Large-N QCD on the lattice from twisted Eguchi–Kawai reduction

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Lattice Seminar
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Introduction

Quantum Chromo-Dynamics (QCD)
Quantum Field Theory describing strong interactions among hadrons

- **Major endeavor**: achieve thorough understanding of QCD from first principles
- Key to describe collider experimental discoveries, e.g., growing hadron spectrum
- QCD $\longrightarrow$ $\mathrm{SU}(N)$ gauge theory with $N = 3$ colors

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- QCD $\longrightarrow$ SU(N) gauge theory with $N = 3$ colors

- **This seminar**: ideal limit $N \longrightarrow \infty$ 't Hooft NPB 72 (1974) 461
- Robust theoretical tool extensively used in QFT
In QCD: predictive power for hadron phenomenology
- **Main message**: new powerful framework for efficient numerical resolution of large-N QCD
- This has allowed significant progress in large-N QCD phenomenology in the last few years

Picture: Proceedings from the Institute for Nuclear Theory: Vol. 12, World Scientific (2002)

The large-N limit of QCD

$N \to \infty$ $\qquad$ $\lambda \equiv g^2 N = \text{fixed}$ $\qquad$ $N_\text{f} = \text{fixed}$ $\qquad$ 't Hooft NPB 72 (1974) 461

$$\to \qquad \mathcal{S}_\text{QCD} = \frac{1}{g^2} \int \mathrm{d}^4 x \left(\frac{1}{4} G^a_{\mu\nu}(x) G^a_{\mu\nu}(x) + g^2 \sum_{\text{f}=1}^{N_\text{f}} \overline{\psi}_\text{f} (\not{D} + m_\text{f}) \psi_\text{f} \right)$$

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$N \to \infty$ path integral via saddle-point approximation $\to$ $1/N$ EXPANSION

$$\langle O \rangle = \frac{1}{Z} \int \mathcal{D}A \mathcal{D}\overline{\psi} \mathcal{D}\psi \, \mathrm{e}^{-N\widetilde{\mathcal{S}}} O \quad \underset{N\to\infty}{\sim} \quad O_0 + O_1 \frac{1}{N} + O_2 \frac{1}{N^2} + \mathcal{O}\left(\frac{1}{N^3}\right)$$

Non-perturbative expansion in $g \to$ framework for **non-perturbative regime**

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Large-N QCD preserves crucial features of QCD:

- typical hadronic scale $\quad \Lambda_\text{QCD} = \mu \, [b_0 \lambda(\mu)]^{-b_1/(2b_0^2)} \, \mathrm{e}^{-1/[2 b_0 \lambda(\mu)]} + \mathcal{O}(\lambda) + \mathcal{O}\left(\frac{1}{N} \right)$
- asymptotic freedom, spontaneous chiral symmetry breaking

Large-N QCD and hadron phenomenology

Main Idea: $1/N = 1/3 \approx$ **small**

- $1/N$ expansion bridges ideal ($N = \infty$) and real-world ($N = 3$) QCD
- Paramount example: **Witten–Veneziano formula** NPB 156 (1979) 269 — NPB 159 (1979) 213

$$m_{\eta'}^2 = \frac{2N_f}{F_\pi^2}\chi_{YM}^{(\infty)} + 2m_K^2 - m_\eta^2 \qquad \chi_{YM}^{(\infty)} \to N = \infty \text{ YM topo. susceptibility}$$

Giusti et al. NPB **628** (2002) 234; Del Debbio et al. PRL **94** (2005) 032003; Cè et al. PLB **762** (2016) 232; Bonati et al. PRD **94** (2016) 085017

Large N: simplified framework to investigate **hadron phenomenology**

- Meson/glueball masses: $\sim \mathcal{O}(N^0) \to$ dynamically relevant as $N \to \infty$
- Baryon masses: $\sim \mathcal{O}(N) \to$ decouple as $N \to \infty$
- Decay rates, mixings, scattering amplitudes $\to$ subleading in $1/N$
  $N = \infty \to$ **theory of stable and non-interacting hadrons**

Large-N QCD and hadron phenomenology

Large-N behavior of meson spectrum/scattering can shed light on open problems

Study of **tetraquark** candidates (e.g., T_{cc}^{+})

- Existence of tetraquarks at large N currently **not fully established**.
- Early arguments $\longrightarrow$ **No** Witten NPB **160** (1979) 57 — Coleman 17$^{\text{th}}$ Erice Lectures (1980) 0011, 3
- Recent developments $\longrightarrow$ **Possible** S. Weinberg PRL **110** (2013) 261601; Knecht et al. PRD **88** (2013) 036016
  Narrow tetraquark poles could show up in meson scattering amplitudes.
- Large-N non-relativistic analysis: $QQ\bar{q}\bar{q}$ bound if $\frac{m}{M} \ll \frac{1}{N}$ Allaman et al. JHEP **11** (2024) 034

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Internal structure of **light scalar mesons** (e.g., $f_0(500)$ a.k.a. σ)

Peláez Phys. Rept. **658** (2016) 1 — Alexandrou et al. PRD **97** (2018) 034506 — Hanlon et al. [arXiv:2603.19192]

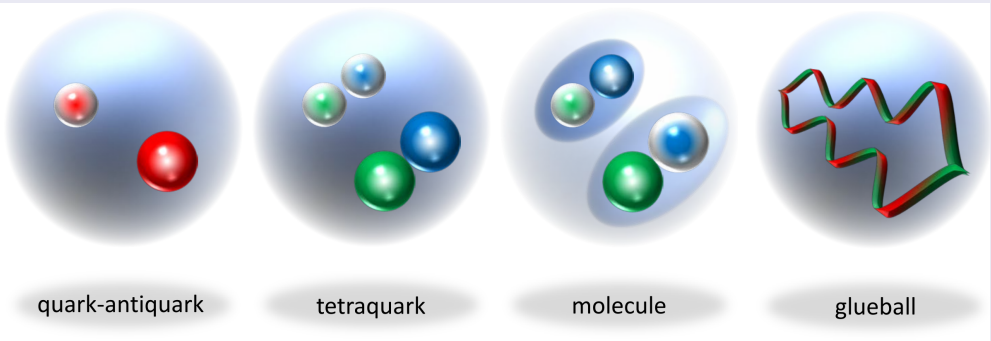

Figure: Peláez [2509.08648]

- Several proposals (dilaton, diquark anti-diquark, meson molecule)
- Behavior of poles at large N strongly depends on actual nature of the resonance.

Peláez PRL **92** (2004) 102001

Jaffe Prog. Theor. Phys. Suppl. **168** (2007) 127–142

Non-perturbative approaches to large-N QCD

Large-N QCD actively studied via several non-perturbative methods.

- Chiral Perturbation Theory (χPT)

Peláez MPLA **19** (2004) 2879-2894 — Peláez [2509.08648] — Cheung et al. [2605.21582]

- Bootstrap approaches

Albert et al. JHEP **09** (2024) 172 — Bocchia et al. [2606.14676] — Albert et al. [2606.19420]

} need inputs from full theory

- First-principles lattice methods Bali et al. JHEP **06** (2013) 071; Hernández et al. EPJC **79** (2019) 10, 865; Tom DeGrand et al. PRD **108** (2023) 094516; Baeza-Ballesteros et al. JHEP **08** (2025) 110

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Topics of this seminar

- New lattice approach to large-N QCD $\rightarrow$ **EGUCHI–KAWAI REDUCTION**
- Mesons masses and χPT low-energy constants in large-N QCD

CB, M. García Pérez, A. González-Arroyo, K.-I. Ishikawa, M. Okawa JHEP **12** (2025) 096

- $\mathrm{I} = 2$ $\pi\pi$ scattering length in large-N QCD

Work in progress (**CB** + Madrid/Hiroshima + F. Romero-López)

Lattice large-N QCD via Eguchi–Kawai reduction

Large-N volume independence

Standard lattice approach: extrapolation towards $N = \infty$ from $N \lesssim \mathcal{O}(10)$.

Alternative approach $\longrightarrow$ large-N volume independence Eguchi–Kawai PRL **48** (1982) 1063

Large-N equivalence of space-time and color degrees of freedom

N^2 = effective volume: $\quad N \to \infty \quad \sim \quad V \to \infty$

Schwinger–Dyson equations of Wilson loops for ordinary and **reduced theory**

$$\boxed{U_\mu(x) \quad \to \quad U_\mu}$$

(volume-reduced theory)

are **equal** up to $\mathcal{O}\left(1/N^2\right)$ **terms**

$\longrightarrow$ possibility of studying the large-N limit on a **1-point lattice**

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Reducing $V = 1$ allows to reach much larger N: **N > 1000**

Eguchi–Kawai reduction: assumptions

Eguchi–Kawai argument $\longrightarrow$ extra terms sub-leading if 2 assumptions are met

- **Large-N Factorization** for gauge-invariant traces

$$\langle \mathrm{Tr}(W_C)\mathrm{Tr}(W_{C'})\rangle \underset{N\to\infty}{\sim} \langle \mathrm{Tr}(W_C)\rangle \langle \mathrm{Tr}(W_{C'})\rangle + \mathcal{O}(1/N^2)$$

$$W_C = \prod_{i\in C} U_i \qquad C = \text{ closed path}$$

Analytically and numerically confirmed

Chatterjee, Commun. Math. Phys. **366** (2019) 203

Garciá Vera & Sommer, EPJC **79** (2019) 35

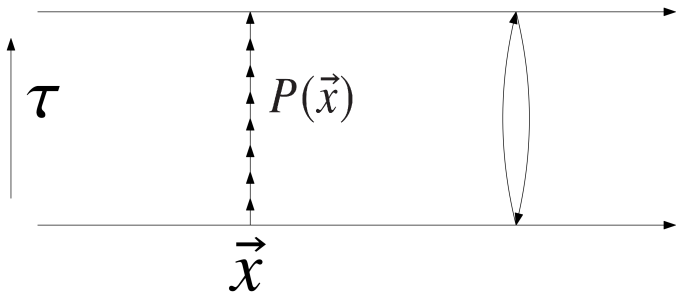
- Large-N lattice theory enjoys **global U(1) symmetry**

More subtle

Center symmetry

Yang–Mills lattice action invariant under $\mathbb{Z}_N$ **center symmetry**

$$U_0(\vec{x}, \tau = \tau_0) \longrightarrow z\, U_0(\vec{x}, \tau = \tau_0) \qquad z \in \boxed{\mathbb{Z}_N \underset{N\to\infty}{\longrightarrow} \mathrm{U}(1)}$$



$$P(\vec{x}) = \mathrm{Tr}\left\{\prod_{\tau} U_0(\vec{x}, \tau)\right\}$$

$\longrightarrow$ **Polyakov loop not invariant under center symmetry**: $P \to zP$

- $L > L_c \quad \to \quad \langle|\sum_{\vec{x}} P|\rangle = 0 \quad \to \quad \mathbb{Z}_N$ symmetric phase
- $L < L_c \quad \to \quad \langle|\sum_{\vec{x}} P|\rangle > 0 \quad \to \quad \mathbb{Z}_N$ spontaneously broken phase

Supplement 1-point lattice with **twisted boundary conditions**
to avoid center symmetry breaking $\Longrightarrow$ **Twisted Eguchi–Kawai** (TEK) formulation

González-Arroyo & Okawa PRD **27** (1983) 2397; JHEP **07** (2010) 043; JHEP **12** (2014) 106

TEK formulation of the gluon action

- $U_\mu(x + aL\hat{\nu}) = \Gamma_\nu U_\mu(x)\Gamma_\nu^\dagger$ — Twisted Boundary Conditions (TBC)
- $U_\mu(x) \longrightarrow U_\mu(x)\Gamma_\mu$ — Change of variable via twist eater Γ_μ
- $U_\mu(x) \longrightarrow U_\mu$ — 1 point $\Longrightarrow$ only $d = 4$ $\mathrm{SU}(N)$ matrices

$$S_{\mathrm{TEK}}[U] = -Nb \sum_{\nu \neq \mu} z_{\nu\mu} \mathrm{Tr}\left\{U_\mu U_\nu U_\mu^\dagger U_\nu^\dagger\right\}$$

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$b = 1/\lambda_0 = \beta/(2N^2)$

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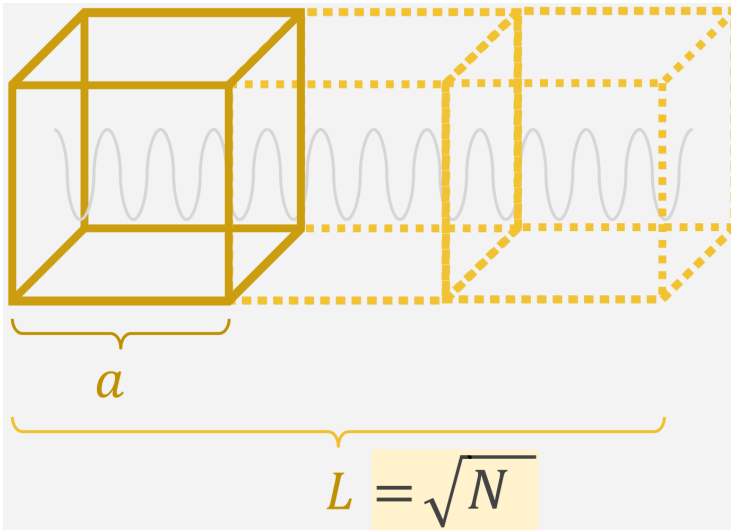
- $k \in \mathbb{N}$ $k(N)/\sqrt{N} \sim$ fixed
- $\left(z_{\nu\mu}\right)^L = 1 \Longrightarrow U_\mu$ effectively L-periodic

$\Longrightarrow$ **Effective size** $L = \sqrt{N} \Longrightarrow V = L^4 = N^2$



Comments on N-dependence in TEK formulation

N-dependence in TEK theory = **effective-volume** dependence

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  The two lattice theories only agree in the large-N limit

$\Longrightarrow$ **TEK only captures the large-N limit**

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Sub-leading terms in $1/N$ expansion: $N = \infty$ (TEK) + standard finite-N results
$\Longrightarrow$ **INTERPOLATION** as opposed to extrapolation.

TEK formulation of the Dirac operator

$$\#\ \text{gluons} \sim O(N^2) \qquad \#\ \text{quarks} \sim O(N_f N)$$

$\longrightarrow$ quark contributions suppressed at large N

Quenched Lattice TEK Dirac operator in **Wilson formulation**:

González-Arroyo & Okawa PLB **755** (2016) 132–137 [1510.05428]

$$D_{\text{TEK}} = \frac{1}{2\kappa}\mathbb{I} - \frac{1}{2}\sum_{\mu}\left[(1+\gamma_\mu)\mathcal{W}_\mu + (1-\gamma_\mu)\mathcal{W}_\mu^\dagger\right]$$

$\mathcal{W}_\mu = U_\mu \otimes \Gamma_\mu^*$ ⟵ twist-eater

κ ⟵ Wilson hopping parameter

In momentum space:

$$\widetilde{D}_{\text{TEK}}(p) = \frac{1}{2\kappa}\mathbb{I} - \frac{1}{2}\sum_{\mu}\left[(1+\gamma_\mu)\mathcal{W}_\mu e^{iap_\mu} + (1-\gamma_\mu)\mathcal{W}_\mu^\dagger e^{-iap_\mu}\right]$$

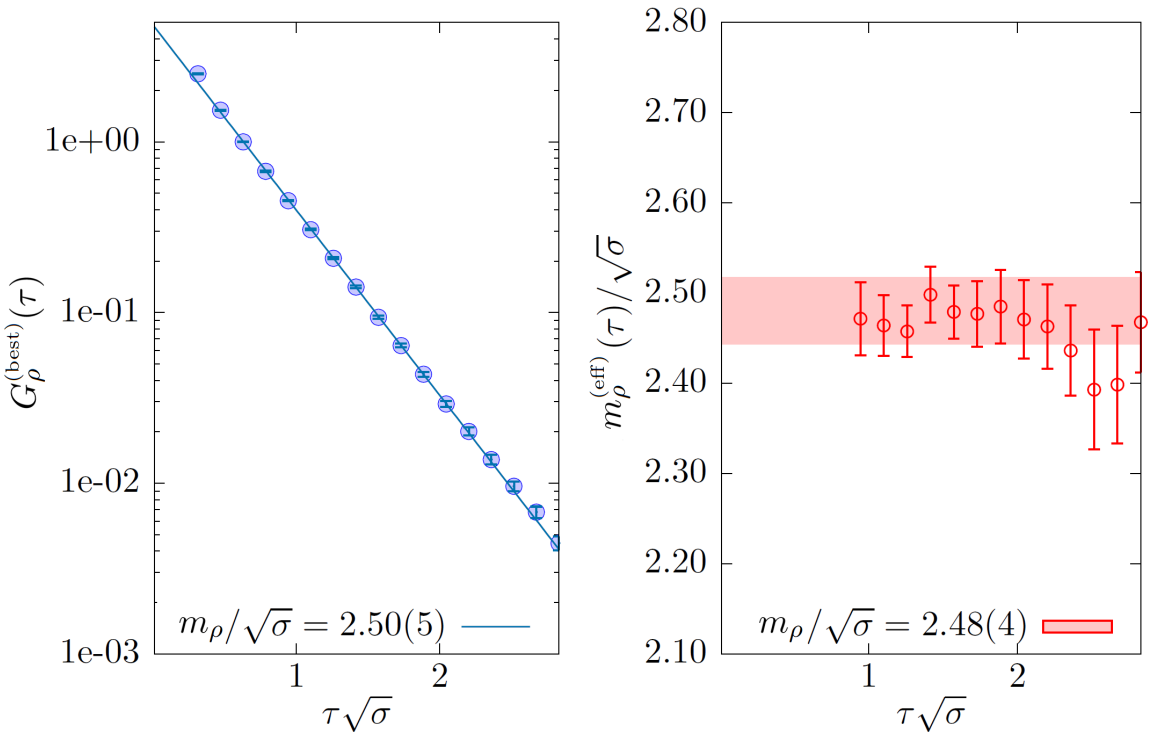
Meson spectrum in large-N QCD

Meson correlation functions

Meson time correlators $\longrightarrow$ anti-transform momentum-space correlators with $\vec{p} = \vec{0}$

$$G_{\mathrm{A}}(\tau) = \sum_{p_0} \mathrm{e}^{\mathrm{i}\tau p_0} \ \langle \mathrm{Tr}\{\Gamma_{\mathrm{A}} \widetilde{D}^{-1}_{\mathrm{TEK}}(p_0) \Gamma_{\mathrm{A}} \widetilde{D}^{-1}_{\mathrm{TEK}}(0)\}\rangle \qquad a p_0 \propto \frac{n}{L} = \frac{n}{\sqrt{N}}$$

Crucial point $\longrightarrow$ lattice momenta discretized in units of $1/L = 1/\sqrt{N}$



- Shown example $\longrightarrow$ ρ meson

$$\Gamma_\rho = \gamma_i \qquad N = 841$$

- GEVP analysis

$$G_\rho^{(\mathrm{best})}(\tau) \underset{\tau\to\infty}{\sim} \exp\{-m_\rho \tau\}$$

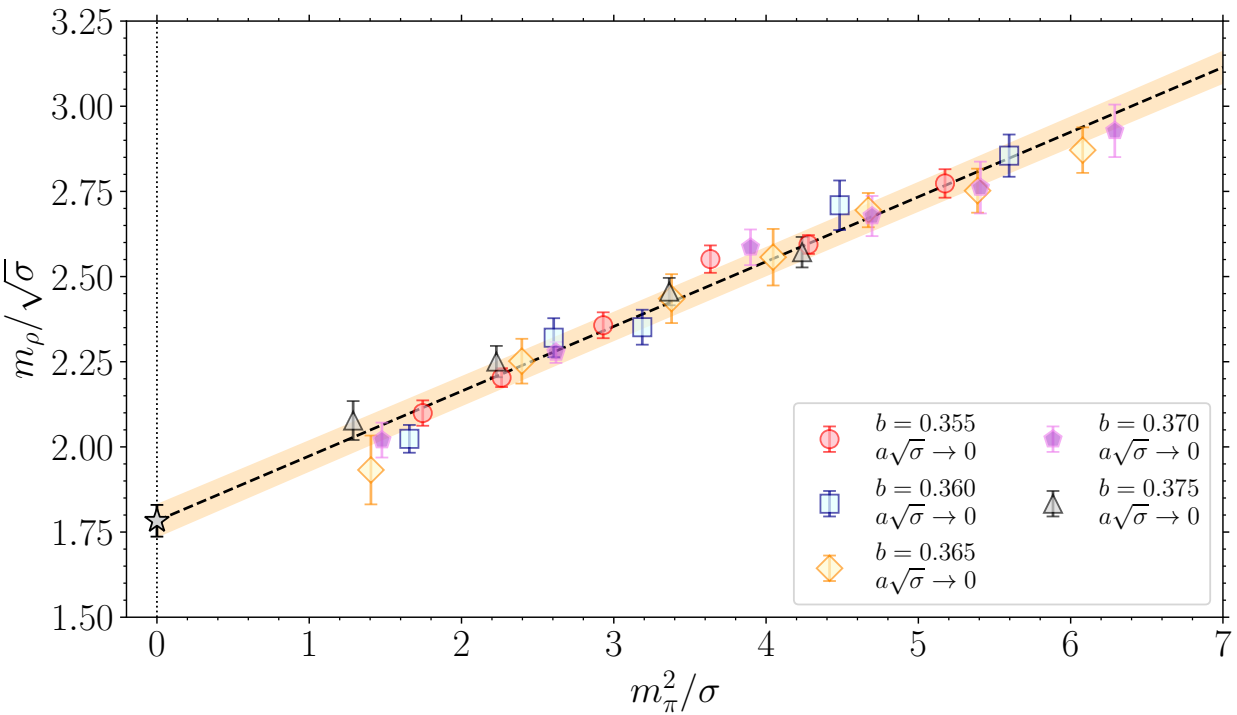
- Fit range $\longrightarrow$ $\tau\sqrt{\sigma} \in [0.9, 2.5]$

The ρ meson mass as a function of the π mass

- 5 lattice spacings: $0.10\ \text{fm} \gtrsim a \gtrsim 0.05\ \text{fm}$ with $\ell = a\sqrt{N} \approx 2\ \text{fm}$
  - For each a: 4 – 7 quark masses with $m_\pi \ell = a m_\pi \sqrt{N} \gtrsim 5$
    - Number of colors: $N = 289,\ 361,\ 529,\ 841$

Joint fit $\longrightarrow$ $m_\rho(a, m_\pi) = m_\rho^{(\chi)}(1 + C_\rho m_\pi^2)(1 + \mathcal{K}a)$

Large-N χPT prediction for $m_\rho(m_\pi)$: Jenkins et al. PRL **75** (1995) 2272



▶ No chiral log at large N

$$\frac{M_\pi^2}{F_\pi^2} \log\left(\frac{M_\pi^2}{\mu^2}\right) \sim \mathcal{O}\left(\frac{1}{N}\right)$$

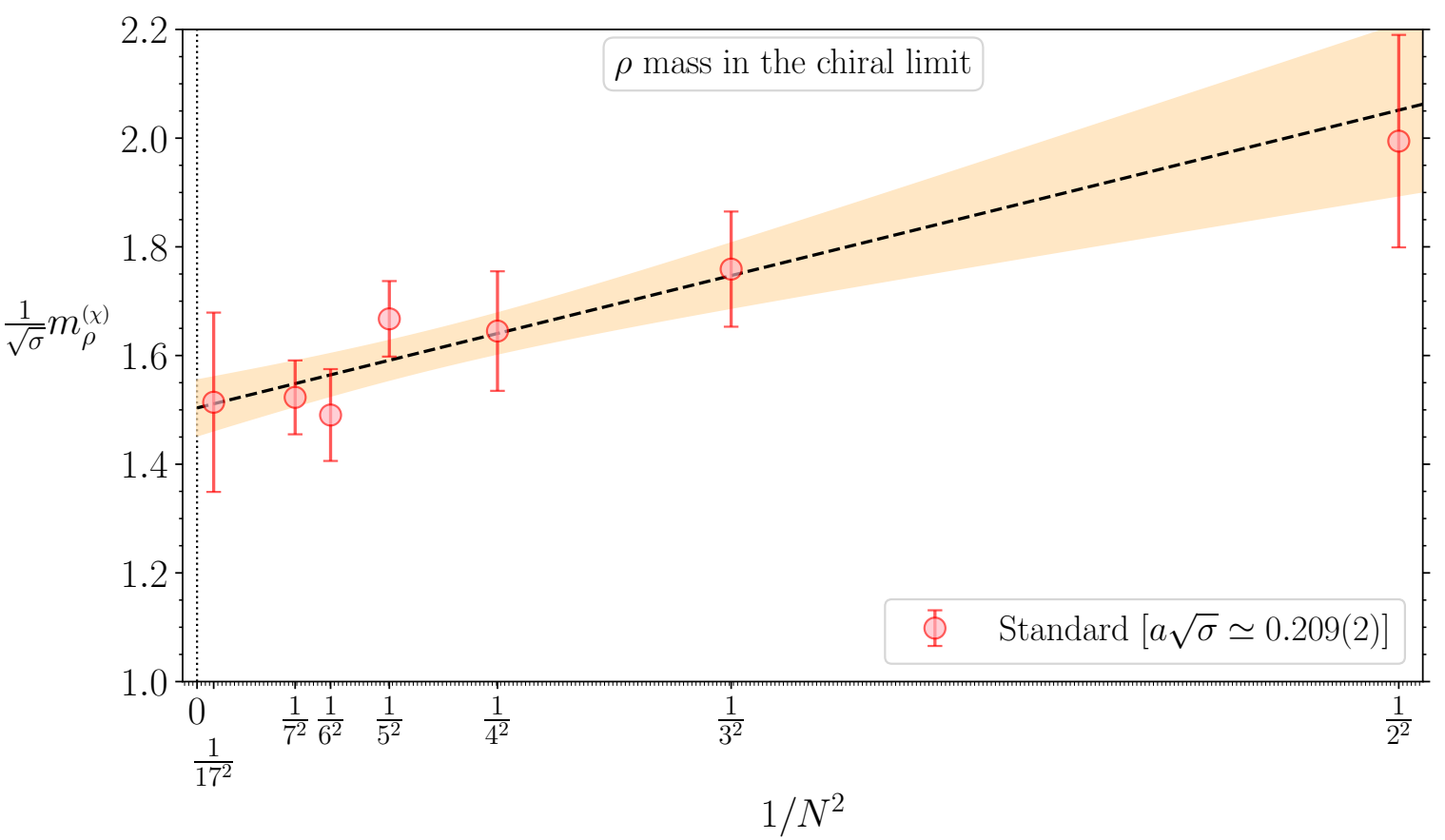
▶ **Points**

Lat. data minus artifacts

▶ **Shaded Area**

$m_\rho^{(\chi)}(1 + C_\rho m_\pi^2)$

Comparison with: Bali et al. JHEP **06** (2013) 071



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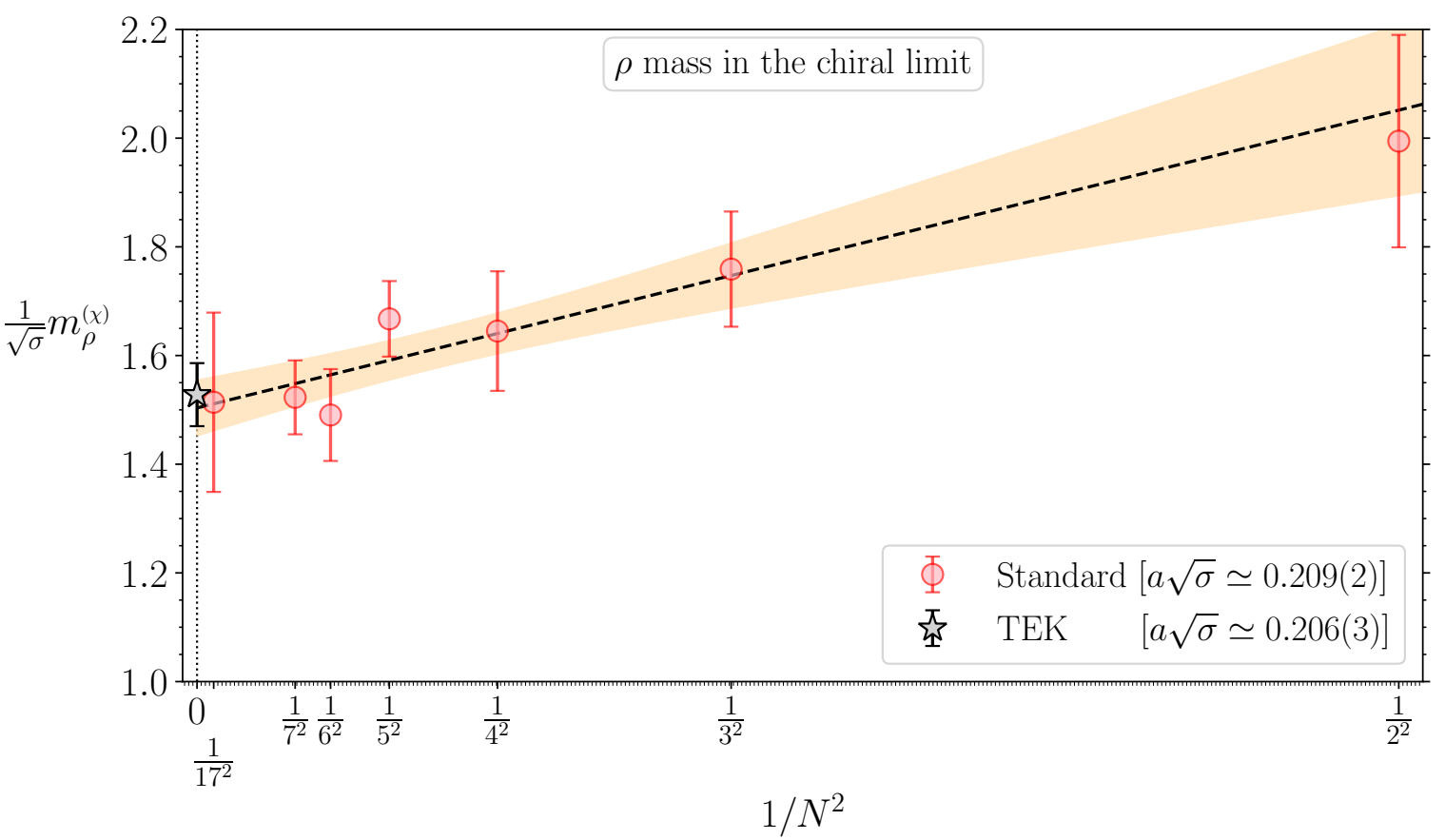

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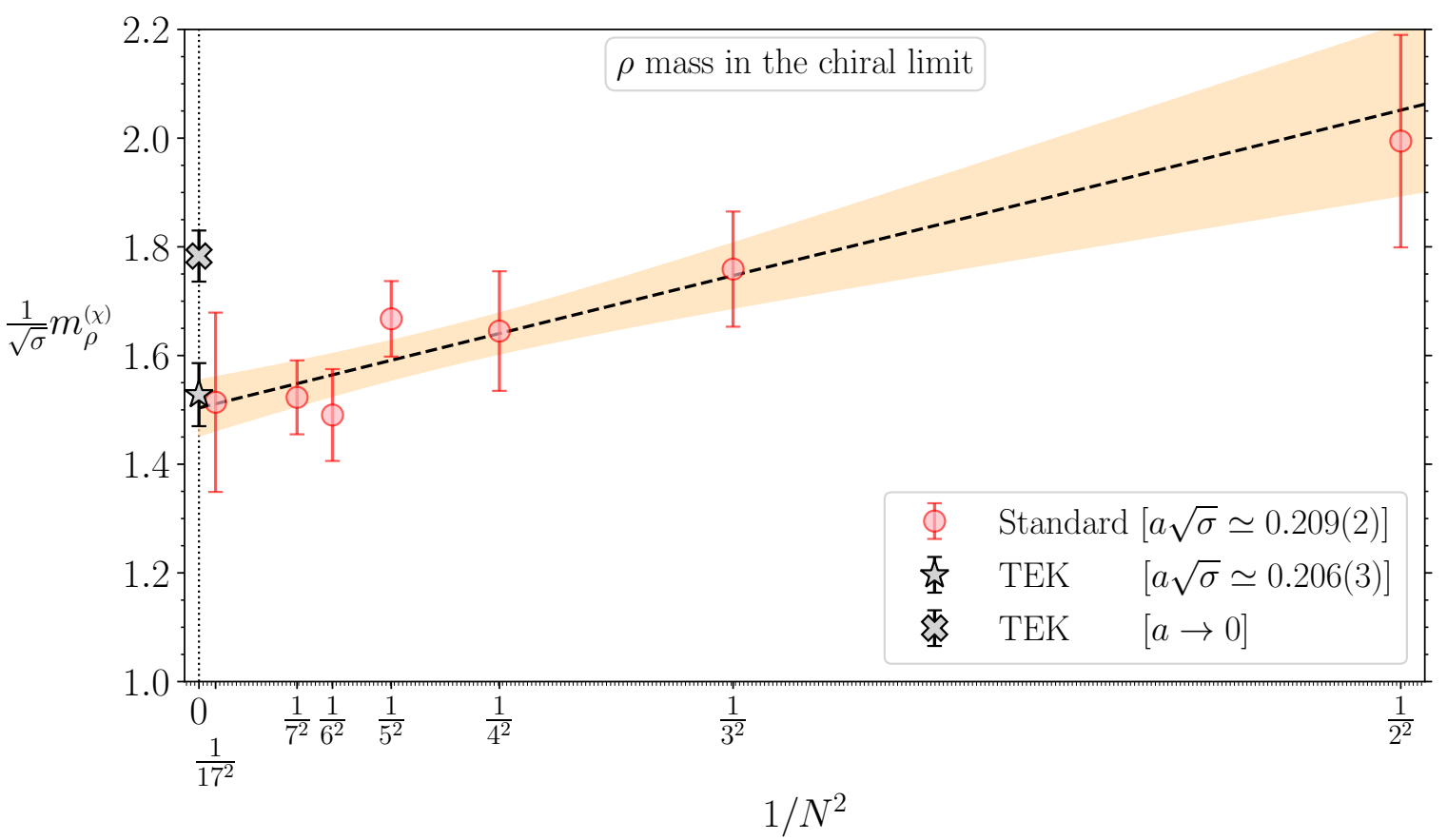

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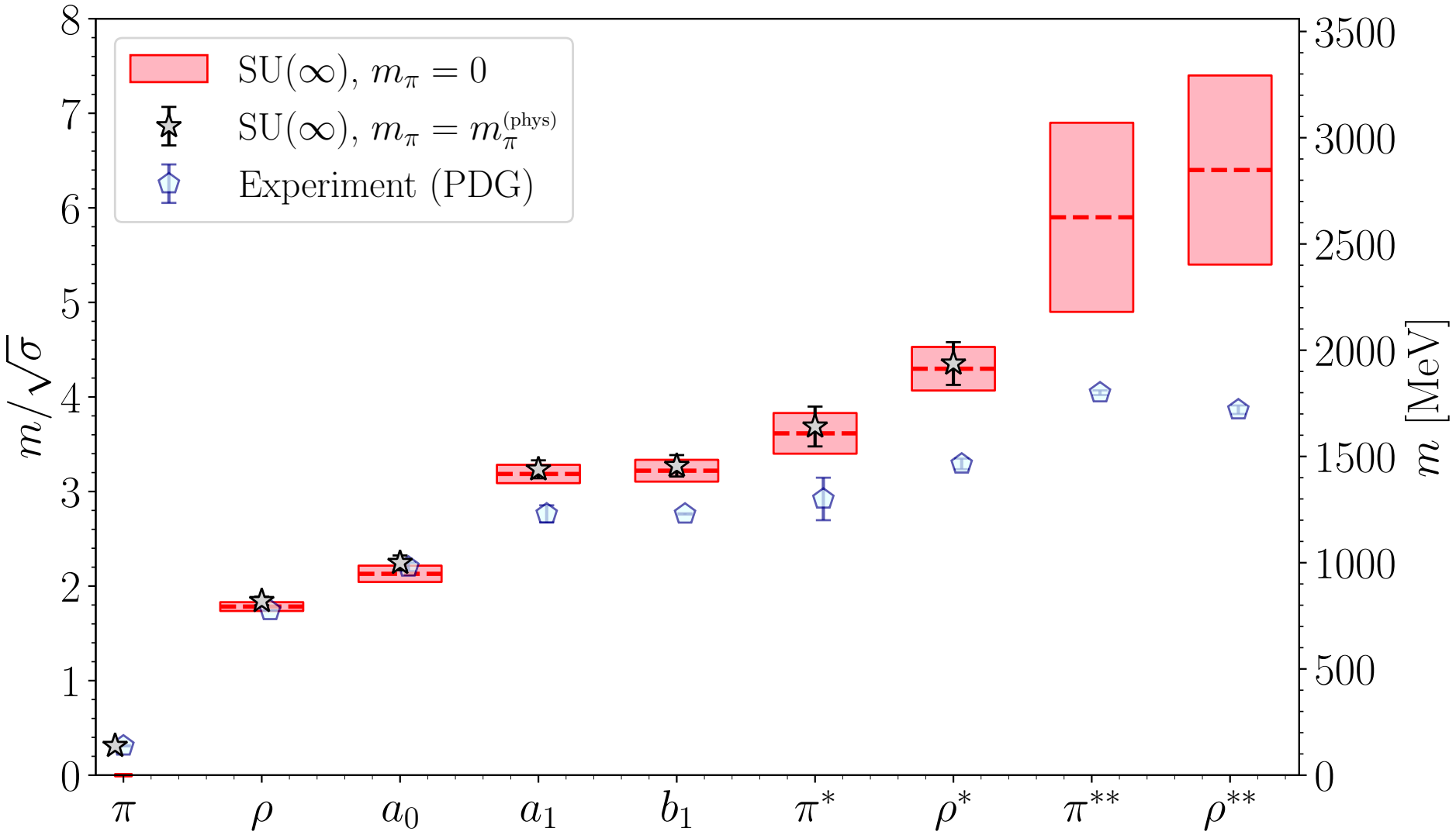
- 5 lattice spacings (3 finer) $\longrightarrow$ **significant improvement** of state of the art

Low-lying meson spectrum at large N

Phys. point convention: $m_\pi^{(\mathrm{phys})} = 135$ MeV PDG $\quad \sqrt{\sigma}\big|_{\mathrm{phys}} = 445$ MeV PLB **854** (2024) 138754



Comparison with PDG gives idea of magnitude of higher-order corrections in $1/N$

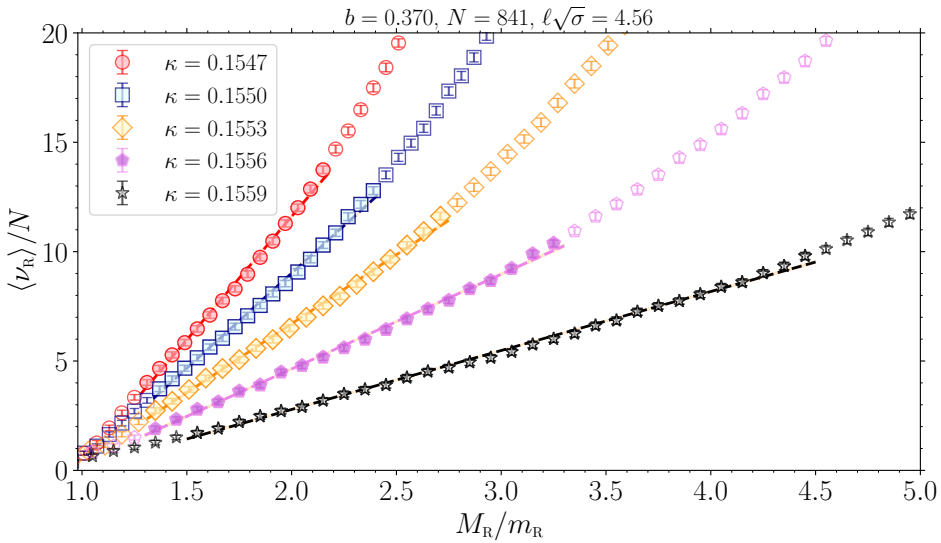
χPT low-energy constants in large-N QCD

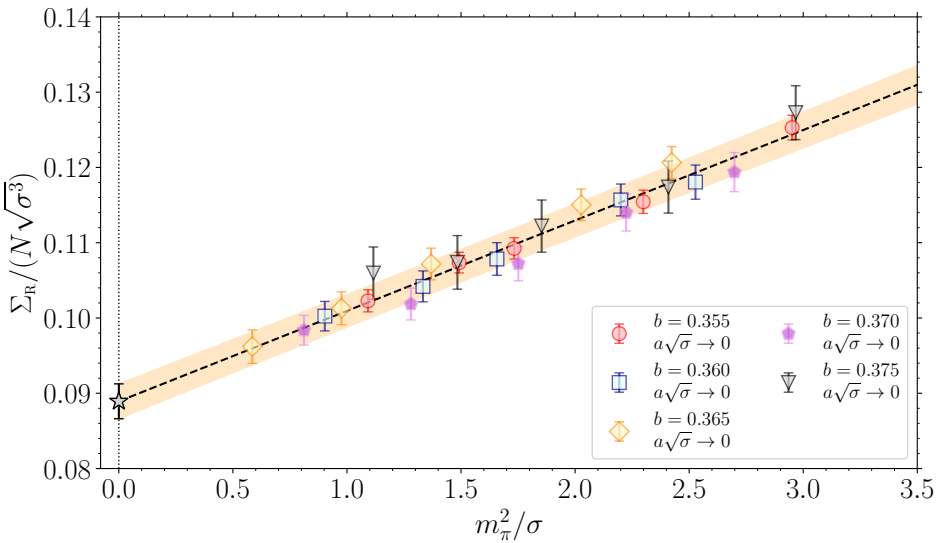
Chiral symmetry breaking at large N

$\Sigma = -\langle\bar{\psi}\psi\rangle \sim O(N) \rightarrow$ order parameter for chiral symmetry breaking

$$\Sigma_{\mathrm{R}}^{(\mathrm{eff})} = \frac{\pi}{2}\frac{\langle \nu_{\mathrm{R}}(M_{\mathrm{R}}, m_{\mathrm{R}})\rangle}{V\sqrt{M_{\mathrm{R}} - m_{\mathrm{R}}}} = \Sigma_{\mathrm{R}}\left[1 + O(m_{\mathrm{R}})\right]$$

Giusti–Lüscher method
JHEP **03** (2009) 013





Chiral condensate Σ_{R}/N extracted from slope of Dirac mode number $\langle\nu\rangle_{\mathrm{R}}$

$\rightarrow$ Non-perturbative renormalization: Z_{P} ($\overline{\mathrm{MS}}$, $\mu = 2$ GeV)

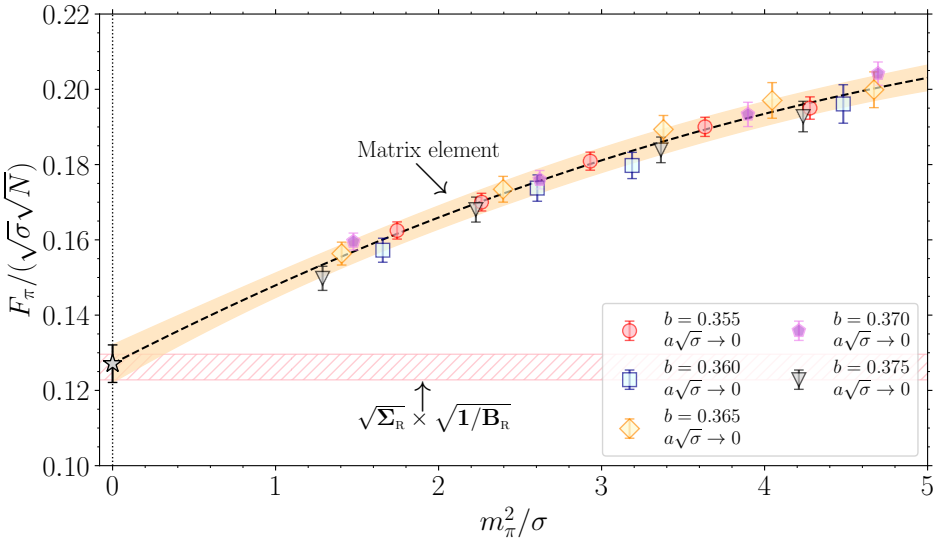
$$\left(\frac{1}{N}\Sigma_{\mathrm{R}}\right)_{N=\infty} = [247(4)\ \mathrm{MeV}]^3 \qquad \left(\frac{1}{N}\Sigma_{\mathrm{R}}\right)^{(\mathrm{FLAG})}_{N=3, N_{\mathrm{f}}=2} = [184(7)\ \mathrm{MeV}]^3$$

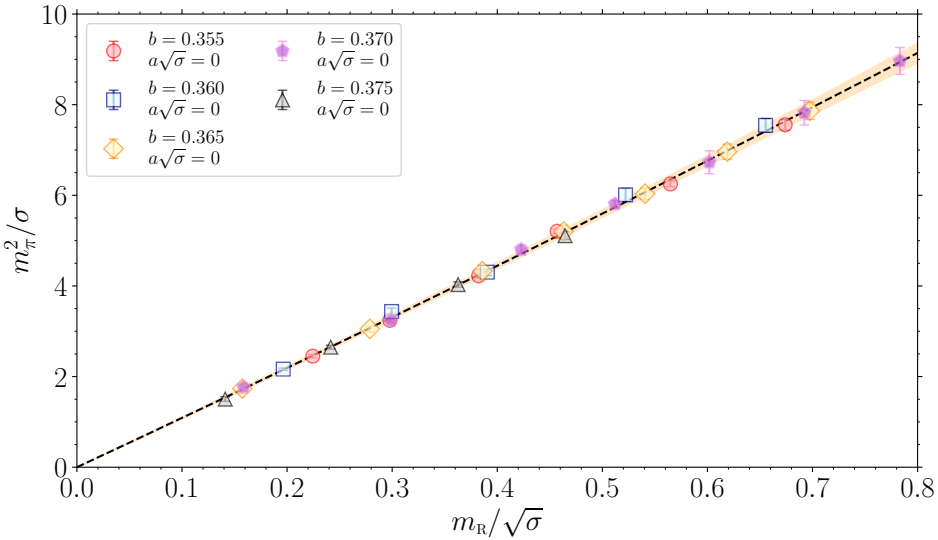
MeV conversion of TEK results: $\sqrt{8t_0} \simeq 0.409$ fm [FLAG 24]

Pion mass and pion decay constant at large N

$$F_\pi = \frac{1}{\sqrt{2} m_\pi} \langle 0 | A_0 | \pi(\vec{p} = \vec{0}) \rangle \sim O(\sqrt{N})$$

m_π^2 versus quark mass m_{R}





$$F_\pi = F \left[1 + \bar{\ell}_3 \xi + O(\xi^2) \right]$$

$$m_\pi^2 = M^2 \left[1 - \frac{1}{2} \bar{\ell}_4 \xi + O(\xi^2) \right]$$

$$\xi = \frac{M^2}{16\pi^2 F^2} \sim O\left(\frac{1}{N}\right) \qquad M^2 = 2 B_{\mathrm{R}} m_{\mathrm{R}} \qquad B_{\mathrm{R}} = \frac{\Sigma_{\mathrm{R}}}{F^2} \sim O(N^0) \qquad \bar{\ell}_3, \bar{\ell}_4 \sim O(N)$$

Large-N χPT predictions: Hernández et al. EPJC **79** (2019) 865

No chiral log at large N

Pion mass and pion decay constant at large N

$$F_\pi = \frac{1}{\sqrt{2} m_\pi} \langle 0 | A_0 | \pi(\vec{p} = \vec{0}) \rangle \sim O(\sqrt{N})$$

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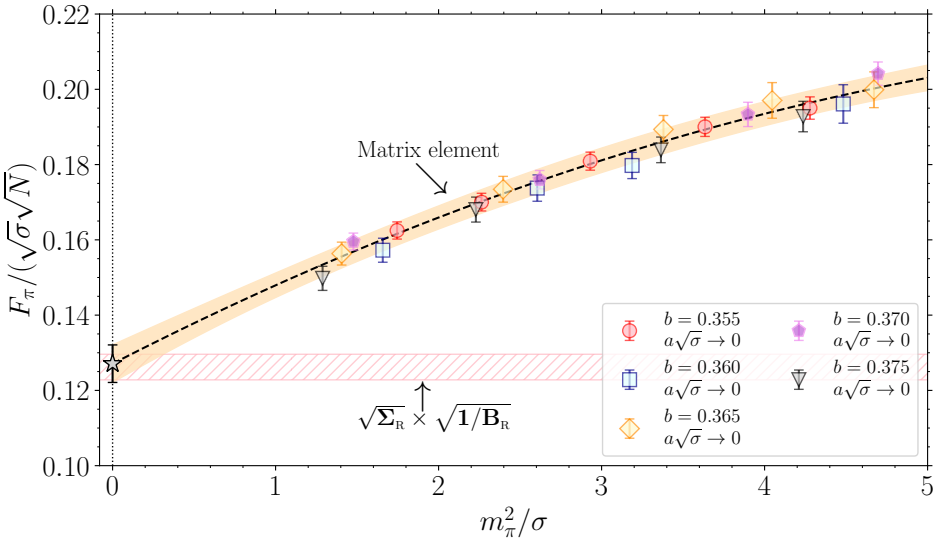

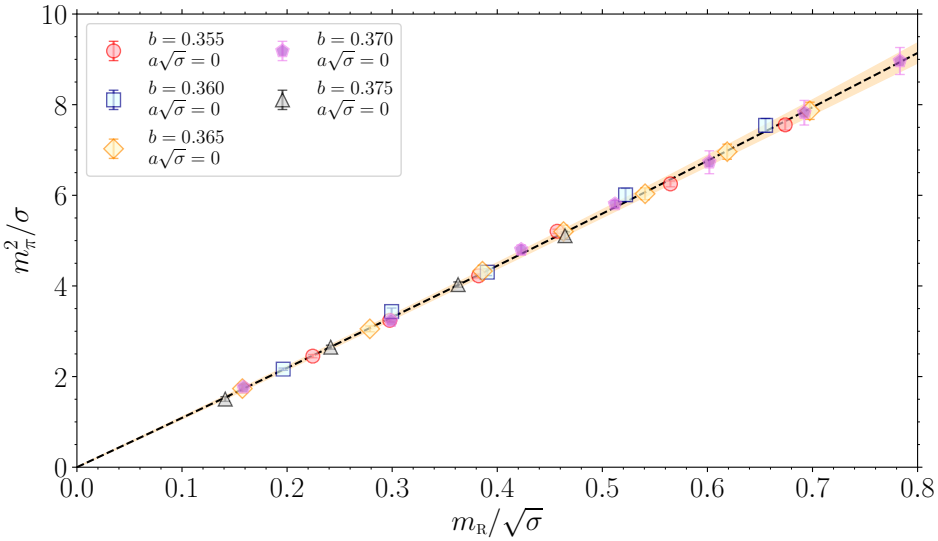

$$\left(\sqrt{\frac{3}{N}} F\right)_{N=\infty} = 122(4)\ \text{MeV}$$

$$\left(\sqrt{\frac{3}{N}} F\right)^{(\text{FLAG})}_{N=3, N_{\mathrm{f}}=2} = 86.6(6)\ \text{MeV}$$

$$\left(\frac{\bar{\ell}_3}{N}\right)_{N=\infty} = -0.034(27)$$

$$\left(\frac{\bar{\ell}_3}{N}\right)^{(\text{FLAG})}_{N=3, N_{\mathrm{f}}=2} = 1.14(27)$$

$$\left(\frac{\bar{\ell}_4}{N}\right)_{N=\infty} = 0.448(54)$$

$$\left(\frac{\bar{\ell}_4}{N}\right)^{(\text{FLAG})}_{N=3, N_{\mathrm{f}}=2} = 1.467(93)$$

I = 2 $\pi\pi$ scattering length in large-N QCD

$\pi\pi$ scattering at large N

Meson scattering at large N: hard with traditional lattice approach

DeGrand [2409.02242]; Baeza–Ballesteros et al. JHEP **06** (2022) 049, **08** (2025) 110

For the first time, TEK allows systematic investigation of this problem.

Clean setup: isospin-2 $\pi\pi$ scattering.

- $N \to \infty$ = free pions $\quad\to\quad a_0 \underset{N\to\infty}{\sim} \boxed{\bar{a}_0}\dfrac{1}{N}$

Target $\quad\to\quad \boxed{\bar{a}_0 = N a_0 \sim \mathcal{O}(N^0)}$

- Finite-volume **energy shift**: $\Delta E = E_{\pi\pi} - 2m_\pi \sim \mathcal{O}(1/N)$
- **Lüscher formula**: ΔE (finite volume) $\leftrightarrow$ a_0 (infinite volume)

Lüscher Commun. Math. Phys. **105** (1986) 153; NPB **354** (1991) 531

$$\Delta E = \frac{4\pi a_0}{m_\pi \ell^3}\left[1 + \mathcal{O}\left(\frac{a_0}{\ell}\right)\right] \qquad a_0/\ell \ll 1$$

$\pi\pi$ scattering at large N

Meson scattering at large N: hard with traditional lattice approach

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Lüscher Commun. Math. Phys. **105** (1986) 153; NPB **354** (1991) 531

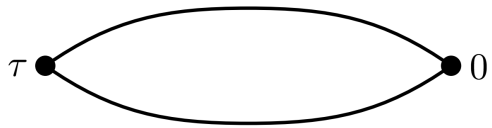
$$N\ell^3\Delta E = \frac{4\pi\bar{a}_0}{m_\pi}\left[1 + \mathcal{O}\left(\frac{\bar{a}_0}{N\ell}\right)\right] \qquad \bar{a}_0/(N\ell) \ll 1$$

- Goal: compute $N\ell^3\Delta E \equiv \Delta\varepsilon \sim \mathcal{O}(N^0), \mathcal{O}(\ell^0)$

Lattice calculation of I = 2 $\Delta\varepsilon$ at large N

$$O_\pi(\tau) = \sum_{\vec{x}} \overline{\psi}(\vec{x},\tau)\gamma_5\psi(\vec{x},\tau)$$

$$G_\pi(\tau) = \quad \langle \mathrm{Tr}\,\{O_\pi(\tau)O_\pi(0)\}\rangle \quad = \langle D(\tau)\rangle$$

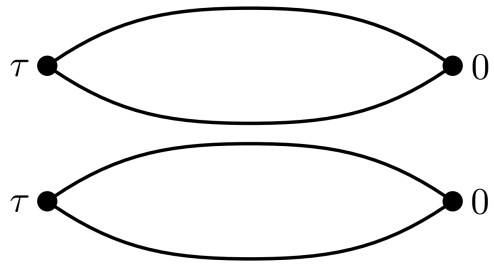


D

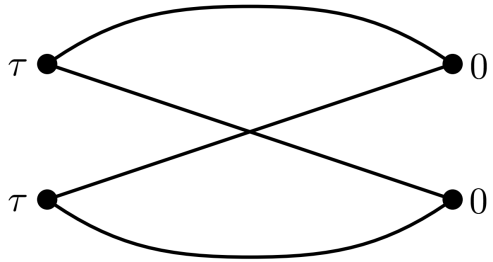
Lattice calculation of I = 2 $\Delta\varepsilon$ at large N

$$O_\pi(\tau) = \sum_{\vec{x}} \overline{\psi}(\vec{x},\tau)\gamma_5\psi(\vec{x},\tau) \qquad O_{\pi\pi}(\tau) \equiv O_\pi(\tau)O_\pi(\tau)$$

$$\begin{aligned} G_\pi(\tau) &= \langle \mathrm{Tr}\,\{O_\pi(\tau)O_\pi(0)\}\rangle &&= \langle D(\tau)\rangle \\ G_{\pi\pi}(\tau) &= \langle \mathrm{Tr}\,\{O_{\pi\pi}(\tau)O_{\pi\pi}(0)\}\rangle &&= 2\langle D^2(\tau)\rangle - \langle C(\tau)\rangle \end{aligned}$$



D^2

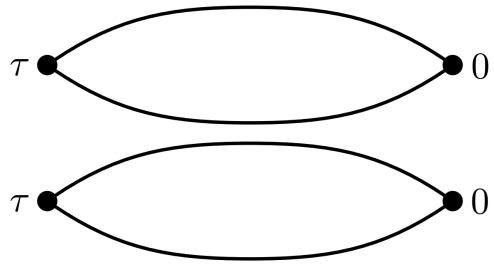


C

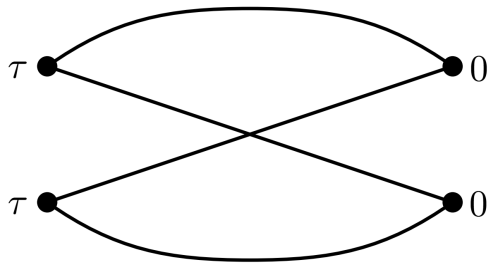
Lattice calculation of I = 2 $\Delta\varepsilon$ at large N

$$\mathcal{O}_\pi(\tau) = \sum_{\vec{x}} \overline{\psi}(\vec{x},\tau)\gamma_5\psi(\vec{x},\tau) \qquad \mathcal{O}_{\pi\pi}(\tau) \equiv \mathcal{O}_\pi(\tau)\mathcal{O}_\pi(\tau)$$

$$\begin{aligned} G_\pi(\tau) &= \langle \mathrm{Tr}\,\{O_\pi(\tau)O_\pi(0)\}\rangle &&= \langle D(\tau)\rangle \\ G_{\pi\pi}(\tau) &= \langle \mathrm{Tr}\,\{O_{\pi\pi}(\tau)O_{\pi\pi}(0)\}\rangle &&= 2\langle D^2(\tau)\rangle - \langle C(\tau)\rangle \end{aligned}$$



$D^2 \sim O[(N\ell^3)^2]$



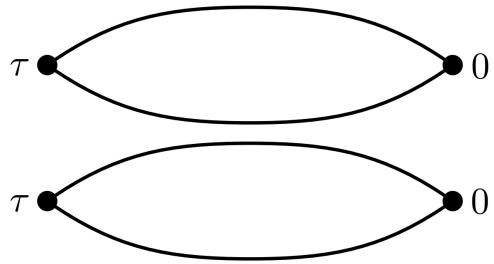
$C \sim O(N\ell^3)$

Large-N factorization $\rightarrow \langle D^2\rangle = \langle D\rangle^2 + O(1/N^2)$

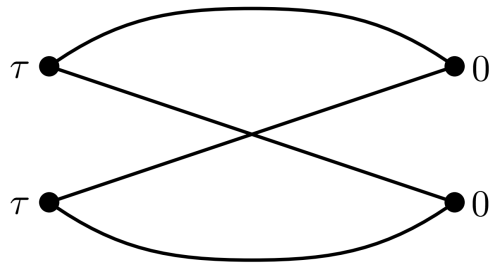
Lattice calculation of I = 2 $\Delta\varepsilon$ at large N

$$\mathcal{O}_\pi(\tau) = \sum_{\vec{x}} \overline{\psi}(\vec{x},\tau)\gamma_5\psi(\vec{x},\tau) \qquad \mathcal{O}_{\pi\pi}(\tau) \equiv \mathcal{O}_\pi(\tau)\mathcal{O}_\pi(\tau)$$

$$G_\pi(\tau) = \langle \mathrm{Tr}\{\mathcal{O}_\pi(\tau)\mathcal{O}_\pi(0)\}\rangle = \langle D(\tau)\rangle$$
$$G_{\pi\pi}(\tau) = \langle \mathrm{Tr}\{\mathcal{O}_{\pi\pi}(\tau)\mathcal{O}_{\pi\pi}(0)\}\rangle = 2\langle D^2(\tau)\rangle - \langle C(\tau)\rangle$$



$D^2 \sim \mathcal{O}[(N\ell^3)^2]$



$C \sim \mathcal{O}(N\ell^3)$

Large-N factorization $\rightarrow \langle D^2\rangle = \langle D\rangle^2 + \mathcal{O}(1/N^2)$

- $G_{\pi\pi}(\tau) \underset{\tau\to\infty}{\sim} 2A_\pi^2 \exp\{-2m_\pi\tau\} = 2G_\pi^2(\tau) + \mathcal{O}(1/N)$ (LO)

Lattice calculation of I = 2 $\Delta\varepsilon$ at large N

$$O_\pi(\tau) = \sum_{\vec{x}} \overline{\psi}(\vec{x},\tau)\gamma_5\psi(\vec{x},\tau) \qquad O_{\pi\pi}(\tau) \equiv O_\pi(\tau)O_\pi(\tau)$$

$$G_\pi(\tau) = \langle \mathrm{Tr}\{O_\pi(\tau)O_\pi(0)\}\rangle = \langle D(\tau)\rangle$$

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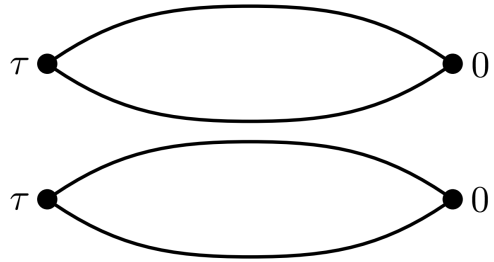
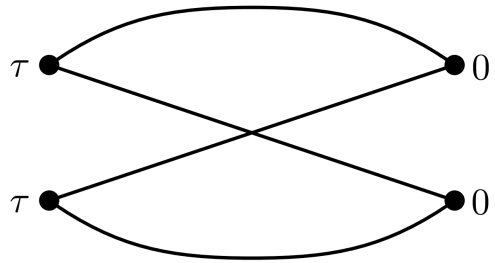


$D^2 \sim O[(N\ell^3)^2]$



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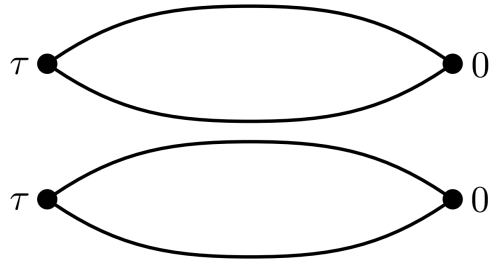
- $G_{\pi\pi}(\tau) \underset{\tau\to\infty}{\sim} 2A_\pi^2 \exp\{-2m_\pi\tau\} = 2G_\pi^2(\tau) + O(1/N)$ (LO)
- $G_{\pi\pi}(\tau) \underset{\tau\to\infty}{\sim} 2A_\pi^2 \exp\{-(2m_\pi + \Delta E)\tau\}$

Lattice calculation of I = 2 $\Delta\varepsilon$ at large N

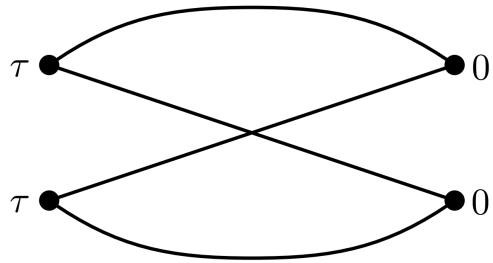
$$O_\pi(\tau) = \sum_{\vec{x}} \overline{\psi}(\vec{x},\tau)\gamma_5\psi(\vec{x},\tau) \qquad O_{\pi\pi}(\tau) \equiv O_\pi(\tau)O_\pi(\tau)$$

$$G_\pi(\tau) = \langle \mathrm{Tr}\{O_\pi(\tau)O_\pi(0)\}\rangle = \langle D(\tau)\rangle$$

$$G_{\pi\pi}(\tau) = \langle \mathrm{Tr}\{O_{\pi\pi}(\tau)O_{\pi\pi}(0)\}\rangle = 2\langle D^2(\tau)\rangle - \langle C(\tau)\rangle$$



$D^2 \sim O[(N\ell^3)^2]$



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- $G_{\pi\pi}(\tau) \underset{\tau\to\infty}{\sim} 2G_\pi^2(\tau)(1-\Delta E\tau) + O(1/N^2)$ (NLO)

Lattice calculation of I = 2 $\Delta\varepsilon$ at large N

$$G_{\pi\pi} = 2\langle D\rangle^2 - \langle C\rangle + O(1/N^2) \qquad G_\pi = \langle D\rangle$$

- $$\frac{G_{\pi\pi}(\tau)}{2G_\pi^2(\tau)} \underset{\tau\to\infty}{\sim} 1 - \Delta E\tau + O\left(\frac{1}{N^2}\right)$$
- $$\frac{G_{\pi\pi}(\tau)}{2G_\pi^2(\tau)} = 1 - \frac{\langle C(\tau)\rangle}{2\langle D(\tau)\rangle^2} + O\left(\frac{1}{N^2}\right) = 1 - \frac{\langle \overline{C}(\tau)\rangle}{2\langle \overline{D}(\tau)\rangle^2 N\ell^3} + O\left(\frac{1}{N^2}\right)$$

$$D \underset{\substack{N\to\infty\\ \ell\to\infty}}{\sim} \overline{D}N\ell^3 \qquad C \underset{\substack{N\to\infty\\ \ell\to\infty}}{\sim} \overline{C}N\ell^3$$

Lattice calculation of I = 2 $\Delta\varepsilon$ at large N

$$G_{\pi\pi} = 2\langle D\rangle^2 - \langle C\rangle + O(1/N^2) \qquad G_\pi = \langle D\rangle$$

- $$\frac{G_{\pi\pi}(\tau)}{2G_\pi^2(\tau)} \underset{\tau\to\infty}{\sim} 1 - \boxed{\Delta E\tau} + O\left(\frac{1}{N^2}\right)$$
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Lattice calculation of I = 2 $\Delta\varepsilon$ at large N

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$$\Longrightarrow \boxed{\frac{\langle \overline{C}(\tau)\rangle}{2\langle \overline{D}(\tau)\rangle^2} \underset{\tau\to\infty}{\sim} \left(N\ell^3\Delta E\right)\tau = \Delta\varepsilon\,\tau} \quad \leftarrow \quad O(N^0), O(\ell^0) \qquad \ell = a\sqrt{N}$$

Lattice calculation of I = 2 $\Delta\varepsilon$ at large N

$$G_{\pi\pi} = 2\langle D\rangle^2 - \langle C\rangle + O(1/N^2) \qquad G_\pi = \langle D\rangle$$

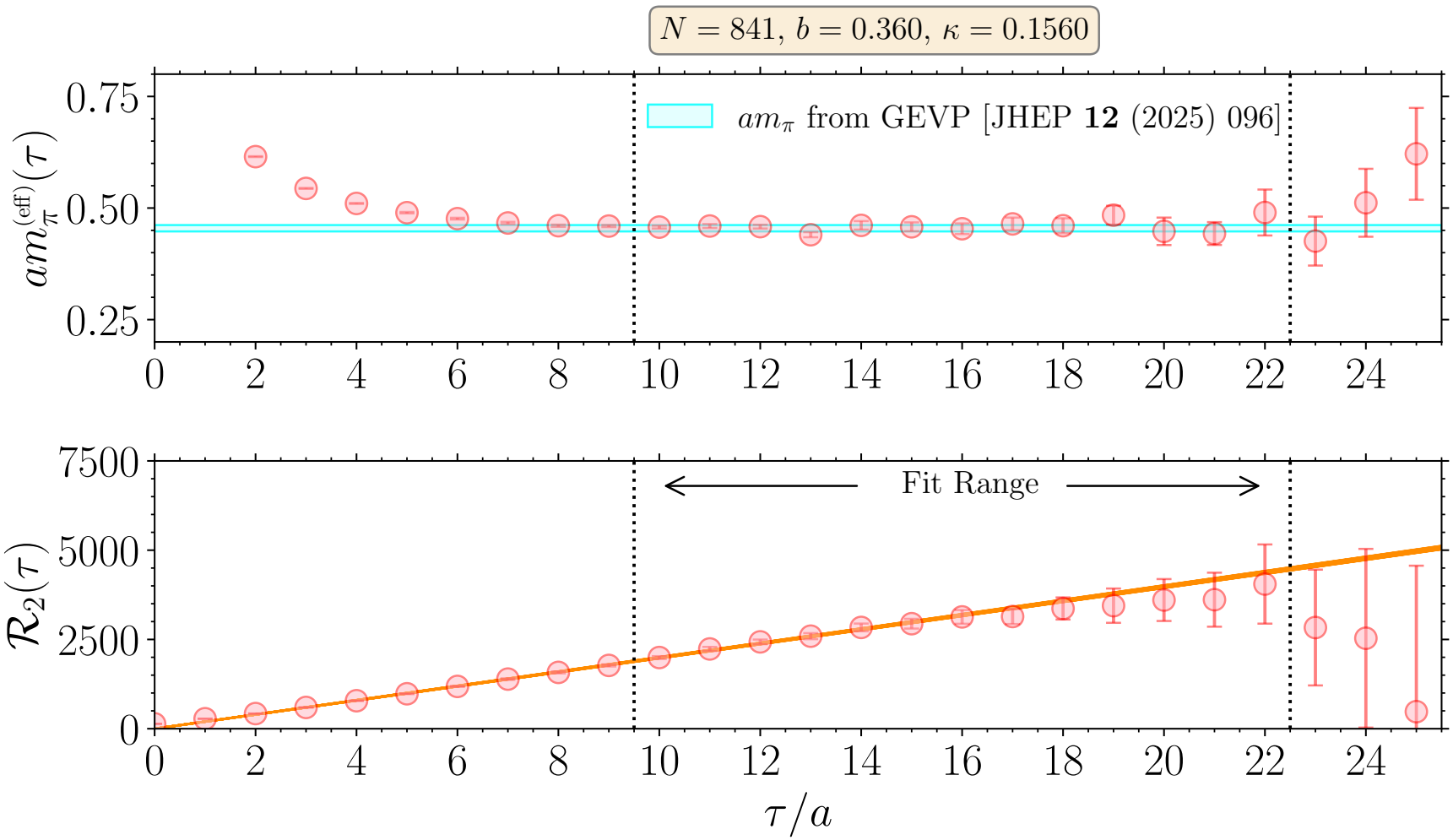
- $$\frac{G_{\pi\pi}(\tau)}{2G_\pi^2(\tau)} \underset{\tau\to\infty}{\sim} 1 - \boxed{\Delta E\tau} + O\left(\frac{1}{N^2}\right)$$
- $$\frac{G_{\pi\pi}(\tau)}{2G_\pi^2(\tau)} = 1 - \frac{\langle C(\tau)\rangle}{2\langle D(\tau)\rangle^2} + O\left(\frac{1}{N^2}\right) = 1 - \boxed{\frac{\langle \overline{C}(\tau)\rangle}{2\langle \overline{D}(\tau)\rangle^2 N\ell^3}} + O\left(\frac{1}{N^2}\right)$$

$$\Longrightarrow \boxed{\frac{\langle \overline{C}(\tau)\rangle}{2\langle \overline{D}(\tau)\rangle^2} \underset{\tau\to\infty}{\sim} \left(N\ell^3\Delta E\right)\tau = \Delta\varepsilon\,\tau} \quad \leftarrow \quad O(N^0), O(\ell^0) \qquad \ell = a\sqrt{N}$$

$$\langle \overline{C}(\tau)\rangle \to \sum_{p_0,q_0,k_0} \mathrm{e}^{\mathrm{i}(p_0-q_0+k_0)\tau} \langle \mathrm{Tr}\{\gamma_5 \widetilde{D}^{-1}_{\mathrm{TEK}}(p_0)\gamma_5 \widetilde{D}^{-1}_{\mathrm{TEK}}(q_0)\gamma_5 \widetilde{D}^{-1}_{\mathrm{TEK}}(k_0)\gamma_5 \widetilde{D}^{-1}_{\mathrm{TEK}}(0)\}\rangle$$

Lattice calculation of I = 2 $\Delta\varepsilon$ at large N

$$\mathcal{R}_2(\tau) \equiv \frac{\langle \overline{C}(\tau) \rangle}{2 \langle \overline{D}(\tau) \rangle^2} \underset{\tau \to \infty}{\sim} \left(N \ell^3 \Delta E \right) \tau = \Delta\varepsilon \, \tau$$

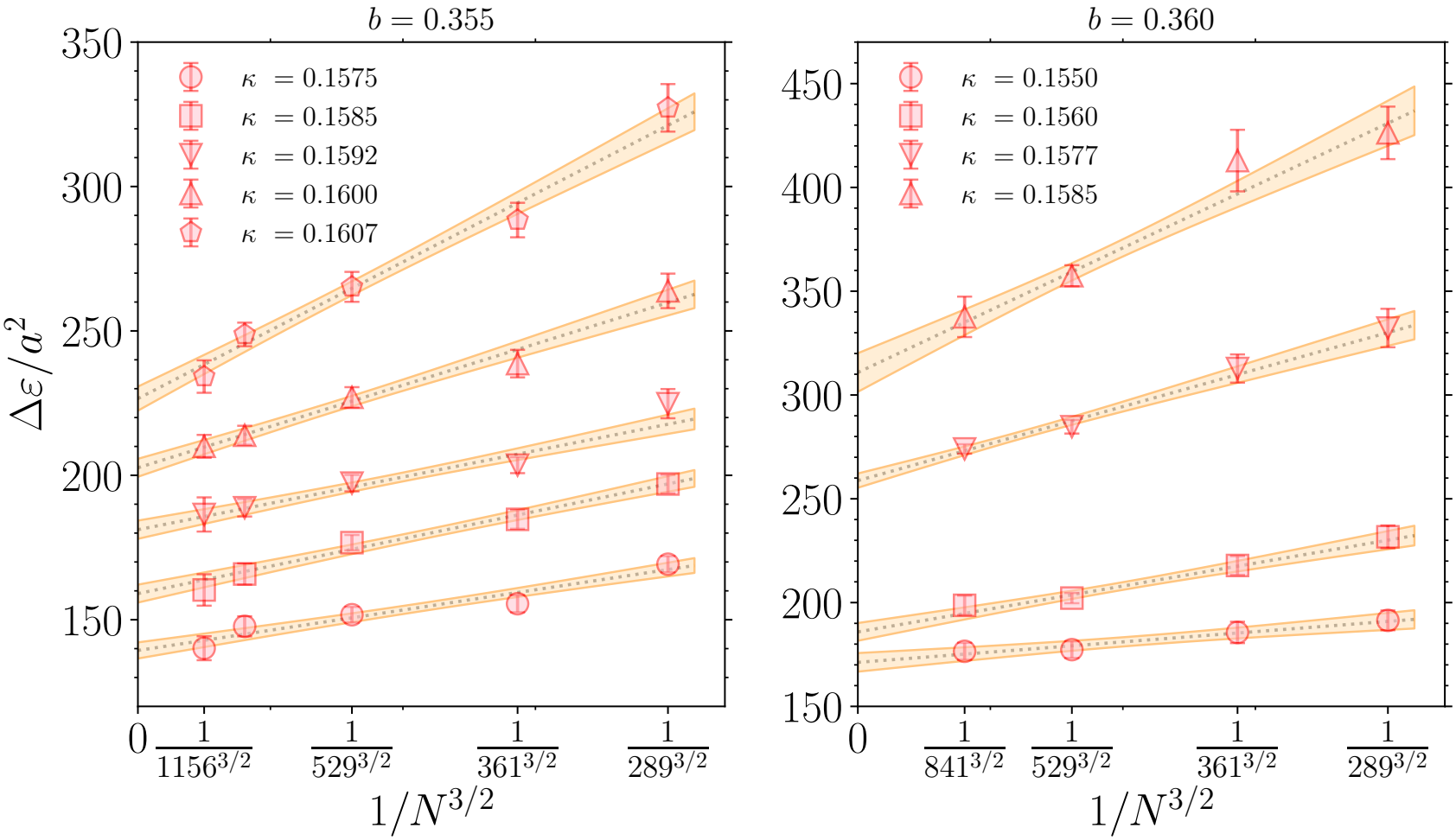


Large-N limit of the energy shift

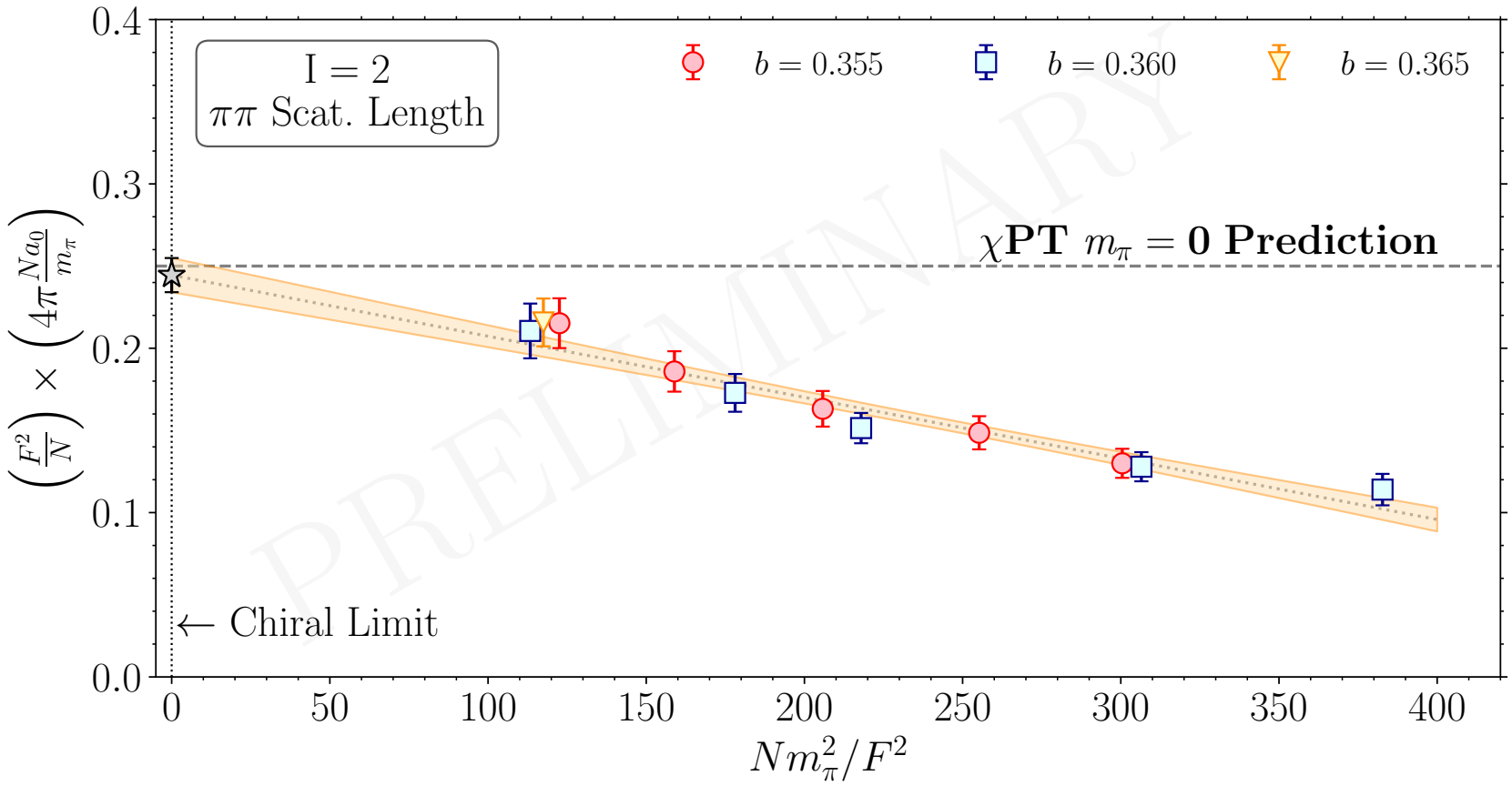
Lüscher formula: $$\Delta\varepsilon = \frac{4\pi}{m_\pi}\overline{a}_0\left[1 + O\left(\frac{1}{N\ell}\right)\right]$$

Large-N limit of the energy shift

Lüscher formula: $\Delta\varepsilon = \dfrac{4\pi}{m_\pi}\bar{a}_0\left[1+O\left(\dfrac{1}{N^{3/2}}\right)\right]$ $\qquad \ell = a\sqrt{N}$



Chiral behavior of a_0 at large N

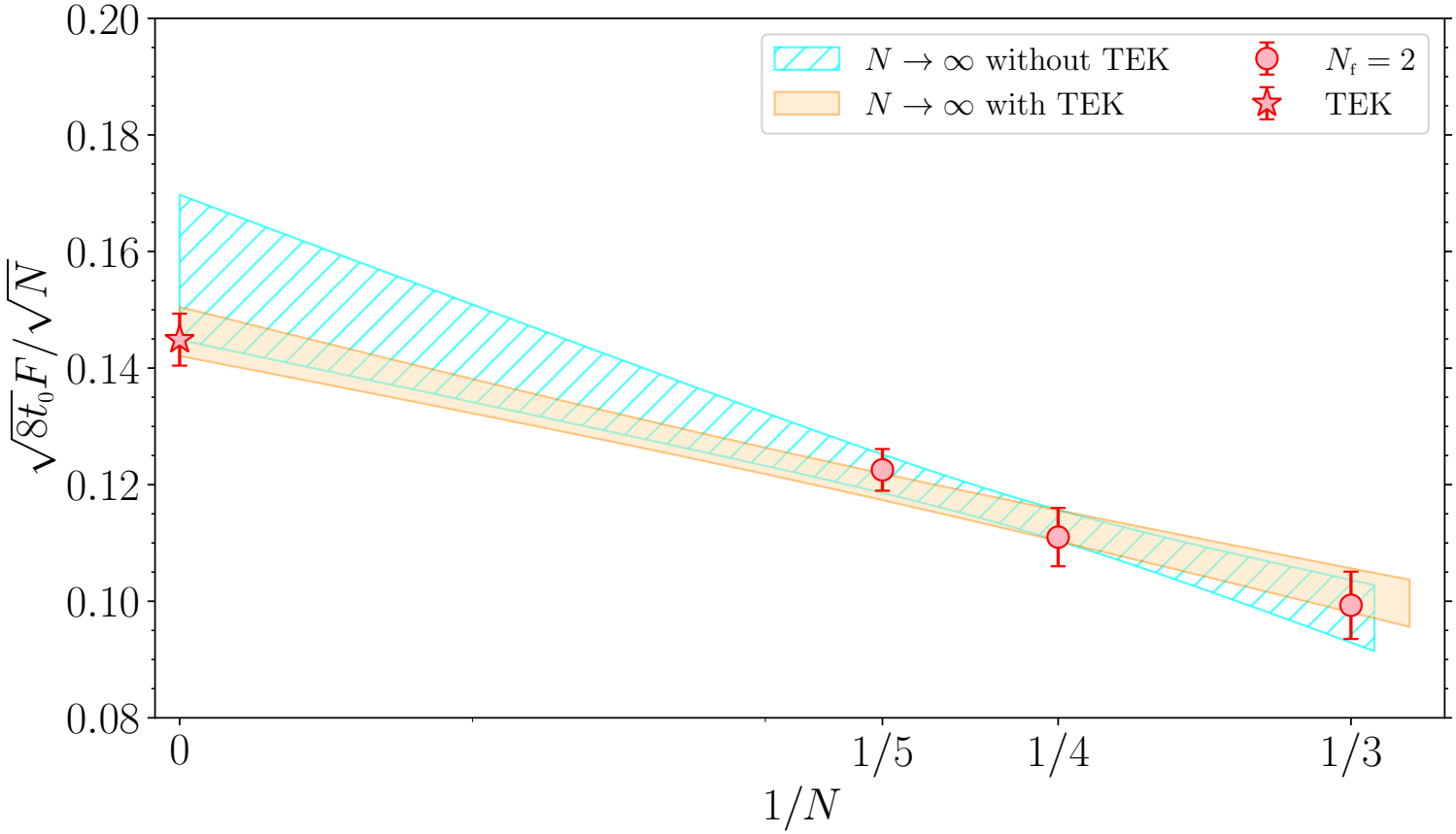


$$\left(\frac{F^2}{N}\right) \times \left(4\pi \frac{N a_0}{m_\pi}\right) = \frac{1}{4}\left[1 - 16\,\frac{L_{\pi\pi}}{N}\,\frac{N m_\pi^2}{F^2} + O(m_\pi^4)\right] \qquad L_{\pi\pi} \sim O(N)$$

↑ Large-N χPT prediction Baeza-Ballesteros et al. JHEP **06** (2022) 046

TEK + traditional method
$\longrightarrow$ precision tool for large-N QCD

$1/N$ expansion of F



In the chiral limit $m_\pi \to 0$:

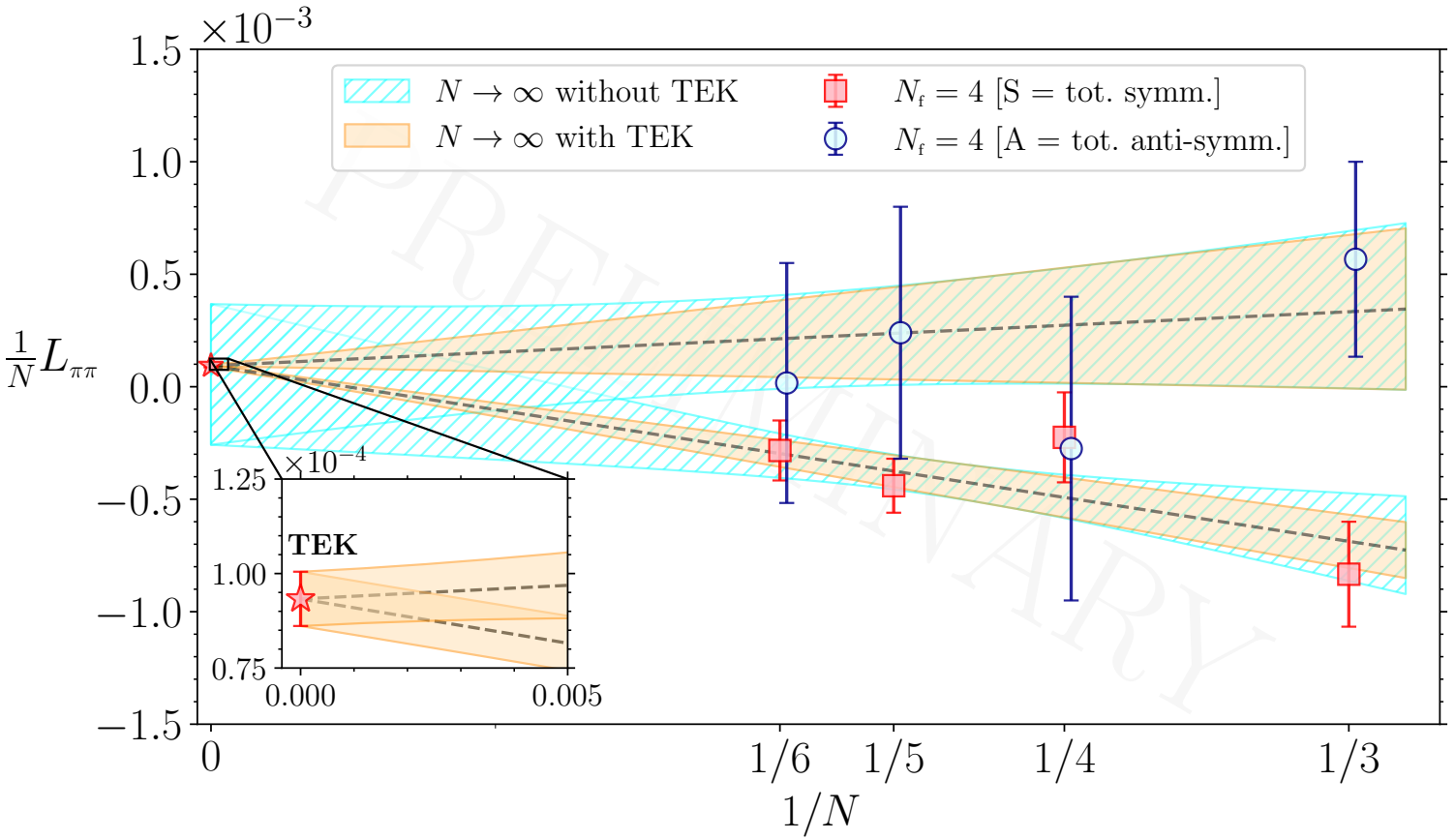
$$\frac{Na_0}{m_\pi} \to \frac{1}{16\pi}\frac{N}{F^2}$$

TEK + $N_f = 2$ finite-N data (standard method) DeGrand et al. PRD **108** (2023) 094516

$$\frac{\sqrt{8t_0}}{\sqrt{N}}F = 0.1463(42)\left[1 - 0.46(7)\frac{N_f}{N} + \mathcal{O}\left(\frac{1}{N^2}\right)\right]$$

$\mathcal{O}(N^0) \to \times 3$ error reduction $\mathcal{O}(1/N) \to \times 3$ error reduction

$1/N$ expansion of $L_{\pi\pi}$



$L_{\pi\pi}$: sub-leading term in chiral expansion of a_0

TEK + $N_f = 4$ finite-N data (standard method) Baeza-Ballesteros et al. JHEP **06** (2022) 049

$$\frac{L_{\pi\pi}}{N} \times 10^5 = 9.33(72)\left\{1 + [-2.2(1.4) \pm 16.4(5.9)N_f]\frac{1}{N} + O\left(\frac{1}{N^2}\right)\right\}$$

A → +
S → −

$O(N^0) \to \times 45$ error reduction $\quad$ $O(1/N) \to \times 15$ error reduction

Conclusions

- TEK successfully reproduces and significantly improves on state of the art
- **TEK + traditional methods** $\longrightarrow$ precision tool for large-N QCD
- Present results $\longrightarrow$ first step of long-term journey along new research avenue: large-N low-energy constants, meson scattering amplitudes and form factors
- **Impact**: TEK large-N results already employed as "experimental data" in recent χPT and bootstrap studies of large-N QCD Cheung et al. [2605.21582] — Albert et al. [2606.19420]

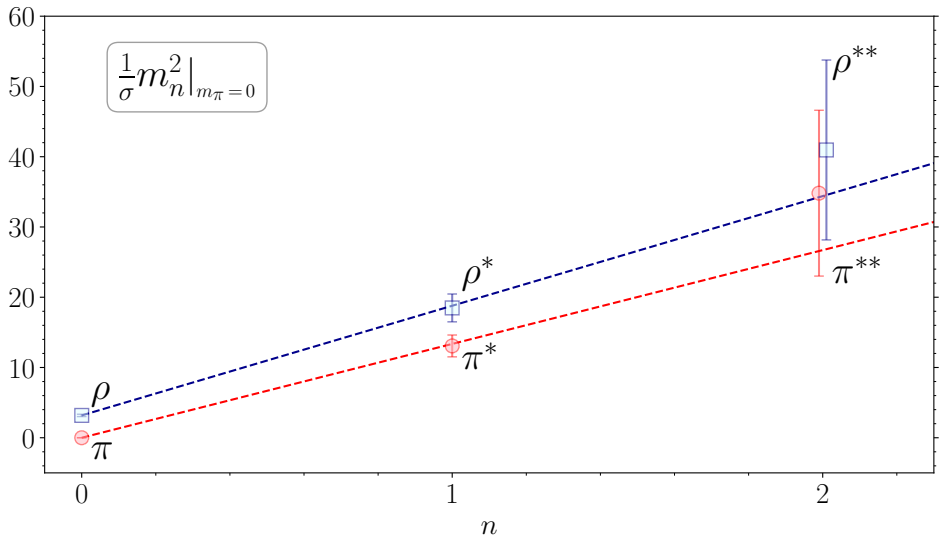
Future outlooks

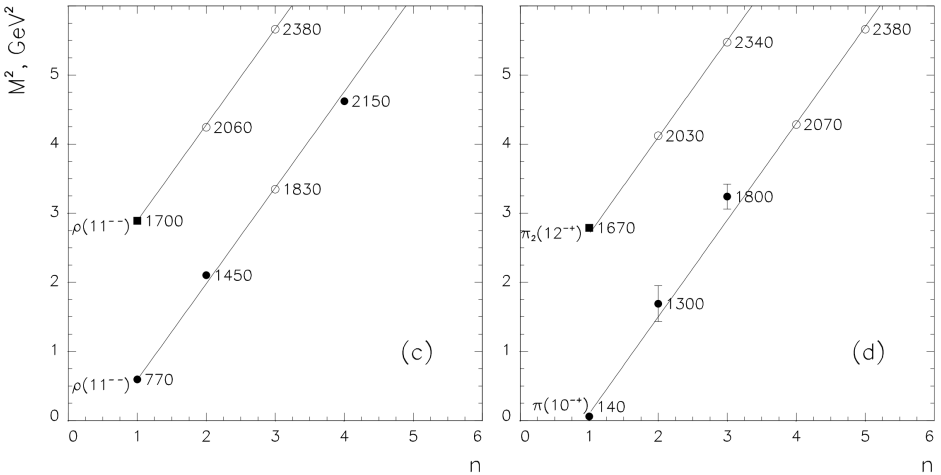
- In the next future I want to pursue an ambitious research program via TEK: systematic study of scattering amplitudes and form factors in large-N QCD
- Future targets: $\mathrm{I} = 0$ $\pi\pi$ scattering, $\rho\rho$ and $\rho\pi$ scattering, π vector form factor, ...
- Ongoing investigation of large-N QCD via TEK with chiral overlap quarks

Extra Slides

Large-N radial Regge trajectories for π and ρ

Radial Regge trajectory $\longrightarrow$ $\left.\frac{m_n^2}{\sigma}\right|_{m_\pi=0} = \left.\frac{m_0^2}{\sigma}\right|_{m_\pi=0} + \frac{\mu_r^2}{\sigma} n$ (m_n = n^{th} excited state mass)





Systematics of $q\bar{q}$ states in the (n, M^2) and (J, M^2) planes

A. V. Anisovich, V. V. Anisovich, and A. V. Sarantsev Phys. Rev. D **62**, 051502(R) – **Published 25 July, 2000**

$\frac{\mu_r}{\sqrt{\sigma}} \simeq 2.65$ Exp. data PRD 62 (2000) 051502

$\frac{\mu_r}{\sqrt{\sigma}} \simeq \sqrt{4\pi} \simeq 3.55$ Large-N eff. model PLB 323 (1994) 41

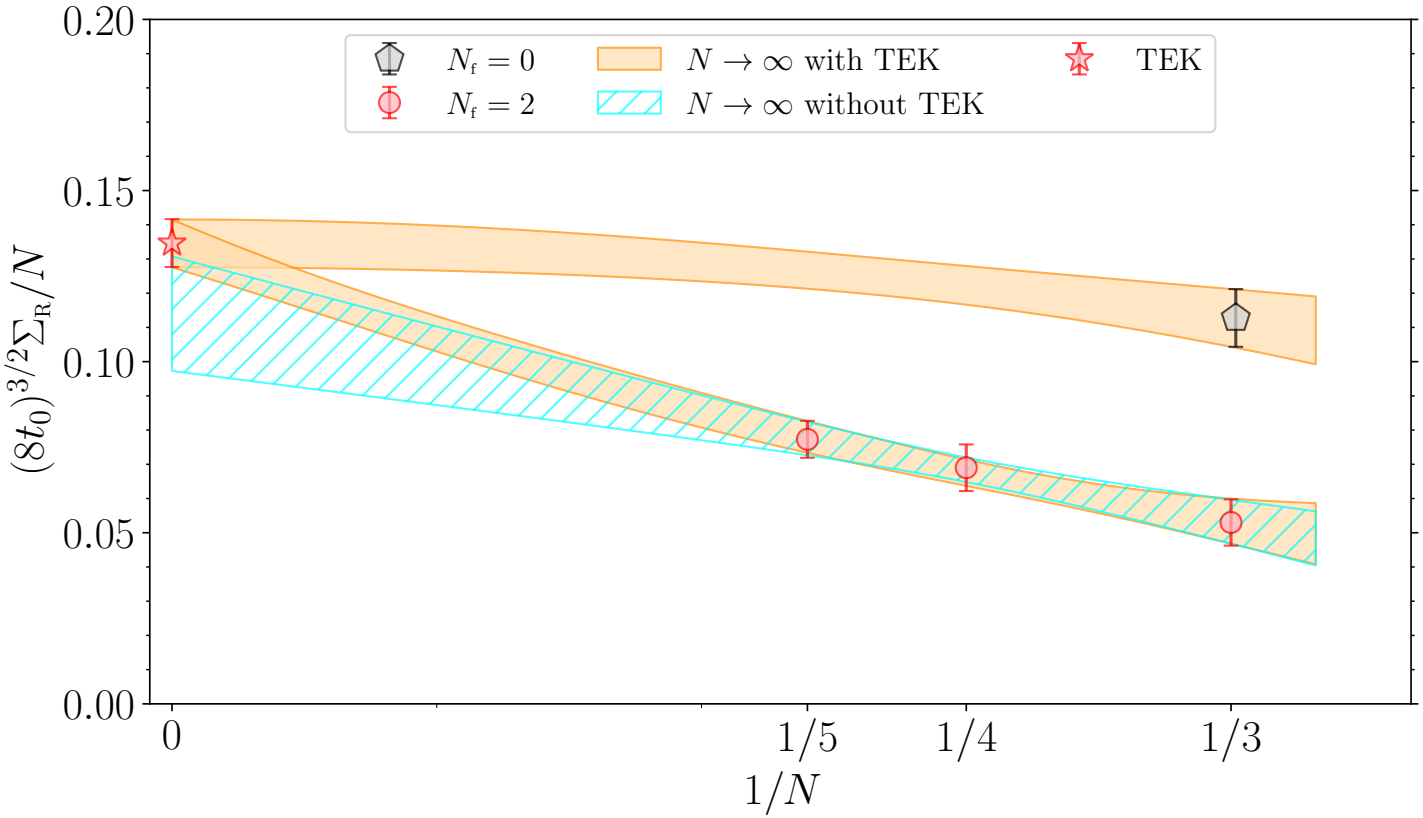
$\frac{\mu_r}{\sqrt{\sigma}} = 3.65(21)$ TEK π channel

$\frac{\mu_r}{\sqrt{\sigma}} = 3.95(24)$ TEK ρ channel

$\mu_r(N=\infty) \simeq 1.94(11)$ GeV $\qquad$ $\mu_r(\text{exp}) \simeq 1.18$ GeV

$1/N$ expansion of Σ_R/N

$N = 3$ quenched $N_f = 0$ result $\longrightarrow$ only affected by $O(1/N^2)$ corrections
$\longrightarrow$ much closer to $N = \infty$ than unquenched $N_f = 2$ ones

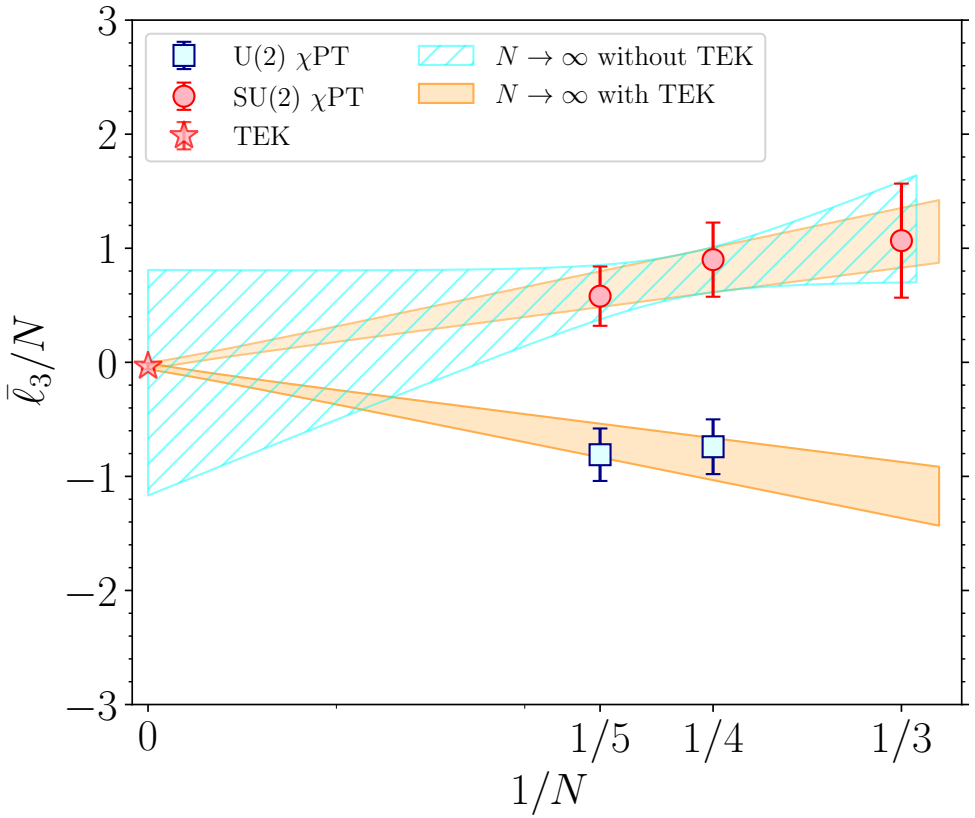


$$\left(\sqrt{8t_0}\right)^3 \frac{\Sigma_R}{N} = 0.1346(70)\left[1 - 1.27(33)\frac{N_f}{N} + \left[-1.38(70) + 0.91(56)N_f^2\right]\frac{1}{N^2} + O\left(\frac{1}{N^3}\right)\right]$$

$1/N$ expansion of $\bar{\ell}_3/N$

$\mathrm{SU}(N_\mathrm{f})$ vs $\mathrm{U}(N_\mathrm{f})$ χPT: NLO LECs differ for $O(1/N)$ terms $\implies$ same $N \to \infty$ limit

$\mathrm{U}(N_\mathrm{f})$ has also light η' $\implies$ should converge faster towards the large-N limit



$$M_\pi^2 = M^2 \left[1 - \frac{M^2}{32\pi^2 F^2} \bar{\ell}_3 + O(M^4) \right]$$

$$M^2 \equiv N_\mathrm{f} \frac{\Sigma_\mathrm{R}}{F^2} m_\mathrm{R} \qquad \bar{\ell}_3 \sim O(N)$$

$$F \equiv \lim_{m_\mathrm{R} \to 0} F_\pi \sim O\left(\sqrt{N}\right)$$

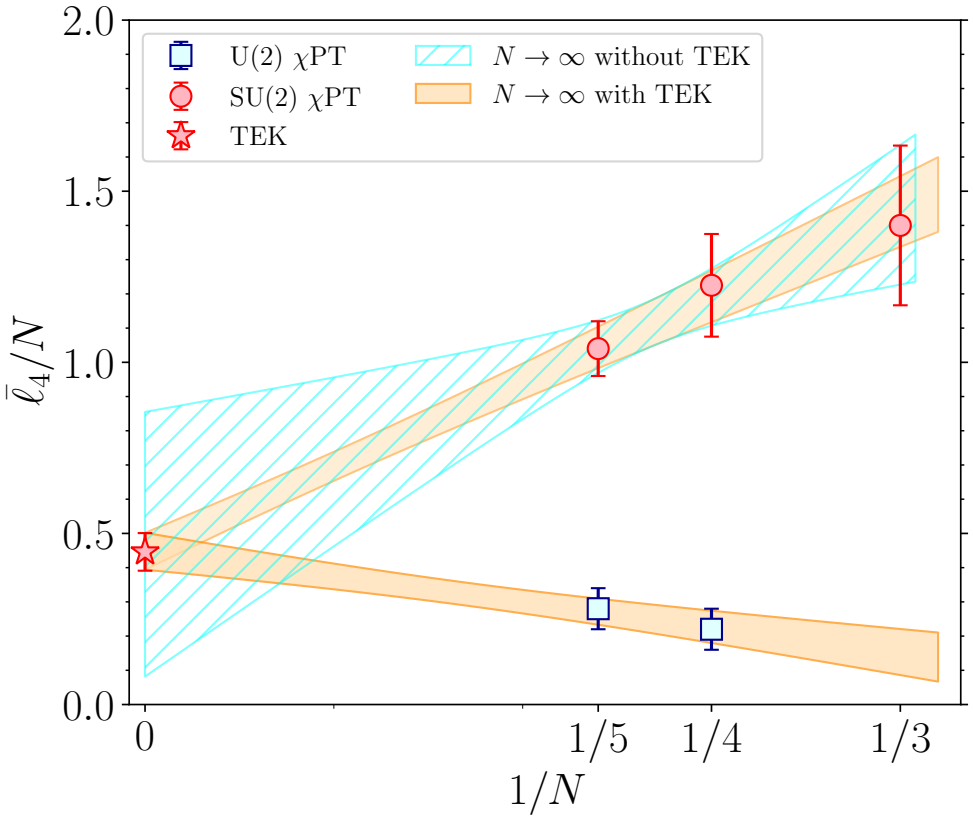
$$\frac{\bar{\ell}_3}{N} = \left[-0.034(27) + 1.69(40) \frac{N_\mathrm{f}}{N} \right]_{\mathrm{SU}(2)}$$

$$\frac{\bar{\ell}_3}{N} = \left[-0.034(27) - 1.63(37) \frac{N_\mathrm{f}}{N} \right]_{\mathrm{U}(2)}$$

$1/N$ expansion of $\bar{\ell}_4/N$

$\mathrm{SU}(N_\mathrm{f})$ vs $\mathrm{U}(N_\mathrm{f})$ χPT: NLO LECs differ for $\mathcal{O}(1/N)$ terms $\implies$ same $N \to \infty$ limit

$\mathrm{U}(N_\mathrm{f})$ has also light η' $\implies$ should converge faster towards the large-N limit



$$F_\pi = F\left[1 + \frac{M^2}{16\pi^2 F^2}\bar{\ell}_4 + \mathcal{O}(M^4)\right]$$

$$M^2 \equiv N_\mathrm{f}\frac{\Sigma_\mathrm{R}}{F^2}m_\mathrm{R} \qquad \bar{\ell}_4 \sim \mathcal{O}(N)$$

$$F \equiv \lim_{m_\mathrm{R}\to 0} F_\pi \sim \mathcal{O}\left(\sqrt{N}\right)$$

$$\frac{\bar{\ell}_4}{N} = \left[0.448(54) + 1.49(19)\frac{N_\mathrm{f}}{N}\right]_{\mathrm{SU}(2)}$$

$$\frac{\bar{\ell}_4}{N} = \left[0.448(54) - 0.44(15)\frac{N_\mathrm{f}}{N}\right]_{\mathrm{U}(2)}$$

Twist choice, center symmetry and number theory

$$z_{\nu\mu} = \exp\left\{2\pi \mathrm{i} \frac{k}{\sqrt{N}} \epsilon_{\nu\mu}\right\} \qquad \sqrt{N} = L, k \in \mathbb{N}$$

$N \to \infty$ at fixed $k \implies$ center symmetry breaking Teper–Vairinhos PLB 652 (2007) 359-369

Intuitively: $z_{\nu\mu} \underset{\substack{N\to\infty \\ k = \text{fixed}}}{\longrightarrow} 1$ (periodic boundaries)

$\implies$ k must be scaled with N to avoid center symmetry breaking JHEP 07 (2010) 043

It is clear that $k/\sqrt{N} \approx$ constant to have $z_{\nu\mu}$ survive the large-N limit. How to choose?

Two optimal solutions have been determined Chamizo & González-Arroyo JPA 50 (2017) 265401

- L prime $\quad k, L$ co-primes $\quad k/L > 0.1 \quad \bar{k}/L > 0.1 \quad \bar{k}k = 1 \pmod N$

  Example: $\quad N = 289 = 17^2 \quad k = 5 \quad \bar{k} = 7 \quad k/L \simeq 0.294 \quad \bar{k}/L \simeq 0.412$

- $N = F_j \quad k = \bar{k} = F_{j-2} \quad F_j = j^{\text{th}}$ Fibonacci $\implies \quad F_{j-2}/F_j \geq 1/\phi^2 \simeq 0.382$

see also Dasilva Golán–García Pérez JHEP 12 (2022) 109

Scale setting

$\sigma \longrightarrow$ string tension

$t_0, t_1 \longrightarrow$ Gradient Flow scales

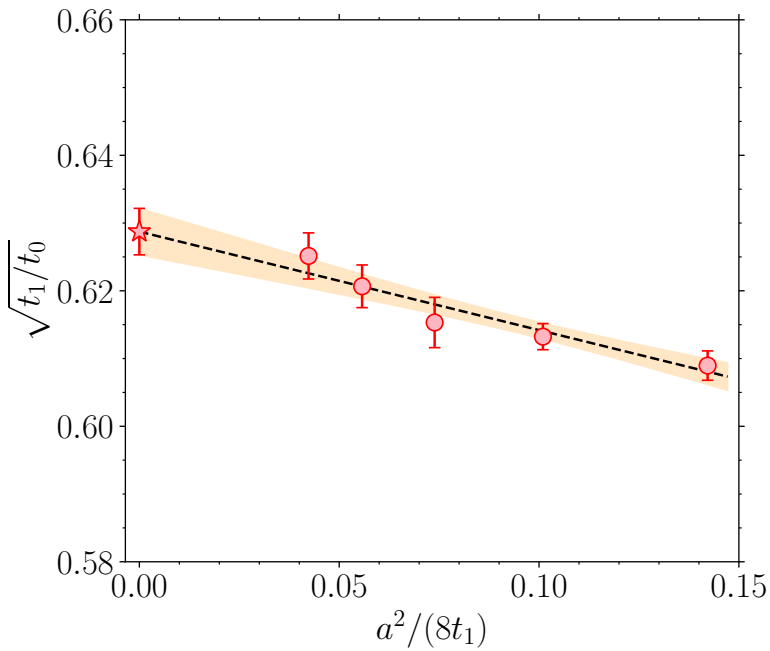
$$\frac{1}{N} t^2 \langle E(t) \rangle = c$$

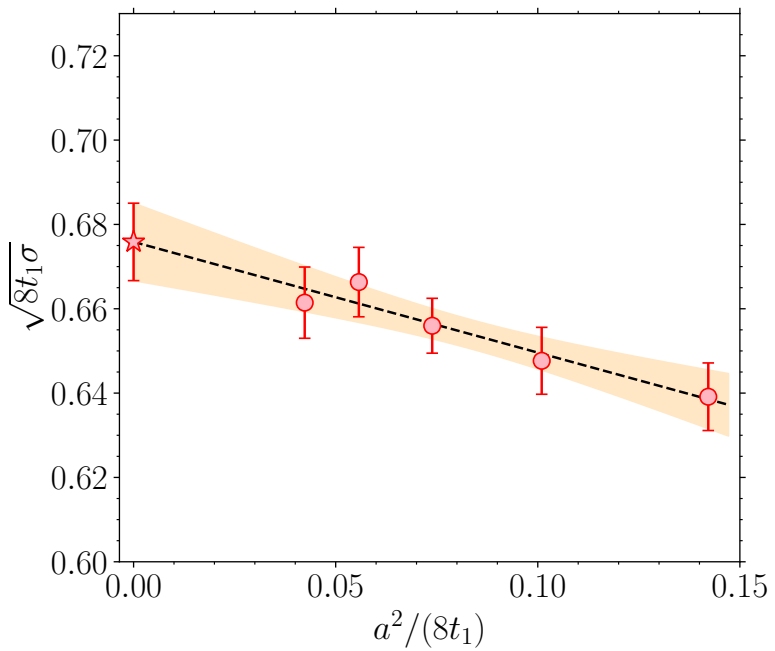
$c = 0.1 \longrightarrow t_0$

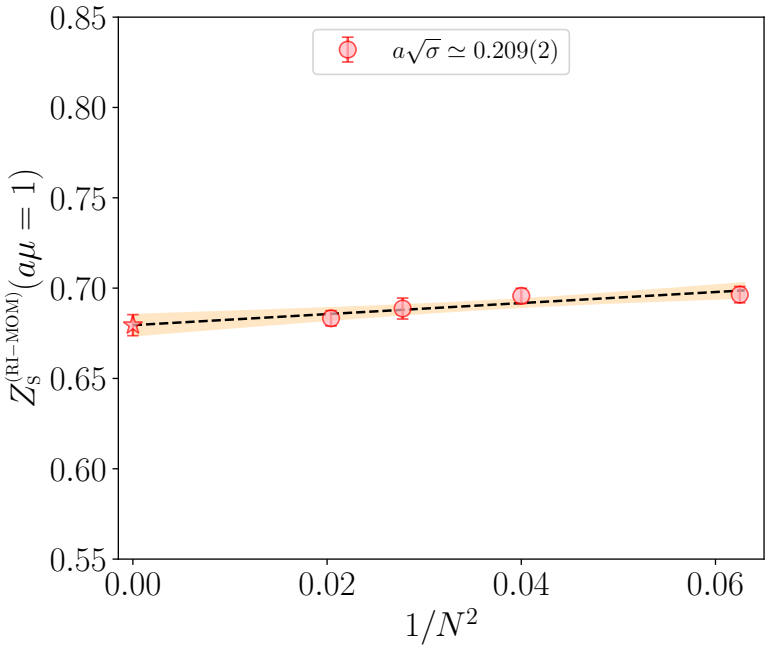
$c = 0.05 \longrightarrow t_1$

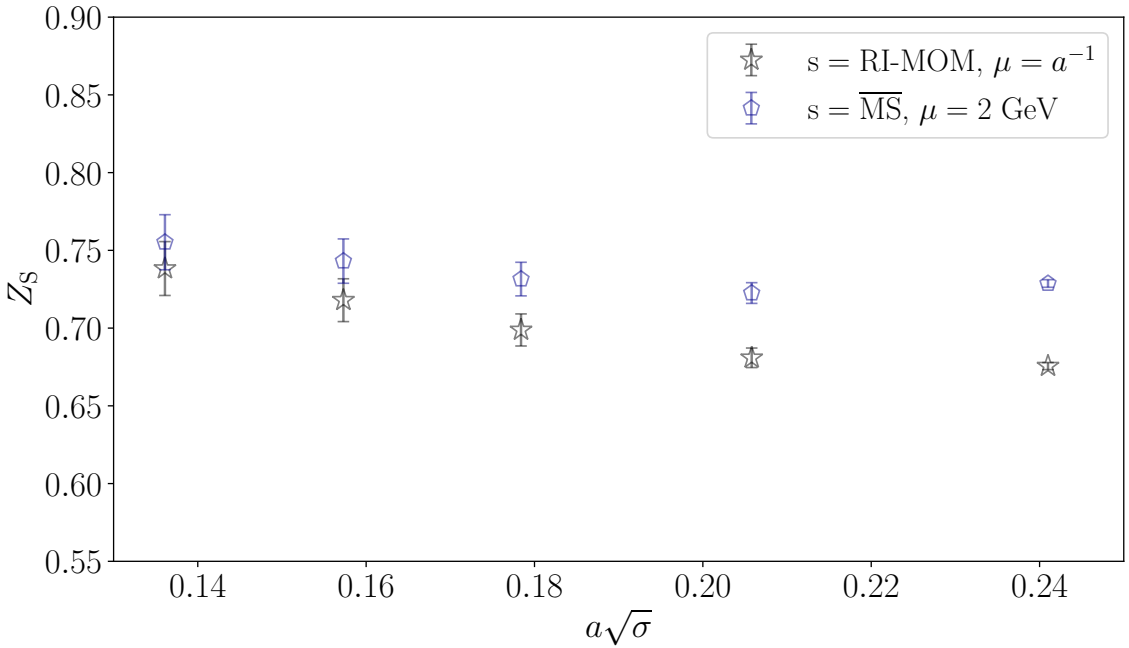
$E \longrightarrow$ clover discr.

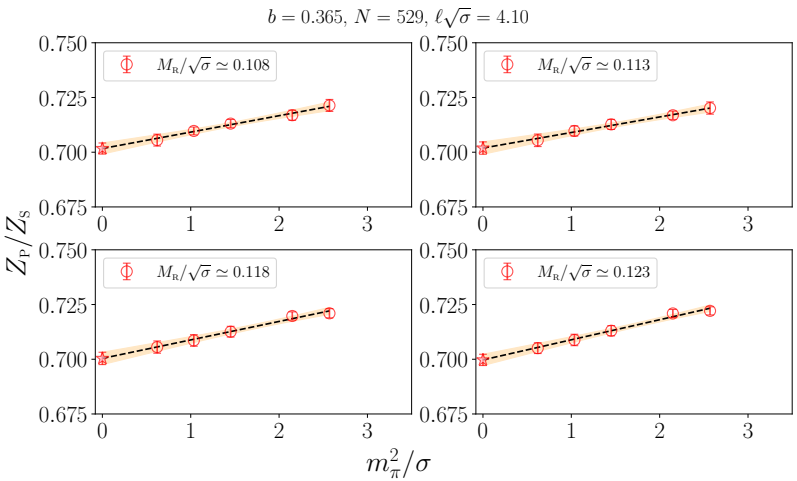
$\sigma \longrightarrow$ smeared Creutz ratios

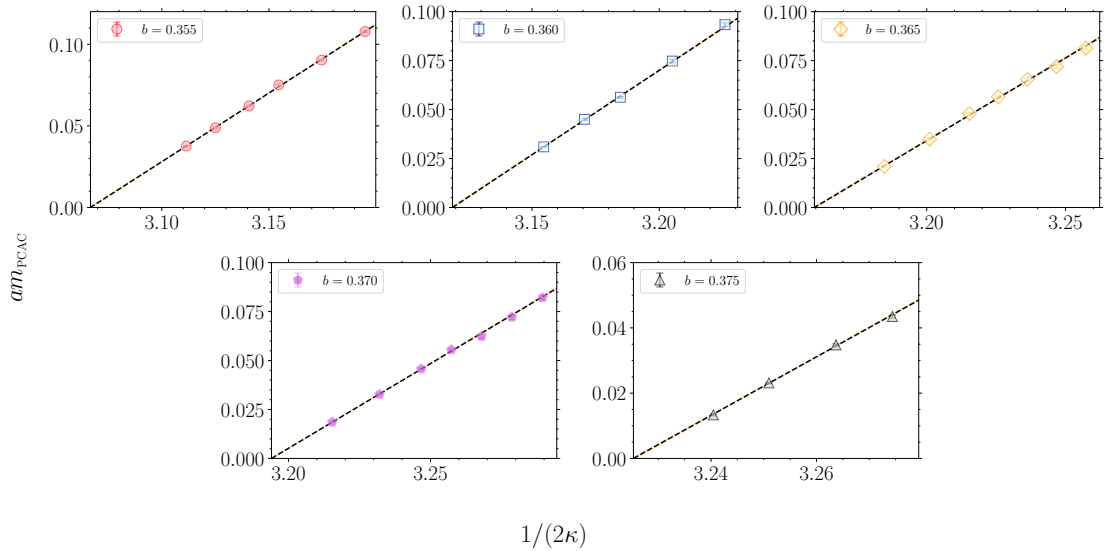




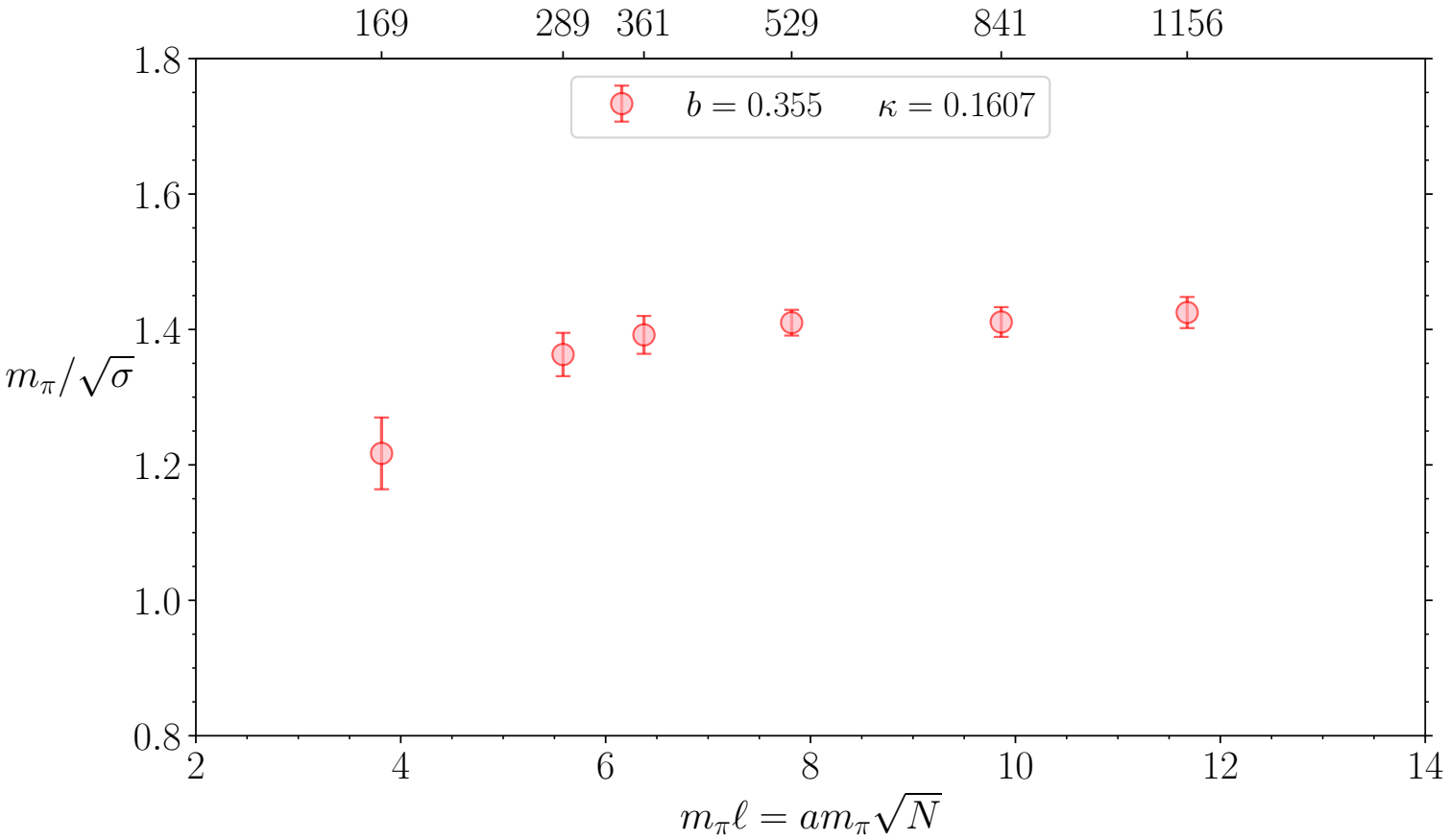








Non-perturbative determination of:
Z_S (top), Z_P/Z_S (bottom left) and $Z_P/(Z_S Z_A)$ (bottom right)



$N = 169 \implies \ell = a\sqrt{N} \approx 1.5$ fm

$N = 1156 \implies \ell = a\sqrt{N} \approx 3.8$ fm