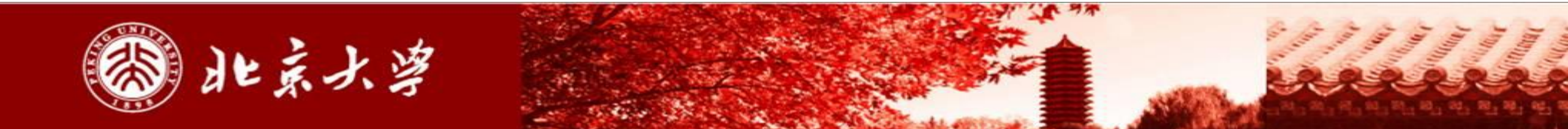
Analyticity Bootstrap of Feynman Integrals

Yan-Qing Ma （Peking University）

yqma@pku.edu.cn

Recent Progress in Amplitudes,
ShanghaiTech University, Nov. 9-12, 2026

arXiv:2610.01154, in collaboration with Xiang Li and Dao-Ming Mu



Outline

I. Introduction

II. Analyticity constraints

III. Reduction exmaples

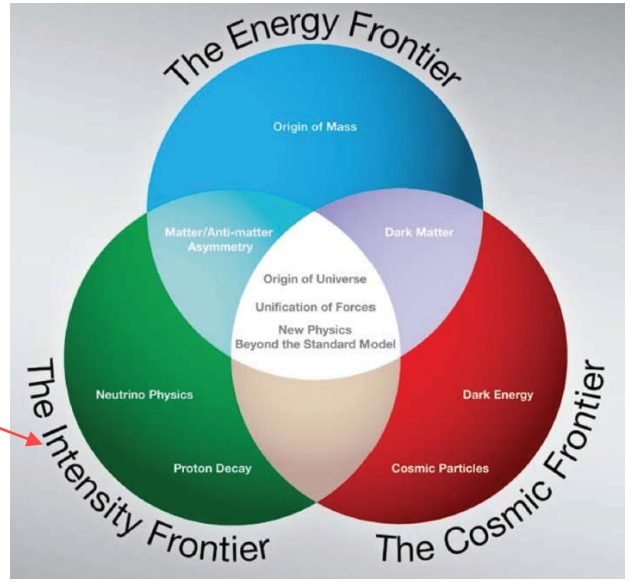
IV. Bootstrapping the DE

V. Summary and outlook

Precision: gateway to discovery

➢ **New particles/physics have not been discovered yet at LHC**

- **Currently main strategy: search anomalous deviations from theory**
- **Interplay between exp. and th.**



To make full use of data: theoretical errors should be much smaller than experimental errors, ideally:

$$Error_{th} < \frac{1}{3} Error_{exp}$$

Feynman integrals: the bottleneck of precision physics

- **Feynman integrals (FIs): encode key information of quantum field theories — precision colliders & gravitational waves**

Integral reduction
all FIs of a family → master integrals (MIs)

→

Differential equations
evaluate the MIs by solving DEs

$$F(\epsilon,\vec{x}) = \sum_{i=1}^{N} c_i(\epsilon,\vec{x}) I_i(\epsilon,\vec{x})$$

$$\partial_{x_i}\vec{I} = A(\epsilon,\vec{x})\vec{I}$$

coefficients: rational in ϵ and kinematics $\vec{x}$ — from, e.g., IBP, intersection number

Computing FIs: the core difficulty of precision calculations

Integration-by-parts reduction: the bottleneck!

➢ The state-of-the-art IBP method: very challenging

- $H + 2j$ production: complete two-loop contribution is missing — Chen, et al., JHEP2022
- $H + t\bar{t}$ production: complete two-loop contribution is missing — Catani, et al., PRL2023
- Black hole scattering: complete 5 PM contribution is missing — Driesse, et al., 2601.16256

➢ Improvements for IBPs

- Reduce the size of IBP systems:
  - Syzygy equations: trimming IBP system
  - Block-triangular form: search simple IBP system

Gluza, Kajda, Kosower, PRD2011
Böhm, Georgoudis, Larsen, Schulze, Zhang, PRD2018
NeatIBP: Wu, et al. CPC2024

Liu, YQM, PRD2019
Guan, Liu, YQM, CPC2020
Blade: Guan, Liu YQM, Wu, CPC2025

- Reduce the number of IBP samplings:
  - Thiele and sparse interpolation
  - Residue-guided rational bases

Firefly: Klappert, Lange. CPC2019

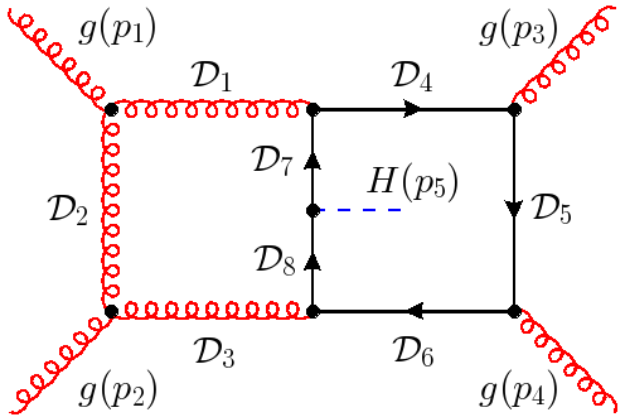
Laurentis, Ita, Page, Sotnikov, JHEP2025

Still need a huge number of samplings!

This talk: analyticity bootstrap (AB)

➢ Teaser: DEs of double pentagon

method	IBP points / prime	t(SetDE) [CPU h]	t(SolveDE) [CPU h]
Blade + AMFlow	372	52.2	0.9
AB + Blade + AMFlow	5	0.7	0.9



nonplanar double pentagon — $gg \to ggH$ via top loop

➢ Analyticity:

- A dimensionally regularized FI has an analytic region at any finite kinematic point
- Finiteness of the analytic region strongly constrains the structure of reduction coefficients
- Asymptotic expansions of non-analytic region (or IBP sampling) fix the remaining parameters

Ansatz compatible with singularity structure → Far fewer free parameters → Fixed by a few IBP samples / asymptotic expansions

Outline

I. Introduction

II. Analyticity constraints

III. Reduction exmaples

IV. Bootstrapping the DE

V. Summary and outlook

Setup: singularities and local expansions

➢ Singularities (for single variable)

- Pole set $\mathcal{S}_A = \{\alpha_1, \alpha_2, \ldots\}$ of $A(x)$:
  - physical singularities (from Landau equations)
  - spurious singularities

$$\partial_x \vec{I} = A(x)\vec{I}$$

- Near $\alpha \in \mathcal{S}_A$ $(z = x - \alpha)$, every MI has a local expansion:

$$I_i(x) = \sum_{\mu \in \Lambda_\alpha} \sum_{p=0}^{p_\mu} z^\mu (\log z)^p \sum_{n=0}^{\infty} b_{i,n}^{(\mu,p)} z^n + \sum_{n=0}^{\infty} a_{i,n} z^n$$

non-analytic regions: $z^\mu (\log z)^p$ analytic region: Taylor series

Analyticity constraints: premises and ansatz

➢ Three premises

$$F(x) = \sum_{i=1}^{N} c_i(x) I_i(x) \qquad \partial_x \vec{I} = A(x)\vec{I}$$

- **Finiteness:** analytic-region contributions of MIs and F finite at every $\alpha \in \mathcal{S}_A$
- **DEs:** the matrix $A(x)$ taken as input (will discussion how to get DEs later on)
- **Poles:** minimal basis ⇒ poles of c_i only in $\mathcal{S}_A$

➢ Most general ansatz

$$F = \sum_{\alpha \in \mathcal{S}_A} \frac{\vec{Q}^{(\alpha)}(x) \cdot \vec{I}(x)}{(x-\alpha)^{n_\alpha}} + \vec{Q}^{(\infty)}(x) \cdot \vec{I}(x)$$

Pole orders and degree caps of numerator are cheap inputs
— independent of ϵ, the phase-space point and the finite-field prime

[Abreu et al.,1812.04586; Heller, Von Manteuffel, 2101.08283]

Analyticity constraints: the numerator module

➢ Finiteness demands the analyticity constraint

$$\vec{Q}^{(\alpha)}(x)\cdot\vec{I}(x)=\mathcal{O}((x-\alpha)^{n_\alpha})$$

$$F=\sum_{\alpha\in\mathcal{S}_A}\frac{\vec{Q}^{(\alpha)}(x)\cdot\vec{I}(x)}{(x-\alpha)^{n_\alpha}}+\vec{Q}^{(\infty)}(x)\cdot\vec{I}(x)$$

➢ Numerator module and its Gröbner basis

- Rows $\vec{Q}^{(\alpha)}$ obeying analyticity constraint: closed under polynomial combinations

  — a **module** under truncated polynomial ring $R[x]/(x-\alpha)^{n_\alpha}$

- Ansatz expressed as the GB $\vec{v}_k^{(\alpha)}$ of the module, with unknow polynomials $P_k^{(\alpha)}(x)$:

$$F=\sum_{\alpha\in\mathcal{S}_A}\sum_k P_k^{(\alpha)}(x)\frac{\vec{v}_k^{(\alpha)}(x)\cdot\vec{I}(x)}{(x-\alpha)^{n_\alpha}}+\vec{Q}^{(\infty)}(x)\cdot\vec{I}(x)$$

- Behavior at infinity: further constraints — restrict $\vec{Q}^{(\infty)}$, correlate the finite-pole parameters

Module from the DE (I): boundary-value independent

➢ Analytic-region expansion in boundary masters

- Near α, each order of the analytic region reduces to boundary masters:

$$\vec{I}(x) = \sum_{r=0}^{\infty} (x-\alpha)^r H_r(\alpha) \vec{b}(\alpha)$$

$$F = \sum_{\alpha \in \mathcal{S}_A} \frac{\vec{Q}^{(\alpha)}(x) \cdot \vec{I}(x)}{(x-\alpha)^{n_\alpha}} + \vec{Q}^{(\infty)}(x) \cdot \vec{I}(x)$$

- Coefficient matrices H_r : determined recursively by the DEs alone

➢ Finiteness gives conditions free of boundary values

- Numerator: $\vec{Q}^{(\alpha)}(x) = \sum_{j=0}^{n_\alpha - 1} (x-\alpha)^j \, \vec{q}_j$
- Analyticity constraint: $\vec{Q}^{(\alpha)}(x) \cdot \vec{I}(x) = \mathcal{O}((x-\alpha)^{n_\alpha})$
- Linearly independence of $\vec{b}$ in $\mathbb{Q}[x]$

$$\sum_{j=0}^{r} \vec{q}_j H_{r-j}(\alpha) = \vec{0}, \qquad 0 \le r < n_\alpha$$

The numerator module is fully determined by the DEs

Module from the DE (II): local construction

➢ Derivative rows

$$\partial_x^m I_i = \vec{r}_i^{[m]}(x) \cdot \vec{I},$$

$$\vec{r}_i^{[0]} = \vec{e}_i, \vec{r}_i^{[m+1]} = \partial_x \vec{r}_i^{[m]} + \vec{r}_i^{[m]} A$$

- Rows $\vec{r}_i^{[m]}$: rational, poles confined to $\mathcal{S}_A$ — pole order at α grows with m

➢ Construct the generating set

- Analytic region of $\partial_x^m I_i$ is $\mathcal{O}(1) \Rightarrow (x-\alpha)^{n_\alpha} \partial_x^m I_i = \mathcal{O}((x-\alpha)^{n_\alpha})$
- Rows $(x-\alpha)^{n_\alpha} \vec{r}_i^{[m]}$ with $m \leq m_{i,\alpha}^{\min}$ generate the module at α
- Fuchsian pole: $m_{i,\alpha}^{\min} = n_\alpha$ suffices, from experience

Buchberger reduction → Gröbner basis $\{\vec{v}_k^{(\alpha)}\}$ as generators

Module from the DE (III): common denominator

➢ One common denominator

$$Q(x) = \prod_{\alpha \in S_A} (x - \alpha)^{n_\alpha}$$

➢ Lifting the local rows

- Lift each row: single pole $(x - \alpha)^{n_\alpha} \to Q(x) \Rightarrow$ prescribed pole power at every zero of Q
- Lifted rows $Q\vec{r}_i^{[m]}$: polynomial, except residual denominators at β with $m > m_{i,\beta}^{\min}$
- Global module: polynomial combinations in which all residual denominators cancel

Global GB $\{\vec{v}_k^{(Q)}\}$

Outline

I. Introduction

II. Analyticity constraints

III. Reduction exmaples

IV. Bootstrapping the DE

V. Summary and outlook

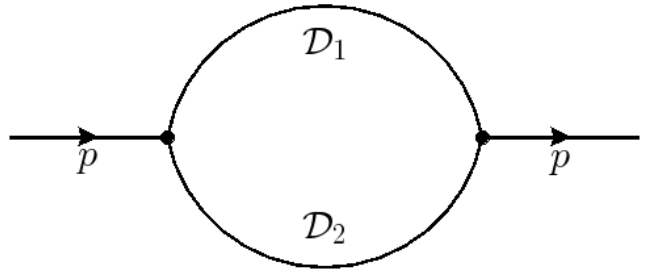
Example: the equal-mass bubble

➢ Masters and DEs

- $\mathcal{D}_1 = \ell^2 - 1, \mathcal{D}_2 = (\ell + p)^2 - 1$
- **MIs:** $\vec{I} = (I_{10}, I_{11})$
- **DEs:**

$$\partial_s I_{10} = 0$$

$$\partial_s I_{11} = \frac{-12 I_{10} + (14 - s) I_{11}}{7 s (s - 4)}$$



one-loop two-point, equal masses, $s = p^2$

➢ Target and generators

$$I_{22} = \frac{(a_0 + a_1 s) I_{10} + (b_0 + b_1 s) I_{11}}{s (s - 4)^2}$$

- **Target** I_{22} **— prescribed** $Q = s(s-4)^2$**, degree caps** $(1,1)$**: four unknown parameters**
- **Generators :**

$$\vec{g}_1 = (Q, 0), \vec{g}_2 = (0, Q),$$

$$Q\, \partial_s I_{11} \Rightarrow \vec{g}_3 = (-12(s - 4), (14 - s)(s - 4))$$

$$s\, Q \partial_s^2 I_{11} \Rightarrow \vec{g}_4 = (9(5s - 14), 2s^2 - 56s + 147)$$

Bubble: from module to one-parameter reduction

➢ Gröbner basis from the generators (degree first)

$$\vec{v}_1 = (-21s + 30, 20s - 35), \vec{v}_2 = (4s^2 - 25s - 90, 105)$$

$$\vec{N}_Q = f_1\vec{v}_1 + f_2\vec{v}_2$$

- Degree caps ⇒ $f_1 = c, f_2 = 0$

➢ Bounded numerator: one-dimensional

$$I_{22} = \frac{c}{s(s-4)^2}[(-21s + 30)I_{10} + (20s - 35)I_{11}]$$

- One IBP point (or other ways) ⇒ $c = -\frac{4}{49}$

4 generic coefficients → 1 unknown parameter

Single-variable reduction: strong compression

➢ Three families, higher dots

family	dots	ansatz parameters	IBP points
equal-mass bubble	10	$16 \rightarrow 1$	$10 \rightarrow 1$
unequal-mass sunset	8	$169 \rightarrow 1$	$26 \rightarrow 1$
planar double box	6	$189 \rightarrow 65$	$14 \rightarrow 4$

- Tests at $\epsilon = 1/7$ over $\mathbb{F}_{2147483629}$
- Prescribed denominators and degree caps kept fixed
- Constraints from analytic regions at finite α plus all regions at infinity

Multivariable generalization

➢ Multi-index derivative rows

- One-variable derivative label $[m]$ → multi-index $[\boldsymbol{\beta}]$
- Includes the mixed derivative rows

tested target	ansatz parameters	IBP points
unequal-mass sunset (s, m_1^2), 8 dots	2406 → 1	378 → 1
double box (s, t), 6 dots	1605 → 62 (only s: 189 → 65)	120 → 4

More variables, less ansatz parameters???

Outline

I. Introduction

II. Analyticity constraints

III. Reduction exmaples

IV. Bootstrapping the DE

V. Summary and outlook

Bootstrapping the DE: setup

➢ Sector-ordered, block-triangular DE

$$\partial_x \begin{pmatrix} \vec{I}_S \\ \vec{I}_{\text{sub}} \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix} \begin{pmatrix} \vec{I}_S \\ \vec{I}_{\text{sub}} \end{pmatrix}$$

- A_{22} known from lower sectors (bottom-up construction)
- A_{11} from a maximal cut — subsector integrals absent

Only the off-diagonal block A_{12} remains to be determined

Bootstrapping the off-diagonal block

➢ Extensions: additional generator

- If α not a pole of A_{11}, the subsector module (with GB $\left\{\vec{v}^{(\alpha)}_{A_{22},l}\right\}$) suffices for the ansatz of A_{12}
- If α is a pole of A_{11}, the singular contributions of $A_{11}\vec{I}_S$ and $A_{12}\vec{I}_{\text{sub}}$ must cancel each other

$$\vec{A}^{(\alpha)}_{11,r}(x)\cdot\vec{I}_S(x)+\vec{u}^{(\alpha)}_{A_{12},r}(x)\cdot\vec{I}_{\text{sub}}(x)=\mathcal{O}((x-\alpha)^{n_\alpha})$$

➢ Syzygy compatibility

- Syzygy relation: $\sum_r s_r(x)\vec{u}^{(\alpha)}_{A_{12},r}(x)=\mathcal{O}((x-\alpha)^{n_\alpha})$
- Leaving a pure-subsector contribution lying in the subsector module

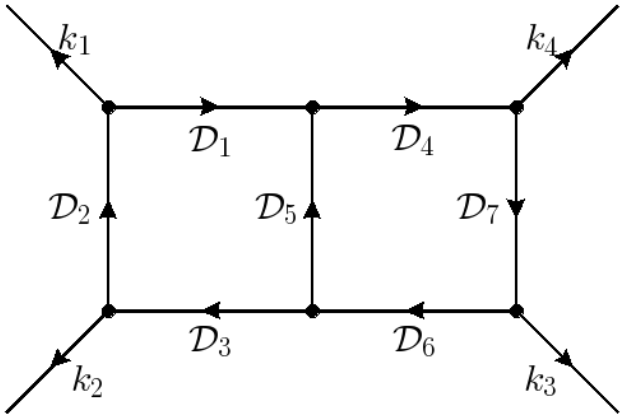
$$\sum_r s_r(x)\vec{u}^{(\alpha)}_{A_{12},r}(x)=\sum_l h_l(x)\vec{v}^{(\alpha)}_{A_{22},l}(x)$$

- Solving for every syzygy → finite generating set of extensions

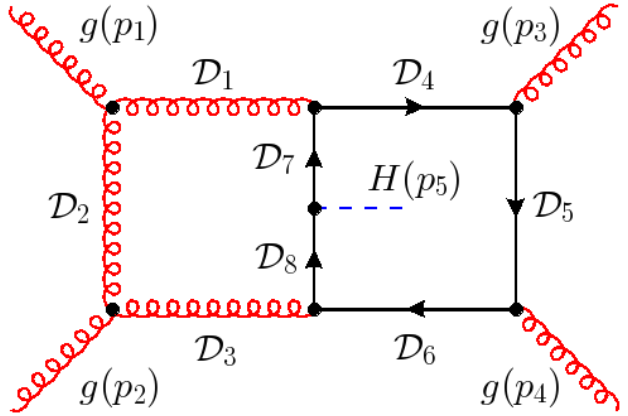
A_{12} ansatz = subsector module + extensions

DE bootstrap: two-loop examples

➢ DEs for two-loop families



massless planar double box



nonplanar double pentagon — $gg \to ggH$ via top loop

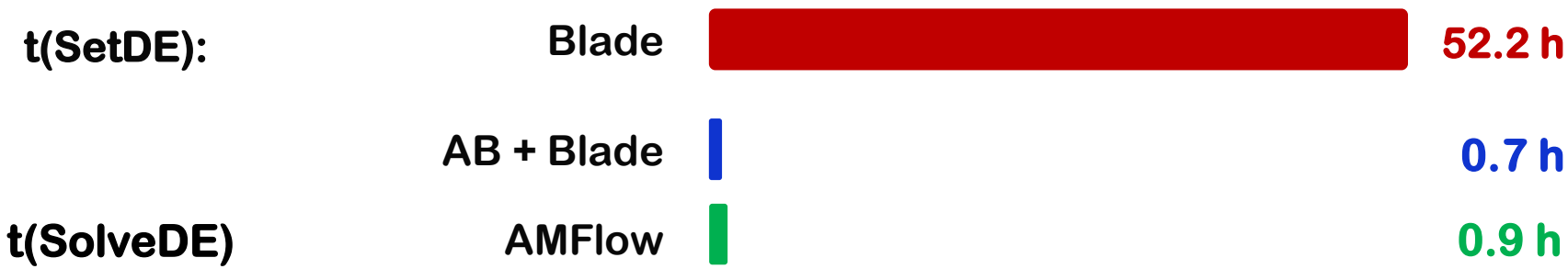
family	variable	masters	target block	ansatz parameters	IBP points
double box	η at $\mathcal{D}_1$	25	all	$212 \to 73$	$8 \to 2$
double pentagon	m_t^2	379	topsector	$168605 \to 10615$	$182 \to 5$

With auxiliary mass flow: cost comparison

➢ Cost for the 379-MI double pentagon

method	IBP points / prime	t(SetDE) [CPU h]	t(SolveDE) [CPU h]
Blade + AMFlow	372	52.2	0.9
AB + Blade + AMFlow	5	0.7	0.9

$\epsilon = 1/3$, 109 finite-field primes; m_t^2 as the AMFlow auxiliary mass, pole orders not taken as input for Blade + AMFlow



Significant efficiency improvement for constructing DE

DE construction drops below the cost of the numerical solution itself

Asymptotic expansions: the idea

➢ Determine the parameters by expansion by regions

- Expand diagonal part of DEs and subsector masters by the method of region at region r:

$$\left(\partial_\eta \vec{I}_S - A_{11}\vec{I}_S\right)\Big|_r = \sum_{i,p,\nu} X^{(r)}_{ip\nu} z^i (\log z)^p \mathcal{J}^{(r)}_\nu$$

$$\vec{I}_{\text{sub}}\Big|_r = \sum_{i,p,\nu} Y^{(r)}_{ip\nu} z^i (\log z)^p \mathcal{J}^{(r)}_\nu$$

- Match powers and logarithms order by order ⇒ linear equations for the ansatz of A_{12}

Trivial ansatz: needs high orders expansion — rapidly expensive

Compact AB ansatz: very low orders already suffice

Leading-order regional expansions suffice

➢ **Double-box top sector: two MIs, 10 + 12 = 22 parameters**

endpoint	ℓ_1 region	ℓ_2 region	regional masters	LO rank
0	ℓ_1 collinear with k_1	ℓ_2 hard	5	8
0	ℓ_1 collinear with k_1 or k_4	ℓ_2 collinear with k_1 or k_4	4	3
1	$\ell_1 + P$ soft	ℓ_2 hard	1	2
1	$\ell_1 + P$ ultrasoft	$\ell_2 + P$ collinear with k_3	2	2
1	$\ell_1 + P$ soft	$\ell_2 + P$ soft	2	4
∞	ℓ_1 hard	ℓ_2 hard	1	2
∞	ℓ_1 hard	ℓ_2 soft	1	2
∞	ℓ_1 soft	ℓ_2 soft	5	6
all	—	—	—	22

LO matching at $\eta = 0, 1, \infty$: fix all 22 parameters

The full block is determined — no needs IBP of the original family

Summary and outlook

- Finiteness of analytical regions: significantly constraining the functional form of reduction coefficients

- Asymptotic expansions of non-analytical regions (or IBP samplings): fix the remaining ansatz parameters

- Full amplitude could also constructed from this analyticity bootstrap

Thank You!