Nonlinear Soliton Dynamics, Wave Engineering, and Physics-Informed Learning in Nonlinear Integrable Systems

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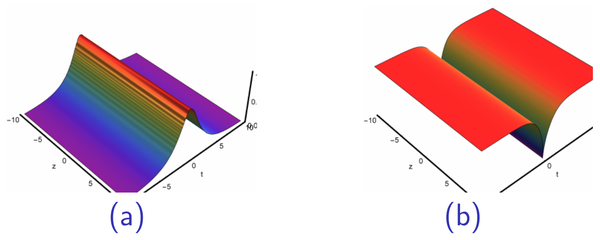
INTRODUCTION

- Nonlinear Integrable Systems are essential for describe wave dynamics, soliton propagation, and nonlinear interactions across a wide variety of physical systems.
- Solitons are stable localized wave arising in nonlinear integrable system.
  - Constant Velocity and Amplitude.
  - Elastic Collision, Phase & Position change
- **Inhomogeneous Media**: Dispersion and Nonlinearity control soliton properties like velocity, amplitude, position.



(a) (b)

Figure: (a) and (b) showing a bright and a dark soliton, respectively.

Governing Equations

- The Fokas Lenells Equation
  An integrable eqaution to describe ultra fast light propagation

$$iU_t + \gamma_1 U_{xx} + \gamma_3 |U|^2 U + \gamma_2 (i\gamma_3 |U|^2 U_x - U_{xt}) = 0$$

- reduced FLE under the gauge transformation

$$U_{xt} - U - 2i|U|^2 U_x = 0$$

- variable FLE with Gain & Loss terms

$$U_{xt} - D(t)U - 2iR(t)|U|^2 U_x = \Gamma(t) U_x$$

  - $D(t)$= Dispersion term.
  - $R(t)$= nonlinearity term.
  - $\Gamma(t)$=Gain/Loss term.
- Boundary conditions determine the soliton family.
  Vanishing Boundary→ Bright Soliton
  Non-vanishing Boundary →Dark soliton

Methodology

- **Hirota Bilinearization with Auxillary functions**: Trilinear→ bilinear equation →Linear equation→ ε- expansion gives $1-,2-,N-$ soliton solutions.
- 2×2 Lax pair + Ricatti equation: →Infinite hierarchy of conserved densities H_n.
- Gauge transformation of the Lax pair→ Landau-Lifshitz spin equation.

$$\Gamma_x = -i\zeta^2\Gamma + \zeta u_x + \zeta q_x^* \Gamma^2.$$

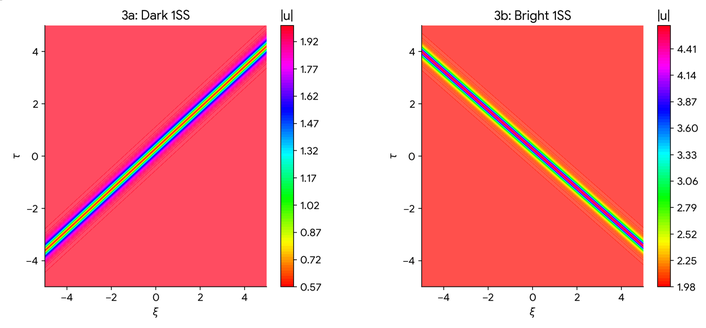
- **Physics Informed Neural Network(PINN)** approach
  FLE is solved using PINN, in which a neural network approximates the solution, and its parameters are optimized by minimizing the loss function $\mathscr{L}$ incorporating the governing PDE residual, initial conditions, boundary conditions and conserved quantities.

$$\mathscr{L} = \mathscr{L}_{IC} + \mathscr{L}_{BC} + \gamma_1 \mathscr{L}_{phys} + \gamma_2 \sum_i \left\| \partial_t \hat{H}_i \right\|^2,$$

RESULTS

Soliton Profiles

The Bright and Dark one soliton solution are explicitly obtained

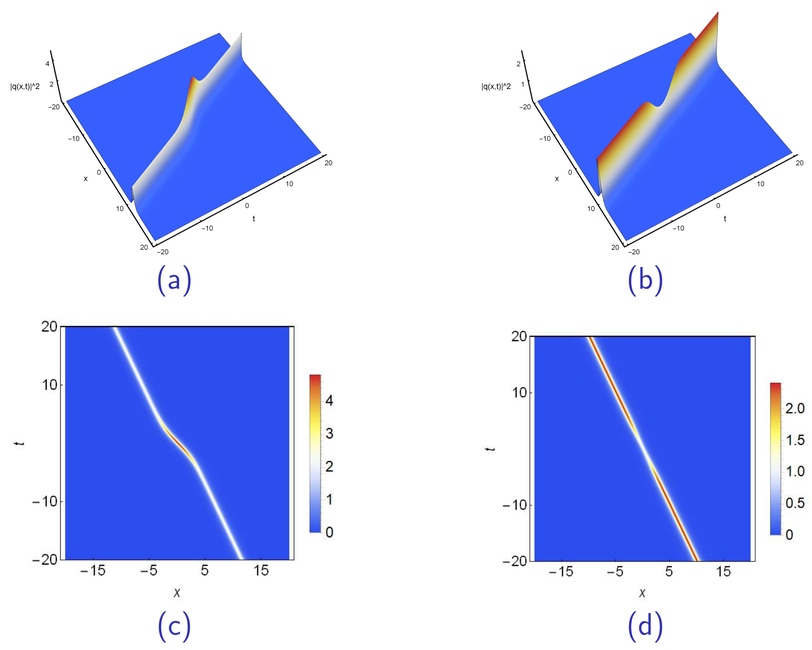


(a) (b)

Figure: (a) and (b) showing adensity ploy of Bright and a Dark soliton, respectively.

Inhomogeneous Propagation

- Soliton velocity and amplitude can be engineered by controlling the dispersion and nonlinearity of the system. [S.Talukdar,et.al.]
- Bright solitons can propagate through Gaussian-shaped dispersion and nonlinearity barriers while preserving their original shape after tunneling.
- The time-dependent gain/loss mechanism governs the amplitude evolution during barrier propagation.



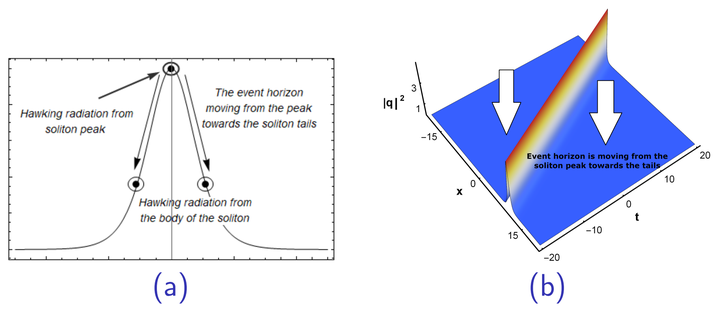
(a) (b)

(c) (d)

Figure: The 3D plot of (a) a 1SS tunneling through the dispersion barrier. In (b) we show the tunneling through a nonlinear barrier. Plots (c) and (d) show the density distributions of the corresponding 3D plots (a), (b) respectively under the same parametric values.

Soliton Geometry and Analogue Black hole

Soliton surface geometry provides a route to investigate analogue gravity. [S. Talukdar et.al.]

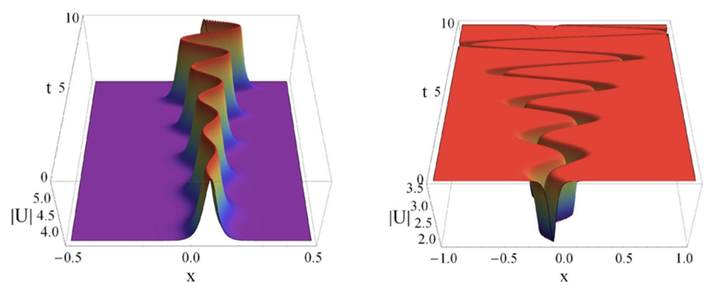


(a) (b)

Figure: Schematic diagram showing the opening of the event horizon at the peak of the soliton and moving towards the soliton tails and eventually closing at the tails.

Soliton Management

$D(t)$ and $R(t)$ are periodic functions that make the propagation of the soliton harmonic from the mean position $x = 0$. The amplitude remains constant along the propagation of the soliton.[Riki Dutta,et.al.]



(a) Anti-Drak (b) Dark

Figure: 3D plot representation of 1SS under SM scheme: $D(t) = R(t) = e^{\sigma t} cos(kt)$ with $\sigma = 0.25$,and $k = \pi$ (a) represents anti-dark soliton and (b) represents dark soliton.

RESULTS

Conserved PINN

A conserved quantity-informed PINN is proposed, demonstrating improved accuracy and performance compared to vanilla PINNs, and is employed to simulate soliton propagation and collision. [GK saharia, et.al.]

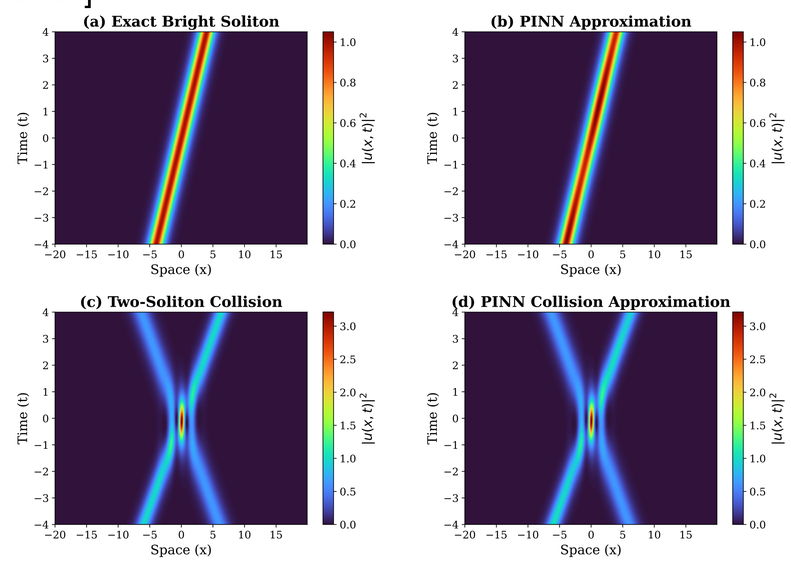


Figure: (a) and (b) showing exact soliton vs PINN and (c) and (d) showing exact soliton collsion vs PINN

Conclusions and Outlooks

- The FLE supports localized wave structures, including bright, dark, and anti-dark solitons, whose dynamics can be controlled through system parameters and inhomogeneous coefficients.
- Integrability establishes a connection between exact soliton solutions, conservation laws, and their equivalence to spin systems.
- Soliton surface geometry provides a route to investigate analogue-gravity connections the corresponding FLE construction is underway.
- Conservation-informed PINNs complement analytical methods for reconstructing nonlinear wave dynamics.

References

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