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Track:VIII

Nonlinear Dynamics Of Mod-max Ads Black Hole: Phase Bifurcation

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Why we formulate black-hole thermodynamics as a dynamical system?

- Black holes are the most intriguing predictions of general relativity and provide a natural setting for exploring the interplay between gravity, thermodynamics, and nonlinear dynamics.

- Conventional black hole thermodynamics primarily describes equilibrium configuration through first law:

$$\mathrm{dM} = \frac{\kappa}{8\pi}\mathrm{dA} + \Omega\mathrm{dJ} + \phi\mathrm{dQ}$$

- However, the equilibrium description alone does not fully reveal-

  - How thermodynamic configurations evolve.
  - Whether equilibrium states are stable or unstable.
  - How different black hole branches emerge or disappear.
  - How critical phenomena and phase transitions are organized.

Why we formulate black-hole thermodynamics as a dynamical system?

- The dynamical system in thermodynamic state enable us to study-
  1. Fixed points and equilibrium configurations.
  2. Local stability and instability.
  3. Evolution of thermodynamic branches
  4. Bifurcations and critical phenomena.

- Our main motivation is to develop a mathematical framework capable of investigating the stability, evolution and structural changes of black hole thermodynamic configurations.

Bifurcation Theory: Development and Motivation in Black Hole Thermodynamics

- Bifurcation theory studies how the behavior of a dynamical system changes when a control parameter is varied.

- We can treat the equilibrium black-hole states as fixed points of a dynamical system. So, by varying the bifurcation parameter, we can study how the number and stability of these equilibrium states change.

- Mathematically,

$$\frac{dz}{d\tau} = F(z, q, h)$$

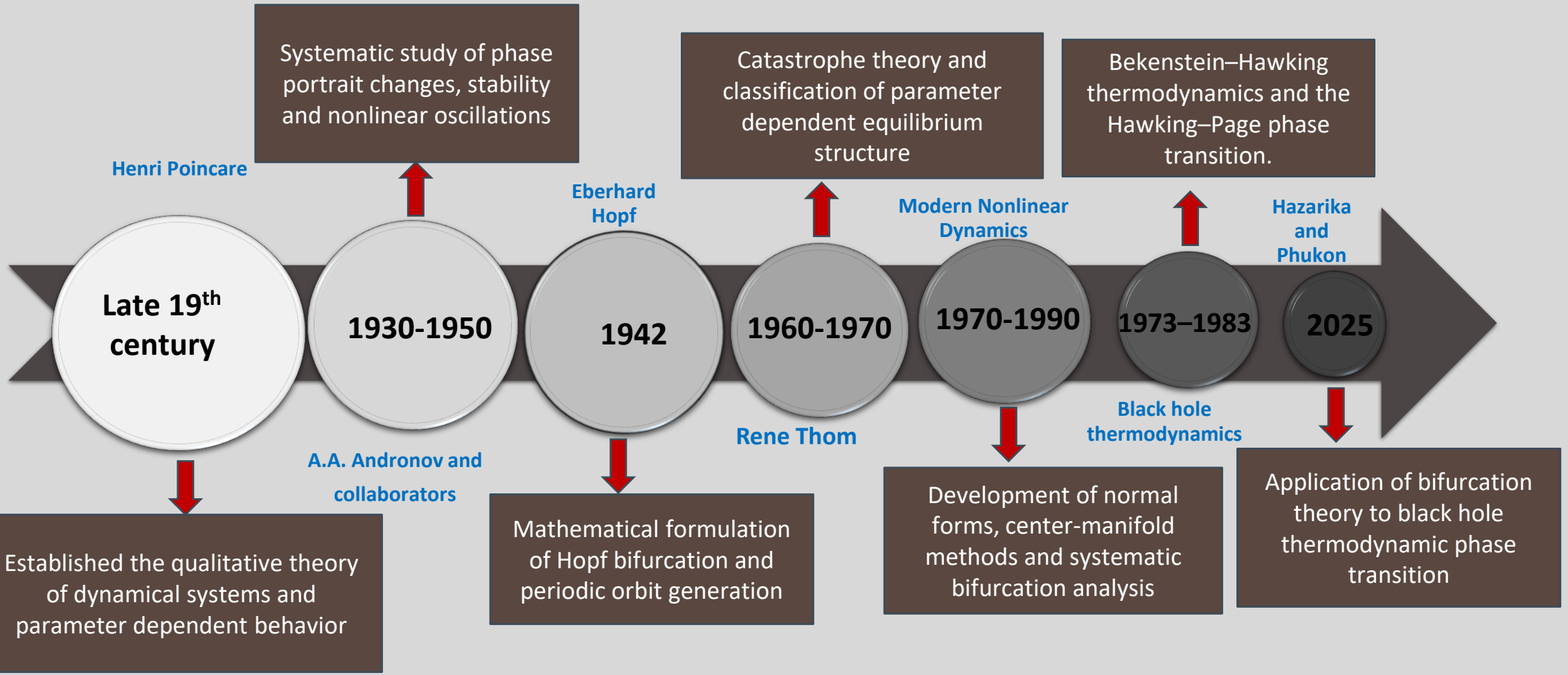
The fixed points satisfy

$$F(z_*, q, h) = 0$$

Equilibrium black hole states ⇔ Fixed points

Control parameter variation → Change in fixed-point structure

Bifurcation Theory: Development and Motivation in Black Hole Thermodynamics



Bifurcations in ModMax AdS black hole thermodynamics

- Bifurcation function is defined as

$$\dot{z} = \frac{dz}{d\tau} = -\frac{d\mathrm{M}}{dz} + h\frac{d\mathrm{S}}{dz}$$

- For ModMax AdS black hole bifurcation function can be written as

$$\dot{z} = \frac{dz}{d\tau} = -\frac{d\mathrm{m}}{dz} + h\frac{d\mathrm{S}}{dz} = \frac{1}{2}\left(\frac{q^2 e^{-\tilde{\eta}}}{z^2} + 4\pi h z - 3z^2 - 1\right)$$

z → is the dimensionless horizon variable.
h>0→ is the control parameter.
τ→is the system's evolution.
ñ → is the ModMax parameter.

z = r_+/l, q = Q/l and ñ denotes the rescaled ModMax parameter.

Bifurcations in ModMax AdS black hole thermodynamics

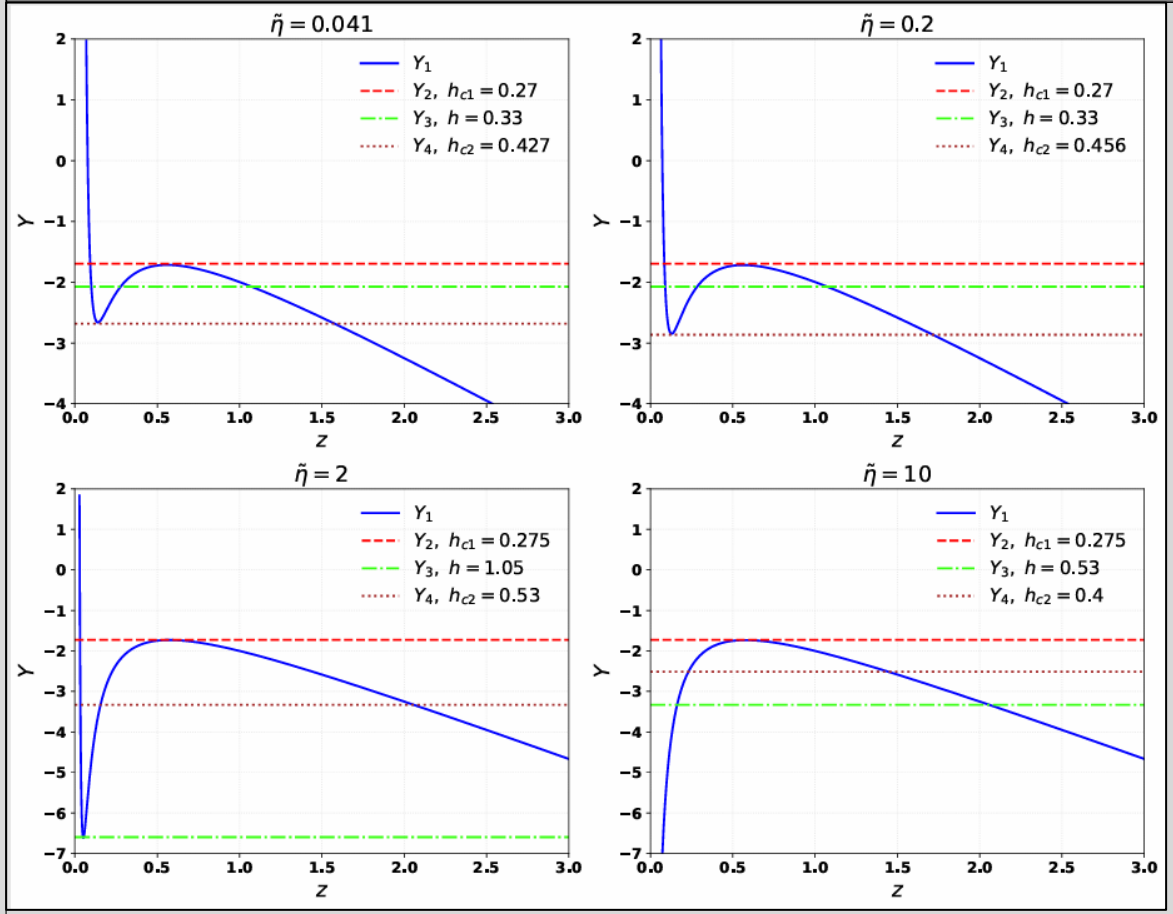


Fig1: Bifurcation structure for different η̃

- Graphical fixed-point condition

$$y_1(z) = -\frac{e^{-\tilde{\eta}}\left(3z^4 e^{\tilde{\eta}} + z^2 e^{\tilde{\eta}} - q^2\right)}{2z^3}$$

$$y_2(z) = -2\pi h$$

- Each intersection gives a fixed point.

- At the critical parameter value, the curves $y_1(z)$ and $y_2(z)$ satisfy the tangency condition, corresponding to a degenerate fixed point and the onset of a saddle-node bifurcation.

Bifurcation Structure and Phase Diagrams of ModMax AdS Black Hole

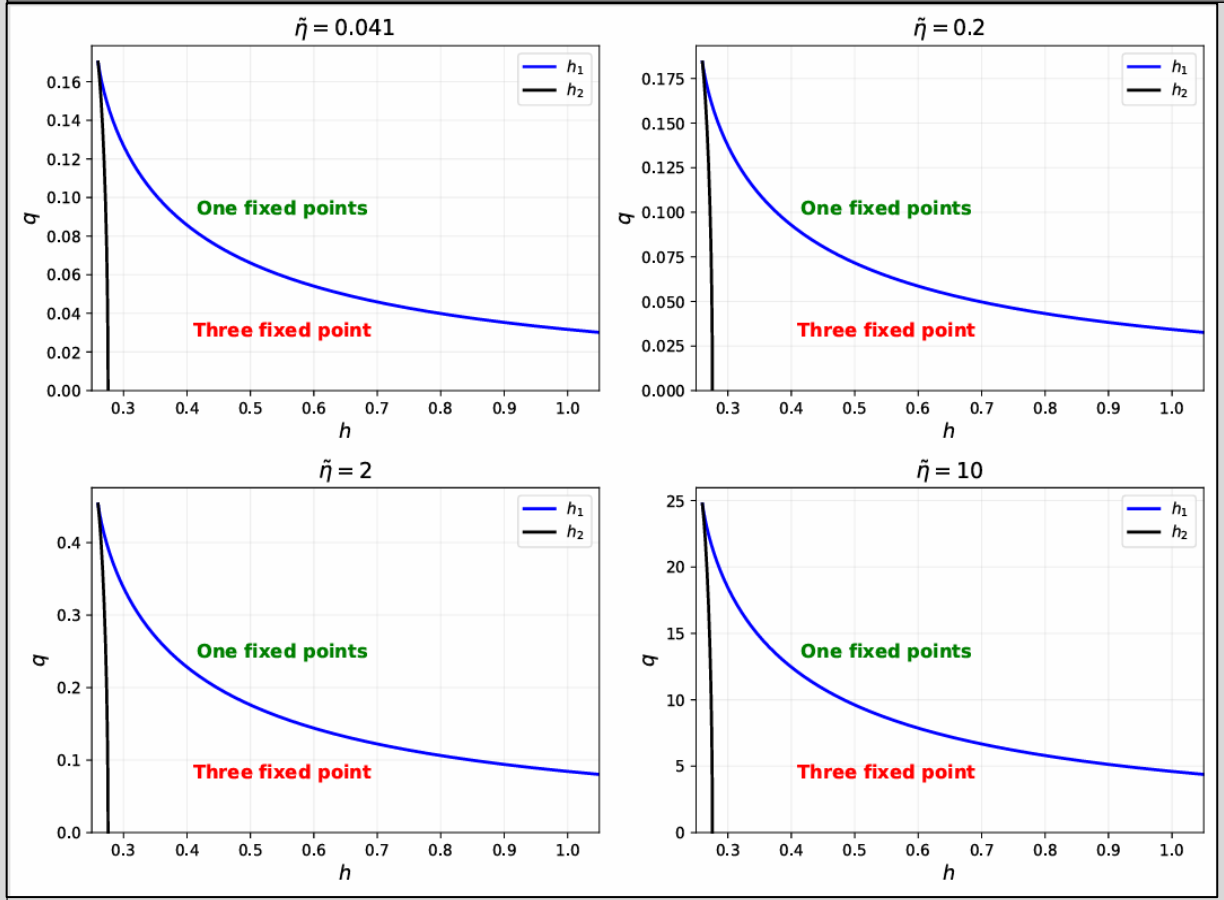


Fig2: Bifurcation curves in the (h, q) parameter space

- The bifurcation structure is characterized by two critical curves

$$h = h_{c1}(q) = 0 \qquad h = h_{c2}(q) = 0$$

- These curves represent the loci of saddle-node bifurcations.
- The two fold branches meet at a codimension-2 cusp in the (h, q) plane.
- **Inside the cusp wedge: three fixed points**
  Outside: one fixed point

Bifurcation Structure and Phase Diagrams of ModMax AdS Black Hole

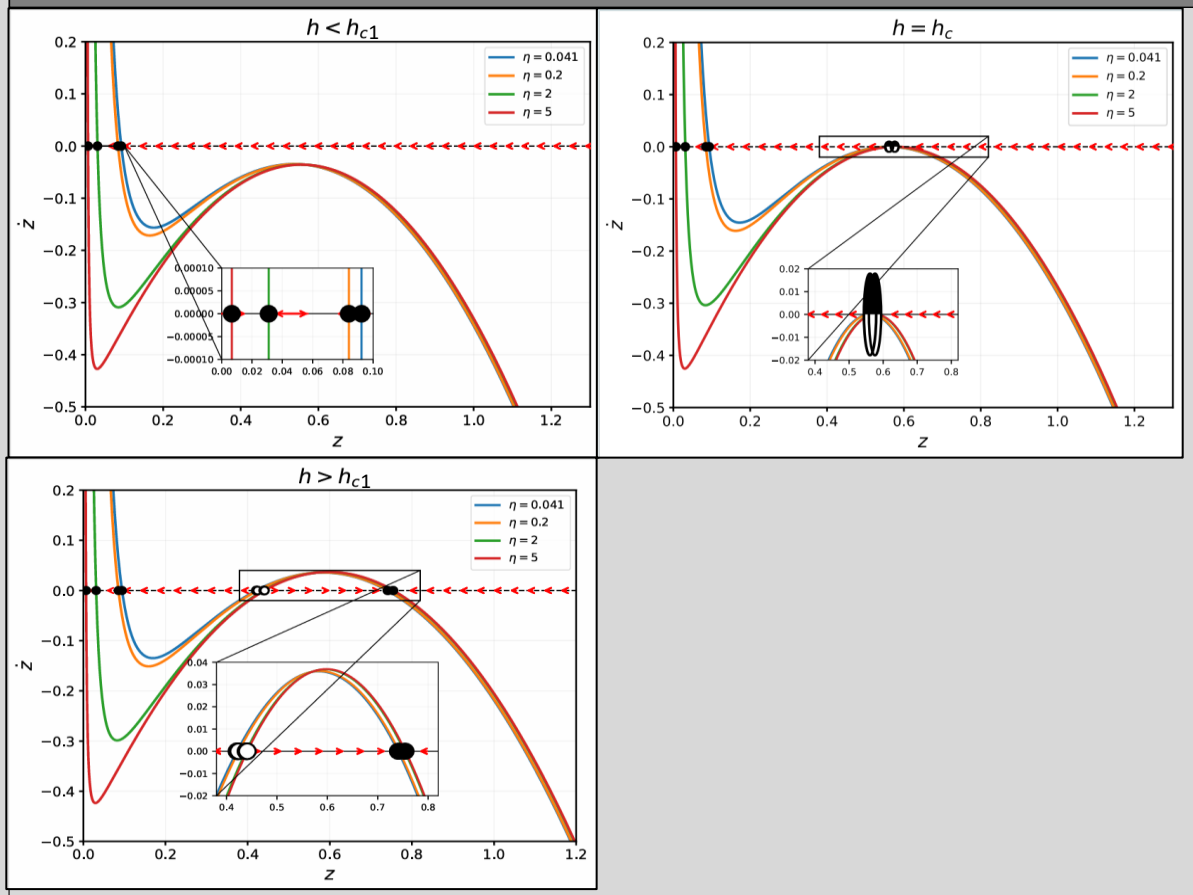


Fig3: Phase portrait of the one-dimensional dynamical system

- The dynamical system is described by

$$\dot{z}=f(z)$$

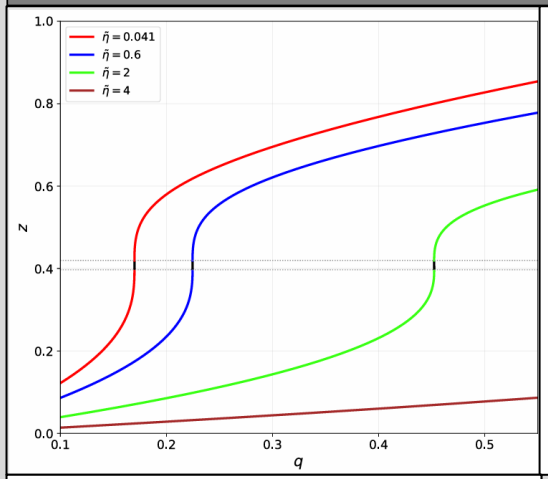
- Fixed points are obtained from the condition

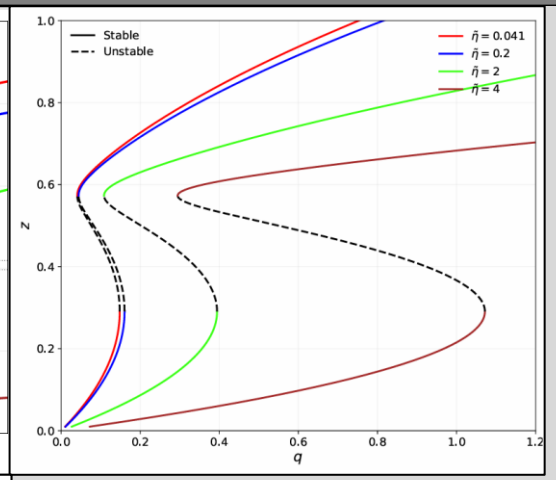
$$f(z^*)=0$$

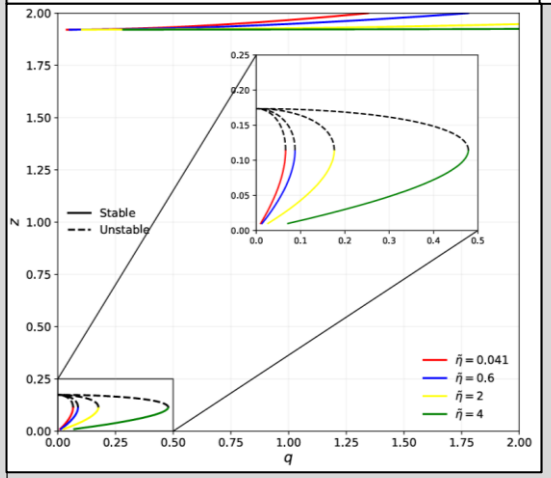
- We consider three different regimes of the control parameter h.

$h< h_{c1}$- One stable fixed point (unique attractor).
$h = h_{c1}$- Two equilibria coalesce into a nonhyperbolic fixed point.
$h> h_{c1}$- Three fixed points: stable–unstable–stable.

Bifurcation Structure and Phase Diagrams of ModMax AdS Black Hole







- The bifurcation diagram illustrates the equilibrium structure in (q, z) plane.
- Different values of the ModMax parameter ñ modify the location and shape of the equilibrium curves.
- The equilibrium configurations consist of three black-hole branches i.e. SBH, IBH and LBH.
- For $h = h_{c1}$, the equilibrium branches exhibit the characteristic cusp-catastrophe behavior.
- For $h > h_{c1}$, the system exhibits a broken-pitchfork like bifurcation structure.

Fig4: Bifurcation diagram in the (q, z) plane for (a)h = 0.26 < hc1, (b)hc1 = 0.275,(c)h = 0.5 > hc1

Limiting Cases of the ModMax Parameter

Saddle-node radii:

$$z_{\pm} = \frac{\sqrt{1 \pm e^{-\frac{\tilde{\eta}}{2}} \sqrt{e^{\tilde{\eta}} - 36q^2}}}{\sqrt{6}}$$

Maxwell limit

$\tilde{\eta} \to 0$, $e^{\tilde{\eta}} \to 1$

$$z_{\min} = \frac{\sqrt{1 - \sqrt{1 - 36q^2}}}{\sqrt{6}} \, , \; z_{\max} = \frac{\sqrt{1 + \sqrt{1 - 36q^2}}}{\sqrt{6}}$$

RN AdS limit- Both saddle-node radii remain finite.

Strong ModMax limit

$\tilde{\eta} \to \infty$, $e^{\tilde{\eta}} \to 0$

$$\lim_{\tilde{\eta} \to \infty} z_{\min} = 0 \quad , \quad \lim_{\tilde{\eta} \to \infty} z_{\max} = \frac{1}{\sqrt{3}}$$

Schwarzschild AdS limit- Lower saddle-node approaches the origin; upper one remains finite.

Increasing $\tilde{\eta}$ suppresses the effective electric charge, $q_{eff} = q\, e^{\frac{\tilde{\eta}}{2}}$, continuously driving the bifurcation structure from RN-AdS towards Schwarzschild-AdS.

Summary and Conclusion

- The ModMax parameter rescales the effective charge $q_{eff} = qe^{-\frac{\tilde{\eta}}{2}}$, shifting the bifurcation curves and fixed-point locations along the charge direction.
- The qualitative bifurcation structure remains unchanged. The system exhibits one and three fixed point regimes.
- The two saddle node branches meet at a codimension-2 cusp, which separates the one and three fixed point regions in parameter space.
- In the three fixed point regime, the stability ordering is stable-unstable-stable, corresponding to the small, intermediate and large black hole branches.
- The ModMax parameter changes the quantitative location and shape of the bifurcation structure while preserving its overall qualitative topology.

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