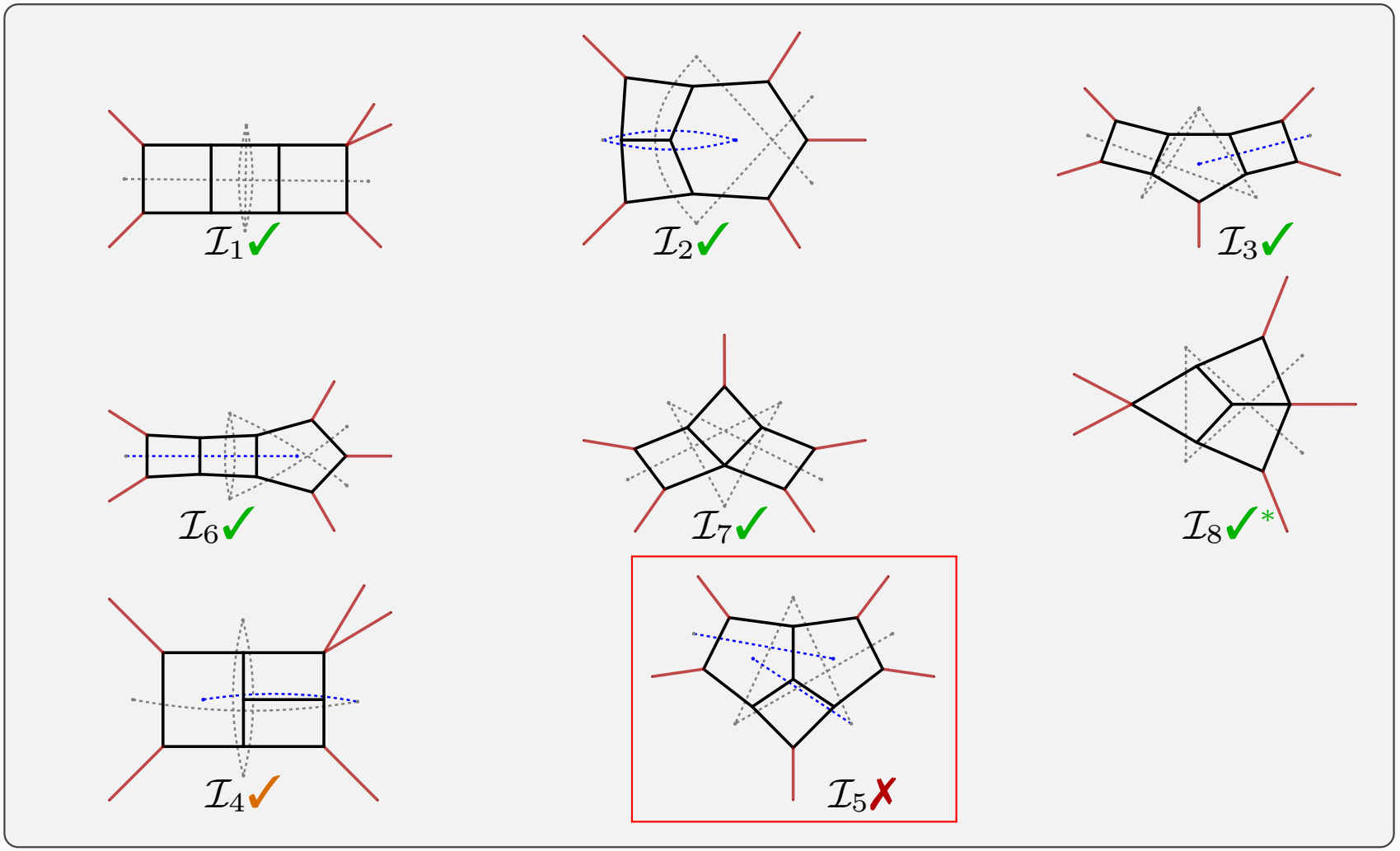
Example: result for $\mathcal{I}_2^{(3)} = \mathcal{I}_6^{(3)}$

$$\mathcal{I}_2^{(3)} = \mathcal{I}_6^{(3)} = \frac{1}{12} L_1 \left(L_3^2 L_2^3 + 3L_3^2 L_4 L_2^2 + 3L_3^2 L_4^2 L_2 + L_4^2 L_5^3 + 3L_3 L_4^2 L_5^2 + 3L_3^2 L_4^2 L_5\right)$$
$$+ \frac{1}{6}(L_1 L_2^3 - L_3 L_2^3 + 6L_1 L_3 L_2^2 + 3L_1 L_4 L_2^2 - 3L_3 L_4 L_2^2 + 3L_1 L_3^2 L_2 + 3L_1 L_4^2 L_2 - 3L_3 L_4^2 L_2$$
$$+12L_1 L_3 L_4 L_2 + L_1 L_5^3 - L_4 L_5^3 + 3L_3^2 L_4^2 + 3L_1 L_3 L_4^2 + 3L_1 L_3 L_5^2 + 6L_1 L_4 L_5^2 - 3L_3 L_4 L_5^2 + 3L_1 L_3^2 L_4$$
$$+ 3L_1 L_3^2 L_5 + 3L_1 L_4^2 L_5 - 3L_3^2 L_4 L_5 + 12L_1 L_3 L_4 L_5)\zeta_2 - \frac{1}{6}(L_2^3 + 3L_4 L_2^2 + 3L_4^2 L_2 + L_5^3 + 3L_1 L_3^2$$
$$+ 3L_1 L_4^2 - 3L_3 L_4^2 + 3L_3 L_5^2 - 3L_3^2 L_4 + 12L_1 L_3 L_4 + 3L_3^2 L_5)\zeta_3 + \frac{1}{4}(-10L_2^2 + 42L_1 L_2 - 10L_3 L_2$$
$$- 20L_4 L_2 + 7L_3^2 + 7L_4^2 - 10L_5^2 + 35L_1 L_3 + 35L_1 L_4 + 20L_3 L_4 + 42L_1 L_5 - 20L_3 L_5 - 10L_4 L_5)\zeta_4$$
$$- (4L_1 + L_2 - 5L_3 - 5L_4 + L_5)\zeta_2 \zeta_3 - 4L_1 \zeta_5 + \frac{1}{8}\left(16\zeta_3^2 + 77\zeta_6\right)$$

- Uniform transcendental function.
- Polynomial in L_i of 6th degree.

Successes and a failure

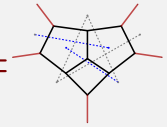
We have successfully applied this approach to almost all integrals:



✓*— some regions evaluate to ${}_pF_{p-1}(\ldots|1)$.

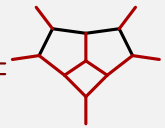
✓ — magic relation: $\mathcal{I}_4 = \mathcal{I}_1$

✗ — $\mathcal{I}_5$ real problem.

"Difficult" integral: $\mathcal{I}_5^{(3)} =$ 



Most of the contributions were expressed via Γ and/or ${}_pF_{p-1}(1)$. **Three** regions out of 163 appeared to be expecially difficult: the integrals seemingly could not be taken in closed form (at least, required 3-fold MB).

Troublesome region $[\mathcal{I}_5^{(3)}]_{99} =$ 

After integration in some of the Feynman parameters, we arrive at 6-fold integral

$$\int\limits_{\mathbb{R}_+^5} dx_1 \dots dx_5 \; x_1^{\alpha_4} x_2^{\alpha_{\bar{9}}-\epsilon} x_3^{\alpha_{\bar{1}\bar{4}\bar{3}}-2\epsilon+1} x_4^{\alpha_{\bar{7}\bar{5}\bar{4}\bar{2}}-2\epsilon} x_5^{\alpha_{79\bar{6}\bar{4}\bar{3}\bar{1}}-\epsilon} x_6^{\alpha_{1\bar{9}\bar{7}}-\epsilon}$$

$$\times \left[x_6+1\right]^{\alpha_{\bar{1}}-2} \cdot \left[x_1+1\right]^{\alpha_{\bar{6}257}+\epsilon-1} \cdot \left[x_5\left(x_3+x_4+1\right)+x_3\right]^{\alpha_{\bar{9}134}+\epsilon-1}$$

$$\times \left[x_3x_6+x_6+1\right]^{\alpha_{\bar{2}}-2} \cdot \left[x_2x_4+x_2+x_4\right]^{\alpha_{9\bar{6}\bar{3}\bar{1}257}+\epsilon-1}$$

$$\times \left[x_1\left(\left(x_2+1\right)x_3\left(x_5+1\right)+x_5\right)+x_2x_3x_5\right]^{\alpha_{\bar{7}136}+\epsilon-1} \cdot P(x)$$

where

$$P(x) = x_1\alpha_2x_3x_6^2 + x_1\alpha_2x_3x_6 + \boxed{120 \text{ terms}} + 2x_6 - \alpha_7 - \epsilon$$

is some big polynomial.

Using HyperInt?

Take $\alpha_i = a_i\epsilon$ with "sufficiently generic" a_i and use HyperInt [Panzer, 2014]. But we first have to express the contribution via locally finite integrals.

Definition

Locally finite integrals are those for which, the ϵ-expansion commutes with the integration. In particular, the integral should be convergent at $\epsilon = 0$.

IBP reduction to locally finite Euler-type integrals

Euler-type integrals

Partially integrated contributions have a form of

$$\int\limits_{\mathbb{R}_+^N} dx_1 \dots dx_N \; x_1^{\nu_1 - 1} \dots x_N^{\nu_N - 1} G_1(x)^{\mu_1} \dots G_M(x)^{\mu_M} P(x),$$

where G_m and P are some polynomials.

NB: we can assume that $\boxed{P(x) = x_1^{n_1} \dots x_N^{n_N}}$ without generality loss.

Unfortunately, the integrals which determine the contributions of the three problematic regions are not locally finite.

Idea: IBP reduction to locally finite integrals

- Set up IBP reduction in parametric space
- Pick sufficiently many locally finite integrals as master-integrals
- Express the contribution via those locally finite masters
- Calculate masters using direct integration with Panzer's `HyperInt`.

Convergent Euler-type integrals

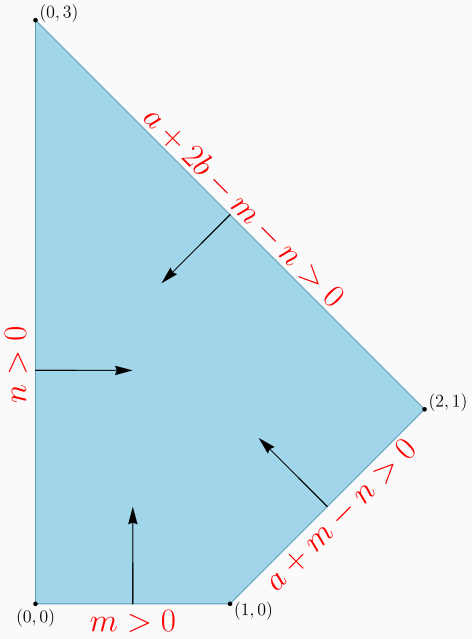
How to determine if the integral is convergent?

- Calculate the Newton polytope $\mathrm{Newt}(G)$ of $G = \prod_k G_k$ (=Minkowski sum of $\mathrm{Newt}(G_k)$)
- For inner normal $\mu = (\mu_1, \ldots, \mu_N)$ rescale $x_n \to x_n \lambda^{\mu_n}$ and find leading asymptotic λ^α of the integrand as $\lambda \to 0$.
- If for each inner normal we have $\alpha > 0$, the integral is convergent. Otherwise, it is divergent.

Example

For which n, m, a and b the integral converges?

$$\int_0^\infty \frac{dx\, dy x^{n-1} y^{m-1}}{(1+x+y)^a (1+xy+x^2)^b}$$



Each face gives rise to one linear inequality:

$$m > 0 \;\&\; n > 0 \;\&\; a+m-n > 0 \;\&\; a+2b-m-n > 0$$

IBP reduction of Euler-type integrals

We consider the infinite family of integrals

$$j(n) = j(n_1, \ldots, n_N) = \int\limits_{\mathbb{R}^N} \frac{dx x^{\nu+n-1}}{\Gamma(\nu+n)} G(x)^{\mu}$$

$$= \int\limits_{\mathbb{R}^N} \frac{dx_1 x_1^{\nu_1+n_1-1}}{\Gamma(\nu_1+n_1)} \ldots \frac{dx_N x_N^{\nu_N+n_N-1}}{\Gamma(\nu_N+n_N)} G_1(x)^{\mu_1} \ldots G_M(x)^{\mu_M}$$

We would like to derive recurrence relations which do not shift μ, therefore, we have to construct the differential operator $Q_i(x)\partial/\partial x_i$ (Q_i are polynomials) with the property:

$$Q_i \partial_i G_l = Q^{(l)} G_l$$

which has a simple geometrical meaning: Q_i is a vector field tangent to $G_l = 0$ surface. **Vector fields themselves form a module, which can be found by standard tools: syzygies and module intersection!**

IBP identities in terms of A and B operators

$$(Q_i(A)B_i - \textstyle\sum_l \mu_i Q^{(l)}(A))j = 0$$

Result for $\mathcal{I}_5^{(3)}$

We have successfully reduced all three troublesome regions to locally finite integrals and applied `HyperInt`.

$$\begin{aligned}\mathcal{I}_5^{(3)} = {} & \frac{1}{12}(L_5^2L_1^3+L_2L_5L_1^3+3L_4L_5^2L_1^2+3L_2L_4L_5L_1^2+6L_3L_4L_5^2L_1+3L_2L_3^2L_4L_1+L_2L_3^3L_4 \\ & +6L_2L_3L_4L_5L_1)L_2+\frac{1}{6}(-L_2L_3^3+L_4L_3^3-3L_1L_2L_3^2+3L_1L_4L_3^2+6L_2L_4L_3^2+6L_1L_2^2L_3-3L_1L_5^2L_3 \\ & -3L_2L_5^2L_3+3L_4L_5^2L_3+3L_2^2L_4L_3+6L_1L_2L_4L_3-3L_2^2L_5L_3+12L_2L_4L_5L_3+3L_1^2L_2^2+3L_1L_2L_5^2 \\ & +6L_1L_4L_5^2+3L_1L_2^2L_4+3L_1L_2^2L_5+6L_1^2L_2L_5+12L_1L_2L_4L_5)\zeta_2+\frac{1}{6}(6L_5^2L_3-L_3^3-3L_1L_3^2+3L_2^2L_3 \\ & +6L_1L_2L_3-6L_1L_4L_3-12L_2L_4L_3+6L_1L_5L_3-6L_4L_5L_3-6L_1L_2^2+3L_1L_5^2+3L_2L_5^2-3L_4L_5^2 \\ & +6L_2^2L_4+6L_1L_2L_4+3L_2^2L_5)\zeta_3+\frac{1}{4}(3L_2^2+41L_1L_2+20L_3L_2+L_4L_2+28L_5L_2-10L_3^2-4L_5^2 \\ & +12L_1L_3+9L_1L_4+9L_3L_4+9L_1L_5-22L_3L_5+31L_4L_5)\zeta_4+(7L_3-4L_1-7L_2+2L_4-4L_5)\zeta_2\zeta_3 \\ & -2(3L_1+L_2+2L_3-2L_5)\zeta_5+c_1\zeta_6+c_2\zeta_3^2+O(v_i)\end{aligned}$$

We failed to find rational numbers $c_{1,2}$ in constant term $c_1\zeta_6+c_2\zeta_3^2$.

Ansatz for amplitude

Substituting the found masters into the amplitude, we obtain

$$\log M_5 = \sum_i \left[\Gamma_0(g) L_i^2 + \Gamma_1(g) L_i L_{i+1} + \Gamma_2(g) L_i L_{i+2}\right] + D_5(g)$$
$$= \sum_i \left[-\tfrac{\Gamma_{\text{oct}}(g)}{4} L_i L_{i+1} + \tfrac{1}{2}\Gamma_0(g)(L_i - L_{i+1})^2 + \Gamma_2(g)(L_i L_{i+2} - L_i L_{i+1})\right]$$

where

$$\Gamma_0(g) = \zeta_2 g^4 - 16\zeta_4 g^6 + O(g^8) \qquad \Gamma_1(g) = -g^2 + 2\zeta_2 g^4 - \tfrac{59}{2}\zeta_4 g^6 + O(g^8)$$
$$\Gamma_2(g) = \zeta_2 g^4 - \tfrac{37}{2}\zeta_4 g^6 + O(g^8) \qquad D_5(g) = -5\zeta_2 g^2 + \tfrac{135}{4}\zeta_4 g^4 + O(g^6)$$

NB: $\Gamma_0(g) + \Gamma_1(g) + \Gamma_2(g) = -\frac{1}{4}\Gamma_{\text{oct}}(g)$ including new $\propto g^6$ terms.

Conclusion and Outlook

Non-DCI integrals

Realistic calculations in QCD: non-DCI integrals. Can the DCI-inspired regularization help? **Probably, yes.**

Non-DCI pentagon (without numerators)

$$
\begin{aligned}
PB_{\text{non-DCI}} &= \frac{1}{s_1^2 s_2 s_5}\left[\frac{1}{4}L_3^2 L_4^2 + \frac{1}{2}(L_3+L_4)^2\zeta_2 + (L_3+L_4)\zeta_3 + \frac{21}{4}\zeta_4\right] \\
&+ \frac{1}{s_1 s_3 s_4 s_5}\left[L_1 L_2 L_3\left(L_4+\frac{L_2}{2}\right) + \left(2L_1L_2 + L_4(L_3-L_2+L_1) - \frac{L_2^2}{2}\right)\zeta_2 + (L_3-L_1)\zeta_3 + \frac{5\zeta_4}{2}\right] \\
&+ \frac{1}{s_1 s_2 s_3 s_4}\left[L_1 L_4 L_5\left(L_3+\frac{L_5}{2}\right) + \left(2L_1L_5 + L_3(L_4-L_5+L_1) - \frac{L_5^2}{2}\right)\zeta_2 + (L_4-L_1)\zeta_3 + \frac{\zeta_4}{2}\right] \\
&+ \frac{1}{s_1^2 s_4 s_5}\left[\frac{1}{4}L_2(2L_4+L_2)L_3^2 + \left(2L_3L_2 + L_4L_2 + L_3L_4 + \frac{1}{2}L_2^2\right)\zeta_2 - (L_3+L_4)\zeta_3 + \frac{21}{4}\zeta_4\right] \\
&+ \frac{1}{s_1^2 s_2 s_3}\left[\frac{1}{4}L_5(2L_3+L_5)L_4^2 + \left(2L_4L_5 + L_3L_5 + L_3L_4 + \frac{1}{2}L_5^2\right)\zeta_2 - (L_3+L_4)\zeta_3 + \frac{21}{4}\zeta_4\right],
\end{aligned}
$$

Conclusion and Outlook

- Dimensional regularization obscures DCI at intermediate steps.
- The DCI-preserving regularization drastically simplifies calculations.
- Several nontrivial examples have already been considered, including two-loop and three-loop pentagon integrals.
- The exponentiation of five-point slightly off-shell amplitude in $\mathcal{N}=4$ SYM is checked up to 3 loops. Leading singularities are governed by $\Gamma_{\text{oct}}(g)$ as conjectured.
- A new ansatz for $\log M_5$ has been suggested. It seems to be capable of surviving further checks at higher loops.
- Future work: consider more applications to non-DCI integrals, consider the 6-point two-loop integrals.

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Thank you!

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