(Line defect) indices of M2-brane SCFTs

Tomoki Nosaka

Shanghai Institute for Mathematics and Interdisciplinary Sciences

Based on collaborations with H. Hayashi (Tokai U.) and T. Okazaki (Southeast U.)

JHEP09(2024)123(arXiv:2406.00413), JHEP12(2024)109(arXiv:2409.13239), *to appear*

@The 2nd Joint Summer School on Theoretical High-Energy Physics (16-28 Aug 2026)

M2 SCFTs

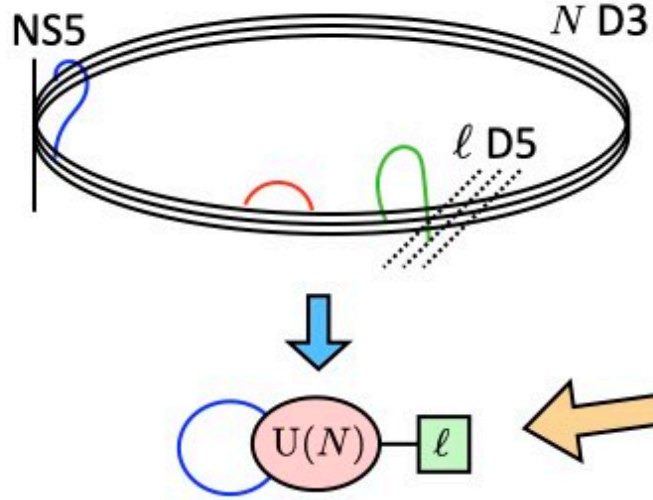
- M2 SCFT = 3d SUSY gauge theory on *N* M2-branes in M theory

(c.f. *N* D3 ⇨ 4d super Yang-Mills)

- Can be constructed systematically via dual IIA/IIB brane setups

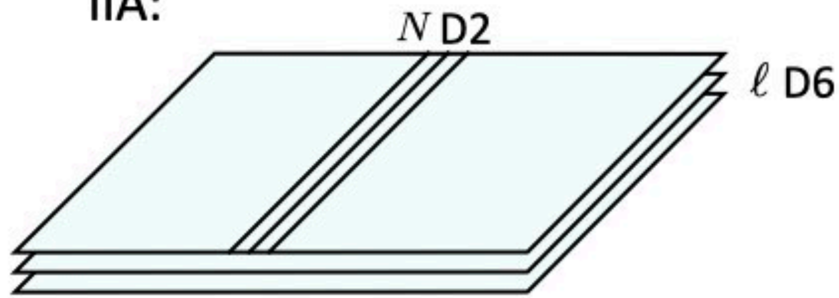
IIB:

	0 1 2	3	4 5 6	7 8 9
D3	✓✓✓	✓		
D5	✓✓✓		✓✓✓	
NS5	✓✓✓			✓✓✓



T-dual in x_3

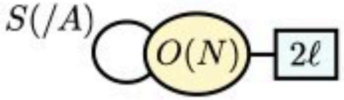
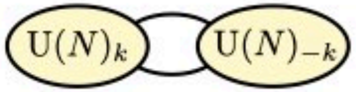
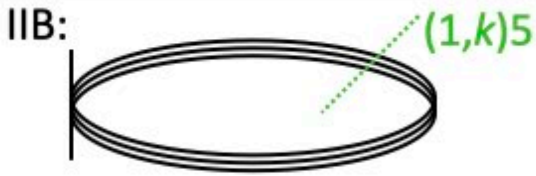
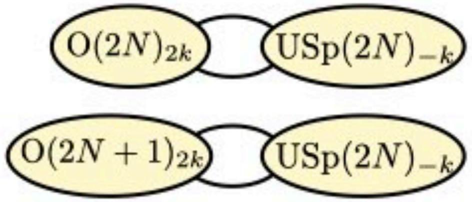
IIA:



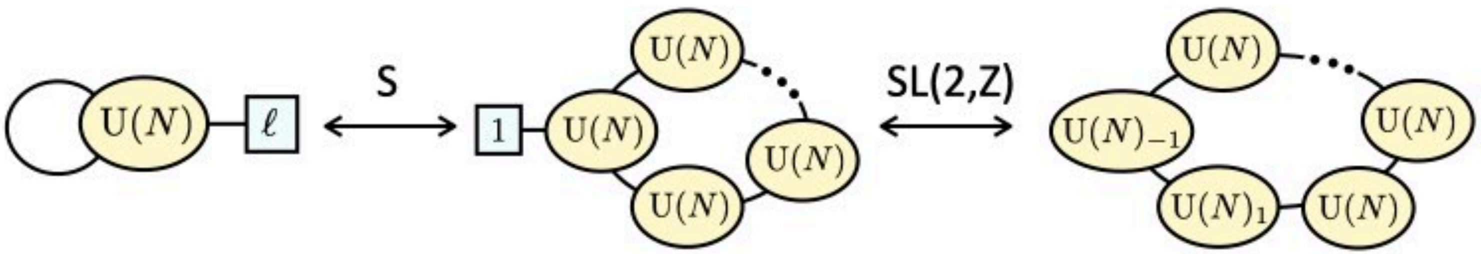
add $S^1(x_{10})$

N M2 on $\mathbb{C}^2 \times \mathbb{C}^2/\mathbb{Z}_\ell$

Other examples

3d theory	Brane setup		N M2 on
		IIA: ND2 + ℓ D6 + $\overset{(\sim)}{\text{O6}}{}^{\pm}$ (or $\overset{(\sim)}{\text{O2}}{}^{\pm}$)	$\mathbb{C}^2 \times \mathbb{C}^2/\hat{D}_{\ell(\pm 2)}$
			
			$\mathbb{C}^4/\mathbb{Z}_k$
			$\mathbb{C}^4/\hat{D}_k$

- Brane pictures suggest various dualities. For example,



Superconformal indices

$$\boxed{I(q,\cdots) = \mathrm{Tr}_{Q|\rangle=Q^\dagger|\rangle=0}((-1)^F q^\Delta \cdots)}$$

(Δ: conformal dimension + R-charge)

$I(q_i)$ is independent of continuous parameters of the theory

⇨ can study strong coupling regime via weak coupling description

⇨ can test IR dualities

$I(q_i)$ itself is also an interesting object:

- Can compare with index in gravity side at large *N*
- Finite *N* corrections ⇨ giant graviton expansion

Giant graviton expansion

There are fewer states with dimension/R-charge Δ > *N* in the finite *N* theory, compared with the large *N* theory.

In gravity side, this can be understood as the volume constraint on "giant graviton":

- $q^\Delta \longleftrightarrow$ graviton with angular momentum Δ in $S^7 \subset \mathrm{AdS}_4 \times S^7$ [McGreevy,Susskind,Toumbas,'00]
- Due to Myers effect, graviton expands to M5-brane wrapped on $S^5 \subset S^7$ with size Δ
- This M5 cannot be bigger than S^7 itself ⇨ Δ ≤ *N*

Alternative viewpoint: $I_{N\mathrm{M2}} = I_{\mathrm{graviton}}[1 + (\text{wrapped M5}) + (2 \text{ wrapped M5}) + \cdots]$ [Arai,Imamura,'19]

$$\boxed{\underbrace{q^N}_{\text{ground state}} \underbrace{\frac{1}{1-q^{-1}}}_{\text{fluctuation modes}}(\cdots) = -\frac{q^{N+1}}{1-q}(\cdots)}$$

Fluctuation modes = index of M5 wvt ⇨

$$\boxed{\frac{I_{N\,\mathrm{M2}}(q)}{I_{\infty\,\mathrm{M2}}(q)} = 1 + \sum_{m=1}^{\infty} q^{mN} I_{m\,\mathrm{M5}}(q')}$$

Main results

- For application to such as giant graviton expansion, it is desirable to calculate $I_{N\mathrm{M2}}(q)$ for finite but large *N* & to high order in *q* ⇨ Difficult

- We calculated $I_{N\mathrm{M2}}(q)$ **exactly for all *N*** in a specialization limit which counts only states preserving two $Q^{(\dagger)}$'s (Coulomb limit)

  *In this limit, index calculation reduces to 1d ideal Fermi gas on S^1

  ⇨ verified structure of giant graviton expansion

- We can also calculate index dressed by Wilson lines by the same method

  ⇨ powerful **recursion relation** with respect to *N*

Plan of talk

1. Superconformal indices

2. Line defect indices

3. Summary and future works

Superconformal index of 3d $\mathcal{N}=4$ theory

$\mathcal{N}=4$ superconformal symmetry:

- Spacetime: $p_\mu, j_{\mu\nu}; k_\mu, h$
- R-symmetry: $SO(4) = SU(2)_H \times SU(2)_C$

Cartans: h, j_{12}, H, C

- SUSY: $\mathcal{Q}^I_\alpha$ with $h = \frac{1}{2},\ j_{12} = \pm\frac{1}{2},\ H = \pm' 1,\ C = \pm'' 1$ $\quad \left(\{\mathcal{Q}^I_\alpha, (\mathcal{Q}^I_\alpha)^\dagger\} = h \pm j_{12} \pm' \frac{H}{2} \pm'' \frac{C}{2}\right)$

Superconformal index:

Choose one $\mathcal{Q}$ of $\{\mathcal{Q}^I_\alpha\}$ s.t.

	h	j_{12}	H	C
$\mathcal{Q}$	$\frac{1}{2}$	$-\frac{1}{2}$	1	1

$\left(\{\mathcal{Q}, \mathcal{Q}^\dagger\} = h - j_{12} - \frac{H+C}{2}\right)$

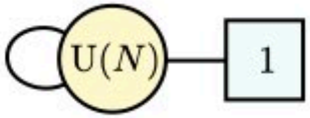
$$\Rightarrow I = \mathrm{Tr}[(-1)^F e^{-\beta\{\mathcal{Q},\mathcal{Q}^\dagger\}} q^{h-\frac{H+C}{4}} t^{H-C} z^{f_C} x^{f_H}]$$

$[\cdot, \mathcal{Q}] = 0 \Rightarrow I$ is β-indep.

$f_{C/H}$: global symmetry (topological U(1) / flavor sym.)

$$\boxed{I \overset{\beta=\infty}{=} \mathrm{Tr}_{\{\mathcal{Q},\mathcal{Q}^\dagger\}=0} (-1)^F q^{h-\frac{H+C}{4}} t^{H-C} z^{f_C} x^{f_H}}$$

U(N) ADHM theory



monopole flux on S^2 — projection to gauge singlet — vector

$$I^{\mathrm{U}(N)\text{-}[1]} = \frac{1}{N!}\sum_{m_i\in\mathbb{Z}}\int \frac{ds_i}{2\pi i s_i}\prod_{i\neq j}\left(1-q^{\frac{|m_i-m_j|}{2}}\frac{s_i}{s_j}\right)\prod_{i,j}\frac{(q^{\frac{1}{2}+\frac{|m_i-m_j|}{2}}t^{2}\frac{s_i}{s_j};q)_\infty}{(q^{\frac{1}{2}+\frac{|m_i-m_j|}{2}}t^{-2}\frac{s_i}{s_j};q)_\infty}$$

$$\times\prod_{i,j}\prod_{\pm}\frac{(q^{\frac{3}{4}+\frac{|m_i-m_j|}{2}}t^{-1}x^{\pm1}\frac{s_i}{s_j};q)_\infty}{(q^{\frac{1}{4}+\frac{|m_i-m_j|}{2}}t x^{\pm1}\frac{s_i}{s_j};q)_\infty}\prod_{i}\prod_{\pm}\frac{(q^{\frac{3}{4}+\frac{|m_i|}{2}}t^{-1}s_i^{\pm1};q)_\infty}{(q^{\frac{1}{4}+\frac{|m_i|}{2}}t s_i^{\pm1};q)_\infty}z^{\sum_i m_i}q^{\frac{1}{4}\sum_i|m_i|}t^{-\sum_i|m_i|}$$

adj. hyp — fund. hyp

[Kim,'09]

$$\left(\frac{1}{(x;q)_\infty}=\prod_{n=0}^{\infty}\frac{1}{1-xq^n}=\mathrm{PE}[x+\underset{\substack{\uparrow\\ \partial}}{qx}+\underset{\substack{\uparrow\\ \partial^2}}{q^2x}+\cdots]\right)$$

$\{m_i\}$-dependence can be derived by applying SUSY localization to $I=Z_{S^2\times S^1}$

Difficult (although it is possible) to evaluate ⇨ Let's take a simplification limit

Coulomb/Higgs limit

Rewrite index as

$$I = \mathrm{Tr}_{\{\mathcal{Q},\mathcal{Q}^\dagger\}=0}[(-1)^F (q^{\frac{1}{4}}t)^{2h-C}(q^{\frac{1}{4}}t^{-1})^{2h-H} z^{f_C} x^{f_H}]$$

Since $2h-C, 2h-H \geq 0$, we can define Coulomb/Higgs indices as [Razamat,Willett,'14]

$$\mathcal{I}_C \equiv \lim_{\substack{q,t\to 0 \\ (q^{\frac{1}{4}}t^{-1}=\mathfrak{t})}} I = \mathrm{Tr}_{\underbrace{\{\mathcal{Q},\mathcal{Q}^\dagger\}=0}_{0=h+j_{12}+\frac{H-C}{2}=\{\mathcal{Q}',(\mathcal{Q}')^\dagger\}},2h-C=0}[\mathfrak{t}^{2h-H} z^{f_C}]$$

counts only 1/4 BPS states with $\mathcal{Q}^{(\dagger)} = (\mathcal{Q}')^{(\dagger)} = 0$

$$\mathcal{I}_H \equiv \lim_{\substack{q,t^{-1}\to 0 \\ (q^{\frac{1}{4}}t=\mathfrak{t})}} I = \mathrm{Tr}_{\underbrace{\{\mathcal{Q},\mathcal{Q}^\dagger\}=0}_{0=h+j_{12}-\frac{H-C}{2}=\{\mathcal{Q}'',(\mathcal{Q}'')^\dagger\}},2h-H=0}[\mathfrak{t}^{2h-C} x^{f_H}]$$

counts only 1/4 BPS states with $\mathcal{Q}^{(\dagger)} = (\mathcal{Q}'')^{(\dagger)} = 0$

In this talk, we mostly focus on Coulomb indices

Coulomb limit = Fermi gas

$$\mathcal{I}_C^{\mathrm{U}(N)\text{-}[1]} = \frac{1}{N!}\sum_{m_i\in\mathbb{Z}}\int\frac{ds_i}{2\pi i s_i}\prod_{i\neq j}\left(1-q^{\frac{|m_i-m_j|}{2}}\frac{s_i}{s_j}\right)\prod_{i,j}\frac{\cancel{(q^{\frac{1}{2}+\frac{|m_i-m_j|}{2}}t^2\frac{s_i}{s_j};q)_\infty}}{(q^{\frac{1}{2}+\frac{|m_i-m_j|}{2}}t^{-2}\frac{s_i}{s_j};q)_\infty}\cancel{\prod_{i,j}\prod_{\pm}\frac{(q^{\frac{3}{4}+\frac{|m_i-m_j|}{2}}t^{-1}x^{\pm 1}\frac{s_i}{s_j};q)_\infty}{(q^{\frac{1}{4}+\frac{|m_i-m_j|}{2}}tx^{\pm 1}\frac{s_i}{s_j};q)_\infty}}$$

$$\times\cancel{\prod_i\prod_{\pm}\frac{(q^{\frac{3}{4}+\frac{|m_i|}{2}}t^{-1}s_i^{\pm 1};q)_\infty}{(q^{\frac{1}{4}+\frac{|m_i|}{2}}ts_i^{\pm 1};q)_\infty}}z^{\sum_i m_i}q^{\frac{1}{4}\sum_i|m_i|}t^{-\sum_i|m_i|}$$

$\to 1$ if $m_i\neq m_j$: monopoles break $\mathrm{U}(N)$ into $\prod_{m=-\infty}^{\infty}\mathrm{U}(\nu_m)$ $\quad(\nu_m=\#\{i\,|\,m_i=m\})$

$$\mathcal{I}_C^{\mathrm{U}(N)\text{-}[1]} = \left\{\sum_{\substack{\nu_m\geq 0\\(\sum_m\nu_m=N)}}\right\}\prod_{m=-\infty}^{\infty}\mathfrak{t}^{|m|\nu_m}z^{m\nu_m}\tilde{\mathcal{I}}_1(\nu_m)$$

$$\tilde{\mathcal{I}}_1(\nu)=\frac{1}{\nu!}\int\left(\frac{d\sigma}{2\pi i\sigma}\right)^\nu\frac{\prod_{i\neq j}^{\nu}(1-\frac{\sigma_i}{\sigma_j})}{\prod_{i,j=1}^{\nu}(1-\mathfrak{t}^2\frac{\sigma_i}{\sigma_j})}=\frac{\mathfrak{t}^{-\nu(\nu-1)}}{\nu!}\int d^\nu\alpha\,\det[\langle\alpha_i|\hat{\rho}_1|\alpha_j\rangle]$$

$\sigma=e^{2\pi i\alpha}$ $\quad[\hat{\alpha},\hat{p}]=\frac{i}{2\pi}$: 1d QM on S^1

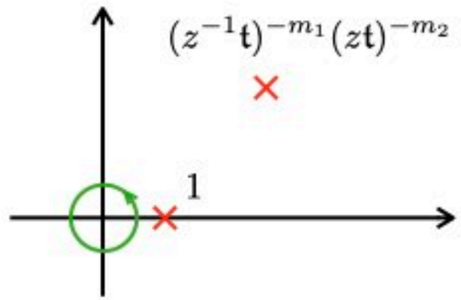
$\hat{\rho}_1=\theta_{\mathbb{Z}}(-\hat{p})\mathfrak{t}^{-2\hat{p}}$ $\quad\theta_{\mathbb{Z}}(z)=\begin{cases}1 & (z\in\mathbb{Z}_{\geq 0})\\ 0 & (z\in\mathbb{Z}_{<0})\end{cases}$

$$\tilde{\mathcal{I}}_1(\nu)\leftrightarrow\mathrm{tr}(\hat{\rho}_1^n)=\frac{1}{1-\mathfrak{t}^{2n}}\ \Rightarrow\ \boxed{\tilde{\mathcal{I}}_1(\nu)=\frac{1}{(\mathfrak{t}^2;\mathfrak{t}^2)_\nu}}$$

Giant graviton expansion

$$\Xi_1(u) = \sum_{N=0}^{\infty} u^N \mathcal{I}_C^{\mathrm{U}(N)\text{-}[1]} = \prod_{m=-\infty}^{\infty} \sum_{\nu=0}^{\infty} (\mathfrak{t}^{|m|} z^m u)^\nu \tilde{\mathcal{I}}_1(\nu) = \prod_{m=-\infty}^{\infty} \frac{1}{(\mathfrak{t}^{|m|} z^m u; \mathfrak{t}^2)_\infty}$$

$$\mathcal{I}_C^{\mathrm{U}(N)\text{-}[1]} = \oint_{|u|=\epsilon} \frac{du}{2\pi i u} u^{-N} \Xi_1(u) = -\sum_{w \neq 0} \mathrm{Res}\left[\frac{u^{-N}}{u} \Xi_1(u), u \to w\right]$$



$u = 1 \to \mathcal{I}_C^{\mathrm{U}(\infty)\text{-}[1]}$

$u = \mathfrak{t}^{-\bigcirc} \to \mathfrak{t}^{\bigcirc N}$: finite *N* corrections

$$\frac{\mathcal{I}_C^{\mathrm{U}(N)\text{-}[1]}}{\mathcal{I}_C^{\mathrm{U}(\infty)\text{-}[1]}} = \sum_{m_1, m_2 \geq 0} \hat{F}_{m_1,m_2} x_1^{m_1 N} x_2^{m_2 N} \qquad (x_1 = z^{-1}\mathfrak{t},\ x_2 = z\mathfrak{t})$$

$$\hat{F}_{m_1,m_2} = \prod_{a=1}^{m_1} \prod_{b=0}^{\infty} \frac{1}{1 - x_1^{-a} x_2^{b}} \prod_{a=1}^{m_2} \prod_{b=0}^{\infty} \frac{1}{1 - x_2^{-a} x_1^{b}} \prod_{a=1}^{m_1} \prod_{b=1}^{m_2} \frac{1}{1 - x_1^{-a} x_2^{-b}}$$

[Gaiotto,Lee,'21][Hayashi,TN,Okazaki,'22]

Claim: $\hat{F}_{m,0} = I_{m\mathrm{M5}}$ with appropriate identification of fugacities

Convention for M2/M5 indices

M2 index

Bosonic symmetry (conformal + R): $\mathrm{SO}(2,3)\times\mathrm{SO}(8)_r$ ➡ Cartans: $h, j_{12}, r_{12}, r_{34}, r_{56}, r_{78}$

$$\{\mathcal{Q},\mathcal{Q}^\dagger\} = h \pm j_{12} - \frac{\pm' r_{12} \pm'' r_{34} \pm''' r_{56} \pm'''' r_{78}}{2}$$

	h	j_{12}	r_{12}	r_{34}	r_{56}	r_{78}
$\mathcal{Q}$	$\frac{1}{2}$	$\pm\frac{1}{2}$	$\pm'\frac{1}{2}$	$\pm''\frac{1}{2}$	$\pm'''\frac{1}{2}$	$\pm''''\frac{1}{2}$

Choose $(-++++)$

➡ $$\boxed{I_{\mathrm{M2}} = \mathrm{Tr}[(-1)^F e^{-\beta\{\mathcal{Q},\mathcal{Q}^\dagger\}} q^{j_{12}+\frac{r_{12}+r_{34}+r_{56}+r_{78}}{4}} (t^{-1}z)^{r_{12}} (tx)^{r_{34}} (t^{-1}z^{-1})^{r_{56}} (tx^{-1})^{r_{78}}]}$$

M5 index

Bosonic symmetry: $\mathrm{SO}(2,6)\times\mathrm{SO}(5)_R$ ➡ Cartans: $H, J_{12}, J_{34}, J_{56}, R_{12}, R_{34}$

$$\{Q,Q^\dagger\} = H \pm J_{12} \pm' J_{34} \pm'' J_{56} - 2(\pm''' R_{12} \pm'''' R_{34})$$

	H	J_{12}	J_{34}	J_{56}	R_{12}	R_{34}
Q	$\frac{1}{2}$	$\pm\frac{1}{2}$	$\pm'\frac{1}{2}$	$\pm''\frac{1}{2}$	$\pm'''\frac{1}{2}$	$\pm''''\frac{1}{2}$

Choose $(---++)$ ➡ $$\boxed{I_{\mathrm{M5}} = \mathrm{Tr}[(-1)^F e^{-\beta\{Q,Q^\dagger\}} p^{H-\frac{R_{12}+R_{34}}{2}} u^{R_{12}-R_{34}} y_1^{J_{12}} y_2^{J_{34}} y_3^{J_{56}}]} \quad (y_1 y_2 y_3 = 1)$$

To do: express six 3d Cartans in terms of 6d Cartans, by matching

(1/2 BPS algebra of 3d preserved by wrapped M5) = (a 1/2 BPS algebra of 6d)

Parameter identification

1/2 BPS algebra in 3d: $\frac{I^{\mathrm{U}(N)\text{-}[1]}}{I^{\mathrm{U}(\infty)\text{-}[1]}} = 1 + (q^{\frac{1}{4}}t^{-1}z^{-1})^N \hat{F}_{1,0} + \cdots$

$(h, j_{12}, r_{12}, r_{34}, r_{56}, r_{78}) = \left(\frac{N}{2}, 0, 0, 0, N, 0\right)$ (= ground state of M5 wrapped on $S^5_{z_{56}=0} \subset S^7$)

preserves $\left\{\mathcal{Q}^I_\alpha \middle| r_{56} = \frac{1}{2}\right\} \supset \boxed{so(3)_{j_{ij}} \times so(6)_{r_{ab}} \times u(1)_{h,r_{56}}}$ $\quad a, b \neq 5, 6$

A 1/2 BPS algebra in 6d: $\left\{Q^I_\alpha \middle| R_{12} = \frac{1}{2}\right\} \supset \boxed{so(6)_{J_{ij}} \times so(3)_{R_{ab}} \times u(1)_{H,R_{12}}}$ $\quad a, b \neq 1, 2$

$\boxed{} = \boxed{} \Rightarrow$ $\boxed{h = \frac{H}{2} - \frac{3R_{12}}{2},\ j_{12} = R_{34},\ r_{12} = J_{12},\ r_{34} = J_{34},\ r_{56} = -R_{12},\ r_{78} = J_{56}}$

[Arai,Fujiwara,Imamura,Mori,Yokoyama,'20]

Writing I_{M2} in terms of 6d Cartans (H, J_{ij}, R_{ab}), we find

$$\hat{F}_{m,0} = I_{m\mathrm{M5}}(p = q^{\frac{1}{4}}t^{\frac{1}{3}}z^{\frac{1}{3}}, u = q^{-\frac{5}{8}}t^{\frac{1}{2}}z^{\frac{1}{2}}, y_1 = t^{-\frac{2}{3}}z^{\frac{2}{3}}, y_2 = t^{\frac{2}{3}}xz^{-\frac{1}{3}}, y_3 = t^{\frac{2}{3}}x^{-1}z^{-\frac{1}{3}})$$

$$\overset{(C)}{\to} I_{m\mathrm{M5}}(p^{\frac{3}{2}}u = x_1^{-1}, py_1 = x_2; p^{\frac{3}{2}}u^{-1} = py_2 = py_3 = 0) \qquad (x_1 = z^{-1}\mathfrak{t},\ x_2 = z\mathfrak{t})$$

: Checked against knows expressions for $I_{1\mathrm{M5}}$ (free tensor), $I_{\infty\mathrm{M5}}$ (graviton) and also $I_{m\mathrm{M5}}$ (unrefined)

Plan of talk

✓ 1. Superconformal indices

2. Line defect indices

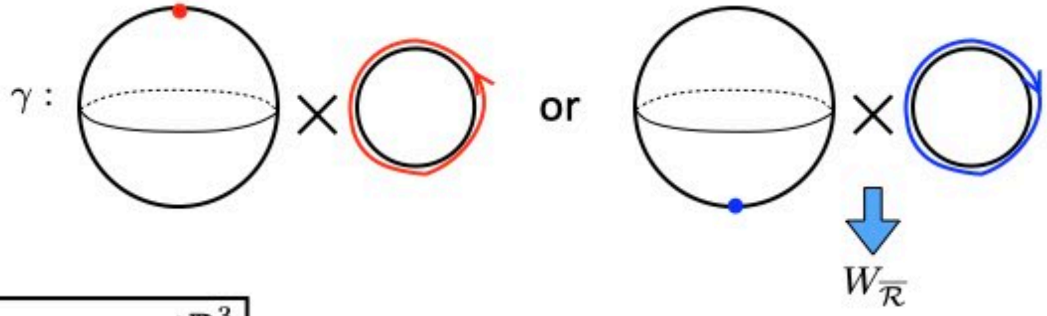
3. Summary and future works

Line defect index

Supersymmetric index can be generalized by adding supersymmetric Wilson loops

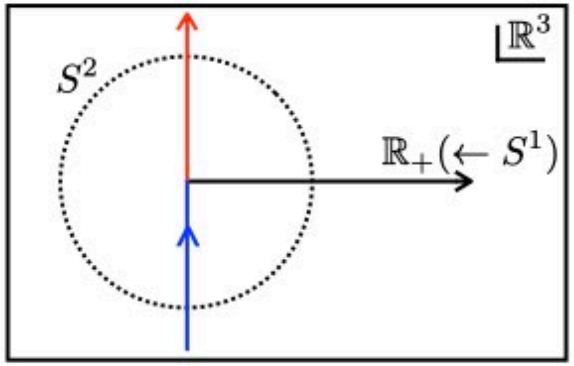
$$I = Z_{S^2\times S^1} \Rightarrow \left\langle \prod_{j=1}^{k} W_{\mathcal{R}_j}(\gamma_j) \right\rangle_{S^2\times S^1}$$

$$W_{\mathcal{R}}(\gamma) = \mathrm{Tr}_{\mathcal{R}}\mathcal{P}\exp\left[\oint_\gamma d\tau\left(i\dot{x}^\mu(\tau)A_\mu + |\dot{x}(\tau)|\sigma\right)\right]$$



Basic choice:

$\langle W_{\mathcal{R}} W_{\overline{\mathcal{R}}}\rangle$ $\Rightarrow$ conformal map



By using SUSY localization formula, we have

$$\left\langle \prod_{j=1}^{k} W_{\overset{(-)}{\mathcal{R}}_j} \right\rangle^{\mathrm{U}(N)\text{-}[1]} = \frac{1}{N!}\sum_{m_i\in\mathbb{Z}}\int \frac{ds_i}{2\pi i s_i}\prod_{i\neq j}\left(1-q^{\frac{|m_i-m_j|}{2}}\frac{s_i}{s_j}\right)\prod_{i,j}\frac{(q^{\frac{1}{2}+\frac{|m_i-m_j|}{2}}t^2\frac{s_i}{s_j};q)_\infty}{(q^{\frac{1}{2}+\frac{|m_i-m_j|}{2}}t^{-2}\frac{s_i}{s_j};q)_\infty}\prod_{i,j}\prod_{\pm}\frac{(q^{\frac{3}{4}+\frac{|m_i-m_j|}{2}}t^{-1}x^{\pm1}\frac{s_i}{s_j};q)_\infty}{(q^{\frac{1}{4}+\frac{|m_i-m_j|}{2}}tx^{\pm1}\frac{s_i}{s_j};q)_\infty}$$

$$\times\prod_i\prod_\pm\frac{(q^{\frac{3}{4}+\frac{|m_i|}{2}}t^{-1}s_i^{\pm1};q)_\infty}{(q^{\frac{1}{4}+\frac{|m_i|}{2}}ts_i^{\pm1};q)_\infty}z^{\sum_i m_i}q^{\frac{1}{4}\sum_i|m_i|}t^{-\sum_i|m_i|}\times\prod_{j=1}^{k}\chi_{\overset{(-)}{\mathcal{R}}_j}(s_1,\cdots,s_N)$$

Coulomb limit = Fermi gas

Consider generating function of line defect indices in different representations

$$\sum_{r,r'\geq 0}^{N} a^r b^{r'} \langle W_{(1^r)} W_{\overline{(1^{r'})}} \rangle^{\mathrm{U}(N)\text{-}[1]} = \frac{1}{N!} \sum_{m_i \in \mathbb{Z}} \int \frac{ds_i}{2\pi i s_i} \cdots \prod_{i=1}^{N} (1+as_i)(1+bs_i^{-1})$$

$$\xrightarrow{(C)} \left\{ \sum_{\substack{\nu_m \geq 0 \\ (\sum_m \nu_m = N)}} \right\} \prod_{m=-\infty}^{\infty} \mathfrak{t}^{|m|\nu_m} z^{m\nu_m} \tilde{\mathcal{I}}_{(1+as)(1+bs^{-1})}(\nu_m)$$

$$\boxed{\tilde{\mathcal{I}}_f(\nu) = \frac{\mathfrak{t}^{-\nu(\nu-1)}}{\nu!} \int d^N\alpha \det[\langle \alpha_i | \hat{\rho}_f | \alpha_j \rangle] \qquad \hat{\rho}_f = f(e^{2\pi i \hat{\alpha}}) \theta_{\mathbb{Z}}(-\hat{p}) \mathfrak{t}^{-2\hat{p}}}$$

$$\Xi_{(1+as)(1+bs^{-1})}(u) = \sum_{N=0}^{\infty} u^N \left\langle \prod_{i=1}^{N} (1+as_i)(1+bs_i^{-1}) \right\rangle_C^{\mathrm{U}(N)\text{-}[1]} = \prod_{m=-\infty}^{\infty} \sum_{\nu=0}^{\infty} (\mathfrak{t}^{|m|} z^m u)^\nu \tilde{\mathcal{I}}_{(1+as)(1+bs^{-1})}(\nu)$$

$$= \Xi_1(u)\Xi_1(abu)$$

$$\Rightarrow \boxed{\langle W_{(1^r)} W_{\overline{(1^r)}} \rangle_C^{\mathrm{U}(N)\text{-}[1]} = \mathcal{I}_C^{\mathrm{U}(r)\text{-}[1]} \mathcal{I}_C^{\mathrm{U}(N-r)\text{-}[1]}}$$: factorized recursion relation!

[Hayashi,TN,Okazaki,'24]

Generalizations

The same strategy (almost) works for the Coulomb limit of other theories, and we find similar recursion relation as in U(N) ADHM theory

[Hayashi,TN,Okazaki, *to appear*]

Higgs limit of U(*N*) ADHM theory

Higgs limit: $q, t^{-1} \to 0$ with $\mathfrak{t} = q^{\frac{1}{4}} t$ fixed.

$$\boxed{\left\langle \prod_{j=1}^{k} W_{\mathcal{R}_j} \right\rangle_H^{\mathrm{U}(N)\text{-}[l]} = \frac{1}{N!} \int \left(\frac{ds}{2\pi i s}\right)^N \frac{\prod_{i\neq j}(1-\frac{s_i}{s_j}) \prod_{i,j}(1-\mathfrak{t}^2 \frac{s_i}{s_j})}{\prod_{i,j}\prod_{\pm}(1-x^{\pm 1}\mathfrak{t}\frac{s_i}{s_j})} \prod_{\alpha=1}^{l} \prod_i \prod_{\pm} \frac{1}{1-\mathfrak{t}(y_\alpha s_i)^{\pm 1}} \prod_{j=1}^{k} \chi_{\mathcal{R}_j}(s_1, \cdots, s_N)}$$

$\mathcal{M}_H = \mathcal{M}_{\mathrm{ADHM}}(\mathrm{SU}(l); N\text{-inst})$ ➡ $\langle \cdots \rangle_H =$ instanton partition function of 5d $\mathrm{SU}(l)$ SYM

➡ Can be calculated by 5d techniques

(Nekrasov formula; topological vertex)

For $\ell = 1$ and anti-symmetric Wilson lines, we find

$$\boxed{\Xi_{(1+as)(1+bs^{-1})}(u) = \sum_{N=0}^{\infty} u^N \Big\langle \underbrace{\prod_{i=1}^{N} (1+as_i)(1+bs_i^{-1})}_{Z_{N\text{-inst}}^{\text{5d U(1)SYM+2hyp}}} \Big\rangle_H^{\mathrm{U}(N)\text{-}[1]} = \frac{\Xi_1(u)\Xi_1(abu)}{\Xi_1(-a\mathfrak{t}u)\Xi_1(-b\mathfrak{t}u)}}$$

[Hayashi,TN,Okazaki, '24]

For example, $\langle W_{\Box} W_{\overline{\Box}} \rangle_H^{\mathrm{U}(N)\text{-}[1]} = \mathcal{I}_H^{\mathrm{U}(N-1)\text{-}[1]} \mathcal{I}^{\mathrm{U}(1)\text{-}[1]} + \mathfrak{t}^2 \mathcal{I}_H^{\mathrm{U}(N-2)\text{-}[1]} (\mathcal{I}_H^{\mathrm{U}(1)\text{-}[1]})^2$

Application: large *N* limit

Extra d.o.f. due to Wilson lines would be captured by $\left\langle \prod_j \mathcal{W}_{\mathcal{R}_j} \right\rangle = \frac{\langle \prod_j W_{\mathcal{R}_j} \rangle}{I}$, whose large *N* limit can be obtanied easily from recursion relations

$$\langle W_{(1^r)} W_{\overline{(1^r)}} \rangle_C^{\mathrm{U}(N)\text{-}[\ell]} = \mathcal{I}_C^{\mathrm{U}(N-r)\text{-}[\ell]} \mathcal{I}_C^{\mathrm{U}(r)\text{-}[\ell]} \Rightarrow \langle \mathcal{W}_{(1^r)} \mathcal{W}_{\overline{(1^r)}} \rangle_C^{\mathrm{U}(\infty)\text{-}[\ell]} = \mathcal{I}_C^{\mathrm{U}(r)\text{-}[\ell]}$$

these results look reasonable

$$\langle \mathcal{W}_{\square} \mathcal{W}_{\overline{\square}} \rangle_H^{\mathrm{U}(\infty)\text{-}[1]} - \langle \mathcal{W}_{\square} \rangle_H^{\mathrm{U}(\infty)\text{-}[1]} \langle \mathcal{W}_{\overline{\square}} \rangle_H^{\mathrm{U}(\infty)\text{-}[1]} = \mathcal{I}_H^{\mathrm{U}(1)\text{-}[1]}$$

???

$\langle \mathcal{W}_{\square} \mathcal{W}_{\overline{\square}} \rangle_{C/H}^{(N=\infty)} \overset{?}{=}$ PE[(fluctuation modes of M2 on $\underset{\mathrm{AdS}_4}{\overset{\mathrm{AdS}_2}{\cap}} \times \underset{Y_7}{\overset{S^1}{\cap}}$)] [Drukker,Plefka,Young,'08][Chen,Wu,'08][Gang,Koh,Lee,'12]

For example, $\langle \mathcal{W}_{\square} \mathcal{W}_{\overline{\square}} \rangle_C^{\mathrm{U}(\infty)\text{-}[1]} = \mathrm{PE}[z^{-1}\mathfrak{t} + z\mathfrak{t}]$: displacement modes in z_{12} and z_{56} directions?

Summary

- Supersymmetric indices of some M2 SCFTs are exactly calculable in two simplification limits: Coulomb/Higgs limit, which still capture interesting features

Fermi gas

5d instanton

- Indices ➡ M2/M5 giant graviton expansion

$$\frac{\mathcal{I}^{(N\mathrm{M2})}_{C/H}}{\mathcal{I}^{(\infty\mathrm{M2})}_{C/H}} = \sum_{m_1=0}^{\infty} x_1^{m_1 N} \mathcal{I}_{m_1\mathrm{M5}}(x_1^{-1}; x_1 x_2) + \cdots \qquad \frac{\mathcal{I}_{m\mathrm{M5}}}{\mathcal{I}_{\infty\mathrm{M5}}} = \sum_{N_1=0}^{\infty} x_1^{m N_1} \mathcal{I}^{(N_1\mathrm{M2})}_{C/H}(x_1^{-1}; x_1 x_2) + \cdots$$

[Hayashi,TN,Okazaki, '24,'25]

- Line defect indices ➡ linear recursion relation $\boxed{\langle W_\lambda W_{\overline{\lambda}} \rangle^{(N)}_{C(/H)} = \sum_{r'=1}^{r} c_{r'}(\mathfrak{t}) \mathcal{I}^{(N-r')}_{C(/H)}}$

This implies $\langle \mathcal{W}_\lambda \mathcal{W}_{\overline{\lambda}} \rangle^{(\infty)}_{C(/H)} = \text{(rational)} = \mathrm{PE}[\text{(finite polynomial)}]$

in some cases

Future Works

- Direct derivation of M5-giant indices $\mathcal{I}_{m\mathrm{M5}}$ (m≥2) from 6d, or gravity side

[Kim,Kim,Kim,Lee,'13] [Eleftheriou,Murthy,Rossello,'23][Deddo,Liu,Pando Zayas,Saskowski,'24]

- 6d/gravity interpretation of full double-sum expansion coefficients $\hat{F}_{m_1,m_2}$

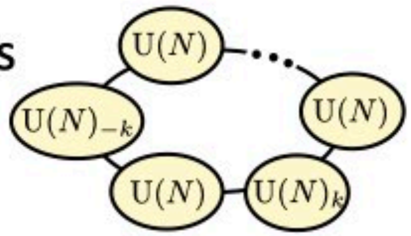
- Gravity interpretation of line defect indices

- Giant graviton expansion of line defect indices = finite *N* & *r* (size of rep.) corrections

[Imamura,Inoue,Sei,Yokoyama;Beccaria,'24~]

- Duality of line defect indices, including vortex lines [Assel,Gomis,'15][Hayashi,TN,Okazaki,'25]

- C/H limits of more general M2 SCFTs



- Full (line defect) indices

Backup slides

Parameter identification (detail)

The 1/2 BPS algebra in 3d: $\left\{\mathcal{Q}^I_\alpha \middle| r_{56} = \frac{1}{2}\right\}$ $\Rightarrow$

$$\boxed{\begin{array}{l} \hat{Z} = h - r_{56} = 0 \\ \{\mathcal{Q}^I_\alpha, (\mathcal{Q}^I_\alpha)^\dagger\} \geq 0 \to \hat{C} = h - \frac{r_{56}}{2} \geq 0 \end{array}}$$

A 1/2 BPS algebra in 6d: $\left\{Q^I_\alpha \middle| R_{12} = \frac{1}{2}\right\}$ $\Rightarrow$

$$\boxed{\begin{array}{l} \check{Z} = H - R_{12} = 0 \\ \{Q^I_\alpha, (Q^I_\alpha)^\dagger\} \geq 0 \to \check{C} = H - 2R_{12} \geq 0 \end{array}}$$

$\boxed{} = \boxed{}$ $\Rightarrow$ $\hat{Z} = \rho\check{Z}, \ \hat{C} = \lambda\check{C} \qquad (\lambda > 0)$

$\Rightarrow$ $h = (2\lambda - \rho)H - (4\lambda - \rho)R_{12}, \quad r_{56} = -2(\rho - \lambda)H + 2(\rho - 2\lambda)R_{12}$

r_{56} and R_{12} are part of larger rotation groups $\Rightarrow$ must be normalized in the same way:

$r_{56} = \pm R_{12}$ $\Rightarrow$ $\rho = \lambda = \frac{1}{2}$

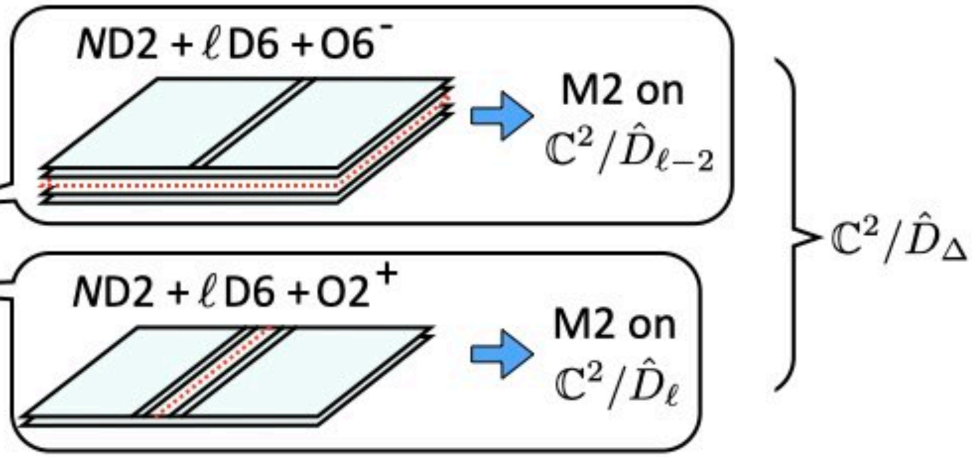
$\Rightarrow$ $\boxed{h = \frac{H}{2} - \frac{3R_{12}}{2}, \ r_{56} = -R_{12}}$

Another example: USp(2N) model

$$\left\langle \prod_{i=1}^{N} f(s_i) \right\rangle^{\mathrm{USp}(2N)+(\mathrm{A/S})\text{-}[2\ell]} = \frac{1}{2^N N!} \sum_{m_1,\cdots,m_N \geq 0} 2^{1-\delta_{m_i,0}} \int \frac{ds_i}{2\pi i s_i}$$
$$\times \prod_{i=1}^{N} \prod_{\pm} (1 - q^{|m_i|} s_i^{\pm 2}) \prod_{i<j} \prod_{\pm} \left(1 - q^{\frac{|m_i - m_j|}{2}} s_i^{\pm 1} s_j^{\mp 1}\right)\left(1 - q^{\frac{|m_i + m_j|}{2}} s_i^{\pm 1} s_j^{\pm 1}\right)$$
$$\times \prod_{i,j}^{N} \prod_{\pm} \frac{(q^{\frac{1}{2}+\frac{|m_i - m_j|}{2}} t^2 s_i^{\mp 1} s_j^{\pm 1}; q)_\infty}{(q^{\frac{1}{2}+\frac{|m_i - m_j|}{2}} t^{-2} s_i^{\pm 1} s_j^{\mp 1}; q)_\infty} \frac{(q^{\frac{1}{2}+\frac{|m_i + m_j|}{2}} t^2 s_i^{\mp 1} s_j^{\mp 1}; q)_\infty}{(q^{\frac{1}{2}+\frac{|m_i + m_j|}{2}} t^{-2} s_i^{\pm 1} s_j^{\pm 1}; q)_\infty}$$
$$\times \mathrm{PE}[(\mathrm{hypers})]$$
$$\times q^{\frac{\Delta}{2} \sum_{i=1}^{N} |m_i|} t^{-2\Delta \sum_{i=1}^{N} |m_i|} \prod_{i=1}^{N} f(s_i)$$

Coulomb indices depend on matter contents only through Δ

$$\Delta = \ell - 2 + 2\epsilon \qquad \epsilon = \begin{cases} 0 & (\text{anti-sym. hyper}) \\ 1 & (\text{sym. hyper}) \end{cases}$$



$$\chi_{\square}^{\mathfrak{usp}(2N)} = \sum_{i=1}^{N} (s_i + s_i^{-1}) \Rightarrow \langle W_\square W_\square \rangle \leftrightarrow f(s) = (1 + a(s + s^{-1}))(1 + b(s + s^{-1}))$$ and take coefficient of *ab*