Kinematic Structure Constants for BCJ Numerators from Open Strings

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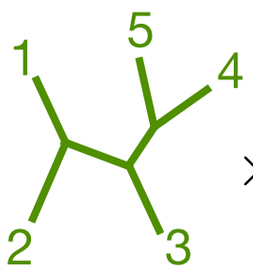
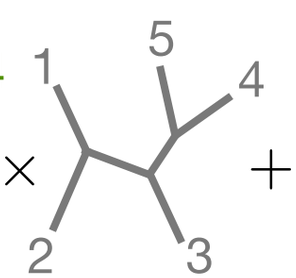
Based on arXiv: 2501.14523, 2005.05177, 1806.09584

The BCJ idea: kinematics behaves as if it had structure constants

- Colour factors are products of Lie-algebra structure constants
- BCJ duality suggests that kinematic numerators can be organised in the same way

[Bern, Carrasco, Johansson 0805.3993]

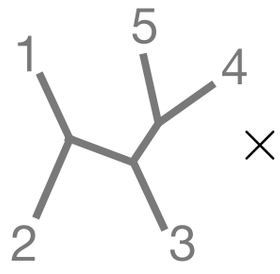
$$\mathcal{A}_{YM} = \frac{1}{s_{12}s_{45}}$$ 

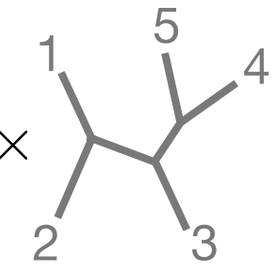
$$\times$$ 

$$+ \cdots$$

$$\| \qquad \|$$

$$f^{12}{}_{\sigma_1} f^{\sigma_1 3}{}_{\sigma_2} f^{\sigma_2 4}{}_{5} \qquad \tilde{f}^{12}{}_{\sigma_1} \tilde{f}^{\sigma_1 3}{}_{\sigma_2} \tilde{f}^{\sigma_2 4}{}_{5}$$

$$\mathcal{A}_{GR} = \frac{1}{s_{12}s_{45}}$$ 

$$\times$$ 

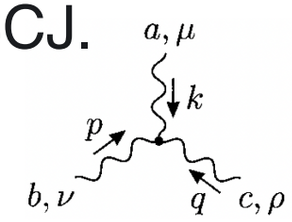
$$+ \cdots$$

Gauge Theory		Gravity Theory
YM	× YM	Einstein gravity + dilaton +B fields
N=4 SYM	× N=4 SYM	N=8 Supergravity
YM	× NLSM	Born–Infeld
NLSM	× NLSM	Special Galileon
BLG	× BLG	3D N=16 Supergravity
Coulomb-branch (S)YM	× YM + ϕ^3	Spontaneously broken Yang–Mills–Einstein (super)gravity
⋮		⋮

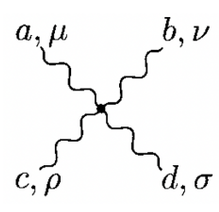
[Bern, Carrasco, Johansson] [Chiodaroli, Gunaydin, Johansson, Roiban] [Bargheer, He, McLoughlin][Cachazo, He, Yuan][Chen, Du] [Bern, de Freitas, Wong][Bjerrum-Bohr, Damgaard, Vanhove] [Stieberger] [Mafra, Schlotterer] [Johansson, Ochirov][Kalin, Mogull] [Du, CF][Carrasco, Mafra, Schlotterer] [Cheung, Shen][Cheung, Remmen, Shen, Wen][Teng, Feng][Nandan, Plefka, Schlotterer, Wen]

Where are the kinematic structure constants?

- The standard Yang–Mills and Einstein–Hilbert actions are not purely cubic
- Their Feynman rules do not directly supply the kinematic structure constants suggested by BCJ.



$$= gf^{abc}[g^{\mu\nu}(k-p)^{\rho} + g^{\nu\rho}(p-q)^{\mu} + g^{\rho\mu}(q-k)^{\nu}]$$



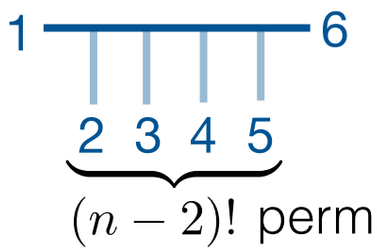
$$= -ig^2[f^{abe}f^{cde}(g^{\mu\rho}g^{\nu\sigma} - g^{\mu\sigma}g^{\nu\rho}) + f^{ace}f^{bde}(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\sigma}g^{\nu\rho}) + f^{ade}f^{bce}(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma})]$$

- In practice: Introduce unknown functions $N_i(k, \epsilon)$ of momenta and polarizations, one for each cubic graph.
- Solve for a set of numerators that reproduces the physical amplitudes and satisfies

$$N_{\begin{smallmatrix}1&&4\\&\diagup\diagdown&\\2&&3\end{smallmatrix}}(k_1, k_2, \dots) + N_{\begin{smallmatrix}2&&4\\&\diagup\diagdown&\\3&&1\end{smallmatrix}}(k_1, k_2, \dots) + N_{\begin{smallmatrix}3&&4\\&\diagup\diagdown&\\1&&2\end{smallmatrix}}(k_1, k_2, \dots) = 0$$

$$f^{12}{}_{\sigma}f^{\sigma 3}{}_{4} + f^{23}{}_{\sigma}f^{\sigma 1}{}_{4} + f^{31}{}_{\sigma}f^{\sigma 2}{}_{4} = 0$$

Jacobi identity (linear relations) $\Rightarrow$ reduce all cubic graphs to a half-ladder basis



$(n-2)!$ permutations

$$\begin{bmatrix} A_{YM}^{color-ordered}(1234) \\ A_{YM}^{color-ordered}(1324) \end{bmatrix} = \begin{bmatrix} \frac{1}{s}+\frac{1}{t} & \frac{-1}{t} \\ \frac{-1}{t} & \frac{1}{u}+\frac{1}{t} \end{bmatrix} \begin{bmatrix} A(1,2,3,4) \\ A(1,3,2,4) \end{bmatrix}$$

propagator matrix

Central question: What is the algebraic origin of kinematic numerators?

- Why do kinematic numerators behave as if they were built from structure constants?
- What are the underlying objects, and what determines their algebra?

From amplitudes to the underlying algebra
(bottom-up)

- Can the algebra generate the numerators, and hence the amplitudes?

From the algebra to explicit numerators
(top-down)

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(top-down)

Short answer:

Integrated open-string vertex operators realize a quantum group (q-deformed Lie algebra) kinematic algebra with computable structure constants.

Example: $f^{1,2}{}_A f^{A,3}{}_B f^{B,4}{}_C f^{C,5}{}_D f^{D,6}{}_E \, tr(T^E T^7)$

$$\sim \frac{q_1 \cdot F_1 \cdot F_2 \cdot F_4 \cdot F_6 \cdot v}{q_1 \cdot k_1} \frac{q_3 \cdot F_3 \cdot F_5 \cdot v}{q_3 \cdot k_3} \frac{(-1)(k_1+k_2)\cdot k_3}{(k_1+k_2+k_4+k_6)\cdot v}$$

What are the underlying objects, and what determines their algebra?

Route 1: Solve for numerators directly from linear relations and try to see what the algebra is (the hard way)

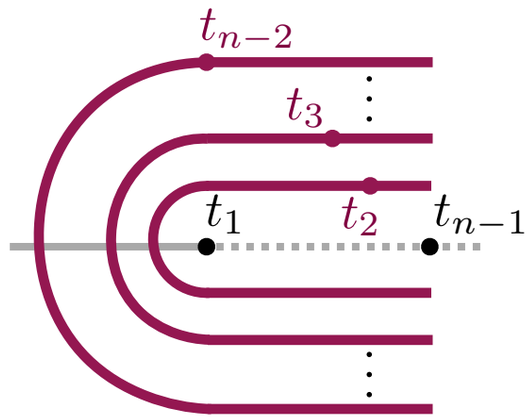
[CF, Vanhove, Wang 1806.09584]

Route 2: Read off numerators from KLT factorisation [Bjerrum-Bohr, Damgaard, Sondergaard, Vanhove 1010.3933]

$$\mathcal{A}_{closed} = \sum_{\sigma\in S_{n-3}} \int \prod_i dz_i^- (z_i^- - z_j^-)^{\alpha' k_i\cdot k_j} f(z_i^-) \times A_{open}(1,\sigma(2),\sigma(3),\ldots,n)$$

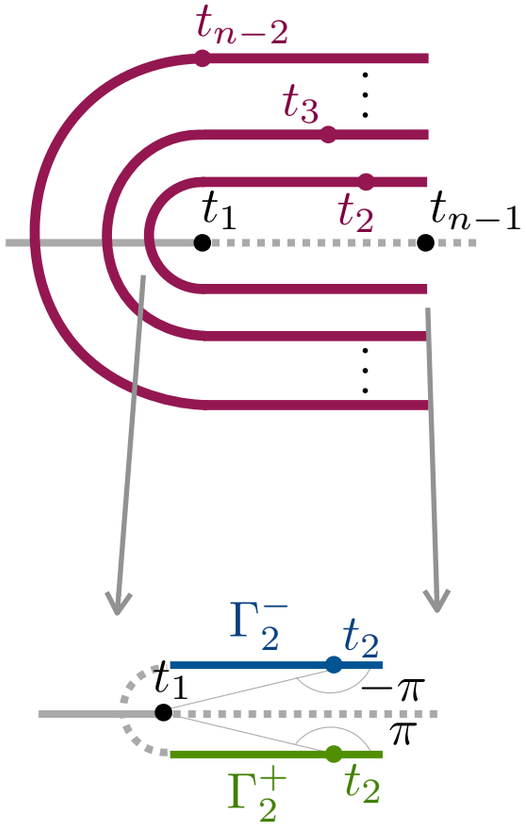
comparing the two

$$\mathcal{A}_{GR} = \sum_{\sigma\in S_{n-2}} f^{1,\sigma(2)}{}_* f^{*,\sigma(3)}{}_* \cdots f^{*,\sigma(n-1)}{}_n \times A_{YM}^{\text{colour-ordered}}(1,\sigma(2),\sigma(3),\ldots,n)$$



Insert a vertex operator and integrate it along the corresponding contour.
What kind of commutator does this contour integral define?

What are the underlying objects, and what determines their algebra?



$$\int_{\Gamma_2^+ - \Gamma_2^-} dt_2 (t_1 - t_2)^{\alpha' k_1 \cdot k_2} : e^{ik_2 \cdot X(t_2)} e^{ik_1 \cdot X(t_1)} :$$

$$= \int_{\Gamma_2^+} dt_2 \underbrace{e^{ik_1 \cdot X(t_1)} e^{ik_2 \cdot X(t_2)}}_{\sim (t_1 - t_2)^{\alpha' k_1 \cdot k_2}} \quad - e^{-i\pi\alpha' k_1 \cdot k_2} \int_{\Gamma_2^-} dt_2 \underbrace{e^{ik_2 \cdot X(t_2)} e^{ik_1 \cdot X(t_1)}}_{\sim (t_2 - t_1)^{\alpha' k_1 \cdot k_2}}$$

$$[E_1, E_2]_q = E_1 E_2 - q^{k_1 \cdot k_2} E_2 E_1$$

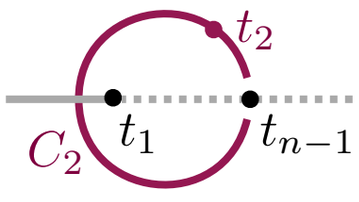
q-deformed Lie algebra

(quantum algebra)

$$E_k = \int_{\text{line interval}} dt\, V_k(t)$$

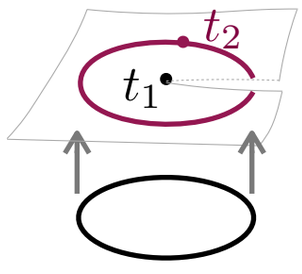
$$q = e^{-i\pi\alpha'}$$

[Dotsenko, Fateev '84,'85]
[Gómez, Sierra '90,'91]



$$= \int_{C_2} dt_2 e^{ik_1 \cdot X(t_1)} e^{ik_2 \cdot X(t_2)}$$

adjoint action

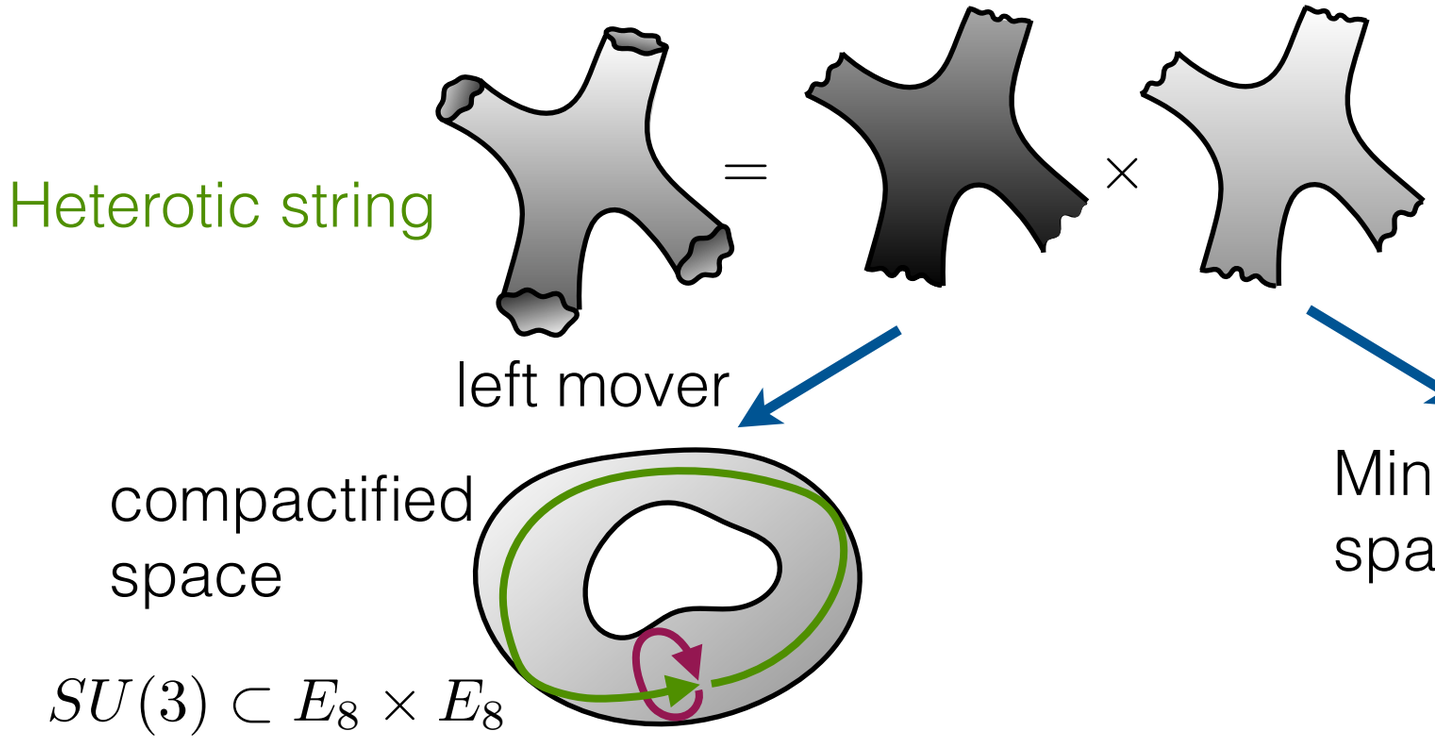


C_2 : a lifted circle on the integrand's covering surface

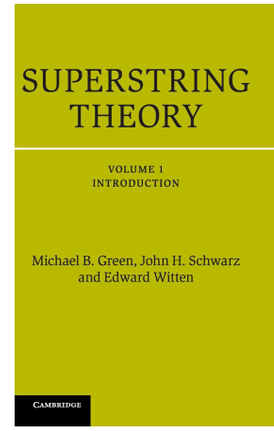
[Lentner, 1702.06431]

Why do kinematic numerators behave as if they were built from structure constants?

The same mechanism already makes colour in string theory



Green-Schwarz-Witten Vol. 1. Sec. 6.4



[Kac, Frenkel '80]
[Goddard, Olive '85]
[Gross, Harvey, Martinec, Rohm '85]

Colour numerator

$$= tr([[[T_1, T_2], T_3] \ldots, T_{n-1}] T_n)$$

$$= f_{1,2}{}^{\sigma_1} f_{\sigma_1,3}{}^{\sigma_2} \ldots f_{\sigma_{n-3},n-1}{}^{\sigma_{n-2}} \times tr(T_{\sigma_{n-2}} T_n)$$

Kinematic numerator

(n-3)-basis

$$= \langle n | \, [[E_{k_1}, E_{k_2}]_q, E_{k_3}]_q \ldots V(t_{n-1}) \, |0\rangle$$

(n-2)-basis

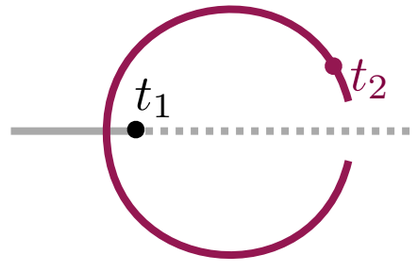
$$= \lim_{k_n^2 \to \text{mass shell}} \langle n | \, [[[E_{k_1}, E_{k_2}]_q, E_{k_3}]_q \ldots E_{k_{n-1}}]_q \, |0\rangle$$

[CF,Vanhove,Wang 1806.09584]

How does the kinematic algebra work?

special cases: integer lattice $m = \alpha' k_i \cdot k_j \in \mathbb{Z}$

Integer exponents remove the branch cuts, so the contour closes.



$$E_k^{\text{scalar}} = \int dt\, e^{ik\cdot X(t)}$$

$$[E_{k_2}^{\text{scalar}}, E_{k_1}^{\text{scalar}}]_q = \int dt_1 \oint dt_2 \underbrace{(t_2 - t_1)^m}_{\text{Koba-Nielsen factor contains the poles}} \underbrace{:e^{ik_2\cdot X(t_2)} e^{ik_1\cdot X(t_1)}:}_{\text{regular at } t_2 = t_1}$$

Residues then give the kinematic structure constants

$m \geq 0$ no pole

$$(t_2 - t_1)^m \implies [E_{k_2}^{\text{scalar}}, E_{k_1}^{\text{scalar}}]_q = 0$$

$m = -1$ simple pole

$$\frac{1}{t_2 - t_1} \implies \int dt_1 e^{i(k_1+k_2)\cdot X(t_1)}$$

$$[E_{k_2}^{\text{scalar}}, E_{k_1}^{\text{scalar}}]_q = 1\, E_{k_1+k_2}^{\text{scalar}}$$

structure constant

$m = -2$ double pole

$$\frac{1}{(t_2 - t_1)^2} \implies \int dt_1 : \frac{d}{dt_2} e^{ik_2\cdot X(t_2)} e^{ik_1\cdot X(t_1)} :\Big|_{t_2\to t_1}$$

$$= \int dt_1 ik_2 \cdot X'(t_1) e^{i(k_1+k_2)\cdot X(t_1)}$$

$$[E_{k_2}^{\text{scalar}}, E_{k_1}^{\text{scalar}}]_q = ik_{2\mu}\, E_{k_1+k_2}^{\text{vector}\mu}$$

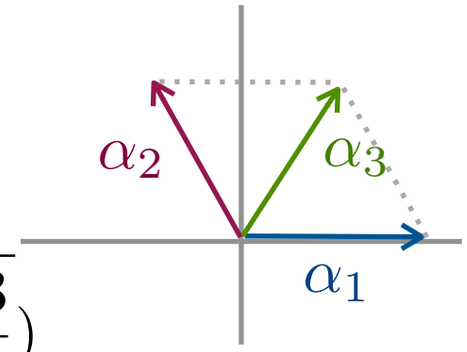
Gluon as a composite of two tachyons

[Goddard, Olive '85]

How does the kinematic algebra work?

example: SU(3)

$$\alpha_1 = (\sqrt{2}, 0)$$
$$\alpha_2 = (\frac{-1}{\sqrt{2}}, \sqrt{\frac{3}{2}})$$
$$\alpha_3 = \alpha_1 + \alpha_2 = (\frac{1}{\sqrt{2}}, \sqrt{\frac{3}{2}})$$



Cartan subalgebra

$$H_{\alpha_1} = \oint \frac{dt}{2\pi i}\, \alpha_1 \cdot i\partial X(t) \quad \leftrightarrow \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$H_{\alpha_2} = \oint \frac{dt}{2\pi i}\, \alpha_2 \cdot i\partial X(t) \quad \leftrightarrow \quad \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

Root vectors

$$E_{\alpha_1} = \oint \frac{dt}{2\pi i} c_{\alpha_1}\, e^{i\alpha_1 \cdot X(t)}$$
$$= \oint \frac{dt}{2\pi i} c_{\alpha_1}\, e^{i\sqrt{2}X^1(t)}$$

$$E_{\alpha_2} = \oint \frac{dt}{2\pi i} c_{\alpha_2}\, e^{i(-X^1/\sqrt{2} + \sqrt{3/2}X^2)}$$

$$E_{\alpha_3} = \oint \frac{dt}{2\pi i} c_{\alpha_3}\, e^{i(X^1/\sqrt{2} + \sqrt{3/2}X^2)}$$

$$\frac{1}{2}(E_{\alpha_1} + E_{-\alpha_1}) \leftrightarrow \frac{1}{2}\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \dots$$

$$\frac{1}{2i}(E_{\alpha_1} - E_{-\alpha_1}) \leftrightarrow \frac{1}{2}\begin{bmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\frac{1}{2}(E_{\alpha_2} + E_{-\alpha_2}) \leftrightarrow \frac{1}{2}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\frac{1}{2i}(E_{\alpha_2} - E_{-\alpha_2}) \leftrightarrow \frac{1}{2}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{bmatrix}$$

[Kac, Frenkel '80]

[Segal '81]

The takeaway: Vertex operators on root lattices provide a concrete realization of most familiar Lie algebras

Higher poles generate a whole tower of operators

$$\alpha' k_1 \cdot k_2 = -m$$

$$= \int dt_1 \oint_{t_1} dt_2 \frac{: e^{ik_2\cdot X(t_2)} e^{ik_1\cdot X(t_1)} :}{(t_2-t_1)^m} \quad = \frac{1}{(m-1)!}\int dt_1 : \partial_{t_2}^{m-1} e^{ik_2\cdot X(t_2)} e^{ik_1\cdot X(t_1)} :\Big|_{t_2\to t_1}$$

$$= \int dt_1 : \left[\frac{ik_{2\mu}}{(m-1)!} X^{(m-1)} + \sum_{a+b=m-1} \frac{ik_{2\mu} ik_{2\nu}}{2!a!b!} X^{(a)\mu} X^{(b)\nu} + \sum_{a+b+c=m-1} \frac{ik_{2\mu} ik_{2\nu} ik_{2\rho}}{3!a!b!c!} X^{(a)\mu} X^{(b)\nu} X^{(c)\rho} + \dots \right] e^{i(k_1+k_2)\cdot X}$$

- The expanded sum can instead be retained as one generator.

$$E_{k_1+k_2} = \frac{1}{(m-1)!}\int dt_1 : \partial_{t_2}^{m-1} e^{ik_2\cdot X(t_2)} e^{ik_1\cdot X(t_1)} :\Big|_{t_2\to t_1}$$

- Equivalently, leave the residue integral unevaluated:

$$E_{k_1+k_2} = \int dt_1 \oint_{t_1} dt_2 \frac{1}{(t_2-t_1)^m} : e^{ik_2\cdot X(t_2)} e^{ik_1\cdot X(t_1)} :$$

A single generator can contain a whole sum of vertex operators.

Beyond integer lattice

- Keep the generators in contour-integral form.
  - At generic momenta, OPE and IBP determine their structure constants.

Generic momenta: a contour-integral basis

The basis we are choosing

$$E^{\text{scalar}}_{k_1} = \int dt_1 e^{ik_1\cdot X(t_1)}$$

$$E^{\text{scalar}}_{k_1+k_2} = \int dt_1 dt_2 \frac{1}{z_2-z_1}(t_2-t_1)^{k_1\cdot k_2} : e^{ik_2\cdot X(t_2)} e^{ik_1\cdot X(t_1)} :$$

$$E^{\text{vector}\,\mu}_{k_1+k_2} = \int dt_1 dt_2 (t_2-t_1)^{k_1\cdot k_2} : X^{(1)\mu}(t_2) e^{ik_2\cdot X(t_2)} e^{ik_1\cdot X(t_1)} :$$

$$E^{\text{scalar}}_{k_1+k_2+k_3} = \int \frac{1}{t_3-t_2}\,\frac{1}{t_3-t_1}\,\prod_{1\le j<i\le n}(t_i-t_j)^{k_i\cdot k_j} : e^{ik_3\cdot X(t_3)} e^{ik_2\cdot X(t_2)} e^{ik_1\cdot X(t_1)} :$$

$$\vdots$$

$$E^{\text{tensor}\,\mu\nu} = \int \prod_{1\le j<i\le n}(t_i-t_j)^{k_i\cdot k_j} : X^{(1)\mu}(t_3) X^{(1)\nu}(t_2) e^{ik_3\cdot X(t_3)} e^{ik_2\cdot X(t_2)} e^{ik_1\cdot X(t_1)} :$$

Formal contour integrals package an unexpanded tower of vertex-operator terms.

generators are identified modulo / quotient by IBP relations

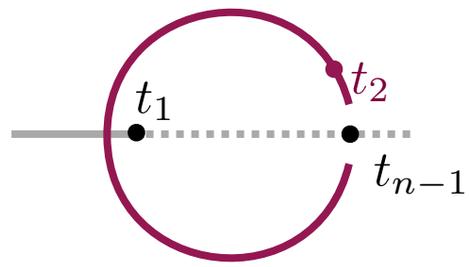
$$\int \prod_{1\le j<i\le n}(t_i-t_j)^{k_i\cdot k_j} : ik_3\cdot X^{(1)}(t_3) X^{(1)\nu}(t_2) e^{ik_3\cdot X(t_3)} e^{ik_2\cdot X(t_2)} e^{ik_1\cdot X(t_1)} :$$
$$+ \int \left(\frac{k_2\cdot k_3}{t_3-t_2}+\frac{k_1\cdot k_3}{t_3-t_1}\right) \prod_{1\le j<i\le n}(t_i-t_j)^{k_i\cdot k_j} : X^{(1)\nu}(t_2) e^{ik_3\cdot X(t_3)} e^{ik_2\cdot X(t_2)} e^{ik_1\cdot X(t_1)} : = 0$$

$$ik_{3\mu} E^{\text{tensor}\mu\nu} + k_2\cdot k_3 E^{\text{vector}\nu}_{(23,2)} + k_1\cdot k_3 E^{\text{vector}\nu}_{(13,2)} = 0$$

Structure constants from OPE and IBP

Non-integer exponents: branch cuts

$$[E_{k_2}, E_{k_1}]_q = \int_C dt_2\, (t_2 - t_1)^{k_1\cdot k_2}\, : e^{ik_2\cdot X(t_2)} e^{ik_1\cdot X(t_1)} : \qquad \in U[[t_1, t_1^{-1}]]$$



$$[\epsilon_{2\mu} E^{\text{vector}\,\mu}_{k_2}, E^{\text{scalar}}_{k_1}]_q$$

$$= \int_C dt_2\, (t_2 - t_1)^{k_1\cdot k_2} \left(: i\epsilon_2 \cdot X^{(1)}(t_2) e^{ik_2\cdot X(t_2)} : \right) \left(e^{ik_1\cdot X(t_1)} \right)$$

$$= \int_C dt_2\, (t_2 - t_1)^{k_1\cdot k_2}\, : \left(\frac{k_1\cdot\epsilon_2}{t_2 - t_1} + i\epsilon_2\cdot X^{(1)}(t_2) \right) e^{ik_2\cdot X(t_2)} e^{ik_1\cdot X(t_1)} :$$

IBP

$$= \int_C dt_2\, (t_2 - t_1)^{k_1\cdot k_2}\, : \left(\frac{-k_1\cdot\epsilon_2}{k_1\cdot k_2} ik_2\cdot X^{(1)}(t_2) + i\epsilon_2\cdot X^{(1)}(t_2) \right) e^{ik_2\cdot X(t_2)} e^{ik_1\cdot X(t_1)} :$$

$$i\frac{-k_1\cdot\epsilon_2\, k_2\cdot X^{(1)}(t_2) + k_1\cdot k_2\, \epsilon_2\cdot X^{(1)}(t_2)}{k_1\cdot k_2}$$

$$= i\frac{k_1\cdot F_2\cdot X^{(1)}(t_2)}{k_1\cdot k_2}$$

$F_{\mu\nu} = k_\mu\epsilon_\nu - \epsilon_\mu k_\nu$

field strength

$$= \frac{(k_1\cdot F_2)_\mu}{k_1\cdot k_2} \int_C dt_2\, (t_2 - t_1)^{k_1\cdot k_2}\, : iX^{(1)\mu}(t_2) e^{ik_2\cdot X(t_2)} e^{ik_1\cdot X(t_1)} :$$

structure constant (gauge invariant)

How structure constants build a HEFT numerator term

$$[\epsilon_1 \cdot E^{\text{vector}}_{k_1}, \epsilon_2 \cdot E^{\text{vector}}_{k_2}]_q = \epsilon_1 \cdot F_2 \cdot E^{\text{vector}}_{k_1+k_2} + \text{tensor}$$

$$E^{\text{vector}\mu}_{k_1+k_2} = \int dt_2 \frac{\text{KN}}{t_{12}} : iX'^{\mu}(t_2) e^{ik_1\cdot X(t_1)+ik_2\cdot X(t_2)} :$$

$$E^{\text{tensor}\mu\nu}_{k_1+k_2+k_3} = \int dt_2 dt_3 \frac{\text{KN}}{t_{12}} : iX'^{\mu}(t_3) iX'^{\nu}(t_2) e^{ik_1\cdot X(t_1)+ik_2\cdot X(t_2)+ik_3\cdot X(t_3)} :$$

$$[\epsilon_1 \cdot F_2 \cdot E^{\text{vector}}_{k_1+k_2}, \epsilon_3 \cdot E^{\text{vector}}_{k_3}]_q = (\epsilon_1 \cdot F_2)_\mu \epsilon_{3\nu} E^{\text{tensor}\mu\nu}_{k_1+k_2+k_3} + \text{vector} + \text{scalar}$$

$$E^{\text{tensor}\mu\nu}_{k_1+k_2+k_3+k_4} = \int dt_2 dt_3 dt_4 \frac{\text{KN}}{t_{12}t_{24}} : iX'^{\mu}(t_4) iX'^{\nu}(t_3) e^{ik_1\cdot X(t_1)+\cdots+ik_4\cdot X(t_4)} :$$

$$[(\epsilon_1 \cdot F_2)_\mu \epsilon_{3\nu} E^{\text{tensor}\mu\nu}_{k_1+k_2+k_3}, \epsilon_4 \cdot E^{\text{vector}}_{k_4}]_q = \text{3-tensor} + (\epsilon_1 \cdot F_2 \cdot F_4)_\mu \epsilon_{3\nu} E^{\text{tensor}\mu\nu}_{k_1+k_2+k_3+k_4} + \text{vector} + \text{scalar}$$

$$[(\epsilon_1 \cdot F_2 \cdot F_4)_\mu \epsilon_{3\nu} E^{\text{tensor}\mu\nu}_{k_1+k_2+k_3+k_4}, \epsilon_5 \cdot E^{\text{vector}}_{k_5}]_q = \cdots + (\epsilon_1 \cdot F_2 \cdot F_4)_\mu (\epsilon_3 \cdot F_5)_\nu E^{\text{tensor}\mu\nu}_{k_1+\cdots+k_5}$$

$$E^{\text{tensor}\mu\nu}_{k_1+\cdots+k_5} = \int dt_2 \ldots dt_5 \frac{\text{KN}}{t_{12}t_{24}t_{35}} : iX'^{\mu}(t_4) iX'^{\nu}(t_5) e^{ik_1\cdot X(t_1)+\cdots+ik_5\cdot X(t_5)} :$$

$$E^{\text{tensor}\mu\nu}_{k_1+\cdots+k_6} = \int dt_2 \ldots dt_6 \frac{\text{KN}}{t_{12}t_{24}t_{35}t_{46}} : iX'^{\mu}(t_6) iX'^{\nu}(t_4) e^{ik_1\cdot X(t_1)+\cdots+ik_6\cdot X(t_6)} :$$

$$[(\epsilon_1 \cdot F_2 \cdot F_4)_\mu (\epsilon_3 \cdot F_5)_\nu E^{\text{tensor}\mu\nu}_{k_1+\cdots+k_5}, \epsilon_6 \cdot E^{\text{vector}}_{k_6}]_q$$

$$= \cdots + (\epsilon_1 \cdot F_2 \cdot F_4 \cdot F_6)_\mu (\epsilon_3 \cdot F_5)_\nu E^{\text{tensor}\mu\nu}_{k_1+\cdots+k_6}$$

$$\int dt_2 \ldots dt_6 \frac{\text{KN}}{t_{12}t_{24}t_{35}t_{46}t_{57}t_{67}} \sim \frac{(-1)(k_1+k_2)\cdot k_3}{(k_1+k_2+k_4+k_6)\cdot v}$$

$$\langle v| (\epsilon_1 \cdot F_2 \cdot F_4 \cdot F_6)_\mu (\epsilon_3 \cdot F_5)_\nu E^{\text{tensor}\mu\nu}_{k_1+\cdots+k_6} e^{iv\cdot X(1)} |0\rangle$$

$$= \quad \epsilon_1 \cdot F_2 \cdot F_4 \cdot F_6 \cdot v \quad \epsilon_3 \cdot F_5 \cdot v \quad \frac{(-1)(k_1+k_2)\cdot k_3}{(k_1+k_2+k_4+k_6)\cdot v}$$

$$= \boxed{\frac{q_1 \cdot F_1 \cdot F_2 \cdot F_4 \cdot F_6 \cdot v}{q_1 \cdot k_1} \quad \frac{q_3 \cdot F_3 \cdot F_5 \cdot v}{q_3 \cdot k_3} \quad \frac{(-1)(k_1+k_2)\cdot k_3}{(k_1+k_2+k_4+k_6)\cdot v}}$$

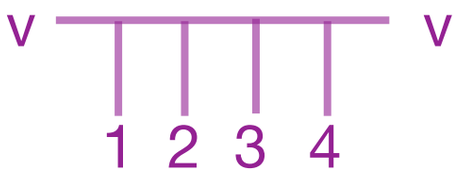
An application: explaining the HEFT fusion rules

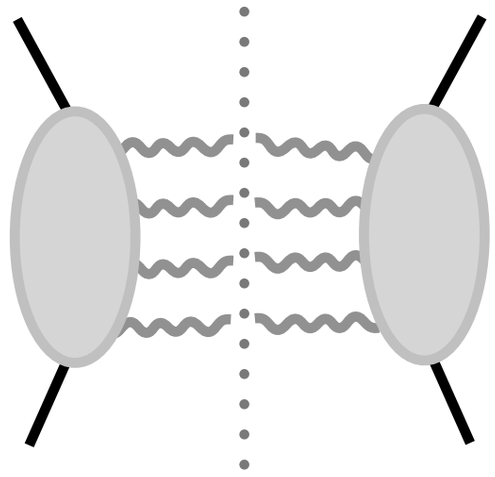
- The kinematic algebra generates numerator terms through nested q-commutators.

- The same construction explains the HEFT fusion patterns and their kinematic evaluation.

Kinematic Hopf Algebra/Fusion Rules

Fusion rules give an efficient algorithm for basis numerators





Two-step structure of fusion rules

(i) combinatorics

$$T_{(12)} \star T_{(3)} = -T_{(123)} + T_{(12),(3)} + T_{(13),(2)}$$

$$(12) \star (\mathbf{3}) \to (12\mathbf{3})$$
$$(12)(\mathbf{3})$$
$$(1\mathbf{3})(2)$$

(ii) Evaluation

universal formula for all terms

$$\left\langle T_{(1\tau_1),(\tau_2),\ldots,(\tau_r)} \right\rangle = \frac{v \cdot F_{1\tau_1} \cdot V_{\Theta(\tau_2)} \cdot F_{\tau_2} \cdots V_{\Theta(\tau_r)} \cdot F_{\tau_r} \cdot v}{(n-2)(v \cdot p_1)(v \cdot p_{1\tau_1}) \cdots (v \cdot p_{1\tau_1\tau_2\cdots\tau_{r-1}})}$$

[Brandhuber, Brown, Chen, Gowdy, Travaglini, Wen 2208.05886]

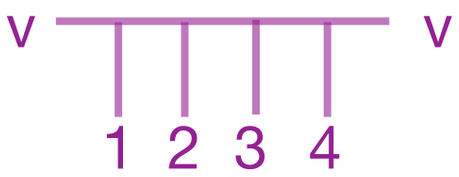
[Brandhuber, Chen, Johansson, Travaglini, Wen 2111.15649]

[Chen, Johansson, Teng, Wang 2104.12726]

[Chen, Johansson, Teng, Wang 1906.10683]

Kinematic Hopf Algebra/Fusion Rules

Fusion rules give an efficient algorithm for basis numerators



For HEFT, the generators give an algebraic explanation of the fusion patterns.

$$(124)(35) \star (6) \;\; \to \cdots + \;\; (1246)(35) \;\; + \ldots$$

$$[(\epsilon_1 \cdot F_2 \cdot F_4)_\mu \, (\epsilon_3 \cdot F_5)_\nu E^{\text{tensor}\mu\nu}_{k_1+k_2+\cdots+k_5}, \epsilon_6 \cdot E^{\text{vector}}_{k_6}]_q$$

$$= \cdots + \;\; (\epsilon_1 \cdot F_2 \cdot F_4 \cdot F_6)_\mu \, (\epsilon_3 \cdot F_5)_\nu E^{\text{tensor}\mu\nu}_{k_1+k_2+\cdots+k_6} \;\; + \ldots$$

The q-commutator appends F_6 to an existing chain via IBP

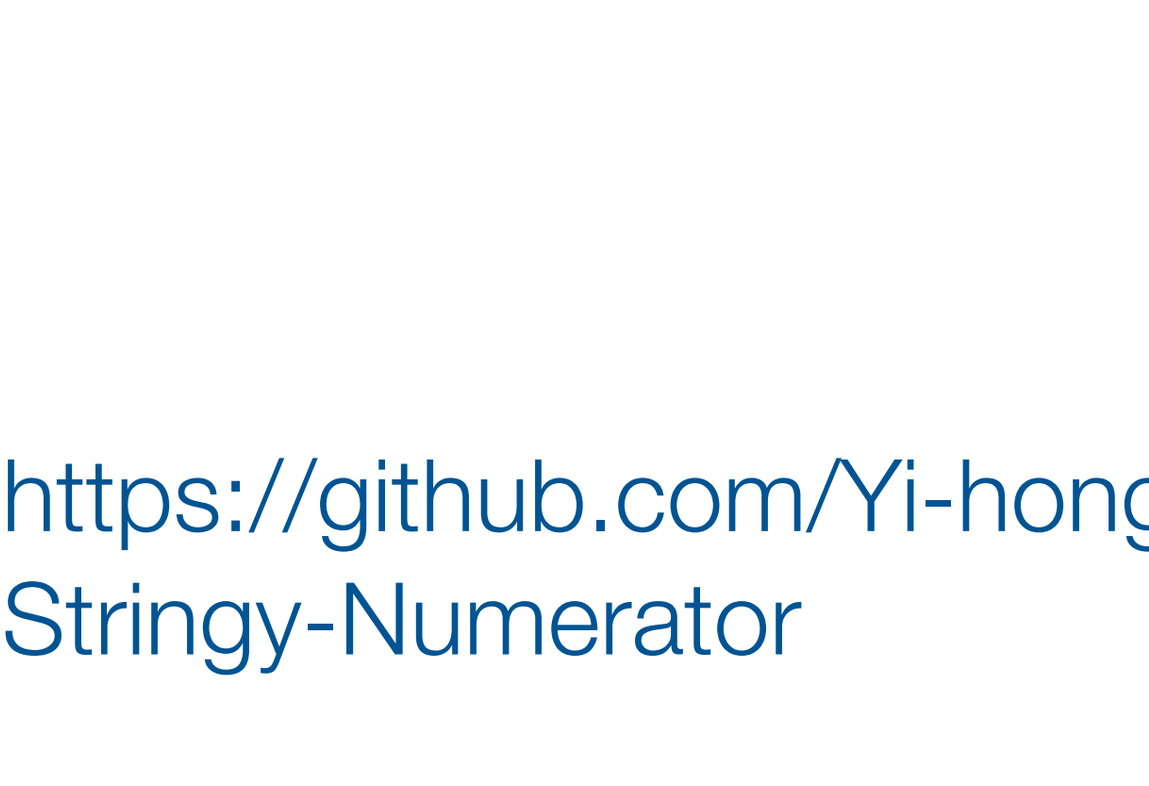
The final matrix element supplies the remaining kinematic factors.

$$\langle v| \, (\epsilon_1 \cdot F_2 \cdot F_4 \cdot F_6)_\mu (\epsilon_3 \cdot F_5)_\nu E^{\text{tensor}\mu\nu}_{k_1+\cdots+k_6} e^{iv\cdot X(1)} \, |0\rangle \;\; = \;\; \epsilon_1 \cdot F_2 \cdot F_4 \cdot F_6 \cdot v \;\; \epsilon_3 \cdot F_5 \cdot v \;\; \frac{(-1)(k_1+k_2)\cdot k_3}{(k_1+k_2+k_4+k_6)\cdot v}$$

$$= \boxed{\frac{q_1 \cdot F_1 \cdot F_2 \cdot F_4 \cdot F_6 \cdot v}{q_1 \cdot k_1} \;\; \frac{q_3 \cdot F_3 \cdot F_5 \cdot v}{q_3 \cdot k_3} \;\; \frac{(-1)(k_1+k_2)\cdot k_3}{(k_1+k_2+k_4+k_6)\cdot v}}$$

Mathematica code

https://github.com/Yi-hongWang/Stringy-Numerator



README

author Chih Hao Fu | author Pierre Vanhove | author Yihong Wang

arXiv 2501.14523 | Language Mathematica

HEFT Numerators from Kinematic Algebra

This algorithm implements the construction of the numerator factor for the multigluon emission from a massive scalar in heavy mass effective field theory from the stringy kinematic algebra.

The algorithm is presented in the paper

HEFT Numerators from Kinematic Algebra 2501.14523 by Chih-Hao Fu, Pierre Vanhove and Yihong Wang

Files

1. functionsHEFT.wl: Contains all the defined functions used for symbolic computations.
2. HEFT-numerators.nb: Mathematica worksheet that includes usage examples demonstrating how to call the functions and their expected outputs. A pdf output of this worksheet is HEFT-numerators.pdf
3. numerator-six-gluons.txt: the numerator for the emission of six gluons. Size of the file 342555.
4. numerator-seven-gluons.txt: the numerator for the emission of seven gluons. Size of the file 3816849.
5. numerator-eight-gluons.txt: the numerator for the emission of eight gluons. Size of the file 47372948.

README

How to Use

Open `functionsHEFT.wl` in Mathematica and evaluate the entire notebook to load all functions.

Functions

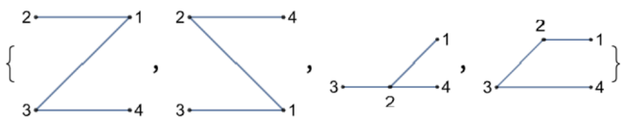
The functions provided are function of `Ngluons` the number of the emitted gluons and `P` the momentum of the incoming massive particle

- `GIAOTree[Ngluons,P]` which generates all the tree graphs as described in section 4.2 of the paper
- `GINum[Ngluons,P]` which generates the numerator using the algorithm in paper
- `QMNum[Ngluons,P]` which generates the numerator using the algorithm described in these papers A new gauge-invariant double copy for heavy-mass effective theory by A. Brandhuber, G. Chen, G. Travaglini and C. Wen and Kinematic Hopf Algebra for Bern-Carrasco-Johansson Numerators in Heavy-Mass Effective Field Theory and Yang-Mills Theory by A. Brandhuber, G. Chen, H. Johansson, G. Travaglini and C. Wen

The gluons have massless momentum k_i and $k_i^2 = 0$, and the two massive scalars have massive momentum p and $p^{'}$ with $p^2 = (p^{'})^2 \neq 0$. We have the Momentum conservation $k_1 + \ldots + k_n = p^{'} - p$

Three gluon example

```
GIAOTree[3, p]
(* lists all the rooted tree graphs for the emission of 3 gluons as described in section 4.2 *
```




```
GINum[3, p]
(* Outputs the numerator factors for the emission of 3 gluons*)
```

A final conceptual point: why “Hopf algebra”?

- Fusion rules were originally formulated in the language of a kinematic Hopf algebra
- The vertex-operator construction explains why such a structure is natural
- Physically, it is the rule for extending an operation from one state to many states

Why Hopf algebra?

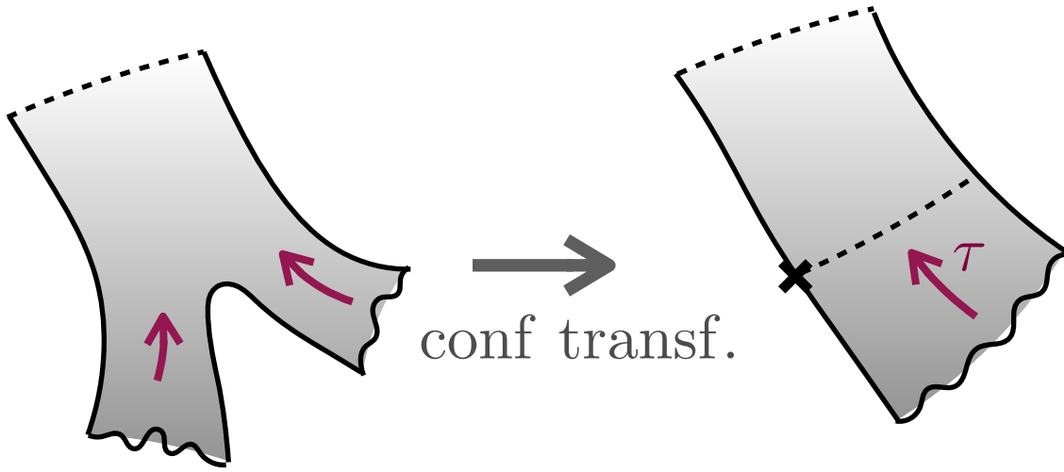
The coproduct specifies the action on multiparticle states

Example: Poincare algebra

	Physicists write	Mathematicians write	
1-particle states	$e^{iJ\theta}\,\lvert v\rangle$	rotation	
	e^{iHt}	time translation	
	$\vdots$	$\vdots$	
2-particle states	$\lvert p_1, p_2\rangle$	$v_{p_1} \otimes v_{p_2}$	
	$J_{total} = J_1 + J_2$	$\Delta(J) = J \otimes 1 + 1 \otimes J$	coproduct (operation)
	$e^{iJ_{total}\theta} = e^{iJ_1\theta} e^{iJ_2\theta}$	$\Delta(e^{iJ\theta}) = e^{iJ\theta} \otimes e^{iJ\theta}$	
2 particle states also form a symmetry representation	$e^{iJ\theta_2} e^{iJ\theta_1}$	$\Delta(e^{iJ\theta_2} e^{iJ\theta_1}) = \Delta(e^{iJ\theta_2})\,\Delta(e^{iJ\theta_1})$	homomorphism
there are group inverses	$e^{-iJ\theta}$	$S(J) = -J$	antipode (operation)
		$S(e^{iJ\theta}) = e^{-iJ\theta}$	

Why Hopf algebra?

Screening contours realise the kinematic coproduct



Mathematicians write

2 particle state $v_{p_1} \otimes v_{p_2}$

coproduct (operation) $\Delta(J) = J \otimes 1 + 1 \otimes J$

$\Delta(e^{iJ\theta}) = e^{iJ\theta} \otimes e^{iJ\theta}$

2 particle states also form a symmetry representation $\Delta(e^{iJ\theta_2} e^{iJ\theta_1}) = \Delta(e^{iJ\theta_2})\,\Delta(e^{iJ\theta_1})$

$|k_2\rangle \otimes |k_1\rangle \to V_2(t_2)V_1(t_1)$

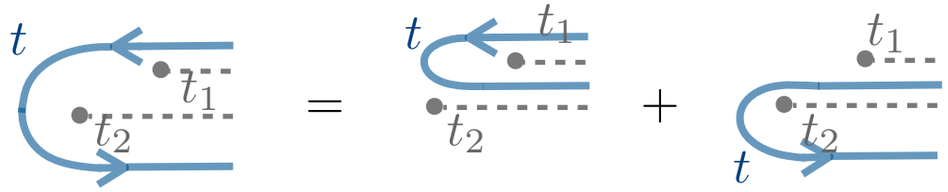
$\simeq V_2(t_2)\,|k_1\rangle$

$\Delta(E_k) = E_k \otimes e^{\alpha' k\cdot H} + 1 \otimes E_k$

q-deformed Lie algebra

$$\int dt V_k(t)\, V_2(t_2)V_1(t_1)$$
$$= \left(\int dt V_k(t)V_2(t_2)\right) V_1(t_1)$$
$$+ \int dt V_2(t_2) q^{k\cdot k_2}\,(V_k(t)V_1(t_1))$$

$V_k V_2 = q^{k\cdot k_2} V_k V_2$ braiding relation



Physical meaning of the generators:

E_k shifts momentum by k and acts on the string oscillator degrees of freedom

[CF,Wang 2005.05177]

[Semikhatov, Tipunin 1101.5810]

[Gómez, Sierra '91]

Summary: an operator origin for BCJ numerators

- **Screening operators realise quantum-group generators.**

  - They shift momentum and act on string oscillator states.

$$E_k = \int_{\text{line interval}} dt\, V_k(t) \qquad\qquad V_p(z) \xrightarrow{E_k} V_{p+k}^{\text{new}}(z)$$

- **OPE and IBP compute the kinematic structure constants.**

  - A common vertex-operator framework for colour and kinematics.

Colour	Kinematics
$[T_a, T_b] = \sum_c f_{ab}{}^c T_c$	$[E_A, E_B]_q = \sum_C f_{AB}{}^C E_C$

- **Nested brackets and external-state evaluation yield BCJ numerators in the field-theory limit.**

  - HEFT: one explicit contribution

$$\frac{q_1 \cdot F_1 \cdot F_2 \cdot F_4 \cdot F_6 \cdot v}{q_1 \cdot k_1} \quad \frac{q_3 \cdot F_3 \cdot F_5 \cdot v}{q_3 \cdot k_3} \quad \frac{(-1)(k_1 + k_2) \cdot k_3}{(k_1 + k_2 + k_4 + k_6) \cdot v}$$

Outlook and open questions

1. A wider spectrum of states

   - The vertex-operator framework extends beyond scalar HEFT.

   - Can higher string states and supersymmetric sectors be organised uniformly?

     SUSY, RNS string, and fermionic vertex operators

2. From screening operators to field-theory generators

   - Can the string generators be realised directly in field-theory terms?

$$E_k^{q-deformed} = E_k^{(0)} + \alpha' E_k^{(1)} + (\alpha')^2 E_k^{(2)} + \cdots \quad \text{(Rigidity Theorem)}$$

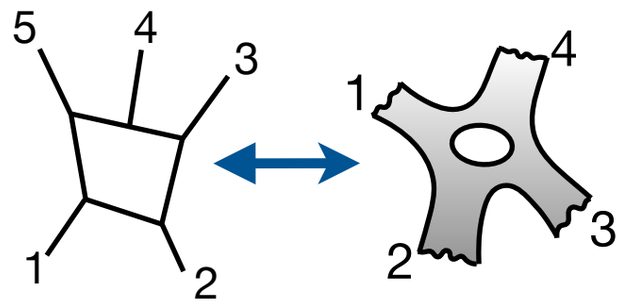
   - Drinfeld doubles and negative root generators [CF, Krasnov1603.02033]

$$A_k = A^\mu \frac{\partial}{\partial x^\mu} = e^{ik\cdot x}\epsilon^\mu \frac{\partial}{\partial x^\mu}$$

$$[A_1, A_2] = [A_1^\mu \frac{\partial}{\partial x^\mu}, A_2^\nu \frac{\partial}{\partial x^\nu}]$$

3. One-loop kinematic algebra

   - Possible route: KZB monodromy and elliptic structures. (Suggested by Drinfeld Kohno theorem)



APPENDIX A

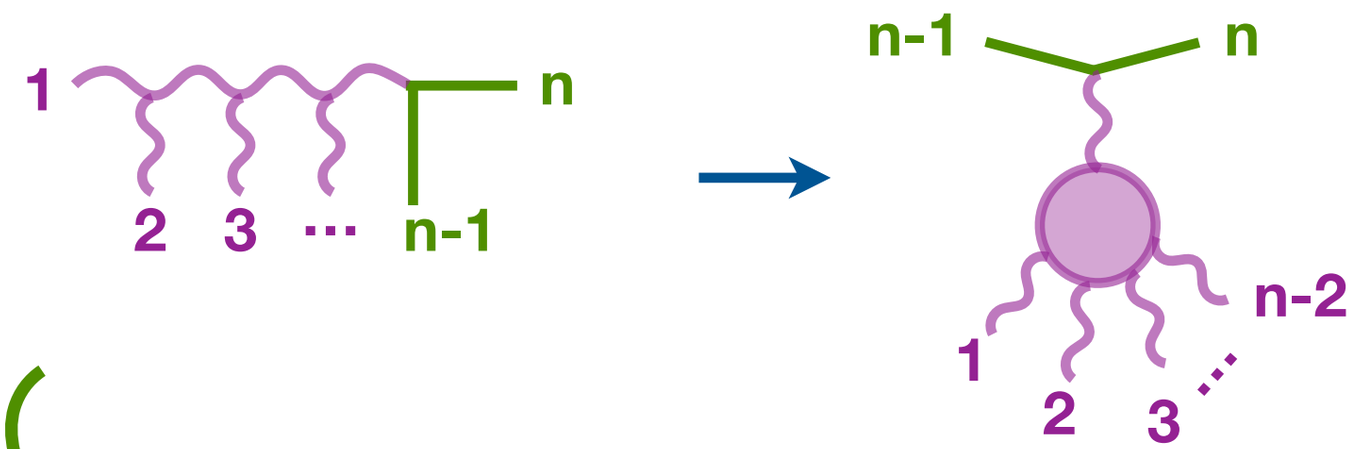
q-commutators

string theory explanation

(n-3)-basis numerators

[Bjerrum-Bohr,Damgaard,Sondergaard,Vanhove 1010.3933]

$$N(123\ldots n) = \frac{1}{n-2} \sum_{\sigma \in S_{n-3}} \mathcal{S}[234\ldots n-2|\sigma(2)\,\sigma(3)\ldots\sigma(n-2)]\; A(1,\sigma,n,n-1)$$



$$(t_1, t_n, t_{n-1}) \to (0, 1, \infty)$$

$$N(123\ldots n) = \lim_{\alpha' \to 0} \frac{1}{n-2} \langle p|\, e^{ip\cdot X(t_n)} \left[\left[\left[E^{\text{vector}}_{k_1}, E^{\text{vector}}_{k_2}\right]_q, E^{\text{vector}}_{k_3}\right]_q \ldots, E^{\text{vector}}_{k_{n-2}}\right]_q |0\rangle$$

leg n-1

$t_n \to 1$

$t_1 \to 0$

$$\ldots \int_C dt_3 i\epsilon_3 \cdot X^{(1)}(t_3) e^{ik_3\cdot X(t_3)} \int_C dt_2 i\epsilon_2 \cdot X^{(1)}(t_2) e^{ik_2\cdot X(t_2)} i\epsilon_1 \cdot X^{(1)}(t_1) e^{ik_1\cdot X(t_1)}$$

What are these T's?



$$N(123\ldots n) = \lim_{\alpha'\to 0} \frac{1}{n-2} \langle p|\, e^{ip\cdot X(t_n)} \left[\left[\left[E^{\text{vector}}_{k_1}, E^{\text{vector}}_{k_2}\right]_q, E^{\text{vector}}_{k_3}\right]_q \ldots, E^{\text{vector}}_{k_{n-2}}\right]_q |0\rangle$$

$$\ldots \int_C dt_3 i\epsilon_3 \cdot X^{(1)}(t_3) e^{ik_3\cdot X(t_3)} \int_C dt_2 i\epsilon_2 \cdot X^{(1)}(t_2) e^{ik_2\cdot X(t_2)} i\epsilon_1 \cdot X^{(1)}(t_1) e^{ik_1\cdot X(t_1)}$$

fusion rules:

$$T_{(1)} \star T_{(2)} = T_{(12)} + T_{(1)(2)}$$

$(1) \star (\mathbf{2})$

$\to (1\mathbf{2})$

$(1)(\mathbf{2})$

$$\left(\int dt_2 \epsilon_2 \cdot X^{(1)}(t_2) e^{ik_2\cdot X(t_2)}\right)\left(\epsilon_1 \cdot X^{(1)}(t_1) e^{ik_1\cdot X(t_1)}\right)$$

$$= \int dt_2 (t_2 - t_1)^{\alpha' k_2\cdot k_1} \frac{1}{t_2 - t_1} \epsilon_1 \cdot F_2 \cdot X^{(1)}(t_2) e^{ik_2\cdot X(t_2)} e^{ik_1\cdot X(t_1)}$$

$$+ \int dt_2 (t_2 - t_1)^{\alpha' k_2\cdot k_1}\, \epsilon_2 \cdot X^{(1)}(t_2) \epsilon_1 \cdot X^{(1)}(t_1) e^{ik_2\cdot X(t_2)} e^{ik_1\cdot X(t_1)}$$

Leading terms after Wick contractions

$$\frac{1}{n-2} \int_C dt_2 \ldots dt_8 \,\text{Koba-Nielsen} \times \text{poles}$$

$$\langle p|\, e^{ip\cdot X(t_n)} i\epsilon_1 \cdot F_4 \cdot F_5 \cdot F_8 \cdot X^{(1)}(t_8)\, i\epsilon_2 \cdot F_6 \cdot X^{(1)}(t_6)\, i\epsilon_3 \cdot F_7 \cdot X^{(1)}(t_7)\, e^{ik_1\cdot X} \ldots e^{ik_8\cdot X} |0\rangle$$

$$(\epsilon_1 \cdot F_4 \cdot F_5 \cdot F_8 \cdot p)\,(\epsilon_2 \cdot F_6 \cdot p)\,(\epsilon_3 \cdot F_7 \cdot p)$$

(heavy massive scalar)

$$T_{(1458),(26),(37)} = \frac{v \cdot F_{1458} \cdot V_1 \cdot F_{26} \cdot V_{12} \cdot F_{37} \cdot v}{8 v\cdot k_1\, v\cdot k_{1458}\, v\cdot k_{124568}}$$

$$V_{\Theta(\tau)\,\mu\nu} = v_\mu k_{\Theta(\tau)\,\nu}$$

string theory explanation

suppose if we shift the polarisations

$$\epsilon \to \epsilon' = \epsilon + \alpha k$$

suppose if we fixe gauge by requiring that it be orthogonal to some vector q

$$(\epsilon + \alpha k) \cdot q = 0$$

$$\to \quad \alpha = -\frac{q \cdot \epsilon}{q \cdot k}$$

$$\epsilon'_\mu = \epsilon_\mu - \frac{q \cdot \epsilon}{q \cdot k} k_\mu = \frac{q \cdot k\, \epsilon_\mu - q \cdot \epsilon\, k_\mu}{q \cdot k}$$

$$= \frac{1}{q \cdot k} (q \cdot F)_\mu$$

those already packed into fieldstrengths are not affected

$$\frac{1}{n-2} \int_C dt_2 \dots dt_8 \,\text{Koba-Nielsen} \times \text{poles}$$

$$\langle p|\, e^{ip \cdot X(t_n)} i\epsilon_1 \cdot F_4 \cdot F_5 \cdot F_8 \cdot X^{(1)}(t_8)\, i\epsilon_2 \cdot F_6 \cdot X^{(1)}(t_6)\, i\epsilon_3 \cdot F_7 \cdot X^{(1)}(t_7)\, e^{ik_1 \cdot X} \dots e^{ik_8 \cdot X}\, |0\rangle$$

$$(\epsilon_1 \cdot F_4 \cdot F_5 \cdot F_8 \cdot p)\,(\epsilon_2 \cdot F_6 \cdot p)\,(\epsilon_3 \cdot F_7 \cdot p)$$

$$\frac{1}{p \cdot k_1} p \cdot F_1 \qquad \frac{1}{q_2 \cdot k_2} q_2 \cdot F_2 \qquad \frac{1}{q_3 \cdot k_3} q_3 \cdot F_3$$

$$T_{(1458),(26),(37)} = \frac{v \cdot F_{1458} \cdot V_1 \cdot F_{26} \cdot V_{12} \cdot F_{37} \cdot v}{8 v \cdot k_1\, v \cdot k_{1458}\, v \cdot k_{124568}}$$

$$V_{\Theta(\tau)\,\mu\nu} = v_\mu k_{\Theta(\tau)\,\nu}$$

$$F_{\mu\nu} = k_\mu \epsilon_\nu - \epsilon_\mu k_\nu$$

APPENDIX B

branch cut & IBP

Evaluating integrals

integration rules

$$\lim_{\alpha'\to 0}\frac{1}{n-2}\langle p|\, e^{ip\cdot X(t_n)}\left[\left[\left[E^{\text{vector}}_{k_1},E^{\text{vector}}_{k_2}\right]_q,E^{\text{vector}}_{k_3}\right]_q\ldots,E^{\text{vector}}_{k_{n-2}}\right]_q|0\rangle$$

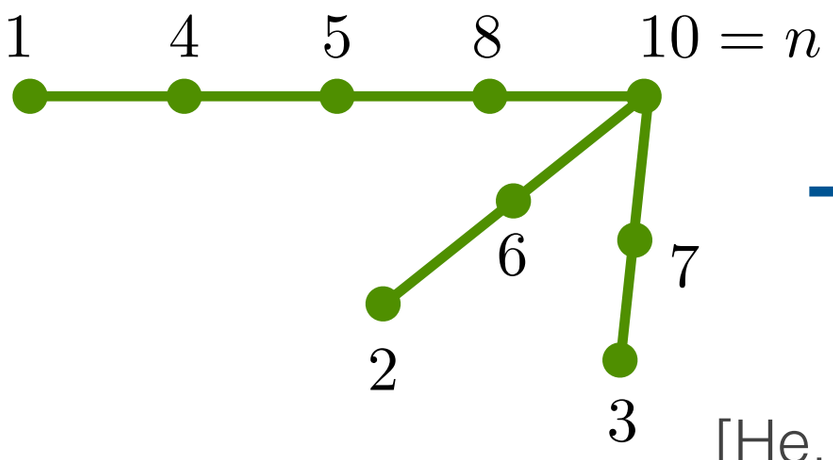
$$=\frac{1}{n-2}\int_C dt_2\ldots dt_8\,\text{Koba-Nielsen}\times\text{poles}$$

$$\langle p|\, e^{ip\cdot X(t_n)} i\epsilon_1\cdot F_4\cdot F_5\cdot F_8\cdot X^{(1)}(t_8)\, i\epsilon_2\cdot F_6\cdot X^{(1)}(t_6)\, i\epsilon_3\cdot F_7\cdot X^{(1)}(t_7)\, e^{ik_1\cdot X}\ldots e^{ik_8\cdot X}|0\rangle$$

$$+\ldots$$

$$\int dt_2\ldots dt_8\prod_{i<j}(t_i-t_j)^{k_i\cdot k_j}\frac{1}{(t_1-t_4)(t_4-t_5)(t_5-t_8)(t_8-t_{10})}\times\frac{1}{(t_2-t_6)(t_6-t_{10})}\times\frac{1}{(t_3-t_7)(t_7-t_{10})}$$

graphical representation



$\longrightarrow\ (1458)(26)(37)$

[He, Teng, Zhang 1812.03369]

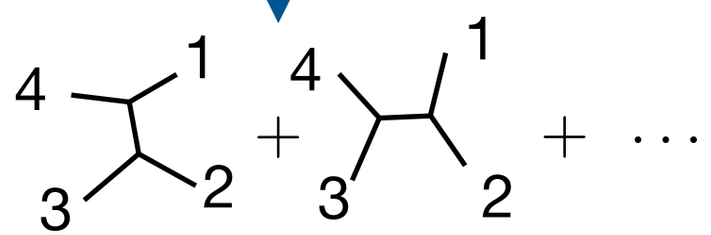
[Baadsgaard,Bjerrum-Bohr,Bourjaily,Damgaard 1506.06137]

propagators of a cubic scalar theory

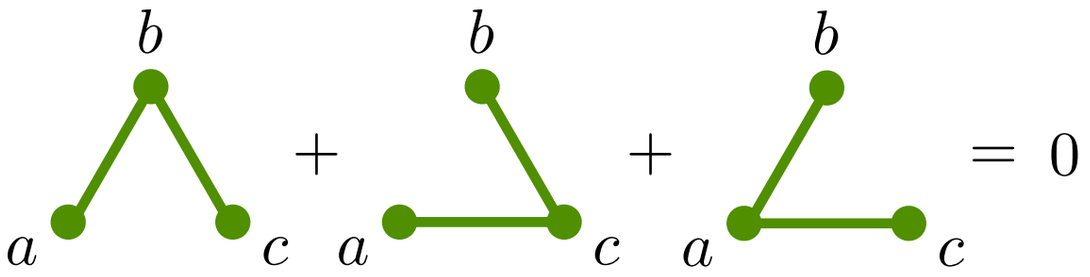
$$\int dt_\ell\prod_{i<j}(t_i-t_j)^{\alpha' k_i\cdot k_j}\times\frac{1}{(t_1-t_2)(t_2-t_3)\ldots(t_{n-2}-t_{n-1})}$$

Parke-Taylor factor

$\alpha'\to 0$



Evaluating integrals



$$= 0$$

$$\frac{1}{(a-b)(b-c)} + \frac{1}{(b-c)(c-a)} + \frac{1}{(c-a)(a-b)}$$

$$= \frac{1}{(a-b)(b-c)(c-a)}\left((c-a)+(a-b)+(b-c)\right)$$

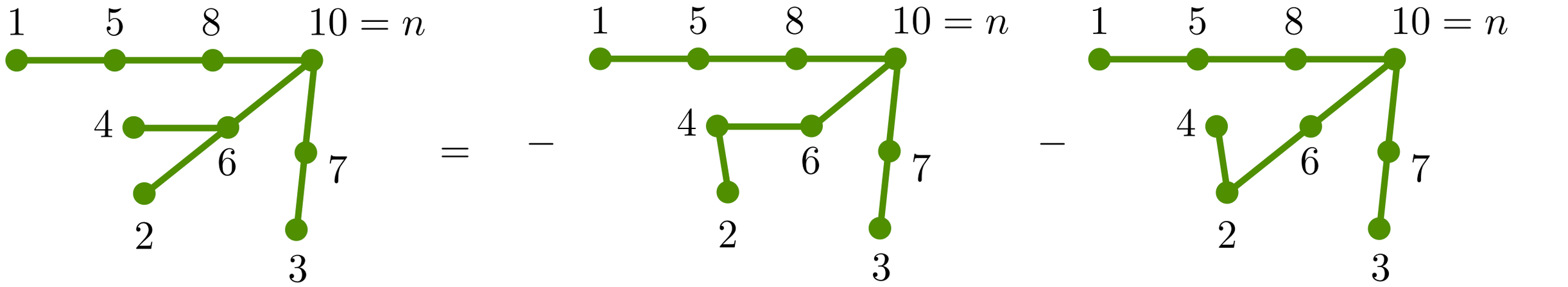
$$= 0$$

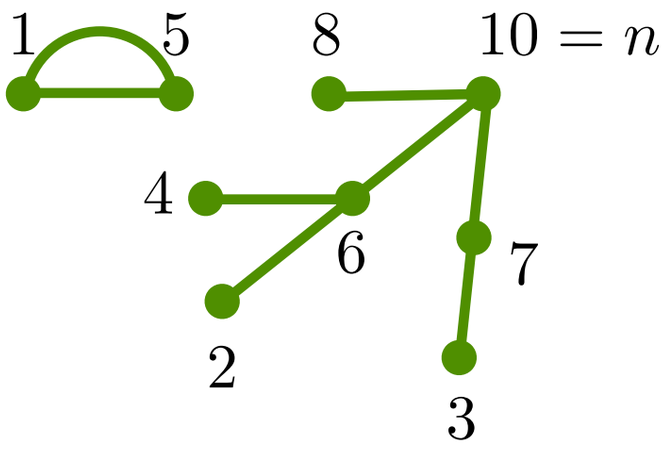
partial fraction

Varchenko's triangle



A. Varchenko





loop/disconnected graphs are sub-leading

Evaluating integrals

integration rules

rules for calculation:

(1) must be in (relative) ascending order



$= 1$



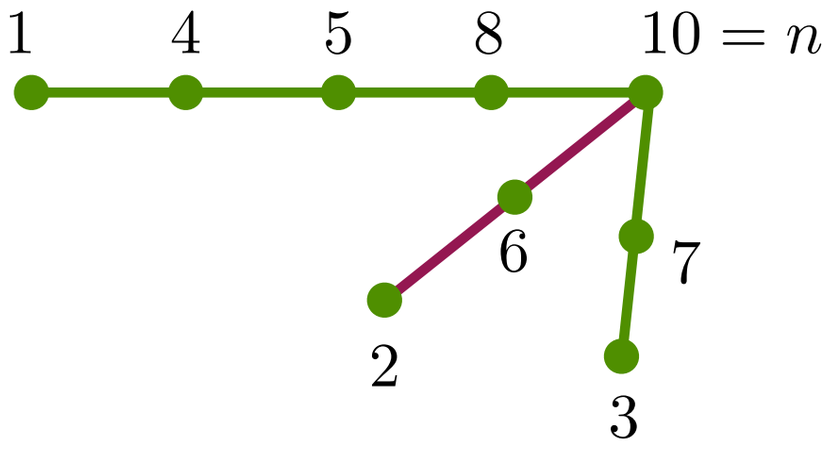
$= 0$

all labels on the left which are smaller

first label of a branch

(2) every branch contributes a factor $\dfrac{k_{\theta(\tau_i(1))} \cdot k_{\tau_i(1)}}{k_{\text{all left to } \tau_i(1)} \cdot p}$

proved by Shapovalov form



$$\frac{k_1 \cdot k_2}{(k_1 + k_4 + k_5 + k_8) \cdot p}$$

$\longrightarrow$ $(1458)(26)(37)$

(3) sum over permutations of branches

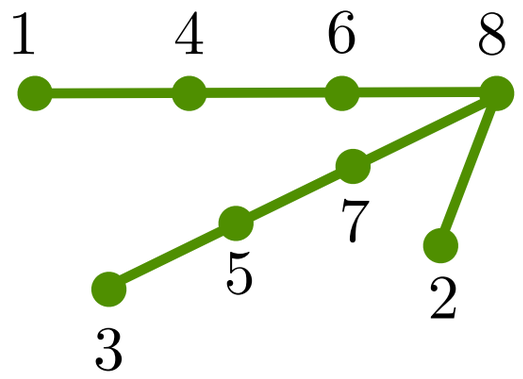
Evaluating integrals

explanation for the formula $\dfrac{k_{\theta(\tau_i(1))}\cdot k_{\tau_i(1)}}{k_{\text{all left to }\tau_i(1)}\cdot p}$

example: $\displaystyle\int_C \prod_{i<j}(t_i-t_j)^{k_i\cdot k_j}\frac{1}{(t_1-t_4)(t_4-t_6)(t_6-t_8)}\times\frac{1}{(t_2-t_8)}\times\frac{1}{(t_3-t_5)(t_5-t_7)(t_7-t_8)}$

integration rules

momentum kernel $\times\ \dfrac{m(146|146)\,m(375|375)\,m(2|2)}{(k_1+k_4+k_6)\cdot p\,(k_1+k_3+k_4+k_5+k_6+k_7)\cdot p}$



numerator translates to

the denominator agrees

$\langle[e_7e_6e_5e_4e_3e_2e_1],[e_2^*,[[e_7e_5e_3]^*,[e_6e_4e_1]^*]]\rangle$

$[e_7e_6e_5e_4e_3e_2e_1]=[e_7\ldots[e_3,[e_2,e_1]]]$

[CF,Wang 2310.19724]

definition of the Shapovalov form

$\langle[e_i,A],B\rangle=\langle A,[f_i,B]\rangle$

definition of simple Lie algebra

$[e_i,f_j]=\delta_{ij}h_k$

$[h_k,e_j]=k_i\cdot k_j e_j$

$[h_k,f_j]=-k_i\cdot k_j f_j$

KK+off-shell BCJ relations

$[f_i,[e_\ell\ldots e_i\ldots e_2e_1]^*]=\delta_{i,\ell}[\ldots e_2e_1]$

$[h_i,[e_6\ldots e_2e_1]]=k_i\cdot(k_1+\cdots+k_6)\,[e_6\ldots e_2e_1]$

Summary of the string theory explanation

$$N(123\ldots n)$$

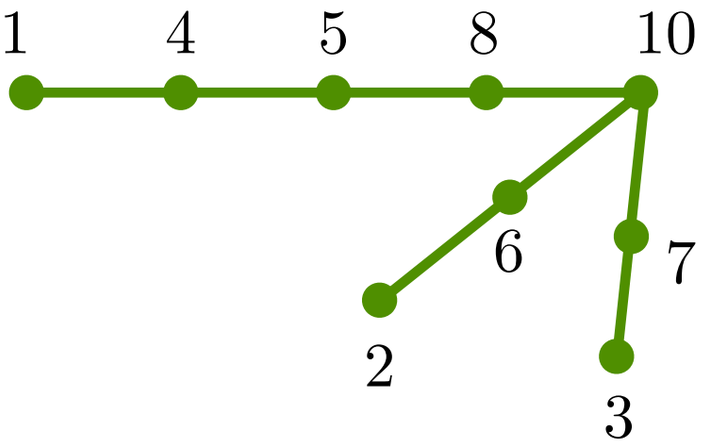
$$= \frac{1}{n-2}\,\langle p|\, e^{ip\cdot X(t_n)} \left[\left[E^{\text{vector}}_{k_1}, E^{\text{vector}}_{k_2}\right]_q \ldots, E^{\text{vector}}_{k_{n-2}}\right]_q |0\rangle$$

$$\left(i\epsilon_2 \cdot X^{(1)} e^{ik_2\cdot X}\right)\left(i\epsilon_1 \cdot X^{(1)} e^{ik_1\cdot X}\right)$$

$$\rightarrow\; i\epsilon_1 \cdot F_2 \cdot X^{(1)} \prod_j e^{ik_j\cdot X} + i\epsilon_1 \cdot X^{(1)} i\epsilon_2 \cdot X^{(1)} \prod_j e^{ik_j\cdot X}$$

$\downarrow \quad \ldots$

$$i\epsilon_1 \cdot F_4 \cdot F_5 \cdot F_8 \cdot X^{(1)}(t_8)\, i\epsilon_2 \cdot F_6 \cdot X^{(1)}(t_6)\, i\epsilon_3 \cdot F_7 \cdot X^{(1)}(t_7)$$



$$\frac{k_{\theta(\tau_i(1))}\cdot k_{\tau_i(1)}}{k_{\text{all left to }\tau_i(1)}\cdot p}$$

multiply

$$\left(\frac{q_1\cdot F_1\cdot F_4\cdot F_5\cdot F_8\cdot p}{q_1\cdot k_1}\right)\left(\frac{q_2\cdot F_2\cdot F_6\cdot p}{q_2\cdot k_2}\right)\left(\frac{q_3\cdot F_3\cdot F_7\cdot p}{q_3\cdot k_3}\right)$$

$$V_{\Theta(\tau)\,\mu\nu} = v_\mu k_{\Theta(\tau)\,\nu}$$

$$(1)\star(\mathbf{2})$$

$$\rightarrow(1\mathbf{2})$$

$$(1)(\mathbf{2})$$

$$(1458)(26)(37)\;+\ldots$$

$\downarrow$

$$(\epsilon_1\cdot F_4\cdot F_5\cdot F_8\cdot p)\,(\epsilon_2\cdot F_6\cdot p)\,(\epsilon_3\cdot F_7\cdot p)$$

$\downarrow$

APPENDIX B

branch cut & IBP

BCJ kinematic algebra in strings

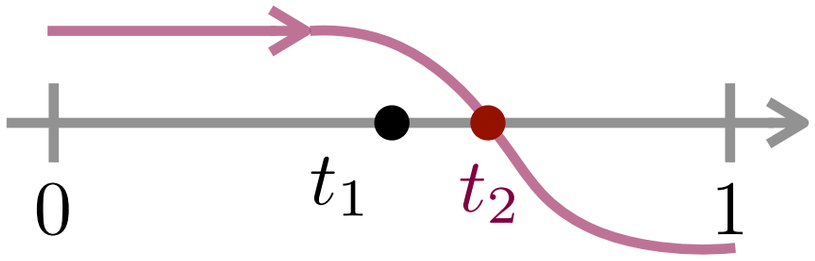
$$E_k := \int_0^1 dt\, e^{ik\cdot X(t)}$$

[CF,Vanhove,Wang 1806.09584]

$$E_{k_2} E_{k_1}$$

$$= e^{ik_2\cdot X(t_2)} e^{ik_1\cdot X(t_1)}$$

$$= (t_2 - t_1)^{\alpha' k_1\cdot k_2} : e^{ik_2\cdot X(t_2)} e^{ik_1\cdot X(t_1)} :$$

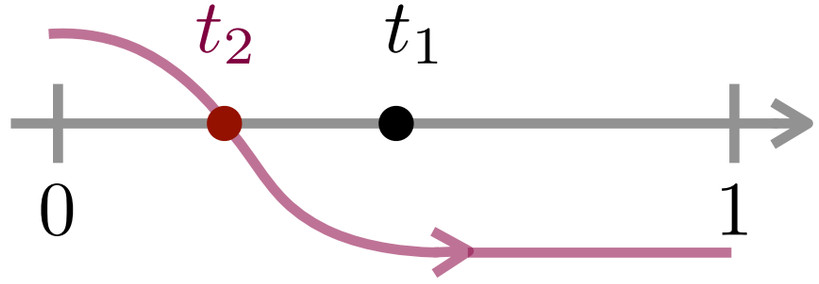


$$e^{i\pi\alpha' k_1\cdot k_2} E_1 E_{k_2}$$

$$= e^{i\pi\alpha' k_1\cdot k_2} e^{ik_1\cdot X(t_1)} e^{ik_2\cdot X(t_2)}$$

$$= e^{i\pi\alpha' k_1\cdot k_2} (t_1 - t_2)^{\alpha' k_1\cdot k_2} : e^{ik_2\cdot X(t_2)} e^{ik_1\cdot X(t_1)} :$$

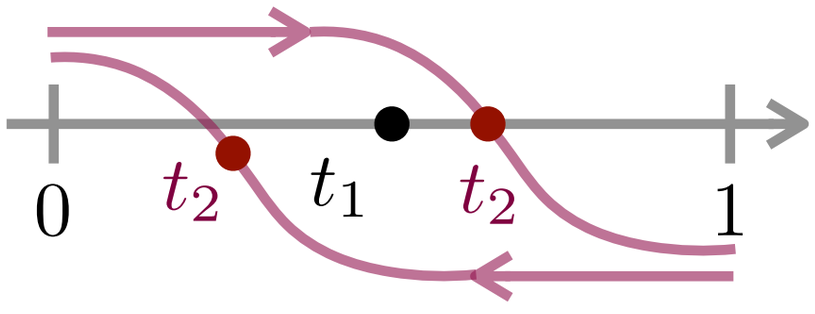
$$= (t_2 - t_1)^{\alpha' k_1\cdot k_2} : e^{ik_2\cdot X(t_2)} e^{ik_1\cdot X(t_1)} :$$

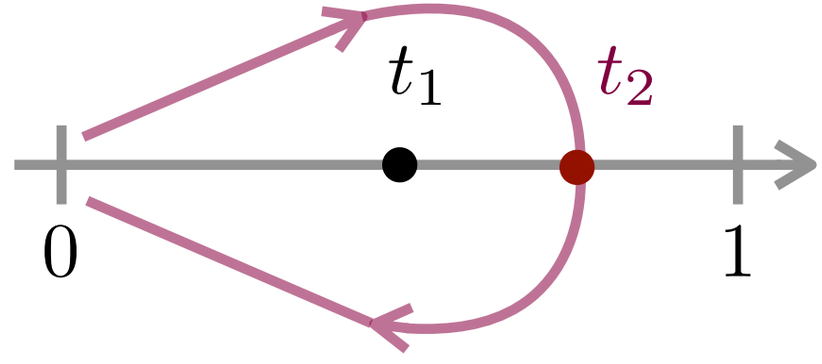


$$[E_{k_2}, E_{k_1}]_q = E_{k_2} E_{k_1} - e^{i\pi\alpha' k_1\cdot k_2} E_1 E_{k_2}$$

α'-weighted/q-deformed Lie algebra

$$q = e^{i\pi\alpha' k_1\cdot k_2}$$





[Dotsenko, Fateev '84,'85]

[Ma, Du, Chen 1109.0685]
[Carrasco,Mafra,Schlotterer 1612.06446]

BCJ kinematic algebra in strings

linear vector space of vertex operators

$$e^{ik\cdot X(t)} \in U[[t, t^{-1}]]$$

$$\exp\left(k\cdot\sum_{n=1}^{\infty}\frac{1}{n}\alpha_{-n}t^n\right)\exp\left(-k\cdot\sum_{n=1}^{\infty}\frac{1}{n}\alpha_n t^{-n}\right)\cdots$$
$$=\cdots v_{-1}t^{-1}+v_0+v_1 t+v_2 t^2\cdots$$

$\alpha_n^\mu, \alpha_{-n}^\nu$

infinite (Laurent) series of t

operator-valued coefficients

How a kinematic algebra generator acts

$$E_k : U[[t_1, t_1^{-1}]] \to U[[t_1, t_1^{-1}]]$$

example 1:

$$[E_{k_2}, E_{k_1}]_q = \int_C dt_2\, e^{ik_2\cdot X(t_2)} e^{ik_1\cdot X(t_1)}$$
$$= \int_C dt_2\, (t_2-t_1)^{k_1\cdot k_2} : e^{ik_2\cdot X(t_2)} e^{ik_1\cdot X(t_1)} :$$

special case: integer lattice $k_1\cdot k_2 = -2$

$$= \oint dt_2\, \frac{1}{(t_2-t_1)^{-k_1\cdot k_2=2}} : e^{ik_2\cdot X(t_2)} e^{ik_1\cdot X(t_1)} :$$
$$= 2\pi i\, \frac{d}{dt_2} : e^{ik_2\cdot X(t_2)} e^{ik_1\cdot X(t_1)} :\Big|_{t_2\to t_1} = 2\pi i k_2\cdot X^{(1)}(t_1) e^{i(k_1+k_2)\cdot X(t_1)}$$

infinite series of t_1 only

$U[[t_1, t_1^{-1}]]$

BCJ kinematic algebra in strings

linear vector space of vertex operators

$$e^{ik\cdot X(t)} \in U[[t, t^{-1}]]$$

$$\exp\left(k\cdot\sum_{n=1}^{\infty}\frac{1}{n}\alpha_{-n}t^n\right)\exp\left(-k\cdot\sum_{n=1}^{\infty}\frac{1}{n}\alpha_n t^{-n}\right)\cdots$$
$$=\cdots v_{-1}t^{-1}+v_0+v_1 t+v_2 t^2\cdots$$

$\alpha_n^\mu, \alpha_{-n}^\nu$ infinite (Laurent) series of t

operator-valued coefficients

$$[E_{k_2}, E_{k_1}]_q = 2\pi i k_2\cdot X^{(1)}(t_1)e^{i(k_1+k_2)\cdot X(t_1)}$$

gluon vertex operator

$$i\epsilon\cdot X^{(1)}(t_1)e^{ip\cdot X(t_1)}$$

$$\epsilon = k_2$$

$$p = k_1 + k_2$$

$k_1\cdot k_2 = -2$ (integer lattice)

$k_1^2 = k_2^2 = 2$ (tachyon mass shell)

gauge invariance

$$\epsilon\cdot p = 0 \quad\Longleftrightarrow\quad k_2\cdot(k_1+k_2) = -2+2 = 0$$

mass shell condition

$$p^2 = 0 \quad\Longleftrightarrow\quad (k_1+k_2)^2 = 2-4+2 = 0$$

$$[E^{\text{scalar}}_{k_2}, E^{\text{scalar}}_{k_1}]_q = 2\pi i k_{2\mu} E^{\text{vector}\,\mu}_{k_1+k_2}$$

structure constant

IBP alternatives

$$\left[E^{\text{vector}}_{k_1}, E^{\text{vector}}_{k_2}\right]_q$$

$$= \int_C dt_2 i\epsilon_2 \cdot X^{(1)}(t_2) e^{ik_2\cdot X(t_2)} i\epsilon_1 \cdot X^{(1)}(t_1) e^{ik_1\cdot X(t_1)}$$

$$= \int_C dt_2 (t_2-t_1)^{k_1\cdot k_2} e^{ik_2\cdot X(t_2)} e^{ik_1\cdot X(t_1)}$$

$$\underbrace{\frac{\epsilon_2\cdot\epsilon_1}{(t_2-t_1)^2}}_{(1)} + \underbrace{\frac{\epsilon_2\cdot k_1}{t_2-t_1}(-1)\frac{\epsilon_1\cdot k_2}{t_2-t_1}}_{(2)}$$

$$+ \underbrace{(-1)\frac{\epsilon_1\cdot k_2}{t_2-t_1} i\epsilon_2\cdot X^{(1)}(t_2)}_{(3)} + \underbrace{\frac{\epsilon_2\cdot k_1}{t_2-t_1} i\epsilon_1\cdot X^{(1)}(t_1)}_{(4)}$$

$$+ \underbrace{i\epsilon_2\cdot X^{(1)}(t_2)\, i\epsilon_1\cdot X^{(1)}(t_1)}_{(5)}$$

$$(1)+(3) \xrightarrow{\text{IBP}} \frac{\epsilon_1\cdot\epsilon_2}{t_2-t_1} ik_2\cdot X^{(1)}(t_2) + (3) = \frac{1}{t_2-t_1} i\epsilon_1\cdot F_2\cdot X^{(1)}(t_2)$$

$$(4)+(5) \xrightarrow{\text{IBP}} -\frac{k_1\cdot\epsilon_2}{k_1\cdot k_2} ik_2\cdot X^{(1)}(t_2)\, i\epsilon_1\cdot X^{(1)}(t_1) + (5) = \frac{1}{k_1\cdot k_2} ik_1\cdot F_2\cdot X^{(1)}(t_2)\, i\epsilon_1\cdot X^{(1)}(t_1)$$

APPENDIX C

Hopf algebra details

Why Hopf algebra?

algebra $A \cong \mathrm{Hom}(k, A)$

associativity

$$\begin{array}{ccc} A \otimes A \otimes A & \xrightarrow{\mu \otimes \mathrm{id}} & A \otimes A \\ \downarrow{\scriptstyle \mathrm{id} \otimes \mu} & & \downarrow{\scriptstyle \mu} \\ A \otimes A & \xrightarrow{\mu} & A \end{array}$$

unital

$$\begin{array}{ccccc} k \otimes A & \xrightarrow{\eta \times \mathrm{id}} & A \otimes A & \xleftarrow{\mathrm{id} \otimes \eta} & A \otimes k \\ & \searrow^{\cong} & \downarrow{\scriptstyle \mu} & \swarrow_{\cong} & \\ & & A & & \end{array}$$

coalgebra $C \cong \mathrm{Hom}(A, k) = A^*$

co-associativity

$$\begin{array}{ccc} C \otimes C \otimes C & \xleftarrow{\Delta \otimes \mathrm{id}} & C \otimes C \\ \uparrow{\scriptstyle \mathrm{id} \otimes \Delta} & & \uparrow{\scriptstyle \Delta} \\ C \otimes C & \xleftarrow{\Delta} & C \end{array}$$

pullback by multiplication induces the co-multiplication

co-unital

$$\begin{array}{ccccc} k \otimes C & \xleftarrow{\varepsilon \times \mathrm{id}} & C \otimes C & \xrightarrow{\mathrm{id} \otimes \varepsilon} & C \otimes k \\ & \nwarrow^{\cong} & \uparrow{\scriptstyle \Delta} & \nearrow_{\cong} & \\ & & C & & \end{array}$$

pullback by unit map induces the co-unit

bialgebra

an algebra with co-multiplication and co-unit as algebra morphisms

$$\begin{array}{ccc} H \otimes H & \xrightarrow{\Delta \otimes \Delta} & (H \otimes H) \otimes (H \otimes H) \\ \downarrow{\scriptstyle \mu} & & \downarrow{\scriptstyle (\mu \otimes \mu)(\mathrm{id} \otimes \tau \otimes \mathrm{id})} \\ H & \xrightarrow{\Delta} & H \otimes H \end{array} \qquad \begin{array}{ccc} k & \xrightarrow{\Delta} & k \otimes k \\ \downarrow{\scriptstyle \eta} & & \downarrow{\scriptstyle \eta \otimes \eta} \\ H & \xrightarrow{\Delta} & H \otimes H \end{array}$$

$$\begin{array}{ccc} H \otimes H & \xrightarrow{\varepsilon \otimes \varepsilon} & k \otimes k \\ \downarrow{\scriptstyle \mu} & & \downarrow{\scriptstyle \mu \cong \mathrm{id}} \\ H & \xrightarrow{\varepsilon} & k \end{array} \qquad \begin{array}{ccccc} H & & \xrightarrow{\varepsilon} & & k \\ & \nwarrow^{\eta} & & \nearrow_{\mathrm{id}} & \\ & & k & & \end{array}$$

Hopf algebra

convolution

$$\star : \mathrm{Hom}(C, A) \times \mathrm{Hom}(C, A) \to \mathrm{Hom}(C, A)$$

$$C \xrightarrow{\Delta} C \otimes C \xrightarrow{f \otimes g} A \otimes A \xrightarrow{\mu} A$$

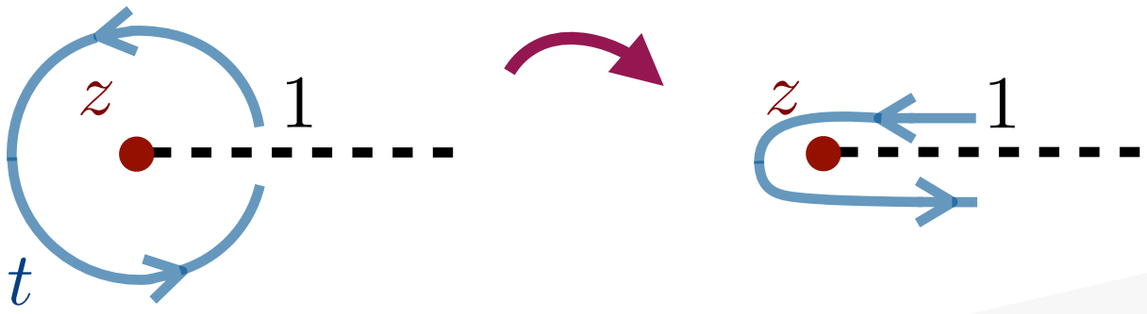
antipode (endomorphism from H to H)

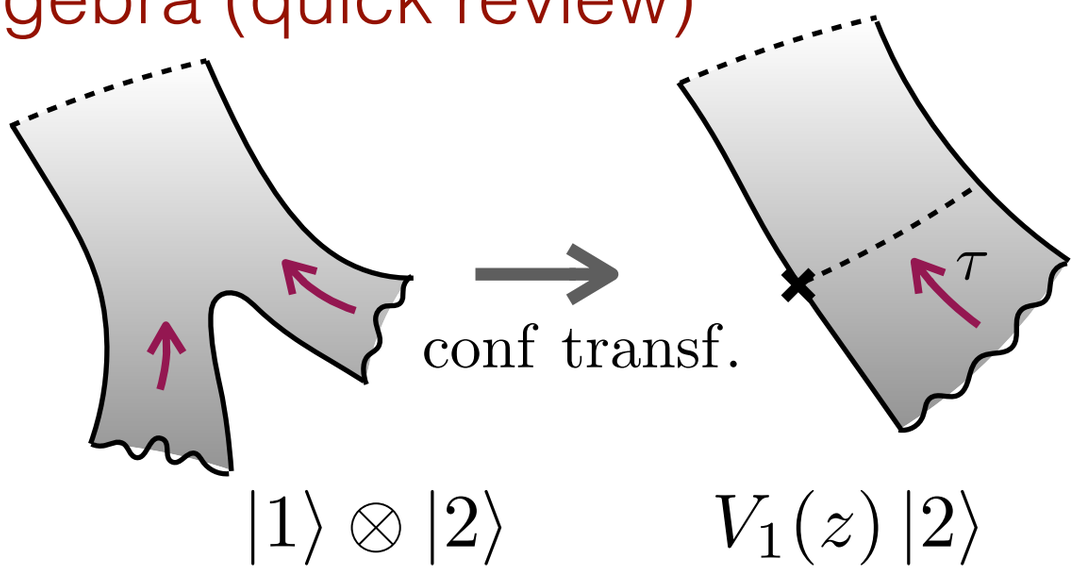
$$S \star \mathrm{id}_H = \mathrm{id}_H \star S = \eta \circ \varepsilon$$

Screening representation of the quantum algebra (quick review)

simple roots $k_1, k_2, \ldots \qquad (k_i^2 = 2)$

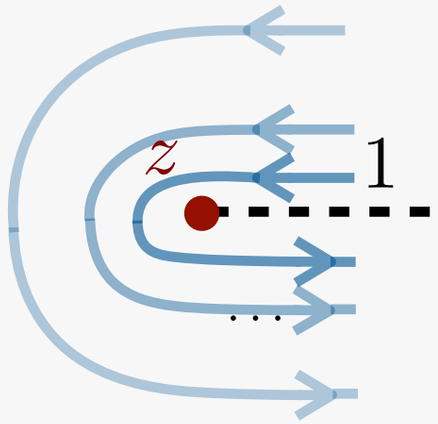
$$F_k = \int_C dt : e^{ik\cdot X(t)} :$$





$|1\rangle \otimes |2\rangle \qquad V_1(z)\,|2\rangle$

$$F_{k_n} \cdots F_{k_2} F_{k_1} V_1(z)$$



$$E_k = \int_C dt : e^{-ik\cdot X(t)} :$$

$$[E_k, F_k] = \int_C dt'(t'-t)^{\overset{-2}{\uparrow}\\ -k\cdot k} : e^{ik\cdot X(t')} e^{ik\cdot X(t)} :$$

$$H_k = \oint dt\, k\cdot\partial X(t)$$

[Dotsenko, Fateev ’84,’85]

[Feigin, Fuks ’82]

[Wakimoto ’86]

[Frenkel, Kac ’80]

Screening representation of the quantum algebra (quick review)

simple roots $k_1, k_2, \ldots \qquad (k_i^2 = 2)$

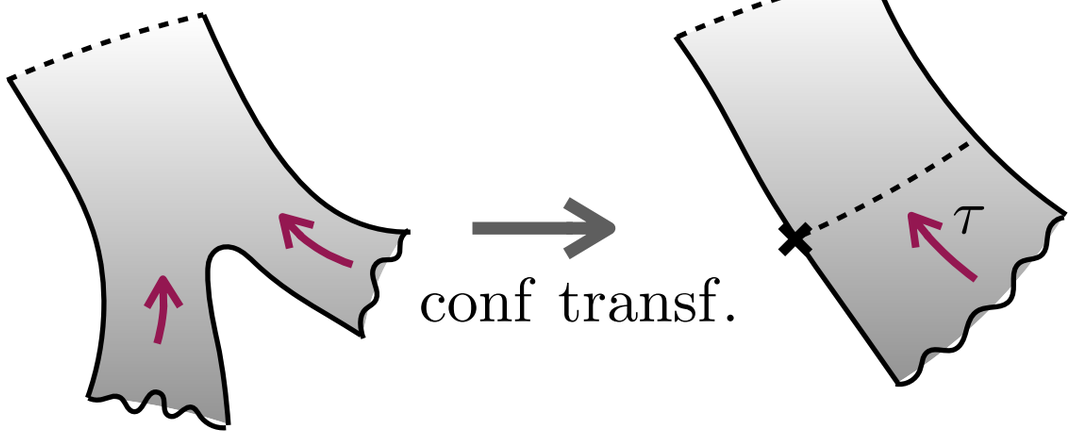
$$F_k = \int_C dt : e^{ik\cdot X(t)} :$$

$$E_k V_p(z) = 0$$

$$\int_C dt'(z-t)^{-k\cdot p} : e^{ip\cdot X(z)} e^{-ik\cdot X(t)} :$$

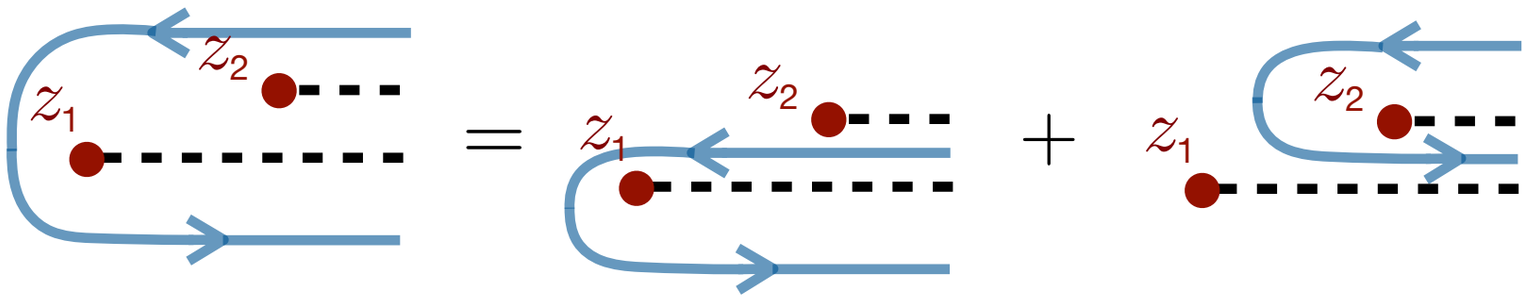
$$\downarrow$$

$+ve.\ integer\ or\ 0$



coproduct

$$\Delta F = F \otimes 1 + K^{-1} \otimes F$$



counit

$\epsilon(F) = 0$

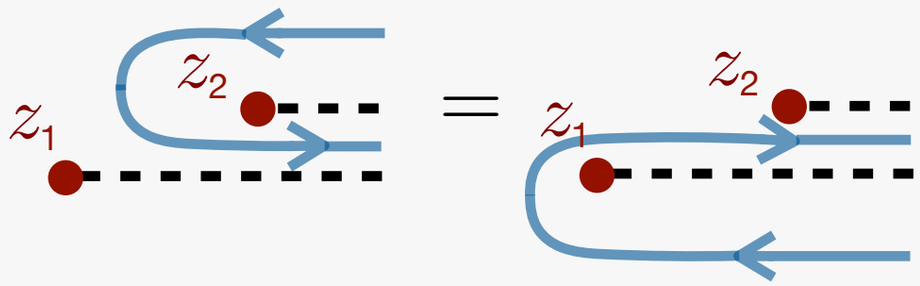
$\epsilon(1) = 1$

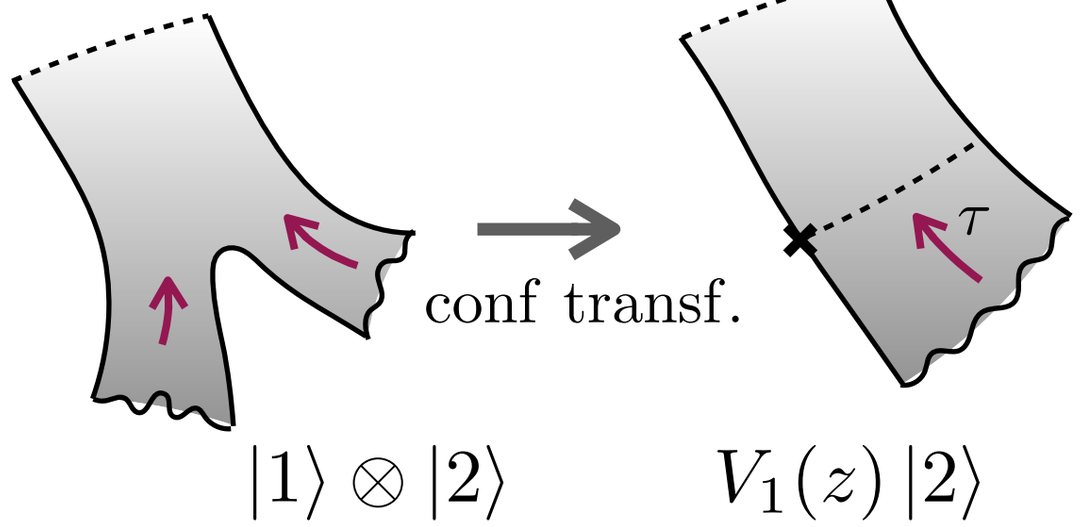
destroys all contours

Screening representation of the quantum algebra (quick review)

antipode

$$S(F) = -KF$$

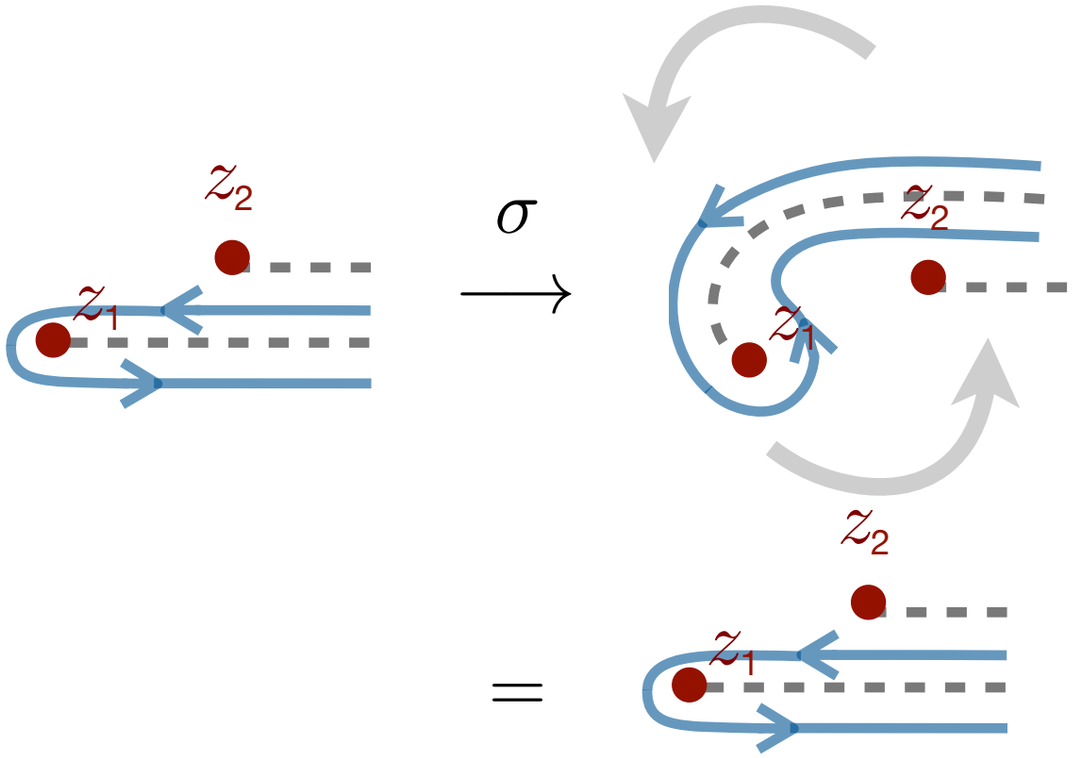




R-matrix

$$|1\rangle \otimes |2\rangle \xrightarrow{\sigma} |2\rangle \otimes |1\rangle \xrightarrow{\mathcal{R}} |1\rangle \otimes |2\rangle \qquad R = \mathcal{R} \circ \sigma$$

$$\mathcal{R} = e^{\frac{H_i \otimes H_i}{2}} \left(1 \otimes 1 + (q_i - q_i^{-1})\right.$$



Yang-Baxter eqn guaranteed by braiding

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}$$

$$(e^{i\pi\alpha' s\cdot k_1} - e^{-i\pi\alpha' s\cdot k_1})$$

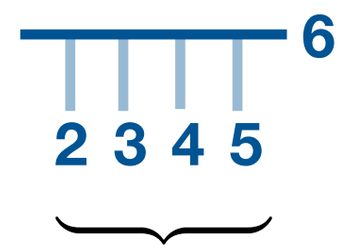
APPENDIX D

other stuffs

Assuming BCJ duality

→ **there is a basis for all cubic graphs**

half-ladders



no need to calculate all cubic graphs (BCJ numerators)

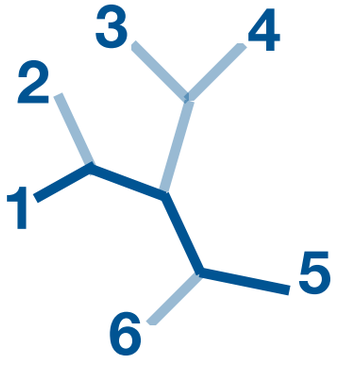
building blocks: half-ladders

$$= N(1,\sigma(2),\sigma(3),\ldots,\sigma(n-1),n)$$

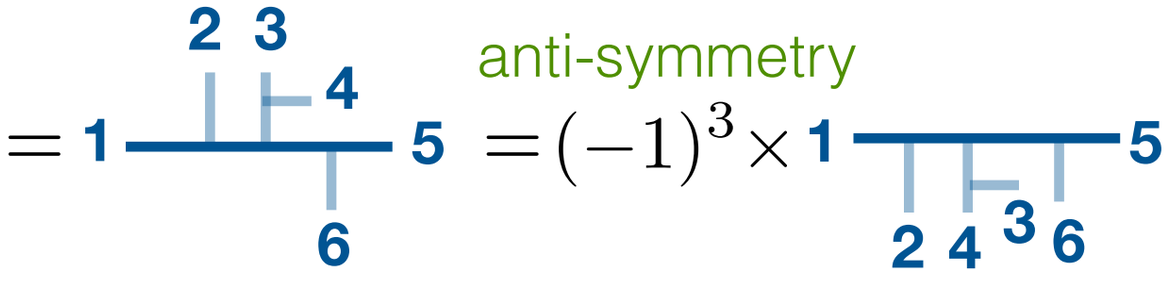
$$\sim f^{1,\sigma(2)}{}_{e_1} f^{e_1,\sigma(3)}{}_{e_2} f^{e_2,\sigma(3)}{}_{e_3} \cdots f^{e_{n-2},\sigma(n-1)}{}_{n}$$

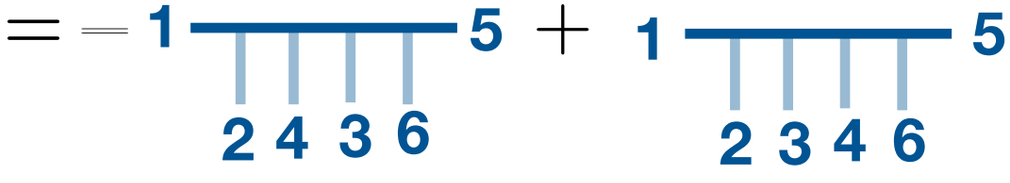
$$\sim tr(\,[\ldots[[T^1,T^{\sigma(2)}],T^{\sigma(3)}],\ldots\ldots T^{\sigma(n-1)}]\,T^n)$$

Example:



anti-symmetry

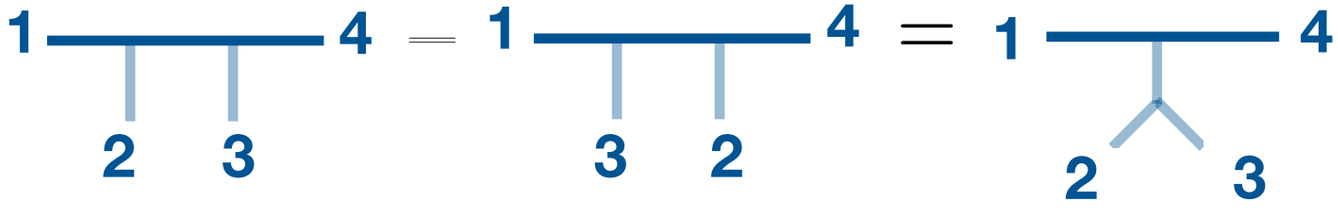




Jacobi identity

$$f^{12}{}_{\sigma} f^{\sigma 3}{}_{4} + f^{23}{}_{\sigma} f^{\sigma 1}{}_{4} + f^{31}{}_{\sigma} f^{\sigma 2}{}_{4} = 0$$

$f^{12}{}_{\sigma} f^{\sigma 3}{}_{4}$ $\quad = -f^{23}{}_{\sigma} f^{\sigma 4}{}_{1}$ $\quad = -f^{13}{}_{\sigma} f^{\sigma 2}{}_{4}$



disentangle sub-trees

Theories known to satisfy BCJ duality

(that are double copies)

bi-adjoint scalar Bern, de Freitas, Wong; Bern, Dennen, Huang; Du, Feng, CF; Bjerrum-Bohr, Damgaard, Monteiro, O'Connell

(s)Yang-Mills/QCD Bern, Carrasco, Johansson; Bjerrum-Bohr, Damgaard, Vanhove; Stieberger; Feng; Mafra, Schlotterer; Johansson, Ochirov;

(s)Einstein gravity Bern, Carrasco, Johansson; Chiodaroli, Gunaydin, Johansson, Roiban; Johansson, Ochirov; Johansson, Kalin, Modul; …

NLSM Chen, Du; Cachazo, He, Yuan; Du, CF; Carrasco, Mafra, Schlotterer; Cheung, Shen

Dirac-Born-Infeld Cachazo, He, Yuan; Cheung, Shen, Wen; Cheung, Remmen, Shen, Wen

special Galileon Cachazo, He, Yuan; Cheung, Shen, Wen; Cheung, Remmen, Shen, Wen; Cachazo, Cha, Mizera

Volkov-Akulov Cachazo, He, Yuan; Bergshoeff, Townsend; Bergshoeff, Kallosh, Ortin, Papadopoulos; Bergshoeff, Coomans, Kallosh, Shahbazi

Einstein-Yang-Mills Chiodaroli, Gunaydin, Johansson, Roiban; Cachazo, He, Yuan; Stieberger, Taylor; Schlotterer; Teng, Feng; Du, Feng, Teng; Bern, de Freitas, Wong; Cheung, Shen, Wen; Nandan, Plefka, Schlotterer, Wen; Du, Feng, CF, Huang

Higgsed super gravity Chiodaroli, Gunaydin, Johansson, Roiban;

Yang-Mills $+F^3 + F^4 + \cdots$ Broedel, Dixon

$R + \phi R^2 + R^3$ gravity Broedel, Dixon

Bagger-Lambert-Gustavsson Bargheer, He, McLoughlin; Huang, Johansson; Sivaramakrishnan

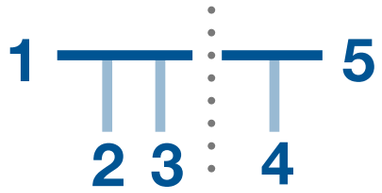
Z-theory Broedel, Schlotterer, Stieberger; Carrasco, Mafra, Schlotterer

open/closed strings Broedel, Schlotterer, Stieberger

⋮

Is there any way to calculate basis numerators efficiently?

brute force solving numerators from lower point amplitudes suggests that



$$N(1,2,3,k) \times N(-k,4,5) \neq N(1,2,3,4,5)$$

$$(\sum_{\sigma} f^{12}{}_{\sigma} f^{\sigma 3}{}_{k})$$

solved from 4 point amplitudes

$$f^{k4}{}_{5}$$

solved from 3 point amplitudes

solved from 5 point amplitudes

Explanation:

need to sum over all intermediate modes

$$A(1,2,3,k^{\text{gluon}})$$

$$A(1,2,3,k^{2-\text{tensor}})$$

$$A(1,2,3,k^{3-\text{tensor}})$$

$$\vdots$$

the kinematic algebra seems to be an infinite dimensional Lie algebra

practical method:

ansatz

$$\begin{aligned} N(1,2,3,\ldots,n) \\ = \; & a \;\; (k_1 \cdot \epsilon_2)(\epsilon_3 \cdot \epsilon_4)(k_5 \cdot k_6)\ldots \\ + \; & b \;\; (k_2 \cdot \epsilon_3)(\epsilon_2 \cdot \epsilon_4)(k_1 \cdot k_6)\ldots \\ + \; & \ldots \end{aligned}$$

solve numerators from YM and then produce gravity amplitudes through the double copy

sacrifice the functionality to use BCJ to calculate YM

can still compute gravity from YM