PANE

Critical Slowing Down of Black Hole Phase Transition and Universal Dynamic Scaling in AdS Black Holes

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Outline

Introduction
- Black Holes
- Black Hole Thermodynamics

What happens at the Critical Point?

Future Prospects

Black Holes

- Predicted by Einstein's theory of General Relativity as regions of spacetime with extremely strong gravity
- Defined by the presence of an event horizon
- Characterized by mass, charge, and spin
- They behave like thermodynamic systems!



Figure: *M87*, EHT collaboration*

A Brief Introduction to Black Hole Thermodynamics

- Hawking's area theorem [*Hawking, 1971*], Bekenstein's entropy area relation [*Bekenstein, 1973*] hinted some connection between black holes and thermodynamics
- In 1973 Bardeen, Carter and Hawking gave the four laws of black hole mechanics [*Bardeen et al.,1973*] which had analogies with the laws of Thermodynamics
- *Zeroth Law:* The surface gravity κ of a stationary Black Hole is constant over the event horizon
- *First Law:* The first law says that

$$dM = \frac{\kappa}{8\pi} dA + \Omega dJ + \Phi dQ \tag{1}$$

A Brief Introduction to Black Hole Thermodynamics

- *Second Law:* It is the Hawking's area theorem which states that the area of a Black Hole horizon can never decrease
- *Third Law:* It is impossible to reduce κ to zero by any finite number of operations

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A Brief Introduction to Black Hole Thermodynamics

- *Second Law:* It is the Hawking's area theorem which states that the area of a Black Hole horizon can never decrease
- *Third Law:* It is impossible to reduce κ to zero by any finite number of operations
- Hawking went on to show that Black Holes emit radiation like black bodies [*Hawking, 1975*]. This discovery strongly suggested that the analogies were not just mere coincidences
- A natural question was if they also undergo processes similar to conventional thermodynamics (like phase transitions?)

- Davies [*Davies, 1977*] and Hut [*Hut, 1977*] first studied "phase transitions" in black holes

A Brief Introduction to Black Hole Thermodynamics

- Davies [*Davies, 1977*] and Hut [*Hut, 1977*] first studied "phase transitions" in black holes
- Cosmological constant was treated as thermodynamic pressure and it was included in the first law [*Kastor et al., 2009*]
- This led to a remarkably rich phase structure
- Analogies with van der Waals fluid: P-V criticality, liquid-gas transition, critical exponents

A Brief Introduction to Black Hole Thermodynamics

- Conventional approach to study phase transitions in black holes is by analyzing the free energies
- However, in recent times, several alternative approaches has been gaining popularity
- Thermodynamic geometry/topology, QNMs [*Liu, 2014*], shadow radius of black holes [*Zhang, 2020*], chaotic dynamics [*Guo et al, 2014*] - all carry signatures of black hole phase transition
- More recently, stochastic dynamics has been used to study thermodynamics and kinetics of black holes

From Stochastic Dynamics to Black Hole Thermodynamics

- Stochastic dynamics provides a framework to describe how thermally fluctuating systems move across a free-energy landscape.
- The **Langevin equation** describes the stochastic evolution of an order parameter, while the **Fokker-Planck equation** describes its probability distribution.
- The approach was first applied in protein-folding! [*Frauenfelder, 1991*]

From Stochastic Dynamics to Black Hole Thermodynamics

- In black hole context, this framework was first applied to the H-P transition [*Li, 2020*] and subsequently to van der Waals type [*Wang, 2020*]
- More recently, this framework was used to study the kinetics of phase transition in the vicinity of critical points [*Mann, 2025*]
- The black hole "slows down" as the black hole approaches near criticality or spinodal!
- The so-called critical slowing down

Critical Slowing Down - Analytical Treatment

The stochastic evolution of the order parameter can be described by the Langevin equation

$$\frac{d^2S}{dt^2} = -\zeta\frac{dS}{dt} - \frac{\partial G}{\partial S} + \eta(t) \tag{2}$$

ζ is the dissipation coefficient and $\eta(t)$ is the Gaussian white noise with vanishing mean, satisfying the fluctuation-dissipation theorem

$$\langle\eta(t)\rangle = 0, \quad \langle\eta(t)\eta(s)\rangle = 2\zeta T\,\delta(t-s). \tag{3}$$

Critical Slowing Down - Analytical Treatment

The stochastic dynamics of the order parameter in the overdamped limit can be approximated as

$$\frac{dS}{dt} = -\frac{G^{(2)}(S_e)}{\zeta}(S - S_e) + \frac{1}{\zeta}\eta(t), \tag{4}$$

Signatures of slowing down can be identified by the divergence of the autocorrelation time as the system approaches criticality

The standard solution to Eq. 50 can be expressed as

$$S(t) - S_e = \big(S(0) - S_e\big)e^{-t/\tau} + \frac{1}{\zeta}\int_0^t e^{-(t-t')/\tau}\,\eta(t')\,dt', \tag{5}$$

Critical Slowing Down - Analytical Treatment

Multiplying both sides by $(S(0) - S_e)$ and taking the ensemble average, we get

$$\langle (S(t) - S_e)(S(0) - S_e) \rangle = \left\langle (S(0) - S_e)^2 \right\rangle e^{-t/\tau} + \frac{1}{\zeta} \int_0^t e^{-(t-t')/\tau} \left\langle \eta(t')(S(0) - S_e) \right\rangle dt'. \tag{6}$$

Taking the mean of the stochastic noise to be zero,

$$\langle (S(t) - S_e)(S(0) - S_e) \rangle \sim e^{-t/\tau} \tag{7}$$

Critical Slowing Down - Analytical Treatment

- The autocorrelation time $\tau = \frac{\zeta}{G^{(2)}(S_e)}$
- It is governed by the curvature of free energy landscape
- As the system approaches criticality, $G^{(2)}(S_e)$ vanishes and τ diverges signalling the onset of critical slowing down

Slowing Down in Kerr AdS Black Holes

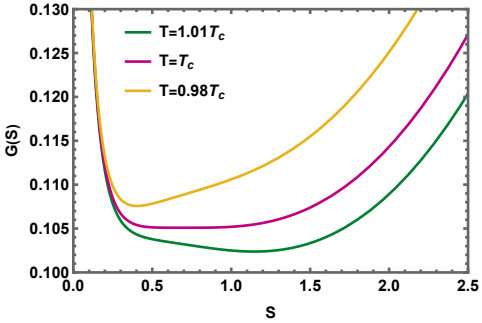


Figure: $G(S)$ for varying T

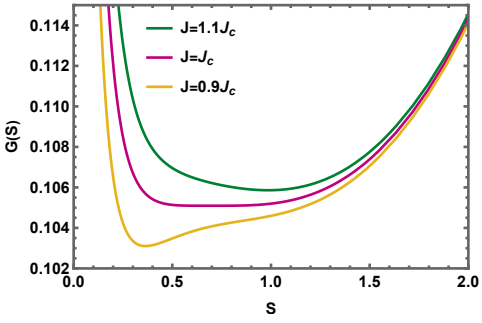


Figure: $G(S)$ for varying J

- The free energy landscape progressively flattens at the critical values of T or J

Slowing Down in Kerr AdS Black Holes

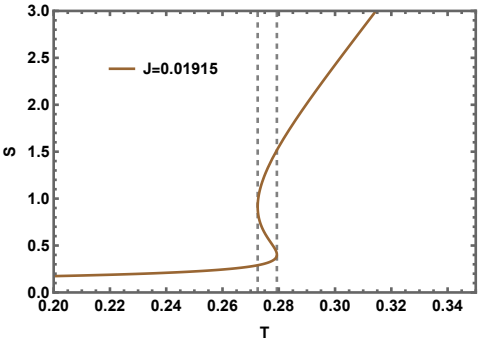


Figure: temperature profile

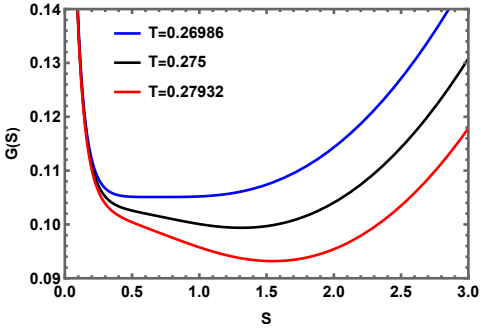


Figure: free energy landscape

- From left to the right spinodal, the SBH well disappears, from right to left LBH well disappears

Simulating the Langevin Equation

- We simulate the Langevin equation governing the evolution of the order parameter on the free energy landscape
- From the resulting stochastic trajectories, we calculate the autocorrelation function and extract the autocorrelation time by fitting it to an exponential decay profile
- We also analyse the variance of the trajectories
- The "Heun Predictor-Corrector" method is used for the simulation

Simulating the Langevin Equation

A Numerical Implementation

In this appendix we explain the numerical techniques used to integrate the Langevin equation and the Fokker-Planck equation

In this appendix, we briefly summarize the numerical methods employed for integrating the Langevin equation and computing the relaxation dynamics near criticality.

The stochastic evolution of the order parameter S is governed by the overdamped Langevin equation

$$\frac{dS}{dt} = -\frac{1}{\zeta}\frac{\partial G(S)}{\partial S} + \eta(t), \tag{A.1}$$

where $G(S)$ denotes the generalized free energy landscape, ζ is the damping coefficient, and $\eta(t)$ represents Gaussian white noise satisfying

$$\langle \eta(t) \rangle = 0, \qquad \langle \eta(t)\eta(t') \rangle = 2\zeta T \delta(t - t'). \tag{A.2}$$

To numerically evolve the stochastic dynamics, we employ the Heun predictor-corrector scheme. For a discretized timestep h, the predictor step is first constructed as

$$\overline{S}_{n+1} = S_n + f(S_n)h + \eta_n, \tag{A.3}$$

where

$$f(S) = -\frac{1}{\zeta}\frac{\partial G}{\partial S}, \tag{A.4}$$

and η_n is sampled from a Gaussian distribution with variance proportional to $2\zeta T h$. The corrected value is then obtained through

$$S_{n+1} = S_n + \frac{h}{2}\left[f(S_n) + f(\overline{S}_{n+1})\right] + \eta_n. \tag{A.5}$$

Numerical Simulation Results

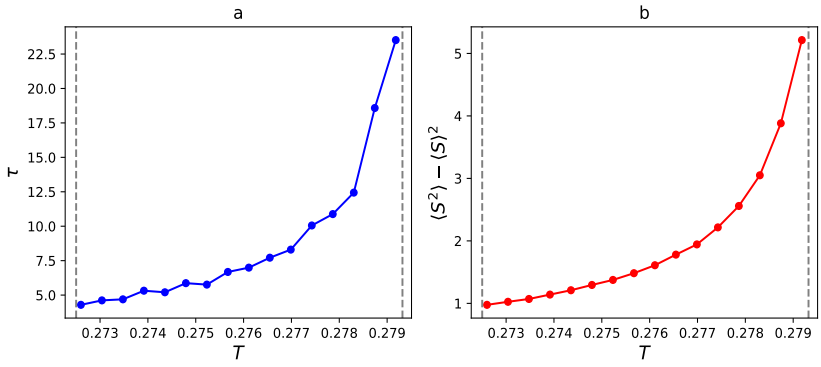


Figure: autocorrelation time and variance near SBH

- τ and variance associated with SBH increase significantly as we go from left to right spinodal

Numerical Simulation Results

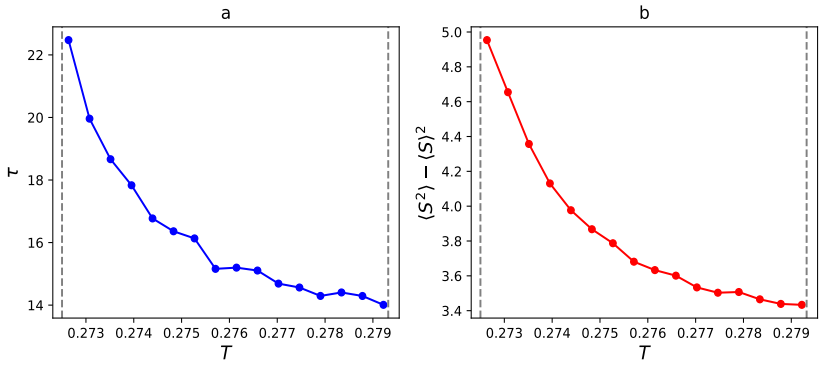


Figure: autocorrelation time and variance near LBH

- τ and variance associated with SBH increase significantly as we go from left to right spinodal

Numerical Simulation Results

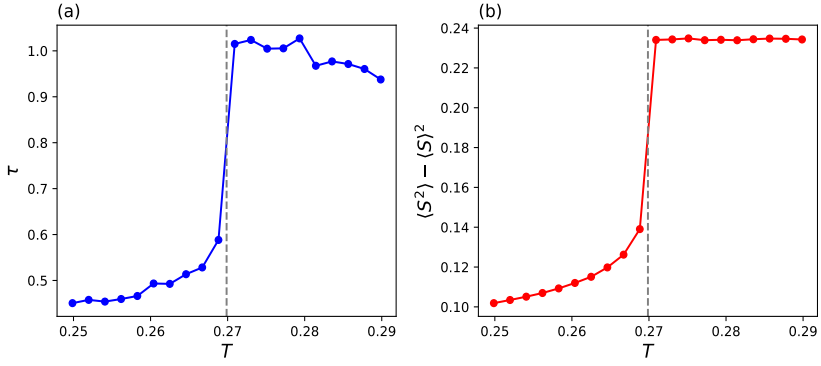


Figure: autocorrelation time and variance around T_c

- τ and variance increase significantly as we approach T_c

Numerical Simulation Results

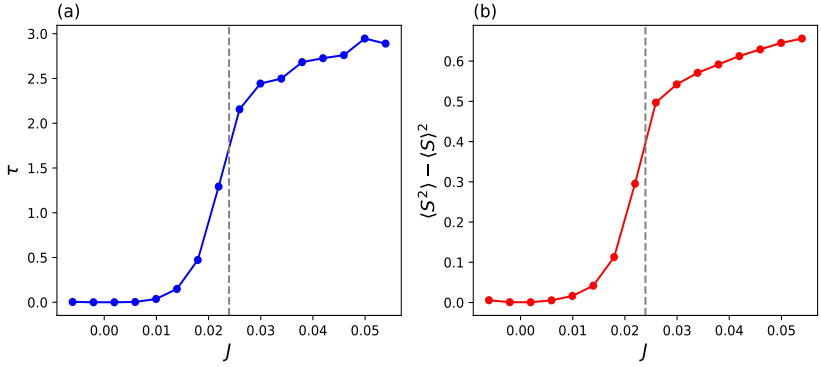
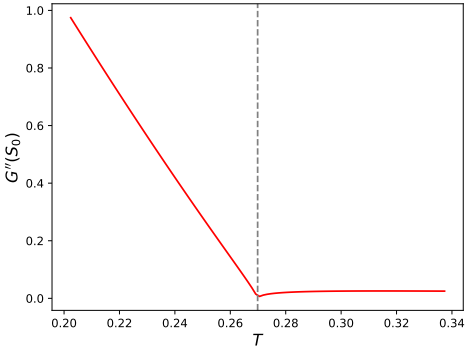


Figure: autocorrelation time and variance around J_c

- τ and variance increase significantly as we approach J_c

Analytical Verification



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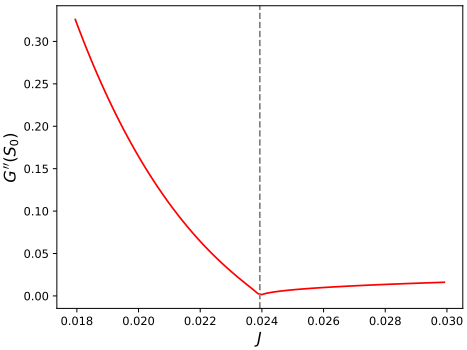

Figure: free energy landscape

- $G''(S_0)$ reaches a minimum at the critical values and then almost flattens

Fokker-Planck Equation

The probability distribution $P(S,t)$ for the order parameter evolves according to the PDE

$$\frac{\partial P(S,t)}{\partial t} = D\frac{\partial}{\partial S}\left\{e^{-\beta G(S)}\frac{\partial}{\partial S}\left[e^{\beta G(S)}P(S,t)\right]\right\} \tag{8}$$

where $D = T/\zeta$ denotes the diffusion coefficient

The stationary solution is given by

$$P_0 \propto e^{-\beta G(S)} \tag{9}$$

Fokker-Planck Equation

The eigenvalues of the FP equation can be calculated by casting it into the form of a self adjoint Schrodinger equation by performing the following spectral expansion

$$P(S,t) = P_0^{1/2}\psi(S)e^{-\lambda t} \tag{10}$$

yielding the Schrodinger type eigenvalue equation

$$-T\frac{d^2\psi}{dS^2} + V(S)\psi = \lambda\psi \tag{11}$$

$V(S)$ is the effective potential determined by the generalised free energy

The autocorrelation time is related as $\tau = 1/\lambda$

Fokker-Planck Equation

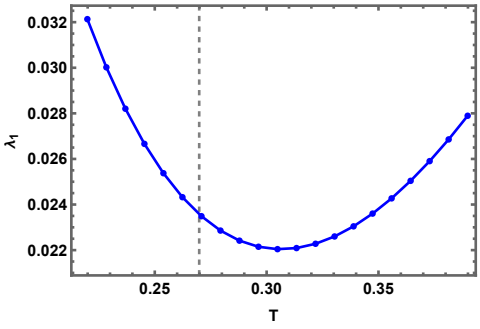

Figure: λ_1 vs T

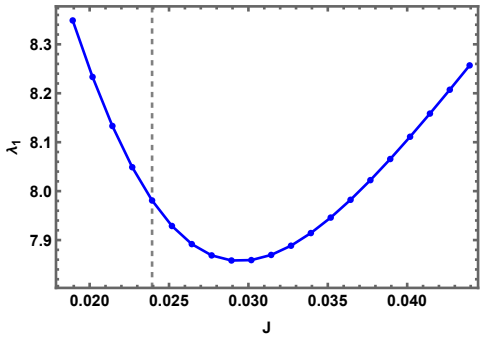

Figure: λ_1 vs J

- Lowest eigenvalue is almost at the critical point

Scaling Law of τ and Universality

- We know that the autocorrelation time τ typically follows a scaling law $\tau \sim |\epsilon|^{-\Delta}$
- With $\epsilon = \frac{T-T_c}{T_c}$, $\epsilon_P = \frac{P-P_c}{P_c}$, $\epsilon_J = \frac{J-J_c}{J_c}$

Scaling Law of τ and Universality

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Scaling Law of τ and Universality

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- But what is Δ?
- And is it universal near the criticality?

Scaling Law of τ and Universality

To determine the "dynamical critical exponent", we fit the numerically obtained autocorrelation time near the critical point to the scaling form

$$\tau = a|\epsilon|^{-\Delta} \tag{12}$$

where a and Δ are treated as fitting parameters.

Scaling Law of τ and Universality

RN-AdS Black Hole

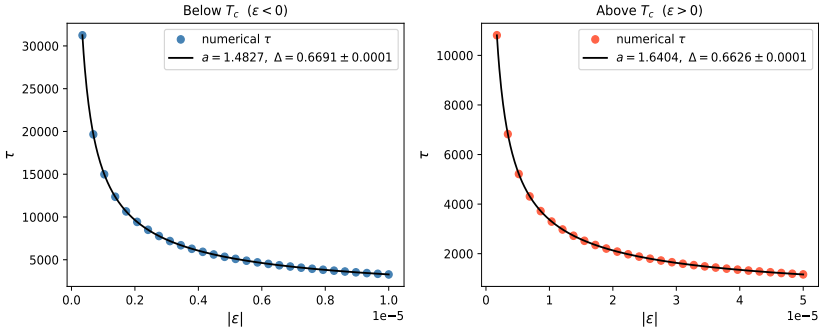

Figure: scaling behaviour of τ near criticality along T

- Δ comes out to be approximately 0.66

Scaling Law of τ and Universality

RN-AdS Black Hole

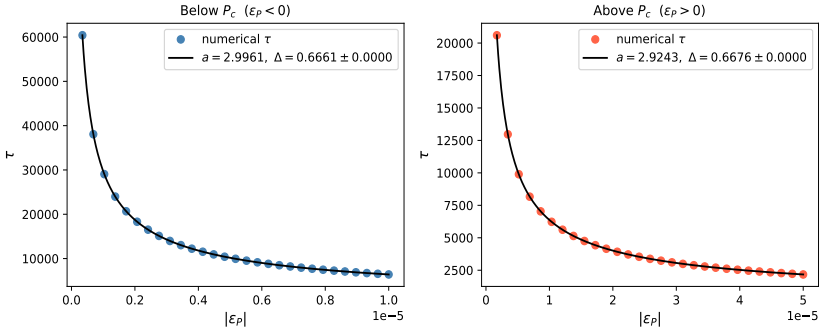

Figure: scaling behaviour of τ near criticality along P

- Δ comes out to be approximately 0.66

Scaling Law of τ and Universality

Kerr-AdS Black Hole

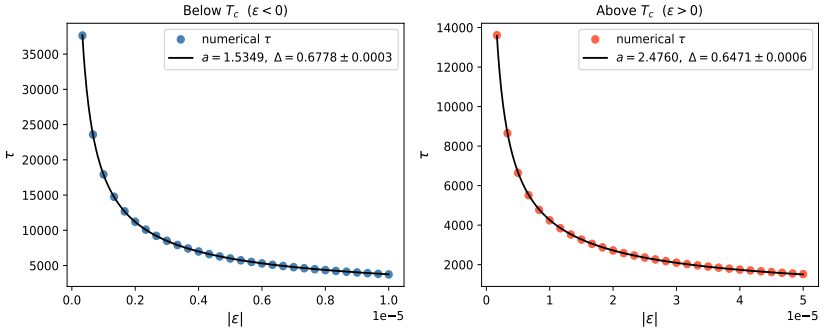

Figure: scaling behaviour of τ near criticality along T

- Δ is found to be approximately 0.66

Scaling Law of τ and Universality

Kerr-AdS Black Hole

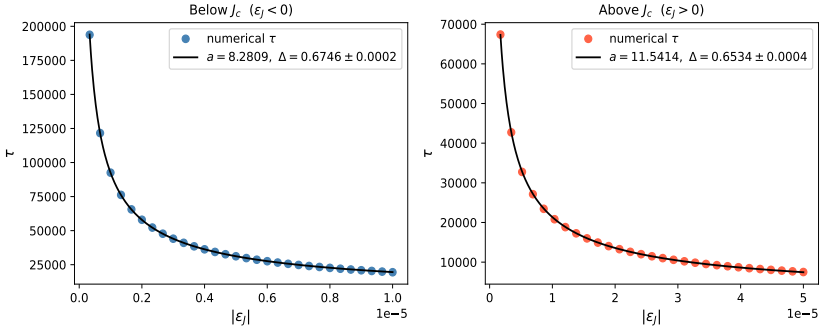

Figure: scaling behaviour of τ near criticality along J

- Δ is found to be approximately 0.66

Scaling Law of τ and Universality

Bardeen Black Hole

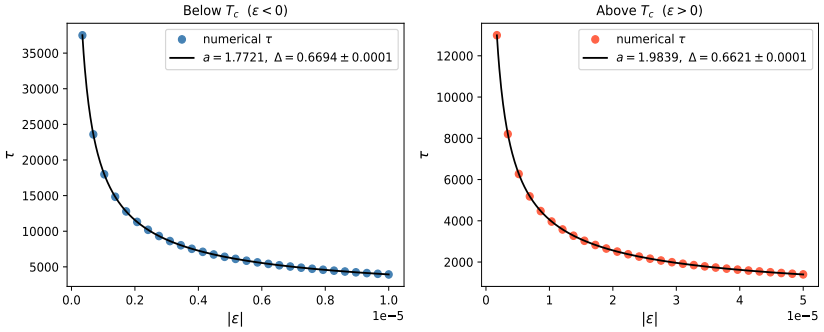

Figure: scaling behaviour of τ near criticality along T

- Δ is found to be approximately 0.66

Scaling Law of τ and Universality

Bardeen Black Hole

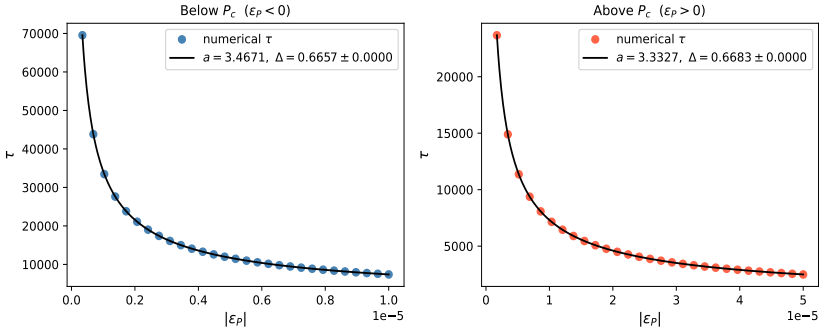

Figure: scaling behaviour of τ near criticality along P

- Δ is found to be approximately 0.66

Scaling Law of τ and Universality

From Fokker-Planck Spectrum

- The slowest relaxation process is governed by the smallest non-zero eigenvalue of the FP operator λ_1

$$\tau \sim 1/\lambda_1$$

- The relaxation time scales as $\tau \sim |\epsilon|^{-\Delta}$. Therefore, λ_1 should obey the following scaling relation

$$\lambda_1 \sim |\epsilon|^{\gamma}$$

- We can extract the exponents from linear fits of the corresponding log-log plots

Scaling Law of τ and Universality

From Fokker-Planck Spectrum

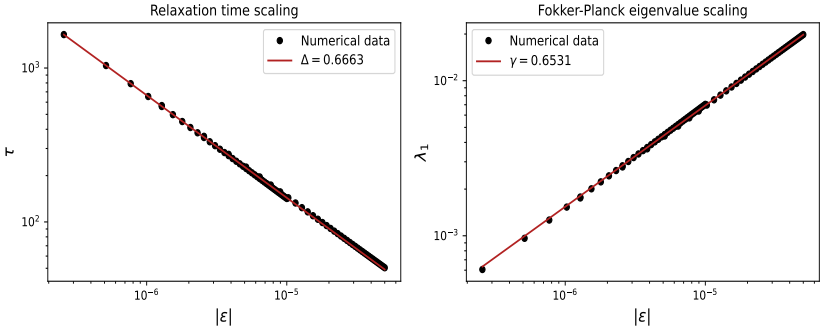

Figure: τ and λ_1 scaling for RN-AdS Black Hole

- $\Delta \approx \gamma \approx 0.66$

Scaling Law of τ and Universality

From Fokker-Planck Spectrum

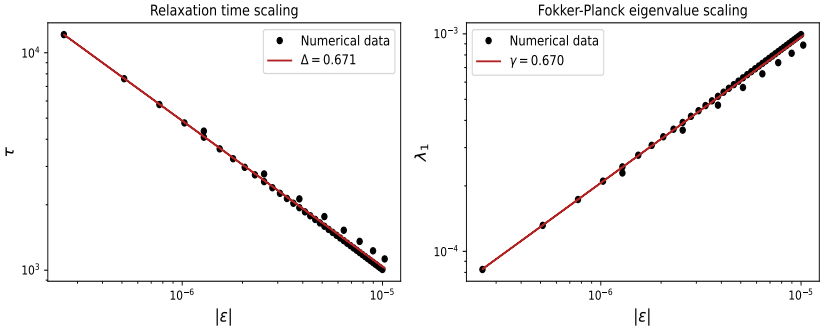

Figure: τ and λ_1 scaling for Kerr-AdS Black Hole

- $\Delta \approx \gamma \approx 0.66$

Scaling Law of τ and Universality

From Fokker-Planck Spectrum

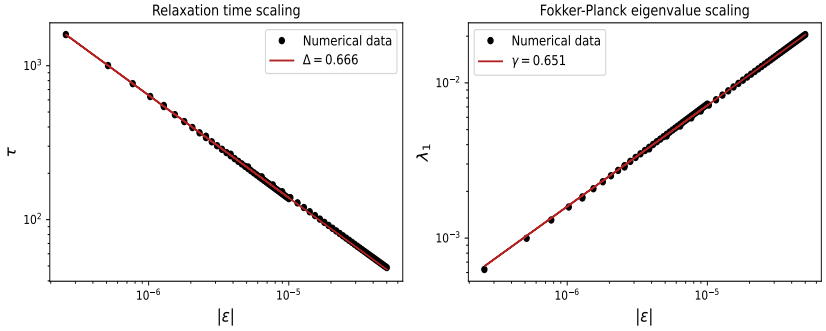

Figure: τ and λ_1 scaling for Bardeen Black Hole

- $\Delta \approx \gamma \approx 0.66$

Equilibrium configurations are determined from

$$\frac{\partial G(x,T)}{\partial x} = 0 \tag{13}$$

At the critical point (x_c, T_c), the free energy develops an inflection point structure. Consequently, the first few derivatives satisfy

$$\left.\frac{\partial G}{\partial x}\right|_c = 0, \qquad \left.\frac{\partial^2 G}{\partial x^2}\right|_c = 0, \qquad \left.\frac{\partial^3 G}{\partial x^3}\right|_c = 0 \tag{14}$$

while the quartic derivative remains nonvanishing,

$$\left.\frac{\partial^4 G}{\partial x^4}\right|_c \neq 0 \tag{15}$$

Expanding the equilibrium condition Eq. 50 around the critical point yields

$$0 = \left.\frac{\partial G}{\partial x}\right|_c + \left.\frac{\partial^2 G}{\partial x \partial T}\right|_c (T - T_c) + \frac{1}{3!}\left.\frac{\partial^4 G}{\partial x^4}\right|_c (\Delta x)^3 + \cdots \tag{16}$$

Retaining the leading contributions, one obtains

$$A\,\epsilon + B\,(\Delta x)^3 \simeq 0 \tag{17}$$

where

$$A = T_c \left.\frac{\partial^2 G}{\partial x \partial T}\right|_c, \qquad B = \frac{1}{3!}\left.\frac{\partial^4 G}{\partial x^4}\right|_c \tag{18}$$

Eq. 63 immediately implies

$$\Delta x \sim |\epsilon|^{1/3} \tag{19}$$

The relaxation dynamics is governed by the curvature of the Gibbs free energy at equilibrium,

$$G''(x_e) = \left.\frac{\partial^2 G}{\partial x^2}\right|_{x=x_e} \tag{20}$$

Expanding this quantity near criticality gives

$$G''(x_e) = \left.\frac{\partial^2 G}{\partial x^2}\right|_c + \left.\frac{\partial^3 G}{\partial x^3}\right|_c \Delta x + \frac{1}{2}\left.\frac{\partial^4 G}{\partial x^4}\right|_c (\Delta x)^2 + \cdots . \tag{21}$$

The first two terms vanish identically, leaving

$$G''(x_e) \sim (\Delta x)^2 \tag{22}$$

Using Eq. 65, one therefore obtains

$$G''(x_e) \sim |\epsilon|^{2/3} \tag{23}$$

The relaxation time is related to the inverse curvature of the free energy,

$$\tau \propto \frac{1}{G''(x_e)} \tag{24}$$

Consequently,

$$\tau \sim |\epsilon|^{-2/3} \tag{25}$$

Hence, $\Delta = \frac{2}{3}$

Hence, $\Delta = \frac{2}{3}$ and $\frac{2}{3} \approx 0.66...$!

Degenerate Bifurcations and Universal Relaxation Scaling in Black Hole Thermodynamics

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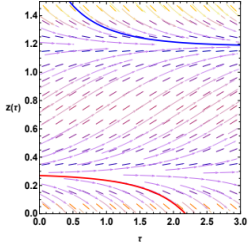
We present a dynamical systems approach to black hole thermodynamic criticality based on bifurcation equations. We construct an effective thermodynamic landscape in which black holes relax toward equilibrium fixed points. To describe this process, we introduce a flow parameter τ, interpreted as a phenomenological relaxation time, which governs the approach toward equilibrium configurations in thermodynamic state space. Near critical points, the thermodynamic flow simplifies into universal mathematical forms, which allows different black holes to be grouped into different universality classes based on their critical behavior. Our analysis further shows critical slowing down, with relaxation timescales determined entirely by the local bifurcation structure.

I. INTRODUCTION

The formulation of black hole thermodynamics [1–4] has witnessed extensive developments and generalizations in different gravitational settings [5–11]. Among the most actively studied topics are black hole phase transitions [12–29], including the Davies transition [12], the Hawking–Page transition [13], extremal phase transitions [14–22], and Van der Waals type behavior in extended phase space [23–29]. Recently, ideas from nonlinear dynamics have also found important applications in black hole physics. Dynamical stability during black hole phase transitions was investigated in Ref. [54], while instabilities of thin black rings were studied in Ref. [55]. Chaotic



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Future Prospects

- We plan to study the Holographic aspects of the critical slowing down using the AdS/CFT correspondence and compare the relaxation timescale with the fast scrambling timescale.

Thank you for listening!