Frontier of Multi-loop and Multi-leg Feynman Integrals and the dawn of AI era

Recent Progress in Amplitudes
ShanghaiTech University
Nov 9th, 2026

Yang Zhang
University of Science and Technology of China

Outline

from the talk on Amplitudes 2026 and the developments afterwards

Introduction

Analytic Feynman integral and the methodology

Analytic Feynman integral frontier

- Frontier 1: 2loop 6point Feynman integrals
- Frontier 2: 3loop 5point Feynman integrals

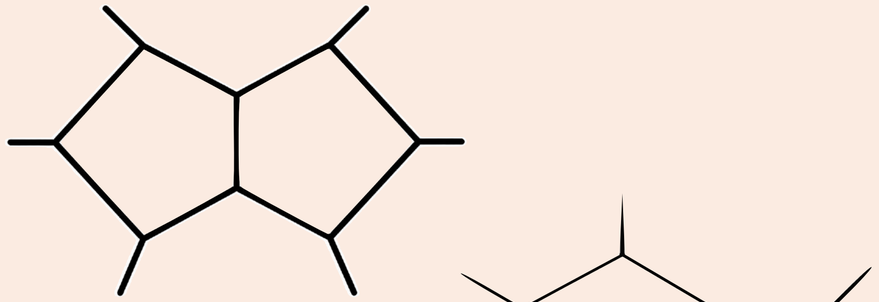
First Applications (a brief overview)

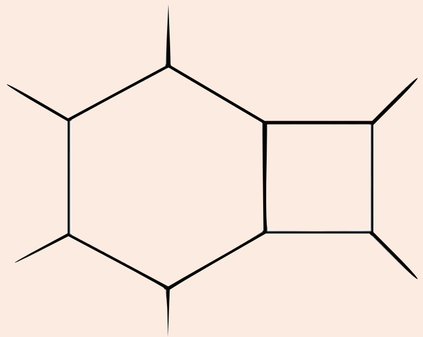
AI Agent for Multi-loop integral computations

Summary and Outlook

Based on

Feynman integral Integral evaluation





✓ analytic computation of all 2loop 6point planar massless integrals

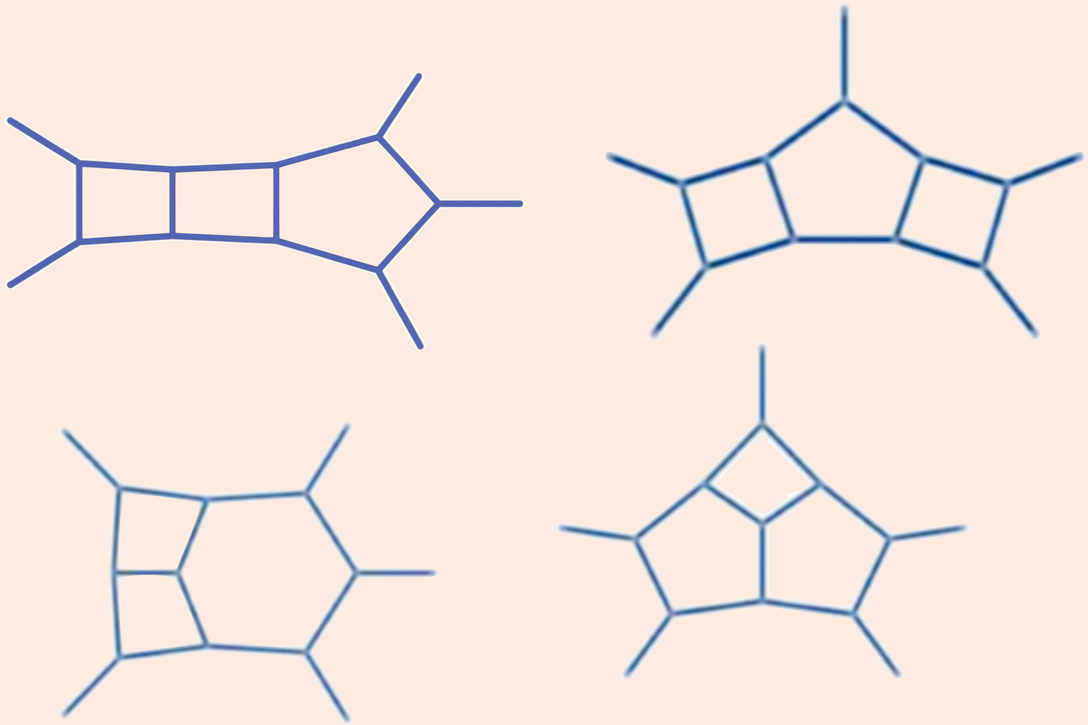
Henn, Matijašić, Miczajka, Peraro, Xu, YZ, Phys. Rev. Lett. 135 (2025) 3, 031601

Liu, Matijasic, Peraro, Xu, Yang, YZ, JHEP 07 (2026) 223

Henn, Matijašić, Miczajka, Peraro, Xu, YZ, JHEP 08(2024) 027

Based on

Feynman integral Integral evaluation



✓ analytic computation of all 3loop 5point planar massless integrals

Liu, Matijasic, Miczajka, Xu, Xu, YZ, Phys.Rev.D 112 (2025) 1, 016021 (Editors' Suggestion)

Chicherin, Wu, Wu, Xu, Zhang, YZ, Phys. Rev. Lett. 137, 111603 (2026)

Based on package development

NeatIBP 1.0, a package generating small-size integration-by-parts relations for Feynman integrals
Wu, Boehm, Ma, Xu, YZ, Comput. Phys. Commun. 295 (2024), 108999

Performing integration-by-parts reductions using **NeatIBP 1.1** + Kira
Wu, Boehm, Ma, Usovitsch, Xu, YZ, Comput.Phys.Commun. 316 (2025) 109798

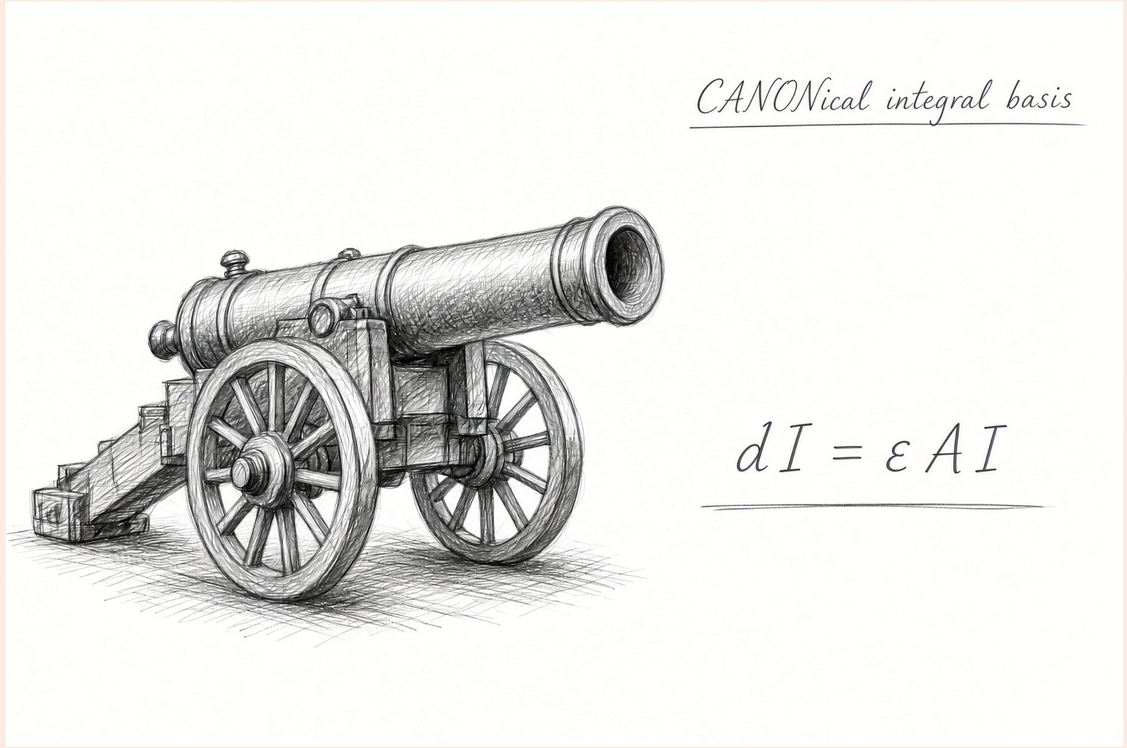
pfd-parallel, *a Singular/GPI-Space package for massively parallel multivariate partial fractioning*
Bendle, Boehm, Heymann Ma, Rahn, Ristau, Wittmann, Wu, Xu, YZ, Comput.Phys.Commun. 294 (2024) 108942

CHESS: CHEbyshev pSeudo-Spectral transport for Feynman integral differential equations

Liu, YZ, arXiv: 2606.26691

Based on AI-assisted development

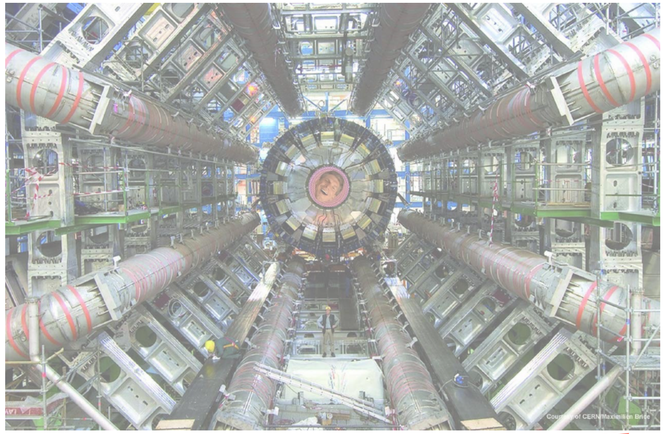
CANON: an AI assistant for canonical differential equations



Choose UT candidates sector by sector using integrands, leading singularities, and DE transformations

Jian, Liu, Ma, Shi, Wu, Wu, Zhang, CANON draft (2026)

Introduction

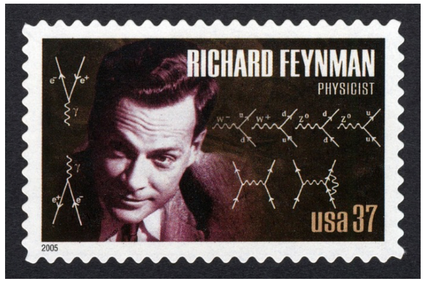


Precision physics

$$\sigma = \sigma^{LO} + \sigma^{NLO} + \sigma^{NNLO}$$

for instance,
Driesse, Jakobsen , Klemm, Mogull, Nega, Plefka, Sauer, Usovitsch
Nature 641 (2025) 8063, 603-607

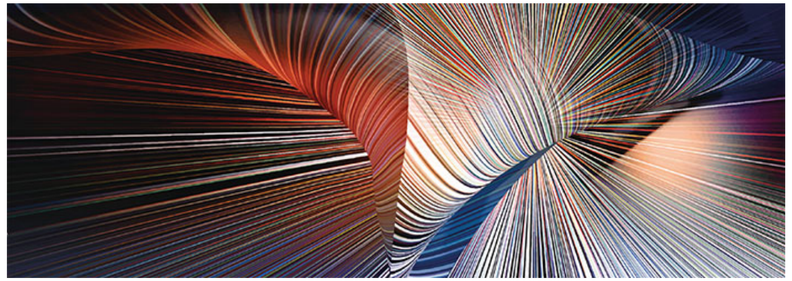




Feynman integrals



Gravitational wave
template computations



(CERN art image for supergravity)

Formal theories

N=8 supergravity UV finiteness

Why *analytic* multi-loop Feynman integrals?

- Historically, analytic results may lead to a lot of phenomenology applications
- Theoretical aspects of quantum field theory

  for example: 2loop spacelike splitting amplitude, factorization violation

  Henn, Ma, Xu, Yan, YZ, Zhu, Phys.Rev.D 112 (2025) 7, 076003

  Buccioni, Fang, Yan arXiv: 2603.27123

- Quantum field theory computation of gravitational wave
- Function space bootstrap for physical observable

  2loop hexagonal Wilson loop + Lagrangian Carrôlo, Chicherin, Henn, Yang, YZ, JHEP 07 (2025) 214

  2loop six-point QCD amplitudes, weight4, Carrôlo, Chicherin, Henn, Yang, YZ, *Phys.Rev.Lett.* 136 (2026) 18, 181602

  3loop pentagonal Wilson loop + Lagrangian Chicherin, Henn, Xu, Zhang,YZ Phys. Rev. D 114, L081905 (2026)

Analytic Feynman integrals to phenomenology applications

Two-loop Five-point Massless integrals

Chicherin, Gehrmann, Henn, Wasser, YZ, Zoia, 2019
Abreu, Dixon, Herrmann, Page, Zeng, 2019



pp → γγγ	Abreu, Page, Pascual, Sonikov 2020
	Chawdhry, Czakon, Mitov, Poncelet 2021
	Abreu, De Laurentis, Ita, Klinkert, Page, Sotnikov 2023
pp → γγj	Agarwal, Buccioni, von Manteuffel, Tancredi 2021
	Chawdhry, Czakon, Mitov, Poncelet 2021
	Badger, Brønnum-Hansen, Chicherin, Gehrmann, Hartanto, Henn, Marcoli, Moodie, Peraro, Zoia 2021
	Buccioni, Chen, Feng, Gehrmann, Huss, Marcoli 2025
pp → γjj	Badger, Czakon, Hartanto, Moodie, Peraro, Poncelet, Zoia 2023
pp → jjj	Abreu, Febres Cordero, Ita, Page, Sotnikov 2021
	De Laurentis, Ita, Klinkert, Sotnikov 2023
	Agarwal, Buccioni, Devoto, Gambuti, von Manteuffel, Tancredi 2023
	De Laurentis, Ita, Sotnikov 2023

Two-loop Five-point integrals with one external mass

Abreu, Ita, Moriello, Page, Tschernow, 2020
Kardos, Papadopoulos, Smirnov, Syrrakos, Wever 2022
Abreu, Ita, Page, Tschernow 2022
Chicherin, Sotnikov, Zoia 2022
Abreu, Chicherin, Page, Sotnikov, Tschernow, Zoia 2024



… …	
pp → Wbb	Badger, Hartanto, Zoia 2021 Hartanto, Poncelet, Popescu, Zoia 2022
pp → Wjj	Abreu, Febres Cordero, Ita, Klinkert, Page, Sotnikov 2022
	De Laurentis, Ita, Page, Sotnikov 2025
pp → Hbb	Badger, Hartanto, Krys, Zoia 2021
	Badger, Hartanto, Poncelet, Wu, *YZ*, Zoia 2024
pp → Wγj	Badger, Hartanto, Krys, Zoia 2022
pp → W/Zbb	Buonocore, Devoto, Kallweit, Mazzitelli, Rottoli, Savoini 2022
	Mazzitelli, Sotnikov, Wiesemann 2024
pp → Wγγ	Badger, Hartanto, Wu, *YZ*, Zoia 2024
pp → ttH	Wang, Xie, Yang, Ye 2024
pp → ttγ	Wang, Yang 2025
…	

adapt Matteo Becchetti's slide in HOCtools-II-2025

Analytic Feynman integral and the methodology

Status of analytic Feynman integrals

with dimensional regulation

Planar massless → nonplanar massless → massive

	4-point	5-point	6-point	7-point	8-point
One-loop	known	known	known	known	known
Two-loop	most known	some results		?	?
Three-loop	some results		?	?	?
Four-loop	some results	?	?	?	?

Frontier

Liu, Matijasic, Miczajka, Xu, Xu, YZ, Phys.Rev.D 112 (2025) 1, 016021, editors' suggestion

Chicherin, Wu, Wu, Xu, Zhang, YZ, Phys. Rev. Lett. 137, 111603 (2026)

Belitsky, Bork, Lee, Onishchenko, Smirnov, arXiv:2605.17057 (2026)

Refer the talk by Lee

Henn, Matijasic, Miczajka, Peraro, Xu, YZ, PRL. 135, 031601

Henn, Matijasic, Miczajka, Peraro, Xu, YZ, JHEP 08(2024) 027

Henn, Peraro, Xu, YZ, JHEP 03 (2022) 056

Abreu, Monni, Page, Usovitsch JHEP 06 (2025) 112

Feynman integrals: analytic functions

in terms of

Goncharov polylogarithm function

$$G(\mathbf{0}_k;z)=\frac{1}{k!}(\log z)^k,\quad G(a_1,\dots,a_k;z)=\int_0^z\frac{dt}{t-a_1}G(a_2,\dots a_k;t)$$

arguments related to Letters, algebraic function of kinematics

well studied function with Hopf algebra structure

elliptic polylogarithm

$$\widetilde{\Gamma}\left(\begin{matrix}n_1,&\dots,&n_r\\a_1,&\dots,&a_r\end{matrix};\xi\middle|\tau\right):=\int_0^\xi dt\,g^{(n_1)}(t-a_1\mid\tau)\,\widetilde{\Gamma}\left(\begin{matrix}n_2,&\dots,&n_r\\a_2,&\dots,&a_r\end{matrix};t\middle|\tau\right),\qquad\widetilde{\Gamma}(;\xi|\tau)=1.$$

hyperelliptic polylogarithm

$$\widetilde{\Gamma}\left(\begin{matrix}i_{n_1},&\dots,&i_{n_k}\\x_1,&\dots,&x_k\end{matrix};z\middle|G\right):=\int_{z_0}^z\omega_{i_{n_1}}(t,x_1\mid G)\,\widetilde{\Gamma}\left(\begin{matrix}i_{n_2},&\dots,&i_{n_k}\\x_2,&\dots,&x_k\end{matrix};t\middle|G\right),\qquad\widetilde{\Gamma}(;z|G):=1.$$

G is Schottky group uniformizing the hyperelliptic curve

Calabi-Yau periods

$$\Pi_\alpha(z)=\int_{\Gamma_\alpha(z)}\Omega(z),\qquad\Gamma_\alpha(z)\in H_n(X_z,\mathbb{Z}).$$

…

Baune, Broedel, Im, Lisitsyn, Zerbini
J. Phys. A: Math. Theor. 57 445202 (2024)

Canonical Differential Equation

$$\frac{\partial}{\partial x_i} I(x,\epsilon) = \epsilon A_i(x) I(x,\epsilon) \qquad \text{Henn 2013}$$

Existence (beyond the polylogarithm) and general algorithm to obtain canonical differential equation

ϵ-collaboration Phys.Rev.Lett. 136 (2026) 24, 241602

Alphabet searching

Effortless, Matijasic, Miczajka to appear
https://github.com/antonela-matijasic/Effortless
BaikovLetter, Jiang, Liu, Xu, Yang, Phys.Lett.B 864 (2025) 139443
SOFIA Correia, Giroux, Mizera 2503.16601
…

iterative integration reorganization (to one-fold integral)

Liu, Matijasic, Miczajka, Xu, Xu, YZ, *Phys.Rev.D* 112 (2025) 1, 016021

$$I^{(0)} = B^{(0)}$$
$$I^{(1)} = B^{(1)} + \int_\gamma (d\tilde{A}) I^{(0)}$$
$$I^{(2)} = B^{(2)} + \int_\gamma (d\tilde{A}) I^{(1)}$$
$$\vdots$$
$$I^{(n)} = B^{(n)} + \int_\gamma (d\tilde{A}) I^{(n-1)}$$

numerical solution

DiffExp, Hidding Comput.Phys.Commun. 269 (2021) 108125
Line, Prisco, Ronca, Tramontano JHEP07(2025)219
CHESS, Liu YZ, arXiv: 2606.26691
…

for Next-to-Next-to-leading order

production of 4 jets, 4 photons …

2loop 6point Feynman integrals

Henn, Matijasic, Miczajka, Peraro, Xu, YZ, *Phys. Rev. Lett. 135 (2025) 3, 031601*

Henn, Matijasic, Miczajka, Peraro, Xu, YZ, *JHEP 08(2024) 027*

Liu, Matijasic, Peraro, Xu, Yang, YZ, *JHEP* 07 (2026) 223

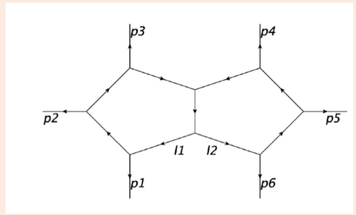
Abreu, Monni, Page, Usovitsch, *JHEP 06 (2025) 112*

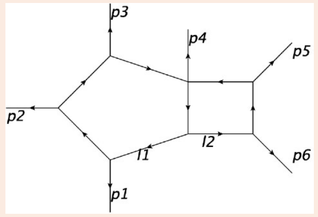
All planar massless 2loop 6point integrals

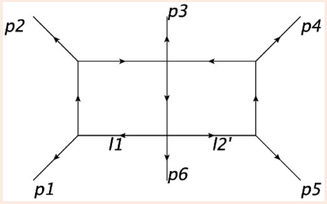
Henn, Matijasic, Miczajka, Peraro, Xu, YZ, *Phys. Rev. Lett. 135 (2025) 3, 031601*

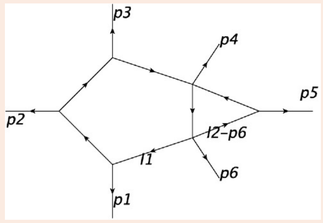
Abreu, Monni, Page, Usovitsch, JHEP 06 (2025) 112

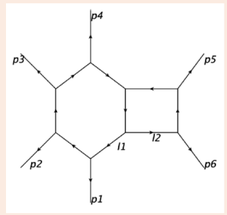
267
pure integrals



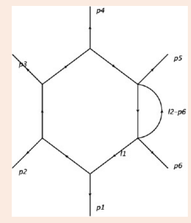








202
pure integrals



Momentum Twistor

external momenta $d=4$

$$Z_i = \begin{pmatrix} \lambda_{i,\alpha} \\ \mu_{i,\dot\beta} \end{pmatrix}, \quad i = 1, \ldots 6$$

$$\tilde\lambda_i = \frac{\langle i,\, i+1\rangle \mu_{i-1} + \langle i+1,\, i-1\rangle \mu_i + \langle i-1,\, i\rangle \mu_{i+1}}{\langle i,\, i+1\rangle \langle i-1,\, i\rangle},$$

A particular parameterization

Badger, Frellesvig, YZ 2013

$$Z = \begin{pmatrix} 1 & 0 & \frac{1}{x_1} & \frac{1}{x_2 x_1} + \frac{1}{x_1} & \frac{1}{x_2 x_1} + \frac{1}{x_2 x_3 x_1} + \frac{1}{x_1} & \frac{1}{x_2 x_1} + \frac{1}{x_2 x_3 x_1} + \frac{1}{x_2 x_3 x_4 x_1} + \frac{1}{x_1} \\ 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & \frac{x_5}{x_2} & x_6 & 1 \\ 0 & 0 & 1 & 1 & x_7 & 1 - \frac{x_8}{x_5} \end{pmatrix}$$

$$x_1 = s_{12}$$

$$x_2 = -\frac{\mathrm{Tr}_+(1234)}{2 s_{12} s_{34}}$$

$$x_3 = -\frac{\mathrm{Tr}_+(1345)}{2 s_{45} s_{13}}$$

$$x_4 = -\frac{\mathrm{Tr}_+(1456)}{2 s_{56} s_{14}}$$

$$x_5 = \frac{s_{23}}{s_{12}}$$

$$x_6 = -\frac{\mathrm{Tr}_+(1532) + \mathrm{Tr}_+(1542)}{2 s_{15} s_{12}}$$

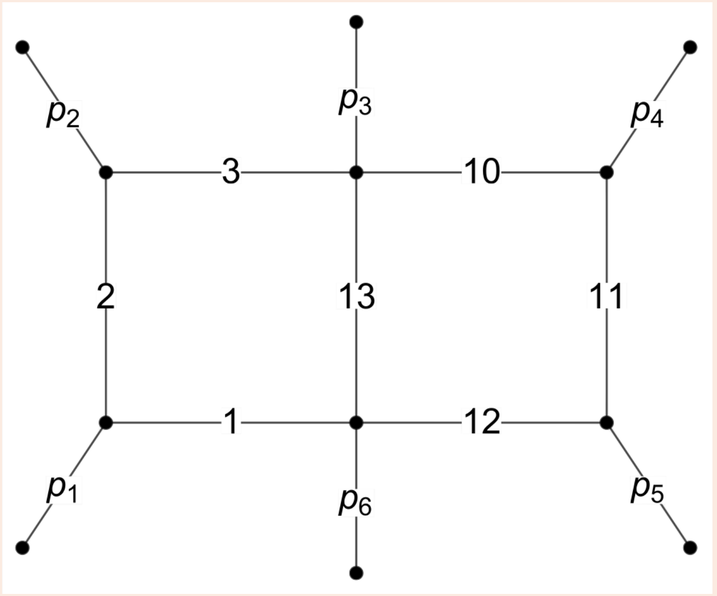
$$x_7 = 1 + \frac{\mathrm{Tr}_+(1542) + \mathrm{Tr}_+(1543)}{2 s_{15} s_{23}}$$

$$x_8 = \frac{s_{123}}{s_{12}}$$

Momentum parametrization rationalizes all pseudo scalars

$$\epsilon_{ijkl} \equiv 4i \epsilon_{\mu\nu\rho\sigma} p_i^\mu p_j^\nu p_k^\rho p_l^\sigma, \quad \epsilon_{ijkl}^2 = G_{ijkl}$$

Uniformly transcendental (UT or pure) basis determination



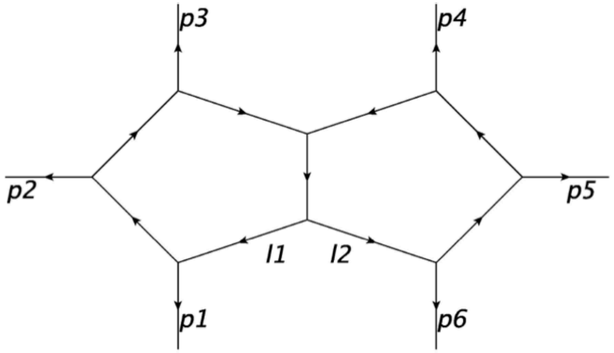
$$I_{\mathrm{db},i} = \int \frac{d^{4-2\epsilon}l_1}{i\pi^{2-\epsilon}} \frac{d^{4-2\epsilon}l_2}{i\pi^{2-\epsilon}} \frac{N_i}{D_1 D_2 D_3 D_{10} D_{11} D_{12} D_{13}}\,, \quad i = 1, \ldots, 7$$

$$\begin{aligned}
N_1 &= -s_{12}s_{45}s_{156}\,,\\
N_2 &= -s_{12}s_{45}(l_1 + p_5 + p_6)^2\,,\\
N_3 &= \frac{s_{45}}{\epsilon_{5126}} G\begin{pmatrix} l_1 & p_1 & p_2 & p_5+p_6 \\ p_1 & p_2 & p_5 & p_6 \end{pmatrix},\\
N_4 &= \frac{s_{12}}{\epsilon_{1543}} G\begin{pmatrix} l_2-p_6 & p_5 & p_4 & p_1+p_6 \\ p_1 & p_5 & p_4 & p_3 \end{pmatrix},\\
N_5 &= -\frac{1}{4}\frac{\epsilon_{1245}}{G(1,2,5,6)} G\begin{pmatrix} l_1 & p_1 & p_2 & p_5 & p_6 \\ l_2 & p_1 & p_2 & p_5 & p_6 \end{pmatrix},\\
N_6 &= \frac{1}{8} G\begin{pmatrix} l_1 & p_1 & p_2 \\ l_2-p_6 & p_4 & p_5 \end{pmatrix} + \frac{D_2 D_{11}(s_{123}+s_{126})}{8}\,,\\
N_7 &= -\frac{1}{2\epsilon}\frac{\Delta_6}{G(1,2,4,5)D_{13}} G\begin{pmatrix} l_1 & p_1 & p_2 & p_4 & p_5 \\ l_2 & p_1 & p_2 & p_4 & p_5 \end{pmatrix}.
\end{aligned}$$

Chiral numerator
(Arkani-Hamed, Bourjaily, Cachazo, Trnka 2011)
/ Gram determinant
correspondence $\Delta_6 = \langle 12\rangle[23]\langle 34\rangle[45]\langle 56\rangle[61] - \langle 23\rangle[34]\langle 45\rangle[56]\langle 61\rangle[12]\,.$

see also Abreu, Monni, Page, Usovitsch *JHEP 06 (2025) 112*, *a systematic way of finding pure basis*

2loop 6point top sector, pure integrals



5 MIs (this sector)
267 MIs (whole family)

Henn, Matijasic, Miczajka, Peraro, Xu, YZ, Phys. Rev. Lett. 135 (2025) 3, 031601

UT integrals list

$$I_1^{\text{DP-a}} = \int \frac{d^{4-2\epsilon}l_1}{i\pi^{2-\epsilon}} \frac{d^{4-2\epsilon}l_2}{i\pi^{2-\epsilon}} \frac{N_1^{\text{DP-a}} - N_4^{\text{DP-a}}}{D_1 \dots D_9} \longrightarrow \text{evanescent}$$

$$I_2^{\text{DP-a}} = \int \frac{d^{4-2\epsilon}l_1}{i\pi^{2-\epsilon}} \frac{d^{4-2\epsilon}l_2}{i\pi^{2-\epsilon}} \frac{N_2^{\text{DP-a}} - N_3^{\text{DP-a}}}{D_1 \dots D_9}$$

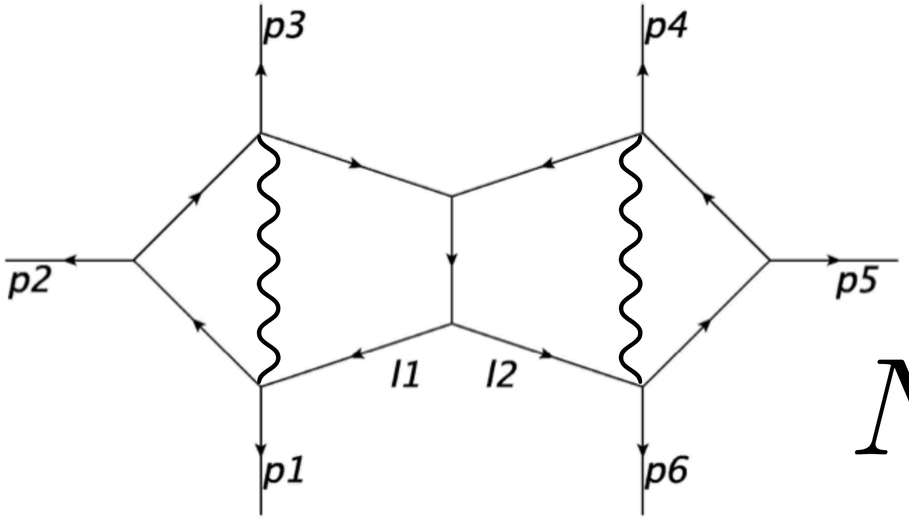
$$I_3^{\text{DP-a}} = F_3 \int \frac{d^{4-2\epsilon}l_1}{i\pi^{2-\epsilon}} \frac{d^{4-2\epsilon}l_2}{i\pi^{2-\epsilon}} \frac{\mu_{12}}{D_1 \dots D_9} \longrightarrow \text{evanescent}$$

$$I_4^{\text{DP-a}} = F_4\epsilon^2 \int \frac{d^{6-2\epsilon}l_1}{i\pi^{3-\epsilon}} \frac{d^{6-2\epsilon}l_2}{i\pi^{3-\epsilon}} \frac{1}{D_1 \dots D_9} \longrightarrow \text{evanescent, 6D weight-6 integral}$$

$$I_5^{\text{DP-a}} = \int \frac{d^{4-2\epsilon}l_1}{i\pi^{2-\epsilon}} \frac{d^{4-2\epsilon}l_2}{i\pi^{2-\epsilon}} \frac{N_1^{\text{DP-a}} + N_4^{\text{DP-a}} + F_5\mu_{12}}{D_1 \dots D_9}$$

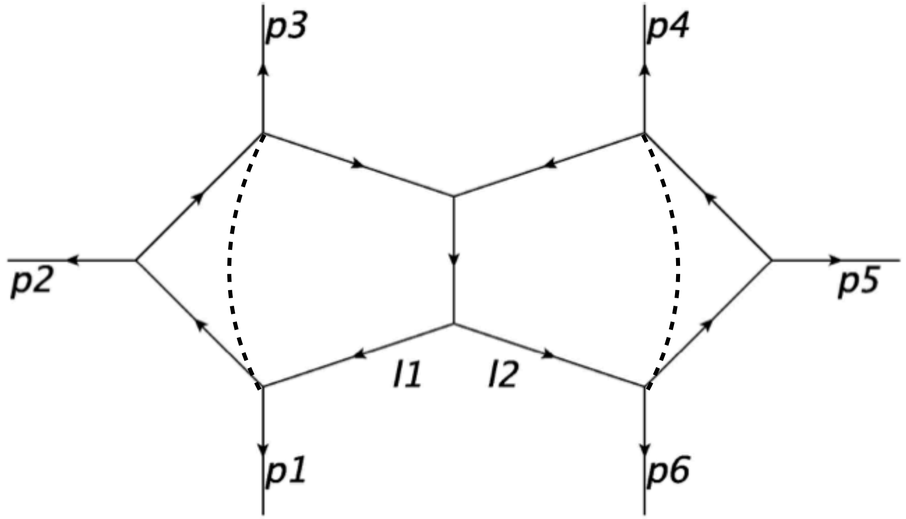
N_1, N_2, N_3 and N_4 are chiral numerators

The weight-4 parts of those integrals known from DCI studies



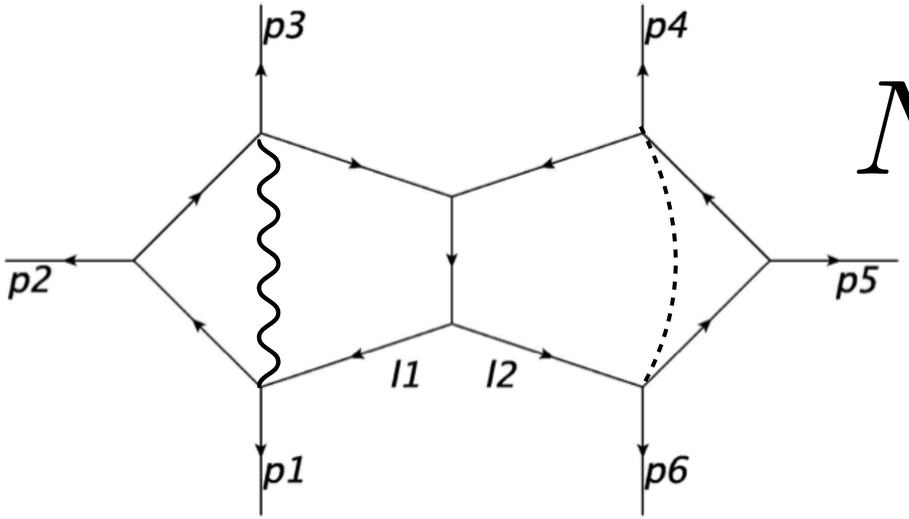
N_1

N_4



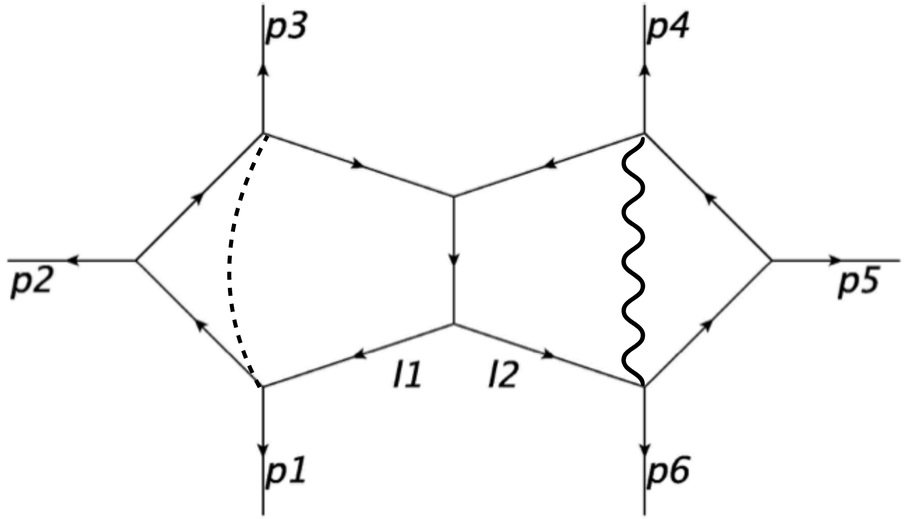
$$ut_2 = I[N_2 - N_3] = -2\tilde{\Omega}_{\rm odd} + O(\epsilon)\,,$$
$$ut_5 = I[N_1 + N_4] = 2\Omega_{\rm even} + O(\epsilon)$$

chiral-numerator integrals are finite and analytically calculated to weight-4 long time ago
Dixon, Drummond, Henn 2011

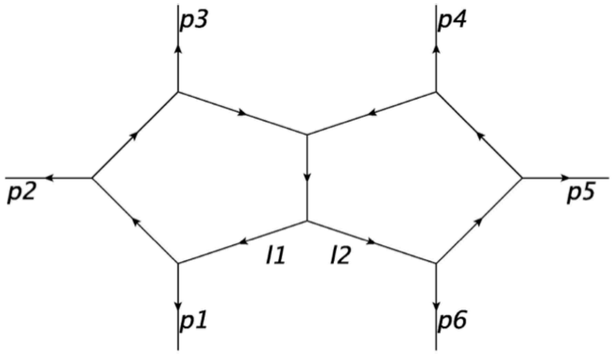


N_2

N_3



2loop 6point top sector, UT integrals



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Henn, Matijasic, Miczajka, Peraro, Xu, YZ, Phys. Rev. Lett. 135 (2025) 3, 031601

UT integrals list

$$I_1^{\text{DP-a}} = \int \frac{d^{4-2\epsilon}l_1}{i\pi^{2-\epsilon}} \frac{d^{4-2\epsilon}l_2}{i\pi^{2-\epsilon}} \frac{N_1^{\text{DP-a}} - N_4^{\text{DP-a}}}{D_1 \dots D_9} \longrightarrow \text{evanescent}$$

$$I_2^{\text{DP-a}} = \int \frac{d^{4-2\epsilon}l_1}{i\pi^{2-\epsilon}} \frac{d^{4-2\epsilon}l_2}{i\pi^{2-\epsilon}} \frac{N_2^{\text{DP-a}} - N_3^{\text{DP-a}}}{D_1 \dots D_9}$$

$$I_3^{\text{DP-a}} = F_3 \int \frac{d^{4-2\epsilon}l_1}{i\pi^{2-\epsilon}} \frac{d^{4-2\epsilon}l_2}{i\pi^{2-\epsilon}} \frac{\mu_{12}}{D_1 \dots D_9} \longrightarrow \text{evanescent}$$

$$I_4^{\text{DP-a}} = F_4\epsilon^2 \int \frac{d^{6-2\epsilon}l_1}{i\pi^{3-\epsilon}} \frac{d^{6-2\epsilon}l_2}{i\pi^{3-\epsilon}} \frac{1}{D_1 \dots D_9} \longrightarrow \text{evanescent, 6D weight-6 integral}$$

$$I_5^{\text{DP-a}} = \int \frac{d^{4-2\epsilon}l_1}{i\pi^{2-\epsilon}} \frac{d^{4-2\epsilon}l_2}{i\pi^{2-\epsilon}} \frac{N_1^{\text{DP-a}} + N_4^{\text{DP-a}} + F_5\mu_{12}}{D_1 \dots D_9}$$

Abreu, Monni, Page, Usovitsch *JHEP 06 (2025) 112*

Those integrals provide no independent weight-4 functions …
not needed for two-loop amplitudes to the $O(\epsilon^0)$ order

This sector has no contribution
to divergent and finite part of two-loop
gauge theory amplitudes

Alphabet, 2loop 6point massless planar integrals, up to weight 4

Even letter $F(s)$

a polynomial in Mandelstam variables
or homogeneously linear in square roots

a Feynman integral's even letters are connected to Landau singularity

Odd letter $\frac{P(s)-\sqrt{Q(s)}}{P(s)+\sqrt{Q(s)}}$ $\log(W) \mapsto -\log(W)$ under the sign change of the square root

"square roots": ϵ_{ijkl}, Δ_6, $\sqrt{\lambda(s_{12}, s_{34}, s_{56})}$

pseudo scalar

leading singularity hexagon

Källin function from massive triangle diagrams

More complicated letter $\frac{P(s)-\sqrt{Q_1(s)}\sqrt{Q_2(s)}}{P(s)+\sqrt{Q_1(s)}\sqrt{Q_2(s)}}$

Matijasic, Miczajka, to appear

Effortless
https://github.com/antonela-matijasic/Effortless

Even letter, Odd letter and the more complicated …

Cluster algebra structure

Bossinger, Drummond, Glew, Gürdoğan, arXiv: 2507.01015
Pokraka, Spradlin, Volovich, Weng, JHEP 03 (2026) 120

156 Even letters

$$s_{12}, \quad s_{123}$$
$$s_{12} - s_{123}$$
$$\cdots$$
$$-s_{12}s_{45} + s_{123}s_{345}$$
$$\cdots$$
$$\sqrt{\lambda(s_{12}, s_{34}, s_{56})}, \quad \epsilon_{ijkl}$$

79 Odd letters

$$\frac{s_{12} + s_{34} - s_{56} - \sqrt{\lambda(s_{12}, s_{34}, s_{56})}}{s_{12} + s_{34} - s_{56} + \sqrt{\lambda(s_{12}, s_{34}, s_{56})}}$$
$$\cdots$$
$$\frac{s_{12}s_{23} - s_{23}s_{34} + s_{23}s_{56} + s_{34}s_{123} - s_{234}(s_{12} + s_{123}) - \epsilon_{1234}}{s_{12}s_{23} - s_{23}s_{34} + s_{23}s_{56} + s_{34}s_{123} - s_{234}(s_{12} + s_{123}) + \epsilon_{1234}},$$
$$\cdots$$
$$\frac{-s_{12}s_{45}s_{234} + s_{34}s_{61}s_{123} + s_{345}(-s_{23}s_{56} + s_{123}s_{234}) - \Delta_6}{-s_{12}s_{45}s_{234} + s_{34}s_{61}s_{123} + s_{345}(-s_{23}s_{56} + s_{123}s_{234}) + \Delta_6}$$
$$\cdots$$

10 More complicated letters

$$\frac{P - \epsilon_{1234}\sqrt{\lambda(s_{12}, s_{34}, s_{56})}}{P + \epsilon_{1234}\sqrt{\lambda(s_{12}, s_{34}, s_{56})}} \quad \cdots$$

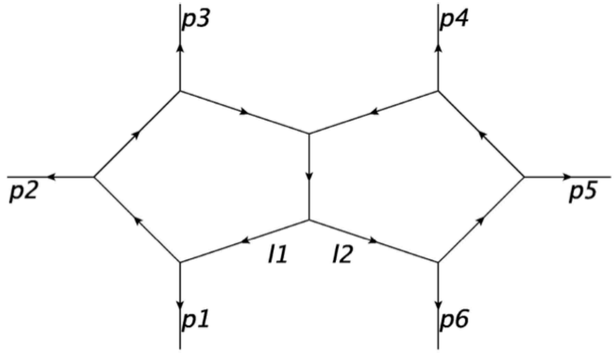
245 letters

up to the weight four

Then the canonical DE is obtained analytically after ~50 times of numeric IBP running

Two-loop Six-point integrals to higher orders of ϵ

Liu, Matijasic, Peraro, Yang, Xu, YZ, arXiv. 2603.16831



6D double pentagon, starts with weight six

$$I_4^{\text{DP-a}} = F_4 \epsilon^2 \int \frac{d^{6-2\epsilon} l_1}{i\pi^{3-\epsilon}} \frac{d^{6-2\epsilon} l_2}{i\pi^{3-\epsilon}} \frac{1}{D_1 \dots D_9} \sim O(\epsilon^2)$$

$$F_4 = \frac{1}{8} \left(G(1,2,3,6) s_{45}^2 + G(1,2,3,5) s_{46}^2 + G(1,2,3,4) s_{56}^2 \right.$$
$$\left. -2\epsilon_{1235}\epsilon_{1236} s_{45} s_{46} - 2\epsilon_{1234}\epsilon_{1236} s_{45} s_{56} - 2\epsilon_{1234}\epsilon_{1235} s_{46} s_{56} \right)^{1/2},$$
$$= \frac{1}{8} \sqrt{\lambda(s_{45}\epsilon_{1236}, s_{46}\epsilon_{1235}, s_{56}\epsilon_{1234})}$$

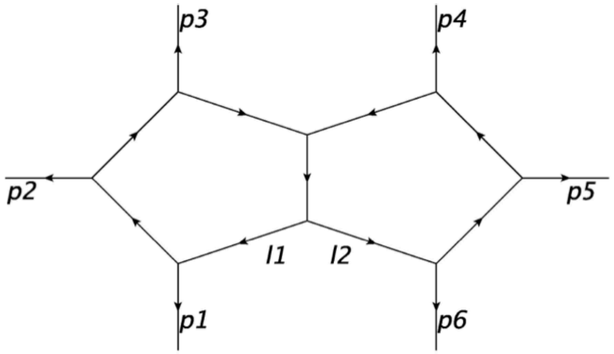
a new square root
not rationalised
by momentum twistor



$$\frac{F_4 + R_i}{F_4 - R_i}, \quad i = 1, \dots 5$$

18 more letters
from the diagonal block

x-point integrals to higher orders of ϵ

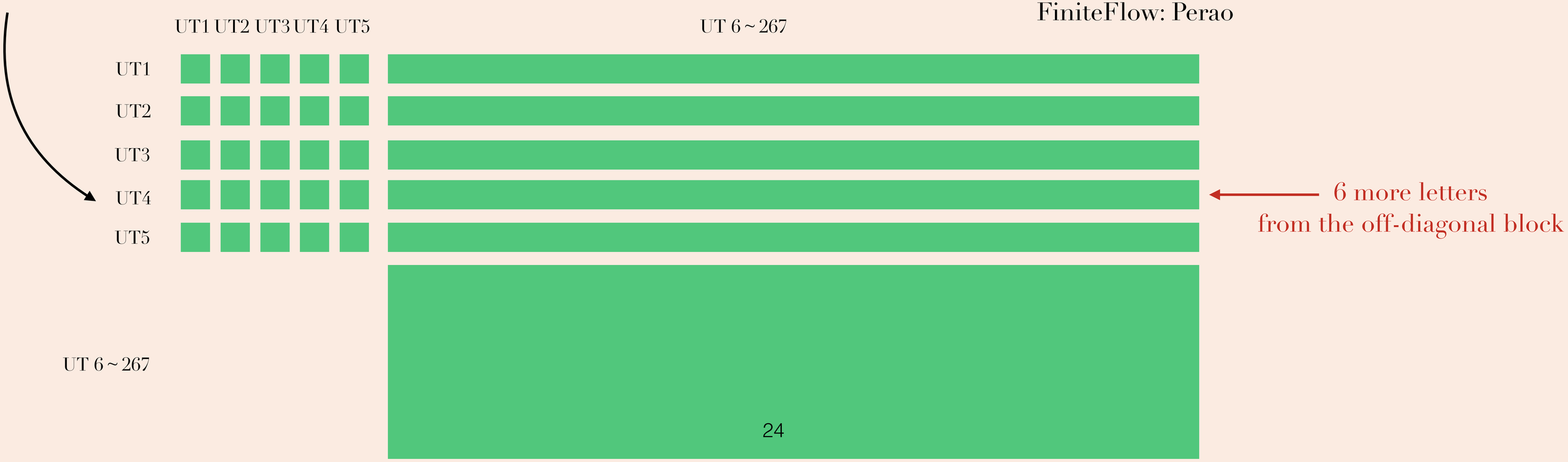


6D double pentagon, starts with weight six

$$I_4^{\text{DP-a}} = F_4 \epsilon^2 \int \frac{d^{6-2\epsilon} l_1}{i\pi^{3-\epsilon}} \frac{d^{6-2\epsilon} l_2}{i\pi^{3-\epsilon}} \frac{1}{D_1 \dots D_9} \sim O(\epsilon^2)$$

Calico: Bertolini, Fontana, Peraro

FiniteFlow: Perao



Two-loop Six-point integrals to higher orders of ϵ

Henn, Matijasic, Miczajka, Peraro, Xu, YZ, Phys. Rev. Lett. 135 (2025) 3, 031601

Liu, Matijasic, Peraro, Yang, Xu, YZ, arXiv: 2603.16831

closed under the dihedral group action

$$245 + 18 + 6 = 269 \text{ letters in total}$$

up to weight-4

Henn, Peraro, Xu, YZ, *JHEP* 03 (2022) 056

2loop 6point massless planar integrals calculated to weight-six

Chebyshev pseudospectral transport, to get weight-6 numerics in one minute

Boundary Values

Euclidean region X_0 : $\{s_{12}, s_{23}, s_{34}, s_{45}, s_{56}, s_{16}, s_{123}, s_{234}, s_{345}\} \to \{-1, -1, -1, -1, -1, -1, -1, -1, -1\}$

Boundary values at the symmetric Euclidean point, are obtained up to weight 4: Zeta values and polylogarithm of roots of unity

$$\epsilon^4 I_{\text{db},1}(X_0) = 1 + \frac{\pi^2}{6}\epsilon^2 + \frac{38}{3}\zeta_3\epsilon^3 + \left(\frac{49\pi^4}{216} + \frac{32}{3}\,\text{Im}\left[\text{Li}_2(\rho)\right]^2\right)\epsilon^4,$$

$$\epsilon^4 I_{\text{db},2}(X_0) = 1 + \frac{\pi^2}{6}\epsilon^2 + \frac{34}{3}\zeta_3\epsilon^3 + \left(\frac{71\pi^4}{360} + 20\,\text{Im}\left[\text{Li}_2(\rho)\right]^2\right)\epsilon^4,$$

$$I_{\text{db},3}(X_0) = I_{\text{db},4}(X_0) = I_{\text{db},5}(X_0) = 0,$$

$$\epsilon^4 I_{\text{db},6}(X_0) = -\left(\frac{\pi^4}{540} + \frac{4}{3}\,\text{Im}\left[\text{Li}_2(\rho)\right]^2\right)\epsilon^4,$$

$$\epsilon^4 I_{\text{db},7}(X_0) = 0.$$

$$\rho = \frac{1}{2} + \frac{\sqrt{3}}{2}i$$

from the ordinary differential equation
spurious pole asymptotic analysis

Physical region, $12 \to 3456$

obtained with AMFlow (about 3 days), ~200 digits, to weight 6

$$\{s_{12}, s_{23}, s_{34}, s_{45}, s_{56}, s_{16}, s_{123}, s_{234}, s_{345}\} \to \left\{\frac{325}{16}, -\frac{181}{16}, \frac{65}{8}, \frac{9}{4}, -\frac{19}{8}, 7, -\frac{109}{16}, \frac{115}{8}\right\}$$

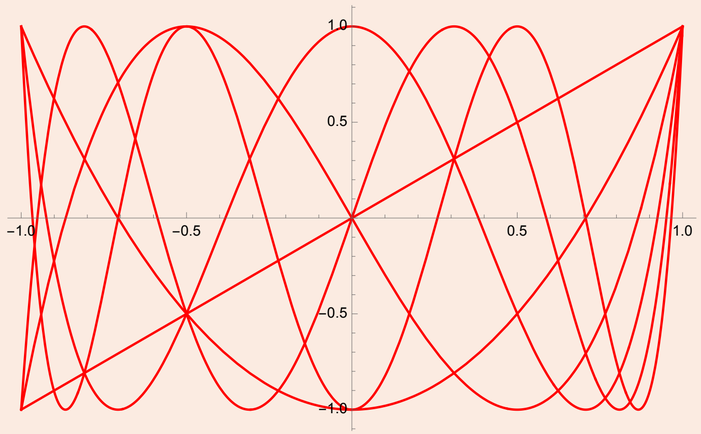
Solution of canonical DE, two-loop six-point integrals

weight-1, 2 — logarithm and classical poly-logarithm

weight-3, 4 — one-fold integration

$$\begin{aligned}\vec{I}^{(4)} &= \vec{I}^{(4)}(\vec{x}_0) + \int_0^1 \mathrm{d}t \frac{\mathrm{d}\tilde{A}}{\mathrm{d}t} \vec{I}^{(3)}(\vec{x}_0) + \int_0^1 \mathrm{d}t_1 \int_0^{t_1} \mathrm{d}t_2 \frac{\mathrm{d}\tilde{A}}{\mathrm{d}t_1}\frac{\mathrm{d}\tilde{A}}{\mathrm{d}t_2} \vec{f}^{(2)}(t_2) \\ &= \vec{I}^{(4)}(\vec{x}_0) + \int_0^1 \mathrm{d}t \left(\frac{\mathrm{d}\tilde{A}}{\mathrm{d}t} \vec{I}^{(3)}(\vec{x}_0) + \left(\tilde{A}(1) - \tilde{A}(t)\right) \frac{\mathrm{d}\tilde{A}}{\mathrm{d}t} \vec{f}^{(2)}(t) \right).\end{aligned}$$

CHEbyshev pSeudo-Spectral transportation (CHESS) Liu and YZ, arXiv:2606.26691



approximate the Feynman integrals by Chebyshev polynomial expansion

An efficient way to solve (canonical) differential equation numerically

Physical region, up to weight 6, all 267 integrals in double pentagon family

Chebyshev Nodes	absolute error	time	RAM
12	$\sim 7.58 \times 10^{-16}$	18.09 s	2.9 GB
24	$\sim 1.36 \times 10^{-31}$	144.76 s	26.1 GB

for Next-to-Next-to-Next-to-leading order

production of 3 jets, 3 photons … with 2loop 6point Feynman integrals

3loop 5point Feynman integrals

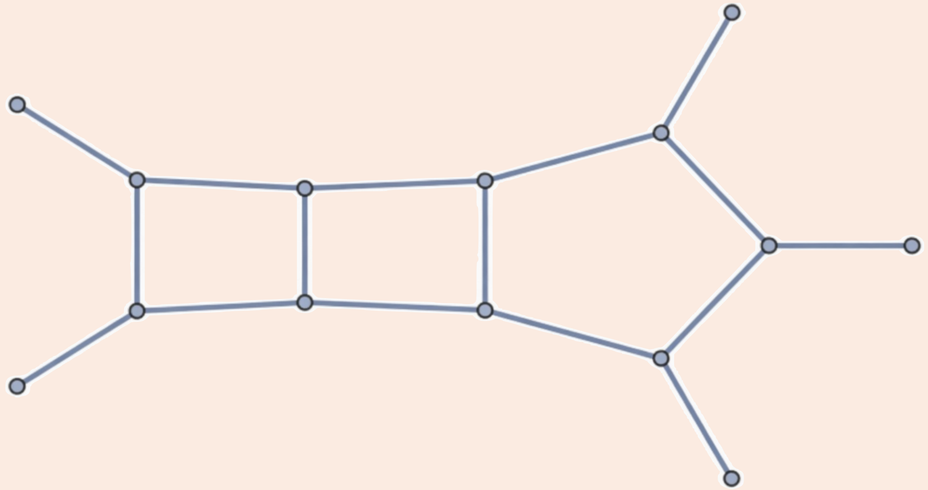
Liu, Matijasic, Miczajka, Xu, Xu, YZ

Phys.Rev.D 112 (2025) 1, 016021, editors' suggestion

Chicherin, Wu, Wu, Xu, Zhang, YZ

Phys. Rev. Lett. 137, 111603 (2026)

3loop 5point planar family



Liu, Matijasic, Miczajka, Xu, Xu, YZ, Phys.Rev.D 112 (2025) 1, 016021
Editors' suggestion

UT basis found!

Baikov analysis
Gram determinant

5 scales $s_{12}, s_{23}, s_{34}, s_{45}, s_{15}$

Canonical differential equation

We use NeatIBP 1.0 to derive the differential equation

~100 million IBPs → 85000 IBPs

316 Master Integrals

weight-1,2,3 classical polylogarithm, weight-4,5,6 one-fold integration
It takes 2 minutes on a laptop to get 10 digits from our analytic solution

Using NeatIBP

"**NeatIBP 1.0**, a package generating small-size integration-by-parts relations for Feynman integrals"
Wu, Boehm, Ma, Xu, YZ, *Comput. Phys. Commun. 295 (2024), 108999*

"Performing integration-by-parts reductions using NeatIBP 1.1 + Kira"
Wu, Boehm, Ma, Usovitsch, Xu, YZ, Comput.Phys.Commun. 316 (2025) 109798

Using algebraic geometry
to find short IBP system,
2 or 3 orders of magnitude shorter
than that from Laporta algorithm.

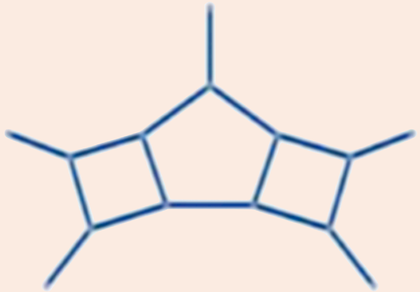
(**selected**)

Studies that used NeatIBP	**Reference**
Differential Equations for Energy Correlators in Any Angle	arXiv:2506.02061
One-loop amplitudes for $t\bar{t}j$ and $t\bar{t}\gamma$ productions at the LHC through $O(\epsilon^2)$	arXiv:2505.10406
Two-loop Feynman integrals for leading color $Wt\bar{t}$ production	JHEP 07 (2025) 001
Two-loop QCD helicity amplitudes for $gg \to gt\bar{t}$ at leading color	JHEP 03 (2025) 070
Full-color double-virtual amplitudes for $q\bar{q} \to b\bar{b}H$	JHEP 03 (2025) 066
Three-loop five-point pentagon-box-box Feynman diagram	arXiv:2411.18697
Two-loop QCD corrections for $pp \to t\bar{t}j$	arXiv:2411.10856
Two-loop amplitudes for $W\gamma\gamma$ production at LHC	JHEP 12 (2025) 221
NLO corrections to $J/\Psi c\bar{c}$ photoproduction	Phys.Rev.D 110 (2024) 9, 094047
Two-loop five-point two-mass planar integrals	JHEP 10 (2024) 167
Two-loop integrals for $t\bar{t}j$ production at hadron colliders in the leading color approximation	JHEP 07 (2024) 073

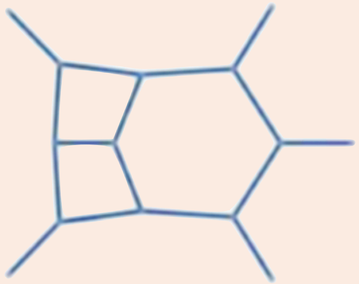
phenomenology application of NeatIBP

Complete result of all 3loop 5point planar massless integrals

Chicherin, Wu, Wu, Xu, Zhang, YZ,
Phys. Rev. Lett. 137, 111603 (2026)



367 MIs



431 MIs



734 MIs

56 letters …

IBP reduction is extremely complicated, we have to use the latest NeatIBP 1.1 version

NeatIBP 1.1: Syzygy + Spanning Cuts

Most complicated IBP system with cuts, consists of $\sim 10^5$ equations, ~100 cuts

Complete result of all 3loop 5point planar massless integrals

Chicherin, Wu, Wu, Xu, Zhang, YZ,
Phys. Rev. Lett. 137, 111603 (2026)

pure basis determination

top sector has 19 MIs



loop-by-loop dlog integrand generation, dimensional shift
module lift method

$$\mathcal{N} = \sum_I c_I ISP^{\alpha_I}$$

$\sum_I c_I \text{ residue}_j\left(\text{ISP}^{\alpha_I}\right) = r_j,$
c_I polynomial in Mandelstam variables, r_j rational numbers

$$\mathcal{N}_4 = \frac{\varepsilon(s_{12}-s_{45})}{(1+2\varepsilon)\epsilon_5} G(\ell_2,\ell_3,k_1,k_2,k_3,k_4),$$
$$\mathcal{N}_{13} = s_{12}s_{34}s_{45}(\ell_2-k_1)^2(\ell_3-k_{123})^2,$$
$$\mathcal{N}_{17} = s_{12}s_{45}(\ell_3+k_5)^2\left((\ell_2-k_{123})^2(\ell_3-k_{12})^2 - (\ell_3-k_{123})^2(\ell_2-k_{12})^2\right),$$
$$\mathcal{N}_{19} = s_{12}s_{15}s_{45}\left((\ell_2-k_{123})^2(\ell_3-k_{12})^2 - (\ell_3-k_{123})^2(\ell_2-k_{12})^2\right).$$

3loop 5point integrals, symbology

Chicherin, Wu, Wu, Xu, Zhang, YZ,
Phys. Rev. Lett. 137, 111603 (2026)

From the canonical differential equation
We confirmed the 3loop 5point letters from Chicherin, Henn, Trnka, Zhang *JHEP* 04 (2025) 022

56 Letters

$$\mathbb{A}_{\rm P}^{\text{3-loop}} = \mathbb{A}_{\rm P}^{\text{2-loop}} \cup \{\widehat{W}_i\}_{i=1}^{5} \cup \{\widetilde{W}_i\}_{i=16}^{20} \cup \{\widetilde{W}_i\}_{i=41}^{55} \cup \{\widetilde{W}_i\}_{i=76}^{80}$$

26 Letters

$$\begin{aligned}
\Delta_4^{(1)} = {} & s_{15}^2 s_{12}^2 + s_{23}^2 s_{12}^2 - 2 s_{15} s_{23} s_{12}^2 - 2 s_{23}^2 s_{34} s_{12} \\
& + 2 s_{15} s_{23} s_{34} s_{12} + 2 s_{15} s_{34} s_{45} s_{12} + 2 s_{23} s_{34} s_{45} s_{12} \\
& + s_{23}^2 s_{34}^2 + s_{34}^2 s_{45}^2 - 2 s_{23} s_{34}^2 s_{45}, .
\end{aligned}$$

$$\begin{aligned}
\widehat{W}_1 &= s_{23} s_{34} - s_{34} s_{45} + s_{45} s_{15} \,, \\
\widetilde{W}_{16} &= \Delta_4^{(1)} \,, \\
\widetilde{W}_i &= \frac{q_i - \sqrt{\Delta_4^{(1)}}}{q_i + \sqrt{\Delta_4^{(1)}}}, \ i \in \{41, 46, 51\} \,, \\
\widetilde{W}_{76} &= \frac{q_{76} - \epsilon_5 \sqrt{\Delta_4^{(1)}}}{q_{76} + \epsilon_5 \sqrt{\Delta_4^{(1)}}} \,,
\end{aligned}$$

Symbols

weight	1	2	3	4	5	6
3loop 5point Symbols	5	20	76	285	1000	2220

small number of symbols
good news for bootstrap

Solution of canonical DE, three-loop five-point integrals

weight-1, 2, 3 logarithm and classical poly-logarithm

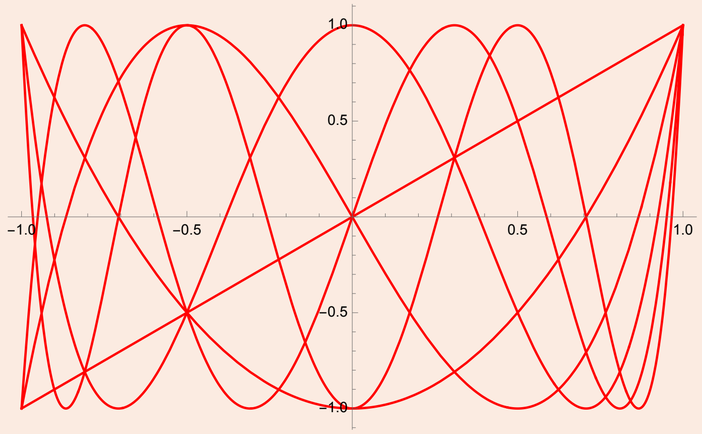
weight-3, 4, 6 one-fold integration

$$\mathrm{d}\tilde{B} = (\mathrm{d}\tilde{A})\tilde{A},$$

$$\begin{aligned}\mathbf{I}^{(n+3)}(x) =& \mathbf{I}^{(n+3)}(x_0) + \int_0^1 \frac{\mathrm{d}\tilde{A}}{\mathrm{d}t}\mathbf{I}^{(n+2)}(x_0)\mathrm{d}t \\ &+ \int_0^1 \left(\tilde{A}(1) - \tilde{A}(t)\right) \frac{\mathrm{d}\tilde{A}}{\mathrm{d}t}\mathbf{I}^{(n+1)}(x_0)\mathrm{d}t \\ &+ \int_0^1 \left(\tilde{A}(t) - \tilde{A}(1)\right) \tilde{A}(t)\frac{\mathrm{d}\tilde{A}}{\mathrm{d}t}\mathbf{I}^{(n)}(t)\mathrm{d}t \\ &+ \int_0^1 \left(\tilde{B}(1) - \tilde{B}(t)\right) \frac{\mathrm{d}\tilde{A}}{\mathrm{d}t}\mathbf{I}^{(n)}(t)\mathrm{d}t.\end{aligned}$$

CHEbyshev pSeudo-Spectral transportation (CHESS)



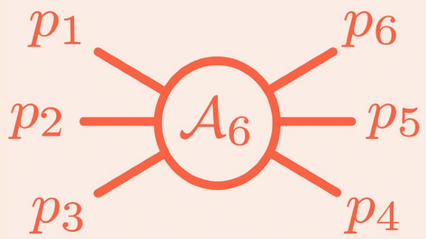




On a MacBook Pro with a 14-core M4 Pro,
we obtain the values with 10 significant digits of the 734 pure integrals,
in about 39 seconds at the following generic point in the Euclidean region.

Applications (A Brief Preview)

2loop 6point amplitude, bootstrap



"Bootstrapping Six-Gluon QCD Amplitudes"

Carrôlo, Chicherin, Henn, Yang, YZ

Phys.Rev.Lett. 136 (2026) 18, 181602

$--++++$ helicity, with fermion loops

weight-4, symbol level

"QCD Scattering Amplitudes and Prescriptive Unitarity"

Carrôlo, Chicherin, Henn, Yang, YZ

JHEP 08 (2026) 100

All MHV-helicity gluon amplitudes, with fermion loops

weight-4, symbol level

2loop hexagonal Wilson loop with Lagrangian insertion

"Hexagonal Wilson loop with Lagrangian insertion at two loops in N=4 super Yang-Mills theory"

Carrôlo, Checherin, Henn, Yang, YZ JHEP 07 (2025) 214

3loop pentagonal Wilson loop with Lagrangian insertion

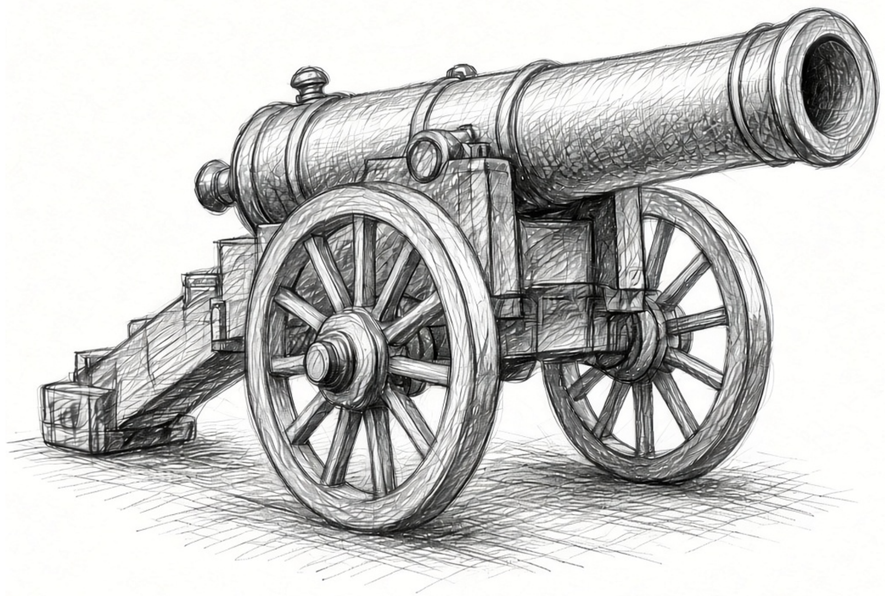
"Three-loop pentagonal Wilson loop with Lagrangian insertion"

Chicherin, Henn, Xu, Zhang, YZ PRD 114, L081905 (2026)

CANON

AI-assisted Feynman integral computations

CANONical integral basis

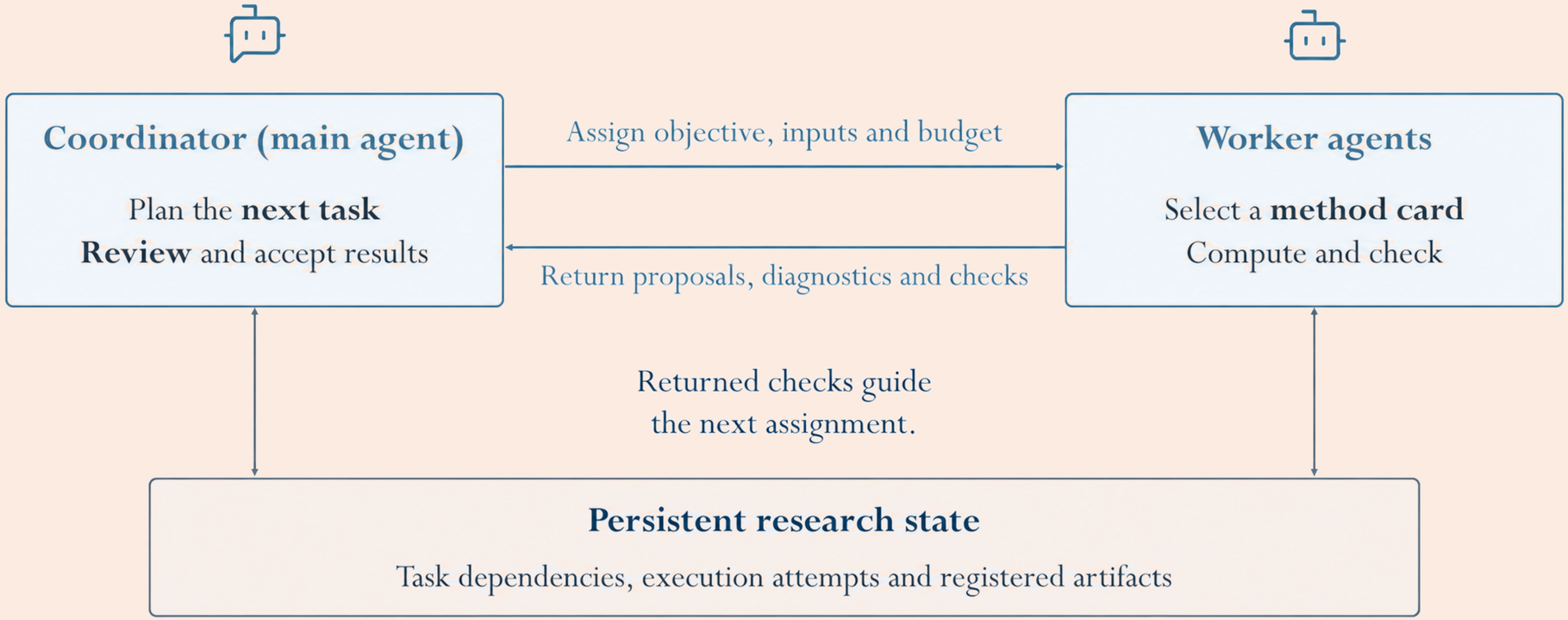


$$d\vec{I} = \varepsilon A \vec{I}$$

Jian, Liu, Ma, Shi, Wu, Wu, Zhang, CANON draft (2026)

CANON: agent-assisted UT basis construction

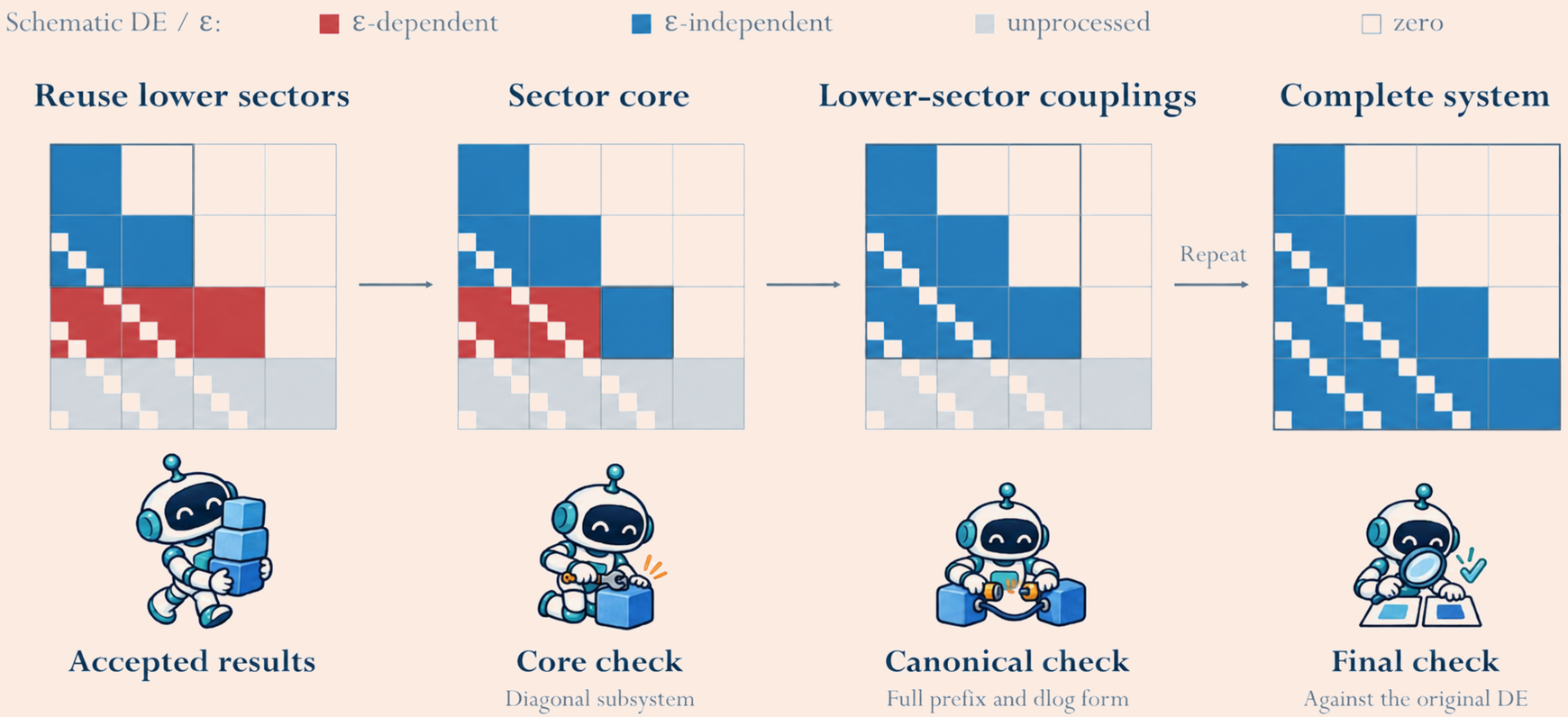
A coordinator assigns tasks; worker agents choose methods, compute candidates, and return diagnostics. Persistent state records dependencies and accepted artifacts.



Jian, Liu, Ma, Shi, Wu, Wu, Zhang, CANON draft (2026)

Sector-by-sector construction with explicit checks

Accepted lower sectors are reused. Each new sector must pass a core check, a canonical-form check, and a final comparison with the original differential equation.



One agent combines several UT-search strategies

Method cards cover dlog integrands, leading singularities, Baikov/Gram information, rational DE transformations, known subsectors, and normalization or repair.

dlog integrands

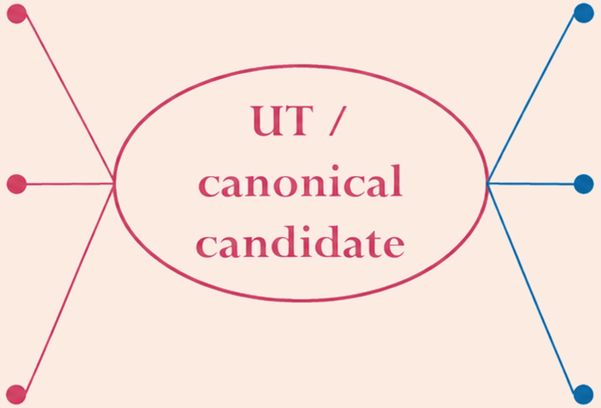
A bounded numerator and dot ansatz
Search with DlogBasis-refine

Leading singularities

Cut residues suggest prefactors
Test the candidate rank with IBP

Baikov / Gram

Dimensional or off-cut information
Test Gram numerators and contact terms



DE transformations

A suitable rational cut system
Try CANONICA / Libra or an ε ansatz

Lee's algorithm
JHEP 04 (2015) 108

Known subsectors

A related basis is available
Transport by symmetry; UTpedia (ongoing)

Normalization and repair

After the core passes its check
Complete the lower-sector couplings

Checks determine whether to continue or revise the proposal.

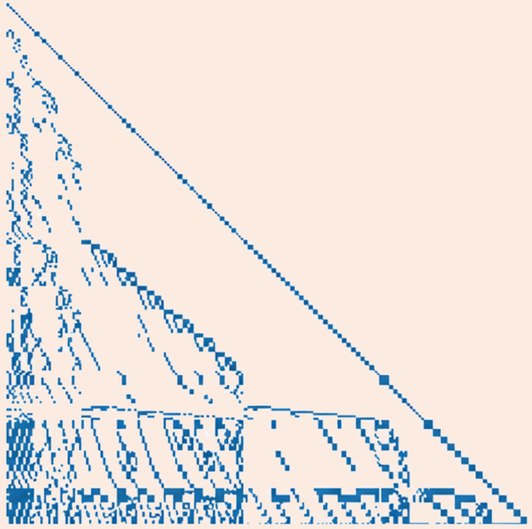
Benchmarks

Massive double box: 37 MIs, 24 sectors, 21 letters, about 12 hours.
Massless pentabox: 61 MIs, 46 sectors, 24 letters, about 1 day.

Unpublished multi-loop examples:

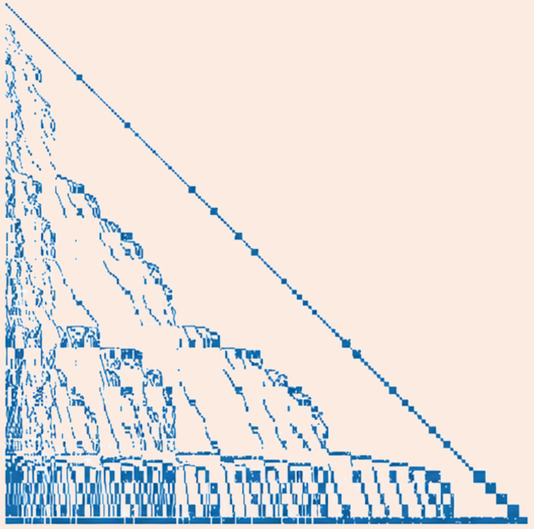
Family A

223



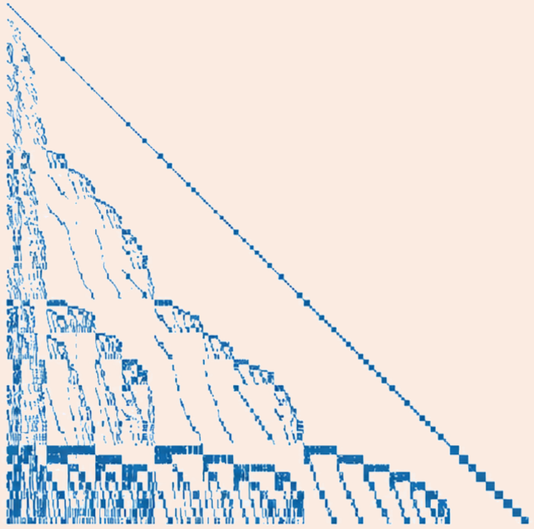
Family B

426



Family C

569



Total time: **~ 1 week (IBP included).** CANON with private symmetry modules.

Summary and Outlook

Analytic computation of

- ✓ all 2loop 6point planar massless integrals
- ✓ all 3loop 5point planar massless integrals

NeatIBP, a powerful package for cutting-edge IBP reduction for precision physics

AI workflow for canonical basis searching