Notes on Feynman Integrals

Yang Zhang

(Dated: August 17, 2026)

Feynman integrals are crucial for both phenomenological and formal studies of perturbative quantum field theories. In these lecture notes, we review the basic aspects of Feynman integrals. The main references are Henn's lecture notes [1] and the textbooks on Feynman integrals by Smirnov [2] and Weinzierl [3].

I. TENSOR FEYNMAN INTEGRAL REDUCTION

An amplitude in perturbative quantum field theory can be generated by public codes such as QGRAF [4, 5], FeynArts [6], FeynCalc [7], and CalcLoop [8]. In this section, we introduce the reduction of tensor Feynman integrals.

In perturbative QFT, a generic Feynman loop integral usually has the form

$$I^{\mu_{11}\cdots\mu_{1,m_1}\cdots\mu_{L1}\cdots\mu_{L,m_L}} = \int \prod_{i=1}^{L} \frac{d^D l_i}{i\pi^{D/2}} \frac{l_1^{\mu_{11}} \ldots l_1^{\mu_{1,m_1}} \ldots l_L^{\mu_{L1}} \ldots l_L^{\mu_{L,m_L}}}{D_1^{\alpha_1} \ldots D_k^{\alpha_k}} . \tag{1}$$

where the μ_{ij} are free Lorentz indices. This is a *tensor* Feynman integral.

Tensor integrals are not Lorentz invariant. Usually, we convert tensor Feynman integrals into *scalar* Feynman integrals.

$$G[\alpha_1, \ldots \alpha_n] = \int \prod_{i=1}^{L} \frac{d^D l_i}{i\pi^{D/2}} \frac{1}{D_1^{\alpha_1} \ldots D_n^{\alpha_n}} . \tag{2}$$

A classical way to perform this conversion is *Passarino-Veltman* reduction,

$$I^{\mu_{11}\cdots\mu_{1,m_1}\cdots\mu_{L1}\cdots\mu_{L,m_L}} = \sum (\text{tensor in external momenta and metric}) \times (\text{scalar integral}) \tag{3}$$

It is best to demonstrate *Passarino-Veltman* reduction with an example:

Example: Consider the one-loop massless box integral with the inverse propagators $D_1 = l^2$, $D_2 = (l - p_1)^2$, $D_3 = (l - p_1 - p_2)^2$, $D_4 = (l + p_4)^2$. The kinematic conditions are

$$p_1 + \ldots + p_4 = 0, \quad p_i^2 = 0, \quad i = 1, \ldots, 4 \tag{4}$$

$$p_1 \cdot p_2 = s/2, \quad p_1 \cdot p_3 = -(s+t)/2, \quad p_1 \cdot p_4 = -t/2 \tag{5}$$

Consider the integral

$$I^\mu \equiv \int \frac{d^D l}{i\pi^{D/2}} \frac{l^\mu}{D_1^{\alpha_1} D_2^{\alpha_2} D_3^{\alpha_3} D_4^{\alpha_4}} \,. \tag{6}$$

In the PV reduction, Lorentz symmetry ensures that

$$I^\mu = p_1^\mu I_1 + p_2^\mu I_2 + p_3^\mu I_3 \tag{7}$$

Contracting with the p_i gives

$$\begin{pmatrix} 0 & \frac{s}{2} & \frac{-s-t}{2} \\ \frac{s}{2} & 0 & \frac{t}{2} \\ \frac{-s-t}{2} & \frac{t}{2} & 0 \end{pmatrix} \begin{pmatrix} I_1 \\ I_2 \\ I_3 \end{pmatrix} = \begin{pmatrix} I[l \cdot p_1] \\ I[l \cdot p_2] \\ I[l \cdot p_3] \end{pmatrix} \tag{8}$$

where

$$I[\dots] \equiv \int \frac{d^D l}{i\pi^{D/2}} \frac{\dots}{D_1^{\alpha_1} D_2^{\alpha_2} D_3^{\alpha_3} D_4^{\alpha_4}} \,. \tag{9}$$

Here the Gram matrix is invertible; therefore,

$$\begin{pmatrix} I_1 \\ I_2 \\ I_3 \end{pmatrix} = \begin{pmatrix} \frac{t}{s(s+t)} & \frac{1}{s} & -\frac{1}{s+t} \\ \frac{1}{s} & \frac{s+t}{st} & \frac{1}{t} \\ -\frac{1}{s+t} & \frac{1}{t} & \frac{s}{t(s+t)} \end{pmatrix} \begin{pmatrix} I[l \cdot p_1] \\ I[l \cdot p_2] \\ I[l \cdot p_3] \end{pmatrix} \tag{10}$$

Thus, I_1, I_2, and I_3 are all determined as linear combinations of scalar integrals without Lorentz indices. This is simply a numerator replacement,

$$\begin{aligned} l^\mu \mapsto & p_1^\mu \left(\frac{t}{s(s+t)} l \cdot p_1 + \frac{1}{s} l \cdot p_2 - \frac{1}{s+t} l \cdot p_3 \right) + p_2^\mu \left(\frac{1}{s} l \cdot p_1 + \frac{s+t}{st} l \cdot p_2 + \frac{1}{t} l \cdot p_3 \right) + \\ & p_3^\mu \left(-\frac{1}{s+t} l \cdot p_1 + \frac{1}{t} l \cdot p_2 + \frac{s}{t(s+t)} l \cdot p_3 \right) \end{aligned} \tag{11}$$

for any denominator indices. Note that $l \cdot p_i$ are *scalars*, not tensors.

Furthermore, in the one-loop case, scalars such as $l \cdot p_i$ and l^2 are always linear combinations of the denominators. For example, here,

$$l^2 = D_1, \quad l \cdot p_1 = \frac{1}{2}(D_1 - D_2), \quad l \cdot p_2 = \frac{1}{2}(D_2 - D_3 + s), \quad l \cdot p_3 = \frac{1}{2}(D_3 - D_4 - s) \tag{12}$$

These scalars are called *reducible scalar products* (RSPs). At two loops or higher, not all scalar products can be reduced to a linear combination of denominators. Such scalars are called *irreducible scalar products* (ISPs).

Then consider the rank-2 tensor integral

$$I^\mu \equiv \int \frac{d^D l}{i\pi^{D/2}} \frac{l^\mu l^\nu}{D_1 D_2 D_3 D_4} \,. \tag{13}$$

Note that the rank-2 Lorentz tensors are $p_i^\mu p_j^\nu$ and $\eta^{\mu\nu}$. The metric tensor cannot be ignored. It appears that we need to solve for $3\times 3+1=10$ coefficients.

However, note that $I^{\mu\nu}$ is a symmetric tensor, so by the symmetry,

$$\begin{aligned}I^{\mu\nu} \equiv{}& p_1^\mu p_1^\nu I_{11} + (p_1^\mu p_2^\nu + p_2^\mu p_1^\nu) I_{12} + (p_1^\mu p_3^\nu + p_3^\mu p_1^\nu) I_{13} + p_2^\mu p_2^\nu I_{22} \\ &+ (p_2^\mu p_3^\nu + p_3^\mu p_2^\nu) I_{23} + p_3^\mu p_3^\nu I_{33} + \eta^{\mu\nu} I_{00}\end{aligned} \tag{14}$$

There are 7 scalar integrals to be determined. Contracting with 7 symmetric tensors, we find the coefficient matrix

$$\begin{pmatrix} 0 & 0 & 0 & \frac{s^2}{4} & -\frac{1}{2}s(s+t) & \frac{1}{4}(s+t)^2 & 0 \\ 0 & \frac{s^2}{2} & -\frac{1}{2}s(s+t) & 0 & \frac{st}{2} & -\frac{1}{2}t(s+t) & s \\ 0 & -\frac{1}{2}s(s+t) & \frac{1}{2}(s+t)^2 & \frac{st}{2} & -\frac{1}{2}t(s+t) & 0 & -s-t \\ \frac{s^2}{4} & 0 & \frac{st}{2} & 0 & 0 & \frac{t^2}{4} & 0 \\ -\frac{1}{2}s(s+t) & \frac{st}{2} & -\frac{1}{2}t(s+t) & 0 & \frac{t^2}{2} & 0 & t \\ \frac{1}{4}(s+t)^2 & -\frac{1}{2}t(s+t) & 0 & \frac{t^2}{4} & 0 & 0 & 0 \\ 0 & s & -s-t & 0 & t & 0 & d \end{pmatrix} \tag{15}$$

This matrix is non-degenerate, so the solution for I_{11}, I_{12}, I_{13}, I_{22}, I_{23}, I_{33} and I_{00} uniquely exists. Again PV reduction reduces the rank-2 tensor integrals to scalar integrals.

We remark that

1. The Gram matrix method does not work if the external momenta are linearly dependent.

2. The Gram matrix method for PV reduction is clearly inefficient for complicated kinematics and high-rank tensors. Important improvements [8] include (1) using the permutation-group action on the tensor indices and (2) choosing a diagonal basis for the external momenta to simplify the Gram-matrix computations.

II. PARAMETRIZATION OF FEYNMAN INTEGRALS

Here we introduce Feynman, Lee–Pomeransky, and Baikov parametrizations.

A. Feynman parametrization

We consider the Feynman parametrization in the $(+-\ldots-)$ metric. Here we review the Feynman parametrization systematically.

The integral under consideration is $G[\alpha_1, \ldots \alpha_n]$. The basic identity for Feynman parameterization is (with $\alpha_j > 0$),

$$\frac{1}{D_1^{\alpha_1}\ldots D_n^{\alpha_n}} = \frac{\Gamma(|\boldsymbol{\alpha}|)}{\Gamma(\alpha_1)\ldots\Gamma(\alpha_n)} \int_0^1 \prod_{i=1}^n dz_i \delta(1-\sum_j z_j) \frac{z_1^{\alpha_1-1}\ldots z_n^{\alpha_n-1}}{(z_1 D_1 + \ldots + z_n D_n)^{|\boldsymbol{\alpha}|}} \tag{16}$$

Here $|\boldsymbol{\alpha}|$ is the sum of α's. Define $l = (l_1, \ldots, l_L)^T$ as an L-component column vector. The denominator of (16) is a quadratic function of l,

$$z_1 D_1 + \ldots + z_n D_n \equiv l^T A l + 2 b^T l + c\,, \tag{17}$$

where A is an $L \times L$ matrix, b is an L-component column vector, and c is a scalar. The quantities A, b, and c are l-independent. Since A is symmetric, it can be diagonalized as $A = O^T J O$, where J is diagonal. Consider the change of loop momenta $l = O^T \tilde{l} - A^{-1} b$. Then

$$z_1 D_1 + \ldots + z_n D_n = \tilde{l}^T J \tilde{l} + c - b^T A^{-1} b\,. \tag{18}$$

Using Wick's rotation $\tilde{l}_i^0 = i \tilde{l}_{i,E}^0$ and $(\tilde{l}_i)^2 = -(\tilde{l}_{i,E})^2$, the integration contour is along the real axis of $\tilde{l}_{i,E}^0$.

$$\begin{aligned} I[\alpha_1, \ldots \alpha_n] &= \frac{(-1)^{|\boldsymbol{\alpha}|}\Gamma(|\boldsymbol{\alpha}|)}{\Gamma(\alpha_1)\ldots\Gamma(\alpha_n)} \int_0^1 \prod_{i=1}^n dz_i\ \delta(1-\sum_j z_j) z_1^{\alpha_1-1}\ldots z_n^{\alpha_n-1} \\ &\quad \times \int \prod_{i=1}^L \frac{d^D \tilde{l}_{i,E}}{\pi^{D/2}} \frac{1}{(\tilde{l}_E^T J \tilde{l}_E - c + b^T A^{-1} b - i\eta)^{|\boldsymbol{\alpha}|}} \qquad (19) \\ &= \frac{(-1)^{|\boldsymbol{\alpha}|}\Gamma(|\boldsymbol{\alpha}| - \frac{DL}{2})}{\Gamma(\alpha_1)\ldots\Gamma(\alpha_n)} \int_0^1 \prod_{i=1}^n dz_i\ \delta(1-\sum_j z_j) z_1^{\alpha_1-1}\ldots z_n^{\alpha_n-1} \frac{F^{LD/2-|\boldsymbol{\alpha}|}}{U^{(L+1)D/2-|\boldsymbol{\alpha}|}} \qquad (20) \end{aligned}$$

where

$$U = \det A, \quad F = -c \det A + b^T A^{\mathrm{adj}} b - i\eta \det A. \tag{21}$$

Here $A^{\mathrm{adj}} = (\det A) A^{-1}$ is the adjugate matrix of A. The quantities U and F are homogeneous polynomials in z of degrees L and $L+1$, respectively.

Note that U is a positive-semidefinite polynomial, but F may not be positive semidefinite. If there is a kinematic region in which F is positive semidefinite, we call this region *Euclidean.* Note that for nonplanar loop integrals, the Euclidean region may not exist.

For the Euclidean region, both U and F are positive semidefinite. The integrand $F^{LD/2-|\boldsymbol{\alpha}|}/U^{(L+1)D/2-|\boldsymbol{\alpha}|}$ is well defined, and the integral must be *real.* For a region where F is not positive semidefinite, we need to include the infinitesimal η and the integral is *complex.*

- Example: massless bubble integral. This is the simplest loop integral beyond the one-loop tadpole integral. The propagators are

$$D_1 = l_1^2, \quad D_2 = (l_1 - p)^2 \,. \tag{22}$$

The kinematic condition is $p^2 = s$.

$$U = z_1 + z_2 \tag{23}$$

$$F = -s z_1 z_2 \tag{24}$$

Note that $s < 0$ is the Euclidean region and the integral is real. $s > 0$ is the *physical* region and the integral is complex.

For this simple integral, direct Feynman parametrization gives the analytical result

$$G[\alpha_1, \alpha_2] = \frac{(-1)^{\alpha_1+\alpha_2}\Gamma\left(\frac{d}{2}-\alpha_1\right)\Gamma\left(\frac{d}{2}-\alpha_2\right)\Gamma\left(-\frac{d}{2}+\alpha_1+\alpha_2\right)(-s)^{\frac{1}{2}(d-2(\alpha_1+\alpha_2))}}{\Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(d-\alpha_1-\alpha_2)} \,. \tag{25}$$

The apparently simplest integral in this sector is ($D = 4 - 2\epsilon$),

$$\begin{aligned} G[1,1] &= \frac{(-s)^{-\epsilon}\Gamma(1-\epsilon)^2\Gamma(\epsilon)}{\Gamma(2-2\epsilon)} \qquad (26)\\ &= \frac{1}{\epsilon} + (-\log(-s) - \gamma + 2) \\ &+\epsilon\left(\frac{1}{12}\left(6\log^2(-s) + 12\gamma\log(-s) - 24\log(-s) - \pi^2 + 6\gamma^2 - 24\gamma + 48\right)\right)+ \\ &\epsilon^2\left(-\frac{1}{6}\log^3(-s) - \frac{1}{2}\gamma\log^2(-s) + \log^2(-s) + \frac{1}{12}\pi^2\log(-s) - \frac{1}{2}\gamma^2\log(-s) + 2\gamma\log(-s)\right. \\ &\left.-4\log(-s) - \frac{7\zeta(3)}{3} + \frac{\gamma\pi^2}{12} - \frac{\pi^2}{6} - \frac{\gamma^3}{6} + \gamma^2 - 4\gamma + 8\right) + O\left(\epsilon^3\right) \qquad (27) \end{aligned}$$

where γ is the Euler gamma constant. The Riemann Zeta function is defined by

$$\zeta(x) \equiv \sum_{n=1}^{\infty} \frac{1}{n^x} \tag{28}$$

We may simplify the expansion by

$$e^{\epsilon\gamma}(-s)^{\epsilon}G[1,1] = \frac{1}{\epsilon} + 2 + \left(4 - \frac{\pi^2}{12}\right)\epsilon + \left(-\frac{7\zeta(3)}{3} + 8 - \frac{\pi^2}{6}\right)\epsilon^2 + \left(-\frac{14\zeta(3)}{3} + 16 - \frac{\pi^2}{3} - \frac{47\pi^4}{1440}\right)\epsilon^3 + O \tag{29}$$

The overall factor $e^{\epsilon\gamma}$ removes all Euler constants. Since this is a one-scale integral, the factor $(-s)^{\epsilon}$ removes the s dependence. The remainder contains the irrational numbers π and $\zeta(3)$ (and more zeta values in the higher-order expansion).

However, the "simplest" integral is not $G[1,1]$ but the magic *uniformly transcendental* (UT) integral ([9]),

$$e^{\epsilon\gamma}(-s)^{\epsilon} sG[1,2] = -\frac{1}{\epsilon} + \frac{\pi^2\epsilon}{12} + \frac{7\zeta(3)\epsilon^2}{3} + \frac{47\pi^4\epsilon^3}{1440} + O\left(\epsilon^4\right) . \tag{30}$$

Note that the ϵ^n order of this expression has the transcendental degree $n+1$. This expression is much more concise than (VI.1).

So why is $G[1,1]$ complicated? The relation between the two integrals is that

$$G[1,1] = \frac{s}{-1+2\epsilon} G[1,2] . \tag{31}$$

The factor $-1+2\epsilon$ complicates the expression for $G[1,1]$.

The main breakthrough of modern loop computation is the invention of UT integrals and the corresponding canonical differential equation [9].

- Massless planar double box. The propagators are

$$l_1^2, \quad (l_1-k_1)^2, \quad (l_1-k_1-k_2)^2, \quad (l_2+k_1+k_2)^2, \quad (l_2-k_4)^2, \quad l_2^2, \quad (l_1+l_2)^2 \tag{32}$$

The polynomials are

$$\begin{aligned} U =& z_1z_4 + z_2z_4 + z_3z_4 + z_7z_4 + z_1z_5 + z_2z_5 + z_3z_5 + z_1z_6 + z_2z_6 + z_3z_6 + z_1z_7 \\ &+ z_2z_7 + z_3z_7 + z_5z_7 + z_6z_7 \\ F =& -sz_1z_3z_4 - sz_1z_6z_4 - sz_2z_6z_4 - sz_3z_6z_4 - sz_1z_7z_4 - sz_6z_7z_4 - sz_1z_3z_5 \\ &- sz_1z_3z_6 - sz_1z_3z_7 - sz_3z_6z_7 - tz_2z_5z_7 \end{aligned} \tag{33}$$

Note that $s<0$, $t<0$ is the Euclidean region and the integral is real. Usually, we call $s>0$, $t<0$ the "physical" region.

- Massless nonplanar crossed box. The propagators are

$$l_1^2, (l_1-k_1)^2, (l_1-k_1-k_2)^2, (l_2)^2, (l_2-k_4)^2, (l_1+l_2)^2, (l_1+l_2-k_1-k_2-k_4)^2 \tag{34}$$

The polynomials are

$$\begin{aligned} U =& z_1z_4 + z_2z_4 + z_3z_4 + z_6z_4 + z_7z_4 + z_1z_5 + z_2z_5 + z_3z_5 + z_1z_6 + z_2z_6 + \\ & z_3z_6 + z_5z_6 + z_1z_7 + z_2z_7 + z_3z_7 + z_5z_7 \\ F =& -sz_1z_3z_4 - sz_3z_6z_4 + sz_2z_7z_4 - sz_1z_3z_5 - sz_1z_3z_6 - sz_1z_3z_7 \\ &- sz_1z_5z_7 + tz_2z_7z_4 - tz_2z_5z_6 \end{aligned} \tag{35}$$

There is no Euclidean region for this integral. This integral is significantly harder to evaluate than its planar counterpart.

The polynomials U and F can also be determined from graph theory [10].

B. Lee-Pomeransky representation

The Lee–Pomeransky representation [11, 12] is a modern variant of Feynman parametrization:

$$G[\alpha_1, \dots \alpha_n] = \frac{(-1)^{|\boldsymbol{\alpha}|}\Gamma(D/2)}{\Gamma\big((L+1)D/2 - |\boldsymbol{\alpha}|\big)\Gamma(\alpha_1)\dots\Gamma(\alpha_n)} \int_0^\infty \prod_{i=1}^n dz_i \; z_1^{\alpha_1 - 1} \dots z_n^{\alpha_n - 1} G^{-D/2} \tag{36}$$

where $G = F + U$ for $\alpha_n > 0$. Note that F and U have different dimensions, and G is apparently a "wrong" expression. How does this formula work?

A standard trick is to insert a trivial integration into the formula and rescale $z_i = z_i' s$.

$$\begin{aligned}
&\int_0^\infty \prod_{i=1}^n dz_i \int_0^\infty ds \delta(s - \sum_{i=1}^n z_i) \; z_1^{\alpha_1 - 1} \dots z_n^{\alpha_n - 1} G^{-D/2} \\
=&\int_0^\infty \prod_{i=1}^n dz_i' \int_0^\infty ds \delta(1 - \sum_{i=1}^n z_i') \; z_1'^{\alpha_1 - 1} \dots z_n'^{\alpha_n - 1} s^{|\boldsymbol{\alpha}|-1} (s^L U(z') + s^{L+1} F(z'))^{-D/2} \\
=&\int_0^\infty \prod_{i=1}^n dz_i' \int_0^\infty ds \delta(1 - \sum_{i=1}^n z_i') \; z_1'^{\alpha_1 - 1} \dots z_n'^{\alpha_n - 1} s^{|\boldsymbol{\alpha}|-1-DL/2} (U(z') + s F(z'))^{-D/2} \\
=&\frac{\Gamma(|\boldsymbol{\alpha}| - DL/2)\Gamma(-|\boldsymbol{\alpha}| + (D+1)L/2)}{\Gamma(D/2)} \int_0^\infty \prod_{i=1}^n dz_i' \; z_1'^{\alpha_1 - 1} \dots z_n'^{\alpha_n - 1} \delta(1 - \sum_{i=1}^n z_i') \frac{F(z')^{LD/2 - |\boldsymbol{\alpha}|}}{U(z')^{(L+1)D/2 - |\boldsymbol{\alpha}|}}
\end{aligned} \tag{37}$$

Here we used the Beta function definition. It is clear that the result is consistent with the original Feynman parametrization.

When an index α_i is zero, we need to modify (36) as follows:

$$\int_0^\infty \frac{dz_i}{\Gamma(\alpha_i)} z_i^{\alpha_i - 1} \Big(\dots \Big) \to \Big(\dots \Big)\Big|_{z_i \to 0} . \tag{38}$$

When an index α_i is negative, we need to modify (36) as follows:

$$\int_0^\infty \frac{dz_i}{\Gamma(\alpha_i)} z_i^{\alpha_i - 1} \Big(\dots \Big) \to (-1)^{\alpha_i} \frac{d^{-\alpha_i}}{dz_i^{-\alpha_i}} \Big(\dots \Big)\Big|_{z_i \to 0} . \tag{39}$$

The main advantage of the Lee–Pomeransky representation is that the kernel is simply one factor $G^{-D/2}$. This means that the $(D-2)$-dimensional integral can be reformulated as a combination of D-dimensional integrals, because

$$G^{-(D-2)/2} = G^{-D/2} G \, . \tag{40}$$

This kind of integration of $G^{(\cdots)}f(z)$ is a focus of mathematical research. The theories of special functions, D-modules, and twisted cohomology are all related to this kind of integration. As we will see later, the Lee–Pomeransky representation has a special role in IBPs.

For example, consider the massless box diagram

$$D_1 = l_1^2, \quad D_2 = (l_1 - p_1)^2, \quad D_3 = (l_1 - p_1 - p_2)^2, \quad D_4 = (l_1 - p_4)^2, \tag{41}$$

with $p_1 \cdot p_2 = s/2, p_1 \cdot p_4 = t/2, p_1 \cdot p_3 = (-s-t)/2$.

The Lee–Pomeransky polynomial is

$$G = z_1 + z_2 + z_3 + z_4 - s z_1 z_3 - t z_2 z_4, \tag{42}$$

Hence,

$$\begin{aligned} G^{(D-2)}[1,1,1,1] = &-\frac{2(D-6)}{(D-2)}\Big(G^D[2,1,1,1] + G^D[1,2,1,1] + G^D[1,1,2,1] \\ &+ G^D[1,1,1,2]\Big) - \frac{2s}{D-2}G^D[2,1,2,1] - \frac{2t}{D-2}G^D[1,2,1,2]\,. \end{aligned} \tag{43}$$

Note that D is *not* an integer and the above dimension-shift identity is regularized by ϵ.

C. Baikov representation

The Baikov representation is dual to Feynman parametrization. This surprisingly simple representation was only discovered in the 1990s. The idea is that instead of integrating over the loop momenta, we integrate over Lorentz-invariant scalar products.

L is the loop order, and the l_i are the loop momenta. We have E independent external vectors, which we label $p_1, ..., p_E$. We assume that the Feynman integrals have been reduced at the integrand level, and we set $m = LE + L(L+1)/2$, which equals the number of scalar products in the configuration. These $LE + L(L+1)/2$ products are defined by

$$x_{ij} = l_i \cdot p_j, \quad 1 \le i \le L, \quad 1 \le j \le E, \tag{44}$$

$$y_{ij} = l_i \cdot l_j, \quad 1 \le i \le j \le L \tag{45}$$

The Baikov representation (I) [11, 13] reads

$$G[n_1, \ldots, n_m] \,=\, C_E^L\, U^{\frac{E-D+1}{2}} \int_\Omega \mathrm{d}\bar{x}\mathrm{d}\bar{y}\; P^{\frac{D-L-E-1}{2}} \frac{1}{D_1^{n_1} \cdots D_m^{n_m}}\,. \tag{46}$$

Note that the D_i are *linear functions* of the scalar products. Thus, the propagators are linearized. Here, P is the Baikov polynomial, which can be written as a Gram determinant,

$$P = \det G \begin{pmatrix} l_1, & \dots & l_L, & p_1, & \dots & p_E \\ l_1, & \dots & l_L, & p_1, & \dots & p_E \end{pmatrix} . \tag{47}$$

Moreover, U and C_E^L are the Gram determinant and constant factor, respectively:

$$U = \det G \begin{pmatrix} p_1, & \dots & p_E \\ p_1, & \dots & p_E \end{pmatrix}, \quad C_E^L = \frac{\pi^{\frac{L-m}{2}}}{\Gamma(\frac{D-E-L+1}{2}) \dots \Gamma(\frac{D-E}{2})} , \tag{48}$$

The Baikov representation (II) [11, 13] reads

$$G[n_1, \dots, n_m] \;=\; J C_E^L \, U^{\frac{E-D+1}{2}} \int_\Omega \mathrm{d}^n z \; P^{\frac{D-L-E-1}{2}} \frac{1}{z_1^{n_1} \cdots z_m^{n_m}} . \tag{49}$$

where $z_i = D_m$. Here J, a rational number, is the Jacobian from (x, y) to z.

Disclaimer: We do not carefully track the overall powers of i in the Baikov representation.

The Baikov representation is well suited to deriving integral relations (IBP, DE, and dimension-shift identities) and to cut analysis. However, it is usually not convenient to evaluate integrals via this expression.

Example: Baikov representation for the one-loop massless box integral.

$$G[\alpha_1, \dots \alpha_4] = \int \frac{d^D l}{i\pi^{D/2}} \frac{1}{D_1^{\alpha_1} \dots D_4^{\alpha_4}} . \tag{50}$$

We set up a vector basis for the D-dimensional spacetime,

$$p_1, p_2, p_4, \omega_1, \dots, \omega_{D-3} \tag{51}$$

where $p_i \cdot \omega_j = 0$, $\omega_i \cdot \omega_j = -\delta_{ij}$. This is an orthogonal decomposition of the "extra" spacetime. Expanding l in this basis gives

$$l^\mu = c_1 p_1^\mu + c_2 p_2^\mu + c_3 p_4^\mu + f_1 \omega_1^\mu + \dots + f_{D-3} \omega_{D-3}^\mu \tag{52}$$

Then we have

$$d^D l = d^3 c \; d^{D-3} f \;\; \det(p_1, p_2, p_4, \omega_1^\mu, \dots, \omega_{D-3}^\mu) \tag{53}$$

$$= d^3 c \; d^{D-3} f \det G \begin{pmatrix} p_1, & p_2 & p_4 \\ p_1, & p_2 & p_4 \end{pmatrix}^{1/2} \tag{54}$$

$$= dx_1 dx_2 dx_3 \; d^{D-3} f \det G \begin{pmatrix} p_1, & p_2 & p_4 \\ p_1, & p_2 & p_4 \end{pmatrix}^{-1/2} \tag{55}$$

where in the second line we used the relation between the Gram determinant and the determinant of the vectors. In the last line we introduced the scalar products,

$$x_1 = l \cdot p_1, \quad x_1 = l \cdot p_2, \quad x_3 = l \cdot p_4, \tag{56}$$

We dropped the powers of the imaginary unit "i" in this derivation.

We define $\lambda = f_1^2 + \ldots + f_{D-3}^2$. Then $l^2 = F(c_1, c_2, c_3)^2 - \lambda$, where F is some quadratic function of the c_i. By simple linear algebra, we find

$$\lambda = -\frac{\det G \begin{pmatrix} l & p_1, & p_2 & p_4 \\ l & p_1, & p_2 & p_4 \end{pmatrix}}{\det G \begin{pmatrix} p_1, & p_2 & p_4 \\ p_1, & p_2 & p_4 \end{pmatrix}} = -\frac{P}{U}. \tag{57}$$

Therefore, the measure becomes

$$\begin{aligned} \int d^D l &\to A_{D-4} \int_0^\infty d\lambda \frac{1}{\lambda^{1/2}} \lambda^{\frac{D-4}{2}} \int dx_1 dx_2 dx_3 U^{-\frac{1}{2}} \\ &= U^{\frac{4-d}{2}} \frac{\pi^{\frac{D-3}{2}}}{\Gamma(\frac{D-3}{2})} \int_0^\infty d\lambda \int dx_1 dx_2 dx_3 P^{\frac{d-5}{2}} \end{aligned} \tag{58}$$

The final transformation uses $d(l^2) = df(c_1, c_2, c_3)^2 - d\lambda$. Defining $y = l^2$, we obtain

$$\int d^D l \to U^{\frac{4-d}{2}} \frac{\pi^{\frac{D-3}{2}}}{\Gamma(\frac{D-3}{2})} \int dy \int dx_1 dx_2 dx_3 P^{\frac{d-5}{2}} \tag{59}$$

where the integration region is defined by $\lambda > 0$ which is an region with a complicated shape in the x_1, x_2, x_3, y space.

Above, we derived the Baikov representation for the one-loop case in a straightforward way. In practice, we need to find the Baikov polynomial P in terms of either the scalar products or the propagators (i.e., Baikov z_i variables).

Note that

$$G \begin{pmatrix} l & p_1, & p_2 & p_4 \\ l & p_1, & p_2 & p_4 \end{pmatrix} = \begin{pmatrix} y & x_1 & x_2 & x_3 \\ x_1 & 0 & \frac{s}{2} & \frac{1}{2}(-s-t) \\ x_2 & \frac{s}{2} & 0 & \frac{t}{2} \\ x_3 & \frac{1}{2}(-s-t) & \frac{t}{2} & 0 \end{pmatrix} \tag{60}$$

- If we use the Baikov representation (46), then

$$\begin{aligned} P = \frac{1}{4}\Big(&- s^2 t y + s^2 x_1^2 + s^2 x_3^2 + 2s^2 x_1 x_3 - s t^2 y + 2st x_1^2 + 2st x_1 x_2 + 2st x_1 x_3 \\ &-2st x_2 x_3 + t^2 x_1^2 + t^2 x_2^2 + 2t^2 x_1 x_2 \Big). \end{aligned} \tag{61}$$

  where $x_1 = l \cdot p_1$, $x_2 = l \cdot p_2$, $x_3 = l \cdot p_4$ and $y = l \cdot l$.

- If we use the Baikov variables z_i in (49), then

$$P = \frac{1}{16}\bigg(s^2t^2 - 2s^2tz_2 - 2s^2tz_4 + s^2z_2^2 + s^2z_4^2 - 2s^2z_2z_4 - 2st^2z_1 - 2st^2z_3 + 2stz_1z_2$$
$$-4stz_1z_3 + 2stz_2z_3 + 2stz_1z_4 - 4stz_2z_4 + 2stz_3z_4 + t^2z_1^2 + t^2z_3^2 - 2t^2z_1z_3\bigg). \qquad (62)$$

The constant factor $U = -st(s+t)/4$.

D. First applications of the Baikov representation

- Dimension shift identity. From the Baikov representation, we see that a $D+2$-dimensional representation can be directly rewritten as a linear combination of D-dimensional integrals.

$$P^{\frac{(D+2)-L-E-1}{2}} \to P^{\frac{D-L-E-1}{2}}P \qquad (63)$$

  In practice, we usually prefer the $D-2 \to D$ dimension-shift identity from the Lee–Pomeransky representation because the *Lee–Pomeransky polynomial G is usually simpler than the Baikov polynomial P.* The backward shift $D+2 \to D$ identities can then be derived by inverting the $D-2 \to D$ shifts.

- Guessing integral-reduction coefficients. When there is no ISP, the top-sector integral-reduction coefficients can be "guessed" from the residues of the Baikov representation. This was the original motivation for the Baikov representation.

  For example, consider the massless double box $G[1,1,1,1]$ and $G[3,1,1,1]$. In the Baikov representation, it is easy to compute that

$$\mathrm{Res}\bigg(G[1,1,1,1])\bigg) = U^{\frac{4-D}{2}} \oint_{(0,0,0,0)} \mathrm{d}^4z\, P^{\frac{D-5}{2}} \frac{1}{z_1\cdots z_4} = \frac{2^{6-d}(s+t)^2(st)^D(-st(s+t))^{-D/2}}{s^3t^3} \qquad (64)$$

$$\mathrm{Res}\bigg(G[3,1,1,1])\bigg) = U^{\frac{4-D}{2}} \oint_0 dz_1 \frac{1}{z_1^3}\left(\frac{1}{16}t^2(s-z_1)^2\right)^{\frac{D-5}{2}}$$
$$= \frac{2^{5-D}(D-6)(D-5)(s+t)^2(st)^D(-st(s+t))^{-D/2}}{s^5t^3} \qquad (65)$$

  Then we guess

$$G[3,1,1,1] = \frac{(D-6)(D-5)}{2s^2}G[1,1,1,1] + \text{triangles} + \text{bubbles}. \qquad (66)$$

  This guess is consistent with the rigorous IBP computation, which will be introduced later.

- Leading singularity. The great advantage of the Baikov representation is that it keeps the pole structures of propagators in Feynman diagrams and the monomials in (49) make the residue computation transparent. Roughly speaking, if we take the residues of all Baikov variables, we obtain the so-called *leading singularity.*

  For example, consider the $(4-2\epsilon)$-dimensional massless one-loop box $G[1,1,1,1]$:

$$\begin{aligned} &JC_E^L U^{\frac{4-D}{2}} \int_\Omega \mathrm{d}^4 z\, P^{\frac{D-5}{2}} \frac{1}{z_1 \cdots z_4} \\ \to &\oint_{(0,0,0,0)} \mathrm{d}^4 z\, P^{\frac{D-5}{2}} \frac{1}{z_1 \cdots z_4} = 4/(st) \end{aligned} \tag{67}$$

  In the last line, by hand-waving arguments, we set $D \to 4$. Here we do not track factors of π or rational constants. The quantity $1/(st)$ is the leading singularity of the $4-2\epsilon$ massless one-loop box $G[1,1,1,1]$.

  The leading singularity is the most divergent part of $G[1,1,1,1]$ for singular external kinematics. Later, we will see that

$$e^{\epsilon\gamma}(-s)^\epsilon G[1,1,1,1] = \frac{1}{st}\left(\frac{1}{\epsilon^2} - \frac{2\log(t/s)}{\epsilon} - \frac{4\pi^2}{3} + O(\epsilon)\right) \tag{68}$$

  The rational function $1/(st)$, which is divergent if $s \to 0$ or $t \to 0$, matches the leading singularity from the Baikov residue computation.

III. INTEGRATION-BY-PARTS (IBP) REDUCTION

Integration-by-parts (IBP) reduction is a crucial step for the loop amplitude computation. Recall that we defined

$$G[n_1, \ldots, n_k] = \int \frac{d^D l_1}{i\pi^{D/2}} \cdots \frac{d^D l_L}{i\pi^{D/2}} \frac{1}{D_1^{n_1} \ldots D_k^{n_k}} \tag{69}$$

For arbitrary $n_i \in \mathbb{Z}$, $i = 1, \ldots, k$, we call the set of all such $G[n_1, \ldots, n_k]$ an integral family.

Suppose that we have used an integrand-reduction process, such as OPP-reduction or Passarino-Veltman reduction, to reduce the tensor structure to a Feynman integral with external momenta only (without polarization vectors or the auxiliary ω vector). Thus the integral contains only external momenta. Furthermore, define E as the number of linearly independent external vectors. To make the IBP reduction work, we must add some irreducible scalar products (ISPs), so that

$$k = LE + \frac{L(L+1)}{2}\,. \tag{70}$$

The goal of IBP reduction is to linearly reduce integrals in a family to a finite number of "simple integrals." Note that for the one-loop case, after OPP reduction, all n_i are either 0 or 1, so there is not much IBP work to do. However, for the two-loop case, even with the OPP plus the Groebner basis reduction, the output contains a large number of integrals and should be reduced by IBPs. Later we will see that for differential equations of Feynman integrals, there are many Feynman integrals with $n_i > 1$. Usually, these integrals are also reduced by IBPs.

In practice, IBP reduction is usually the most time- and RAM-consuming part of an amplitude computation.

A. Fundamental IBPs

For an integral family, the fundamental IBPs are

$$0 = \int \frac{d^D l_1}{i\pi^{D/2}} \cdots \frac{d^D l_L}{i\pi^{D/2}} \frac{\partial}{\partial l_i^\mu} \frac{v^\mu}{D_1^{n_1} \ldots D_k^{n_k}} \tag{71}$$

Here $i = 1, \ldots, L$, and the vectors v can be chosen from $\{p_1, \ldots p_E, l_1 \ldots l_L\}$. Hence there are $L(L+E)$ fundamental IBPs. Note that each fundamental IBP generates infinitely many IBP relations because the n_i are arbitrary integers.

The derivatives in (71) should be rewritten as a linear combination of the D_i. This involves some simple computations.

B. Sectors

As a linear system, the IBPs have the structure of a large upper block-triangular matrix. To see this, we introduce the concept of sectors of integrals.

For an integral $G[n_1, \ldots n_k]$, we define its sector as $(s_1, \ldots s_k)$. Here

$$s_i = \begin{cases} 1 & n_i > 0 \\ 0 & n_i \leq 0 \end{cases} \tag{72}$$

The concept of sector is equivalent to the topology of Feynman diagrams.

For two *distinct* sectors $S^{(1)} = (s_1^{(1)}, \ldots, s_k^{(1)})$ and $S^{(2)} = (s_1^{(2)}, \ldots, s_k^{(2)})$,

$$s_i^{(1)} \geq s_i^{(2)}, \quad 1 \leq i \leq n, \tag{73}$$

we call $S^{(1)}$ a super-sector of $S^{(2)}$ and $S^{(2)}$ a sub-sector of $S^{(1)}$.

Usually we think that integrals in the super sectors are more "complicated." Hence, if $S^{(2)}$ is a sub-sector of $S^{(1)}$, we define

$$I_1 \succ I_2, \quad \forall I_1 \in S^{(1)}, \quad I_2 \in S^{(2)} \tag{74}$$

However, the super–sub-sector ordering of the sectors is not a total ordering. We need to extend it by hand to a total ordering of all the sectors.

For two integrals in the same sector, we also define an ordering to formally indicate their complexity. We call the unique integral I in a sector with the n_i either zero or one the "corner integral." Then we define the corner integral as the simplest integral in the sector. We then order the other integrals by their distance from the corner integral.

By the procedure above, we have a total ordering of all integrals in one family. The total ordering is not unique. However, we emphasize that, *traditionally*, two crucial features are shared by all total orderings:

1. The super-sub-sector relation is respected by an integral ordering.

2. The corner integral is the lowest integral for a given sector.

We have the total ordering of all integrals in a family, and the IBP relations between integrals in a family. The linear coefficients must be rational functions of Mandelstam variables, masses and the spacetime dimension. As usual, we list the IBP relation coefficients in a matrix, where each row corresponds to a relation and each column to an integral *sorted in the total integral ordering, from higher to lower.*

Therefore, the IBP coefficient matrix is block upper-triangular.

C. IBP reduction and Master integrals

The goal of IBP reduction is to express "complicated" Feynman integrals as linear combinations of "simple" Feynman integrals. Given enough IBP relations, the standard Laporta algorithm [14] applies Gaussian elimination on the IBP coefficient matrix to obtain the IBP-reduction result.

Since the matrix is block upper-triangular, we carry out the Gaussian elimination block by block. Furthermore, we may also carry out the Gaussian elimination in parallel for several sectors without the "super-sub" sector relation. Laporta algorithm is implemented in [15–17]. See [18] for the block triangular form of the IBP relations and [19] for the application of computational algebraic geometry in IBP reduction.

Since the coefficients contain the Mandelstam variables, masses, and the spacetime dimension, Gaussian elimination is computationally demanding.

If an integral cannot be expressed as a linear combination of lower integrals, i.e., the corresponding matrix column does not have a pivot after Gaussian elimination, then we call this integral a "master integral". Note that the master integral definition depends on the ordering choice of the integral ordering.

Smirnov and Petukhov [20] proved that, for a given integral family, the number of master integrals is always finite. Different integral orderings do not change this number. Lee [11] claimed that the number of master integrals in one sector equals the number of

$$\{\text{critical points of } F\} \equiv \{\frac{\partial F}{\partial z_i} = 0 \text{ and } \ F \neq 0\} \tag{75}$$

where F is the Lee-Pomeransky polynomial or the Baikov polynomial on cut. However, this claim has counterexamples.

D. Zero sectors

We treat any scaleless integral in dimensional regularization as zero. This statement is to be understood as an analytic continuation.

For instance, the massless tadpole and Feynman integrals with no propagator scale are all zero in dimensional regularization. Furthermore, if all integrals in one sector are zero, we call this sector a "zero sector."

From partial fractions, we see that if the corner integral in a sector is zero, then all integrals in this sector are zero and thus this sector is a zero sector [21]. It is important to identify zero sectors in an integral family, so that we have the IBP reduction boundary.

A practical way to find zero sectors is Lee's criterion [22]. If, under a transformation,

$$z_i \mapsto (1 + k_i\omega) z_i \tag{76}$$

the Lee–Pomeransky polynomial transforms as

$$G \mapsto (1 + \omega) G \tag{77}$$

or, equivalently,

$$\sum_i k_i z_i \frac{\partial G}{\partial z_i} = G \tag{78}$$

Then, clearly,

$$\int dz_1 \dots dz_n G(z_1, \dots z_n)^{-D/2} = \int dz'_1 \dots dz'_n G(z'_1, \dots z'_n)^{-D/2} \tag{79}$$

$$= (1 - \frac{D}{2}\omega + \sum_{i=1} \omega k_i) \int dz_1 \dots dz_n G(z_1, \dots z_n)^{-D/2} \tag{80}$$

which means the Feynman integral is zero.

IV. SYMMETRIES

Frequently, different Feynman integrals are related by symmetries. These may arise from graph symmetries or the symmetry of Symanzik polynomials (Pak algorithm) [23].

The key idea is that each sector S corresponds to a Lee–Pomeransky polynomial G_S. Let S_1 and S_2 be two sectors with the same height, and denote their Feynman parameters by $x_1, \dots, x_n$, respectively. The sectors S_1 and S_2 are symmetric if the two Lee–Pomeransky polynomials differ by a permutation:

$$G_{S_1}(x_1, \dots, x_n) = G_{S_2}(x_{\sigma(1)}, \dots, x_{\sigma(n)}) \tag{81}$$

with $\sigma \in S_n$. The existence of such a permutation can be found by the Pak algorithm [23]. The permutation σ of the Feynman parameters implies the corresponding permutation of the propagators.

For example, consider the two-loop massless double box diagram with the propagators:

$$\begin{gathered} D_1 = l_1^2, \quad D_2 = (l_1 - p_1)^2 \quad D_3 = (l_1 - p_1)^2 \quad D_4 = (l_1 - k_1 - k_2)^2, \\ D_5 = (l_2 + k_1 + k_2)^2, \quad D_6 = (l_2 - k_4)^2, \quad D_6 = l_2^2 \quad , D_7 = (l_1 + l_2)^2 \end{gathered} \tag{82}$$

as well as two ISPs,

$$D_8 = (l_1 + k_4)^2, \quad D_9 = (l_2 + k_1)^2 \,. \tag{83}$$

First, consider the graph symmetry. This diagram has left–right and up–down symmetries. For the left–right symmetry, we see that

$$k_1 \to k_4, \quad k_2 \to k_3, \quad k_3 \to k_2, \quad k_4 \to k_1, \tag{84}$$

and

$$D_1 \to D_6, \quad D_2 \to D_5, \quad D_3 \to D_4, \quad D_4 \to D_3, \quad D_5 \to D_2 \quad D_6 \to D_1 \quad D_7 \to D_7 \tag{85}$$

as well as the transformation of ISPs. We see that it induces the transformation,

$$l_1 \to l_2, \quad l_2 \to l_1 \tag{86}$$

This implies that

$$\begin{aligned} & G[n_1, n_2, n_3, n_4, n_5, n_6, n_7, n_8, n_9](p_1, p_2, p_3, p_4) \\ &= G[n_6, n_5, n_4, n_3, n_2, n_1, n_7, n_9, n_8](p_4, p_3, p_2, p_1) \end{aligned} \tag{87}$$

However, we know that a scalar Feynman integral only depends on Lorentz invariants. It is clear that $p_4 \cdot p_3 = p_1 \cdot p_2$ and $p_4 \cdot p_2 = p_1 \cdot p_3$. Therefore

$$G[n_1, n_2, n_3, n_4, n_5, n_6, n_7, n_8, n_9] = G[n_6, n_5, n_4, n_3, n_2, n_1, n_7, n_9, n_8] \tag{88}$$

This relation provides symmetry relations in the double-box sector, such as

$$G[1,1,1,1,1,1,1,0,-1] = G[1,1,1,1,1,1,1,-1,0]\,. \tag{89}$$

From the polynomial viewpoint, the sector's Symanzik polynomial, given in (33), has the following symmetry under the Pak algorithm:

$$z_1 \to z_6, \quad z_2 \to z_5, \quad z_3 \to z_4, \quad z_4 \to z_3, \quad z_5 \to z_2, \quad z_6 \to z_1, \quad z_7 \to z_7 \tag{90}$$

as well as

$$z_1 \to z_3, \quad z_2 \to z_2, \quad z_3 \to z_1, \quad z_4 \to z_6, \quad z_5 \to z_5, \quad z_6 \to z_4, \quad z_7 \to z_7 \tag{91}$$

We then obtain the corresponding momentum map.

Symmetry can also relate different sectors. From (88), we see that

$$G[1,1,0,1,1,0,1,0,0] = G[0,1,1,0,1,1,1,0,0] \tag{92}$$

$$G[1,1,1,0,1,0,1,0,0] = G[0,1,0,1,1,1,1,0,0] \tag{93}$$

This means that the sector $(1,1,0,1,1,0,1,0,0)$ is equivalent to $(0,1,1,0,1,1,1,0,0)$.

Sometimes the symmetry induces complicated relations between integrals if the symmetry group is large. For instance, consider the two-loop uniformly massive subset diagram with the propagators

$$D_1 = l_1^2 - m^2, \quad D_2 = l_2^2 - m^2, \quad D_3 = (l_1 + l_2 + p)^2 - m^2, \tag{94}$$

as well as the ISPs,

$$D_4 = (l_1 + p)^2, \quad D_5 = (l_2 + p)^2, \tag{95}$$

with $p^2 = s$. Consider the symmetry,

$$l_1 \to -l_1 - l_2 - p, \quad l_2 \to l_2 \tag{96}$$

This transformation does not change the integral measure. Under this symmetry,

$$\begin{aligned} D_1 \to D_3, \quad D_2 \to D_2, \quad D_3 \to D_1 \\ D_4 \to (l_1 + l_2)^2 = D_1 + D_2 + D_3 - D_4 - D_5 + 3m^2 + s, \quad D_5 \to D_5 \end{aligned} \tag{97}$$

Therefore, we have complicated symmetry relations such as

$$\begin{aligned} G[1,1,1,-1,0] = G[0,1,1,0,0] + G[1,0,1,0,0] + G[1,1,0,0,0] \\ -G[1,1,1,-1,0] - G[1,1,1,0,-1] + (3m^2 - s)G[1,1,1,0,0] \end{aligned} \tag{98}$$

For more complicated examples, after running the Pak algorithm, one can use a Gröbner basis to find the corresponding momentum map [24].

V. DIFFERENTIAL EQUATION

The differential-equation method is a modern approach to evaluating Feynman integrals. One advantage is that, for numerical computation, it provides results for many different kinematic points.

On the analytic side, Henn's canonical differential equation [9] is a milestone of the differential-equation method. It is often possible to get *analytic* results for complicated loop integrals from canonical differential equations.

On the numerical side, Ma's group's AMFlow [25–28] is a mainstream numerical method for evaluating Feynman integrals based on differential equations.

Suppose that the integral family depends on Mandelstam variables s_{ij}'s and mass parameters m_i^2. Collectively, we call them x_i. For simplicity, we assume that all of these x_i have the same unit $[energy]^2$. Let I be a column vector of master integrals. The derivatives of the Feynman integrals in Mandelstam variables and mass variables are again the integrals in this integral family. Therefore, by IBP reduction, we have

$$\frac{\partial}{\partial x_i} I = A_i(x, D) I \,. \tag{99}$$

where D is the dimension of spacetime. $A_i(x, D)$ is a square matrix whose entries are rational functions in x and D.

The differential equation has the following integrability condition,

$$\frac{\partial}{\partial x_j} A_i - \frac{\partial}{\partial x_i} A_j - [A_j, A_i] = 0\,, \tag{100}$$

otherwise, there is no solution to the differential equation. Furthermore, dimensional analysis provides the Euler relation,

$$\sum_i x_i A_i = \mathrm{diag}\{[I_1]/2, \ldots, [I_k]/2\} \tag{101}$$

where $[I_j]$ is the energy dimension of I_j. The conditions (100) and (101) are very useful for checking the correctness of differential equations.

If we have a different integral basis,

$$\tilde{I} = TI \tag{102}$$

Then in the $\tilde{I}$ basis, the differential equation reads,

$$\frac{\partial}{\partial x_i} \tilde{I} = \tilde{A}_i \tilde{I}\,. \tag{103}$$

where A_i transforms as a connection,

$$\tilde{A}_i = T A_i T^{-1} + \partial_i T T^{-1} \tag{104}$$

Note that this is not a homogeneous transformation, as is also the case for a connection in Riemannian geometry.

A. Deriving the differential equation

The derivative of Feynman integrals in Mandelstam variables is not straightforward, since the original Feynman integrals do not have the explicit dependence in Mandelstam variables. We need to build differential operators in the external momenta that effectively serve as derivatives with respect to Mandelstam variables.

Take the four-point massless kinematics as an example. We have $p_1^2 = p_2^2 = p_4^2 = 0$, $p_1 \cdot p_2 = s/2$, $p_1 \cdot p_4 = t/2$ and $p_2 \cdot p_4 = -(s+t)/2$. We make an ansatz for the ∂/∂_t operator,

$$\mathcal{O}_t = (c_{11} p_1^\mu + c_{12} p_2^\mu + c_{14} p_4^\mu) \frac{\partial}{\partial p_1^\mu} + (c_{21} p_1^\mu + c_{22} p_2^\mu + c_{24} p_4^\mu) \frac{\partial}{\partial p_2^\mu} + (c_{41} p_1^\mu + c_{42} p_2^\mu + c_{44} p_4^\mu) \frac{\partial}{\partial p_4^\mu} \tag{105}$$

We require that

$$\mathcal{O}_t(p_1^2) = \mathcal{O}_t(p_2^2) = \mathcal{O}_t(p_4^2) = 0, \quad \mathcal{O}_t(p_1 \cdot p_2) = 0, \quad \mathcal{O}_t(p_1 \cdot p_4) = \frac{1}{2}, \quad \mathcal{O}_t(p_2 \cdot p_4) = -\frac{1}{2} \tag{106}$$

The solution for the c's is not unique. However, the different constructions are physically equivalent because the integrals themselves are Lorentz invariant. One simple construction is

$$\partial/\partial t = \mathcal{O}_t = \left(\frac{1}{2t}p_1^\mu + \frac{1}{2(s+t)}p_2^\mu + \frac{s+2t}{2t(s+t)}p_4^\mu\right)\frac{\partial}{\partial p_4^\mu} \tag{107}$$

Note that there is a derivative only with respect to the fourth momentum. Similarly,

$$\partial/\partial s = \mathcal{O}_s = \left(\frac{1}{2s}p_1^\mu + \frac{2s+t}{2s(s+t)}p_2^\mu + \frac{1}{2(s+t)}p_4^\mu\right)\frac{\partial}{\partial p_2^\mu} \tag{108}$$

Acting on the propagators produces an increase in propagator power and a new numerator factor. We thus see explicitly that the derivatives are still in the original integral family.

Another way is to directly consider the non-vector form of the Feynman integrals. For example, it is straightforward to get the derivatives in the Baikov representation [29].

$$\partial_x G[n_1,\dots n_k] = \partial_x\left(U^{\frac{E-D+1}{2}}\int dz_1\dots dz_k\frac{P^{\frac{D-L-E-1}{2}}}{z_1^{n_1}\dots z_k^{n_k}}\right) \tag{109}$$

where x is a Mandelstam variable or an external mass squared. The only term that looks unusual is

$$\partial_x P^{\frac{D-L-E-1}{2}} \tag{110}$$

for which the degree appears to change. However, we proved in ref. [29] that

$$\partial_x P = \left(\sum_{i=1}^{k} a_k \frac{\partial}{\partial z_i}P\right) + bP \tag{111}$$

for any Feynman diagram. This identity is a statement of ideal membership. After using this relation and IBPs, we see that

$$\begin{aligned}\partial_x\left(\int dz_1\dots dz_k\frac{P^{\frac{D-L-E-1}{2}}}{z_1^{n_1}\dots z_k^{n_k}}\right) &= \int dz_1\dots dz_k\left(\sum_{i=1}^{k}\frac{a_k}{z_1^{n_1}\dots z_k^{n_k}}\frac{\partial}{\partial z_i}P^{\frac{D-L-E-1}{2}}\right) + b\frac{P^{\frac{D-L-E-1}{2}}}{z_1^{n_1}\dots z_k^{n_k}}\\ &= \int dz_1\dots dz_k P^{\frac{D-L-E-1}{2}}\left(\frac{b}{z_1^{n_1}\dots z_k^{n_k}} - \sum_{i=1}^{k}\frac{\partial}{\partial z_i}\frac{a_k}{z_1^{n_1}\dots z_k^{n_k}}\right)\end{aligned} \tag{112}$$

However, although this method does not have a differential-operator construction step, it generally generates more complicated expressions. Currently, its usage is limited to some special examples.

Note that it is easy to obtain derivatives with respect to internal mass parameters.

We consider the massless box example with

$$D_1 = l^2, \quad D_2 = (l-p_1)^2, \quad D_3 = (l-p_1-p_2)^2, \quad D_4 = (l+p_4)^2\,. \tag{113}$$

We select the master integrals

$$I = \{G[1,1,1,1], G[1,0,1,0], G[0,1,0,1]\} \tag{114}$$

It is clear that

$$\partial_t I_1 = \frac{G[0,1,1,2]}{2(s+t)} + \frac{sG[1,0,1,2]}{2t(s+t)} + \frac{G[1,1,0,2]}{2(s+t)} - \frac{(s+2t)G[1,1,1,1]}{2t(s+t)} - \frac{sG[1,1,1,2]}{2(s+t)} \tag{115}$$

$$= \frac{-6s+Ds-2t}{2t(s+t)}G[1,1,1,1] - \frac{2(D-3)}{st(s+t)}G[1,0,1,0] + \frac{2(D-3)}{t^2(s+t)}G[0,1,0,1] \tag{116}$$

where in the second step we used the IBP reduction.

Finally we get the differential equation matrices,

$$A_s = \begin{pmatrix} -\frac{-Dt+2s+6t}{2s(s+t)} & \frac{2(D-3)}{s^2(s+t)} & -\frac{2(D-3)}{st(s+t)} \\ 0 & \frac{D-4}{2s} & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{117}$$

and similarly,

$$A_t = \begin{pmatrix} \frac{Ds-6s-2t}{2t(s+t)} & -\frac{2(D-3)}{st(s+t)} & \frac{2(D-3)}{t^2(s+t)} \\ 0 & 0 & 0 \\ 0 & 0 & \frac{D-4}{2t} \end{pmatrix} \tag{118}$$

It is easy to check the integrability condition

$$\partial_t A_s - \partial_s A_t - [A_t, A_s] = 0 \tag{119}$$

and the Euler condition,

$$sA_s + tA_t = \begin{pmatrix} \frac{D-8}{2} & 0 & 0 \\ 0 & \frac{D-4}{2} & 0 \\ 0 & 0 & \frac{D-4}{2} \end{pmatrix} \tag{120}$$

We have the following observations:

- The differential equation has poles in s, t and $s+t$. However, we know that the bubble diagrams depend on s and t respectively. The box function would have singularities if $s \to 0$ or $t \to 0$, but should not have singularity if $s+t \to 0$. In the future, we will see that the apparent $s+t$ pole does not enter the solutions (Feynman integrals).

- There are double poles. Usually, a double pole in a differential equation is a bad sign, which means the solution may have intrinsic singular points. A Feynman integral should not have intrinsic singularities at kinematic points. However, after a basis change this double pole disappears.

VI. UT INTEGRALS AND POLYLOGARITHMS

In this lecture, we study a milestone method for computing Feynman integrals: canonical differential equations for Feynman integrals with uniformly transcendental (UT) weights [9].

The outline of this method is as follows:

- The ϵ^n order of UT Feynman integrals is obtained by integrating the ϵ^{n-1} order of these integrals, and the Feynman integrals are then expressed analytically as *iterated integrals*.
- The mathematical properties of iterated integrals were studied long ago by the mathematician Kuo-Tsai Chen.
- Specifically, many beautiful properties of iterated integrals can be captured by the so-called *symbol* [30], which was introduced in the context of $\mathcal{N} = 4$ super-Yang-Mills theory.
- If the iterated integrals have certain simple integration kernels, the result is given by *polylogarithmic* functions.

In practice, order by order in ϵ, a Feynman integral can be expressed in terms of *polylogarithmic*, *elliptic*, *Calabi–Yau*, and *hyperelliptic* functions.

These lectures provide an introduction to UT integrals, canonical differential equations, symbols, and polylogarithms. The main references are [1, 31].

We begin with the simplest example: the massless bubble integral.

Example VI.1. *This is one of the simplest loop integrals. The propagators are*

$$D_1 = l_1^2, \quad D_2 = (l_1 - p)^2 \,. \tag{121}$$

with $p^2 = s$*. From the Symanzik polynomials,* $U = z_1 + z_2$*,* $F = -s z_1 z_2$*,* $s < 0$ *is the Euclidean region, and* $s > 0$ *is the physical region. By a direct computation,*

$$G[\alpha_1, \alpha_2] = \frac{(-1)^{\alpha_1+\alpha_2}\Gamma\left(\frac{d}{2} - \alpha_1\right)\Gamma\left(\frac{d}{2} - \alpha_2\right)\Gamma\left(-\frac{d}{2} + \alpha_1 + \alpha_2\right)(-s)^{\frac{1}{2}(d-2(\alpha_1+\alpha_2))}}{\Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(d - \alpha_1 - \alpha_2)} \,. \tag{122}$$

The apparently simplest integral in this sector is ($d = 4 - 2\epsilon$)*,*

$$\begin{aligned} I_{bub} \equiv e^{\epsilon\gamma}(-s)^{\epsilon} G[1,1] &= \frac{1}{\epsilon} + 2 + \left(4 - \frac{\pi^2}{12}\right)\epsilon + \left(-\frac{7\zeta(3)}{3} + 8 - \frac{\pi^2}{6}\right)\epsilon^2 \\ &+ \left(-\frac{14\zeta(3)}{3} + 16 - \frac{\pi^2}{3} - \frac{47\pi^4}{1440}\right)\epsilon^3 + O\left(\epsilon^4\right) \end{aligned} \tag{123}$$

where γ is the Euler gamma constant. The Riemann Zeta function is defined by

$$\zeta(z) \equiv \sum_{n=1}^{\infty} \frac{1}{n^z} \tag{124}$$

Since this is a one-scale integral, the factor $(-s)^\epsilon$ removes the s dependence. The factor $e^{\epsilon\gamma}$ removes Euler constants.

However, the "simplest" integral is not $G[1,1]$ but the magic uniformly transcendental (UT) integral ([9]),

$$I'_{bub} \equiv e^{\epsilon\gamma}(-s)^\epsilon s G[1,2] = -\frac{1}{\epsilon} + \frac{\pi^2\epsilon}{12} + \frac{7\zeta(3)\epsilon^2}{3} + \frac{47\pi^4\epsilon^3}{1440} + O\left(\epsilon^4\right) . \tag{125}$$

This expression is much more concise than (VI.1)*. If we define π to have transcendental degree one and ζ_n to have transcendental degree n, then the ϵ^n order of this expression has transcendental degree $n+1$.*

So why is I_{bub} complicated? The IBP relation between the two integrals is that

$$I_{bub} = \frac{1}{-1+2\epsilon} I'_{bub} \, . \tag{126}$$

The factor $1/(-1+2\epsilon)$ complicates the ϵ expansion. We say that I'_{bub} is an integral with uniform transcendental weights.

This example suggests that a UT integral has a simpler analytic expression, whereas a non-UT integral in the same sector tends to have a more complicated expression. For analytic computations, we therefore seek UT integrals.

The formal definition of UT integrals is given by Henn [9]: First, the transcendental degree $\mathcal{T}$ is defined by

$$\begin{gathered} \mathcal{T}(\text{rational number}) = 0, \quad \mathcal{T}(\text{rational function}) = 0 \\ \mathcal{T}(\pi) = 1, \quad \mathcal{T}(\zeta(n)) = n, \quad \mathcal{T}(\log x) = 1, \quad \mathcal{T}(\text{Li}_n(x)) = n \\ \mathcal{T}(H(a_1, \ldots a_n; x)) = n, \quad \mathcal{T}(G(a_1, \ldots a_n; x)) = n, \end{gathered} \tag{127}$$

where $\text{Li}_n(\ldots)$, $H(\{a_1, \ldots a_n\}, x)$ and $G(\{a_1, \ldots a_n\}, x)$ are the weight-n classical, harmonic, and Goncharov polylogarithmic functions, respectively:

- Classical polylogarithms

$$\text{Li}_1(z) \equiv -\log(1-z), \quad \text{Li}_n(z) \equiv \int_0^z \frac{dt}{t} \text{Li}_{n-1}(t) \tag{128}$$

Note that Li_n has the branch cut from 1 to $+\infty$. When $z < 1$, $\mathrm{Li}_n(z)$ is real. This integral is convergent since $\mathrm{Li}_n(z) = 0$, $\forall n$. The pole $1/t$ does not provide a monodromy group around $z = 0$ again because $\mathrm{Li}_n(0) = 0$.

$$\frac{\partial}{\partial z}\mathrm{Li}_n(z) = \frac{1}{z}\mathrm{Li}_{n-1}(z) \tag{129}$$

- Harmonic polylogarithms (HPLs). We first define three rational functions,

$$f_{-1}(z) = \frac{1}{z+1}, \quad f_0(z) = \frac{1}{z}, \quad f_1(z) = \frac{1}{1-z} \tag{130}$$

Then define,

$$\begin{aligned} H(-1;z) &\equiv \int_0^z dt f_{-1}(t) = \log(1+z), \\ H(0;z) &\equiv \log(z), \\ H(1;z) &\equiv \int_0^z dt f_1(t) = -\log(1-z), \end{aligned} \tag{131}$$

The definition of $H(0,z)$ may appear unusual; however, all of these functions satisfy $\partial_z H(a;z) = f_a(z)$, $a = -1, 0, 1$.

We then define recursively

$$H(a_1, \ldots, a_n; z) \equiv \int_0^z dt f_{a_1}(t) H(a_2, \ldots, a_n; t) \tag{132}$$

with the special case when $a_1 = \ldots = a_n = 0$

$$H(\vec{0}_n; z) \equiv \frac{1}{n!}\log^n(z) \tag{133}$$

The vector $(a_1, \ldots, a_n)$ is called the *weight vector*, and $H(a_1, \ldots, a_n; z)$ has weight n. These functions satisfy

$$\partial_z H(a_1, \ldots, a_n; z) = f_{a_1}(z) H(a_2 \ldots, a_n; z) \tag{134}$$

It is easy to check that

$$H(\vec{0}_{n-1}, 1; z) = \mathrm{Li}_n(z) \tag{135}$$

Except for the case when $a_1 = \ldots = a_n = 0$,

$$H(a_1, \ldots, a_n; 0) = 0 \tag{136}$$

All HPLs are real in the range $z \in (0, 1)$. Depending on the weight vector, an HPL may have the branch cuts $(-\infty, 0)$, $(-\infty, -1)$, $(1, \infty)$ or a union of these.

HPLs are commonly used functions in the field of scattering amplitudes. With the weight $n \leq 3$, all HPLs can be converted to classical polylogarithms. But when $n > 3$, this property does not hold. Mathematica cannot handle generic HPLs; however, these functions can be easily handled by the Mathematica package HPL [32].

- Goncharov polylogarithms (GPLs). These are more complicated functions than HPLs and are also common in scattering amplitudes, especially for complicated kinematics. Like HPLs,

$$G(a_1, \ldots, a_n; z) \equiv \int_0^z dt \frac{1}{t - a_1} G(a_2, \ldots, a_n; t) \tag{137}$$

with the exception

$$G(\vec{0}_n; z) = \frac{1}{n!} \log^n(z) \tag{138}$$

Unlike for HPLs, here the weight vector $(a_1, \ldots, a_n)$ can take arbitrary values. If $\vec{a}_n \neq \vec{0}$, then by simple calculus,

$$G(\lambda \vec{a}_n; \lambda z) = G(\vec{a}_n; z) \tag{139}$$

The integration of rational functions in terms of GPLs can be handled with the Maple package HypInt [33], while the numerical evaluation of GPLs can be performed with the C package GiNaC [34]. It is convenient to use the package Polylogtools [35] to manipulate the GPLs.

Note that as in any graded algebra, the element 0 can be associated with arbitrary transcendental weight.

Second, a function f with uniform transcendental weight n is *pure* if

$$\mathcal{T}(f) = n, \quad \mathcal{T}(\partial_x f) = n - 1 \tag{140}$$

for any kinematic variable x. For example, $\log^2(x)$ is UT and pure, while $x \log(1 - x)$ is UT but not pure. Nowadays, in the literature, by the abuse of notation, “UT” usually means UT and pure.

Third, a UT Feynman integral has the ϵ expansion:

$$I = \epsilon^k \sum_{i=0}^{\infty} I^{(n)} \epsilon^n \tag{141}$$

with

$$\mathcal{T}(I^{(n)}) = n \tag{142}$$

Again, usually we require that $I^{(n)}$ is pure also. Here k is an integer to make the series formally from the order ϵ^0.

We note that the transcendental weight of Euler's gamma constant γ is not defined. Multiplying an L-loop Feynman integral by $e^{\epsilon L\gamma}$ cancels γ out at all orders of ϵ.

VII. CANONICAL DIFFERENTIAL EQUATION

For an integral basis $\vec{J}$ of an integral family, we know that $\vec{J}$ satisfies a first-order differential equation:

$$\partial_i \vec{J} = M_i(x,\epsilon)\vec{J}, \tag{143}$$

where M_i is an $m \times m$ square matrix. Recall that under a basis transformation $\vec{J} \to T\vec{J}$, the DE matrix transforms as a connection,

$$M_i \to T M_i T^{-1} + (\partial_i T)T^{-1}\,. \tag{144}$$

(143) is called a *canonical differential equation* if M_i is proportional to ϵ.

$$M_i(x,\epsilon) = \epsilon M_i(x) \tag{145}$$

A UT (and pure) integral basis

$$\vec{I} = \epsilon^k \sum_{n=0}^{\infty} \vec{I}^{(n)} \epsilon^n, \tag{146}$$

such that $\mathcal{T}(\vec{I}^{(n)}) = n$ and $\mathcal{T}(\partial \vec{I}^{(n)}) = n-1$, must be associated with a canonical DE.

To see this, assume that $\partial_i \vec{I} = A_i(x,\epsilon)\vec{I}$ and

$$A_i(x,\epsilon) = \sum_{j=-\infty}^{0} \epsilon^j \bar{A}_i^{(j)}(x) + \epsilon A_i^{(1)}(x) + \sum_{j=2}^{\infty} \epsilon^j \bar{A}_i^{(j)}(x) \tag{147}$$

The differential equation for an integral basis in the sense of Laporta can contain only rational functions of the kinematic variables. To get a UT basis, we further request that the transformation from the Laporta basis to a UT basis can only have algebraic functions in the kinematic variables.

Compare the ϵ^n order of the differential equation. On the left-hand side,

$$\mathcal{T}(\partial_i \vec{I}^{(n)}) = n-1\,. \tag{148}$$

On the right-hand side, the only weight-$(n-1)$ term is $A_i^{(1)}(x)\vec{I}^{(n-1)}$. Therefore,

$$\partial_i \vec{I}^{(n)} = A_i^{(1)}(x)\vec{I}^{(n-1)} \,. \tag{149}$$

Summing over $n = 0, \ldots, \infty$ and renaming $A_i^{(1)}(x)$ as $A_i(x)$, we find

$$\partial_i \vec{I} = \epsilon A_i(x)\vec{I} \,. \tag{150}$$

For a canonical DE, the proportionality in ϵ provides significant advantages because this differential equation can be solved perturbatively in ϵ.

- The integrability condition for the differential equation splits as

$$\partial_j A_i = \partial_i A_j, \quad [A_i, A_j] = 0 \tag{151}$$

  This means that the A_i are total derivatives:

$$A_i = \partial_i \tilde{A} \tag{152}$$

  Thus, the canonical DE can be written in exterior differential form:

$$d\vec{I} = \epsilon(d\tilde{A}(x))\vec{I} \,. \tag{153}$$

  where $d\tilde{A}(x)$ is an $m \times m$ matrix whose entries are one-forms.

- $\tilde{A}(x)$ should have a further decomposition

$$\tilde{A}(x) = \sum_{l=1}^{N} a_l \log(W_l) \tag{154}$$

  where each a_l is a constant $m \times m$ matrix, and the W_l are originally required to be rational or algebraic functions of the kinematic variables. Each W_l is called a *symbol letter* or simply a *letter*. The set of all W_l is called the *alphabet* . As the name suggests, symbol letters are the building blocks of Feynman integrals, as we will see.

  For more general cases, the letters can be elliptic [36], hyperelliptic [37] or Calabi-Yau functions.[38]. See [39] the systematic treatment for generalized canonical differential equation.

- With the canonical DE, order by order in ϵ, we obtain the beautiful structure,

$$\begin{aligned}
\partial_i \vec{I}^{(0)} &= 0 \\
\partial_i \vec{I}^{(1)} &= A_i(x)\vec{I}^{(0)} \\
\partial_i \vec{I}^{(2)} &= A_i(x)\vec{I}^{(1)} \\
\partial_i \vec{I}^{(3)} &= A_i(x)\vec{I}^{(2)} \\
&\dots
\end{aligned} \tag{155}$$

  Hence, $I^{(n)}$ is obtained simply by integrating $A_i(x)I^{(n-1)}$. From the structure of $\tilde{A}$, A_i should be "simple," so the iterated integration can likely be performed analytically, and the UT Feynman integral can be calculated analytically to an arbitrary order of ϵ.

  In particular, we see that the leading order $I^{(0)}$ is a constant vector. In practice, its entries are usually rational numbers.

- There are advantages to choosing a UT basis for the IBP reduction. Let the I_i form a UT basis and let J be an integral in the family:

$$J = \sum_i c_i(\bar{x}, \epsilon) I_i \tag{156}$$

  If J is also a UT integral with the matched degree of transcendental weights, then the $c_i(\bar{x}, \epsilon)$ are *rational numbers*. This property is important since we can use it to "optimize" a UT basis. We perform an IBP reduction of a large list of Feynman integrals to the UT basis; if the c_i are all rational numbers, then we have found new UT integrals. We can then replace some of the I_i's by simpler UT integrals to get a simpler basis.

  For generic J, the reduction coefficients also have useful features [40]. In particular, the c_i corresponding to a UT basis can be dramatically simplified with a multivariate partial fraction.

However, a potential concern remains: if we know from their expressions that Feynman integrals are UT, then we do not need to solve the canonical DE. The real power of the canonical DE is that we can predict (guess) a UT integral basis, set up a canonical DE and then solve it. This is the content of the next section.

VIII. TO DETERMINE A UT BASIS, I

There are many ways to determine a UT basis. Roughly speaking, the ideas come from two directions: (1) hints from formal theory studies like $\mathcal{N}=4$ Super-Yang-Mills theory and modern field theory methods such as generalized unitarity; and (2) the mathematical theory of differential equations. In practice, they are both useful.

A. Leading singularity analysis

For years in the study of the planar $\mathcal{N}=4$ Super-Yang-Mills theory, people preferred "nice" Feynman integrals with *constant leading singularity.* The amplitudes in this theory can usually be expressed as

$$\begin{aligned}\mathcal{A}=\sum&(\text{color factor})\times(\text{Parke–Taylor factor})\\ \times&(\text{integral with constant leading singularity})\end{aligned} \tag{157}$$

After the invention of UT integrals and canonical differential equations, it was discovered that integrals with constant leading singularity are likely to be UT integrals.

Leading singularity can be calculated from $4D$ generalized unitarity analysis, i.e., by setting the propagators on shell and computing the residue [41, 42]. This computation is simple and provides valuable information for the search for UT integrals.

Example VIII.1. *Consider the one-loop massless box diagram with*

$$D_1=l_1^2,\quad D_2=(l_1-p_1)^2,\quad D_3=(l_1-p_1-p_2)^2,\quad D_4=(l_1+p_4)^2 \tag{158}$$

with $p_i^2=0$, $i=1,2,3,4$. $(p_1+p_2)^2=s$, $(p_1+p_4)^2=t$ *and* $(p_2+p_4)^2=-s-t$. *The master integrals in the Laporta sense are*

$$G[1,1,1,1],\quad G[1,0,1,0],\quad G[0,1,0,1] \tag{159}$$

The differential equation matrices are,

$$A'_s=\begin{pmatrix}-\frac{s+t\epsilon+t}{s(s+t)} & -\frac{2(2\epsilon-1)}{s^2(s+t)} & \frac{2(2\epsilon-1)}{st(s+t)}\\ 0 & -\frac{\epsilon}{s} & 0\\ 0 & 0 & 0\end{pmatrix},\quad A'_t=\begin{pmatrix}-\frac{s\epsilon+s+t}{t(s+t)} & \frac{2(2\epsilon-1)}{st(s+t)} & -\frac{2(2\epsilon-1)}{t^2(s+t)}\\ 0 & 0 & 0\\ 0 & 0 & -\frac{\epsilon}{t}\end{pmatrix} \tag{160}$$

- *Sector* $(1,1,1,1)$. *We first check the leading singularity of* $G[1,1,1,1]$ *by the generalized unitarity cut. The cut equation*

$$D_1 = D_2 = D_3 = D_4 = 0 \tag{161}$$

has two solutions at $l_1 = l^*$ *and* $l_1 = \bar{l}^*$. *It is useful to apply spinor-helicity formalism to express the two solutions and compute the residue.*

$$l_1 = a_1 p_1 + a_2 p_2 + a_3 \frac{\langle 43 \rangle}{\langle 13 \rangle} \lambda_1 \tilde{\lambda}_4 + a_4 \frac{\langle 13 \rangle}{\langle 43 \rangle} \lambda_4 \tilde{\lambda}_1 \,. \tag{162}$$

There are two maximal cut solutions at,

$$\begin{aligned} l^* : &\quad a_1 = 0, \quad a_2 = 0, \quad a_3 = -1, \quad a_4 = 0 \\ \bar{l}^* : &\quad a_1 = 0, \quad a_2 = 0, \quad a_4 = 0, \quad a_4 = \frac{s}{s+t} \end{aligned} \tag{163}$$

The multivariate residues are,

$$\pm \frac{1}{st} \tag{164}$$

Therefore we guess that $stG[1,1,1,1]$ *is a UT integral.*

- *Sector* $(1,0,1,0)$. *We try to compute the 4D leading singularity like the box. However, a short computation ends up with*

$$\oint da_2 \oint da_3 \frac{1}{a_3} \tag{165}$$

There is no multivariate residue defined for a_2 *and* a_3. *It is difficult to convert* $G[1,0,1,0]$ *to a UT integral by multiplying by a function in* s *and* t.

However, for the bubble diagram, or any kind of two-point Feynman integral, the trick is to consider $2D$ *leading singularity. Consider the bubble integral with* $d = 2 - 2\epsilon$. *The* $2D$ *leading singularity computation is similar: we can parameterize* $l_1^{[2D]}$ *as,*

$$l_1^{[2D]} = c_1(p_1 + p_2) + c_2 p^{\perp} \tag{166}$$

where $p^{\perp}$ *is a 2D vector such that* $p^{\perp} \cdot (p_1 + p_2) = 0$. *It is convenient to set* $(p^{\perp})^2 = -s$. *A two-fold computation provides that* $G^{(2-2\epsilon)}[1,0,1,0]$ *has the leading singularity,*

$$\pm \frac{1}{s} \tag{167}$$

Thus, we guess that $sG^{(2-2\epsilon)}[1,0,1,0]$ *is a UT integral. By the dimension recursion relation,*

$$sG^{(2-2\epsilon)}[1,0,1,0] = -2sG^{(4-2\epsilon)}[2,0,1,0] = 2(1-2\epsilon)G^{(4-2\epsilon)}[1,0,1,0] \tag{168}$$

Thus, we guess that $(1-2\epsilon)G[1,0,1,0]$ *is UT.*

- *Sector* $(0,1,0,1)$*. We guess that* $(1-2\epsilon)G[0,1,0,1]$ *is UT.*

Using the three UT candidates $stG[1,1,1,1]$*,* $(1-2\epsilon)G[1,0,1,0]$ *and* $(1-2\epsilon)G[0,1,0,1]$*, we unfortunately find that the new DE is still not proportional to* ϵ*. This is because each candidate is UT, but their transcendental weights do not match at the same* ϵ *order. This is easily fixed by considering*

$$stG[1,1,1,1], \quad \frac{1-2\epsilon}{\epsilon}G[1,0,1,0], \quad \frac{1-2\epsilon}{\epsilon}G[0,1,0,1] \tag{169}$$

We can do some "cosmetic" work to get dimensionless UT integrals,

$$I_1 = e^{\epsilon\gamma}(-s)^{\epsilon} stG[1,1,1,1], \quad I_2 = e^{\epsilon\gamma}(-s)^{\epsilon}\frac{1-2\epsilon}{\epsilon}G[1,0,1,0], \quad I_3 = e^{\epsilon\gamma}(-s)^{\epsilon}\frac{1-2\epsilon}{\epsilon}G[0,1,0,1]\,. \tag{170}$$

Indeed the new DE is canonical. With $x \equiv t/s$*,*

$$A_x = \begin{pmatrix} -\frac{\epsilon}{x(x+1)} & -\frac{2\epsilon}{x+1} & \frac{2\epsilon}{x(x+1)} \\ 0 & 0 & 0 \\ 0 & 0 & -\frac{\epsilon}{x} \end{pmatrix}. \tag{171}$$

It is a classical trick to double the propagator of bubble-type integrals to get UT integrals. However the new fashion is to consider reducible integrals to replace bubble-type integrals. In this example,

$$sG[1,0,1,1], \quad tG[0,1,1,1] \tag{172}$$

are also UT integrals, based on the leading singularity computations.

Example VIII.2. *Consider the two-loop massless box family with inverse propagators,*

$$\begin{aligned} D_1 = l_1^2, \quad D_2 = (l_1-p_1)^2, \quad D_3 = (l_1-p_1-p_2)^2, \quad D_4 = (l_2+p_1+p_2)^2 \\ D_5 = (l_2-p_4)^2, \quad D_6 = l_2^2, \quad D_7 = (l_1+l_2)^2, \quad D_8 = (l_1+p_4)^2, \quad D_9 = (l_2+p_1)^2, \end{aligned} \tag{173}$$

The last two are irreducible scalar products and we consider the sector $(1,1,1,1,1,1,1,0,0)$ *and all its subsectors. By the standard IBP process (with the symmetry), there are* 8 *master integrals. We search for UT integrals sector by sector:*

- *Sector* $(1,1,1,1,1,1,1,0,0)$*. There are two master integrals and we need to find two UT integrals.*

Note that the two-loop $4D$ *integration is* 8*-fold, while in this sector we only have* 7 *denominators. With the abuse of multivariate residue definition, we can pick up* 7 *integration variables first, and treat the rest one as a free variable "*y*". If for generic values of* y*,*

$$D_1 = \ldots = D_7 = 0 \tag{174}$$

has a solution, then we compute the 7*-fold residue as a function of* y*. Then we continue to compute the residue in* y*. This kind of computation for* $G[1,1,1,1,1,1,1,0,0]$ *provides values:*

$$\frac{1}{s^2 t}, \quad -\frac{1}{s^2 t}, \quad 0, \tag{175}$$

So the leading singularity of $G[1,1,1,1,1,1,1,0,0]$ *can be defined as* $\frac{1}{s^2 t}$*.*

Note that in this step, we have to exhaust all possible ways of computing the multivariate residues. Clearly, this is a complicated procedure, and the complete treatment is given [42]. We note that this multivariate-residue computation is a typical algebraic geometry problem, and the leading singularity computation can be efficiently carried out with the so-called "primary decomposition" and "transformation law" [43].

We guess that $s^2 t G[1,1,1,1,1,1,1,0,0]$ *is a UT integral. Similarly,* $s^2 G[1,1,1,1,1,1,1,-1,0]$ *is also a UT integral candidate.*

Alternatively, we can compute the leading singularity in a loop-by-loop fashion. The left loop is a massless box with external legs $p_1, p_2, -l_2 - p_1 - p_2, l_2$*, which is the so-called "hard" two-mass box. The leading singularity of this box is*

$$\frac{1}{s(l_2 + p_1)^2} \tag{176}$$

For the right loop with the propagators l_2^2*,* $(l_2 - p_4)^2$ *and* $(l_2 + p_1 + p_2)^2$*. Note that from* (176)*, we get a new box in* l_2*. Taking the singularity again, we see that the leading singularity of* $G[1,1,1,1,1,1,1,0,0]$ *is* $1/(s^2 t)$*. This analysis is much easier than the full leading singularity analysis.*

- *Sector (0, 1, 0, 1, 1, 1, 1, 0, 0). This is a box-bubble diagram. We can perform a loop-by-loop analysis to compute the right box residue first, and get a bubble. For the one-loop bubble analysis, we know that it is UT if one of its propagators is doubled. Therefore, we guess that* $stG[0,2,0,1,1,1,1,0,0]/\epsilon$ *is UT.*

- *Sector (0, 1, 1, 0, 1, 1, 1, 0, 0). This is a slashed box diagram. This one is tricky. It is not easy to find the leading singularity from the* $4D$ *residue computation. It can be determined by other methods. The UT candidate is* $(s+t)G[0,1,1,0,1,1,1,0,0]$.

- *Sector (0, 1, 0, 1, 0, 1, 1, 0, 0). This is a triangular-bubble diagram. The analysis would be similar to the box-bubble diagram, and the UT candidate is* $sG[0,2,0,1,0,1,1,0,0]/\epsilon$.

- *Sector (1, 0, 1, 1, 0, 1, 0, 0, 0). This is a factorized bubble-bubble diagram. From the one-loop result, we know that* $G[2,0,1,2,0,1,0,0,0]/\epsilon^2$ *must be a UT integral.*

- *Sector (0, 0, 1, 0, 0, 1, 1, 0, 0) and (0, 1, 0, 0, 1, 0, 1, 0, 0). These are the sunset integrals. From the* $2D$ *leading singularity analysis, we see that* $sG[0,0,2,0,0,2,1,0,0]/\epsilon^2$ *and* $tG[0,2,0,0,2,0,1,0,0]/\epsilon^2$ *are UT candidates.*

With some cosmetic work, the UT candidates are,

$$\begin{aligned}
I_1 &= s^2 e^{2\gamma\epsilon} G[1,1,1,1,1,1,1,-1,0](-s)^{2\epsilon},\\
I_2 &= s^3 x e^{2\gamma\epsilon} G[1,1,1,1,1,1,1,0,0](-s)^{2\epsilon},\\
I_3 &= s(x+1) e^{2\gamma\epsilon} G[0,1,1,0,1,1,1,0,0](-s)^{2\epsilon},\\
I_4 &= -\frac{s^2 x e^{2\gamma\epsilon} G[0,2,0,1,1,1,1,0,0](-s)^{2\epsilon}}{\epsilon},\\
I_5 &= \frac{s^2 e^{2\gamma\epsilon} G[2,0,1,2,0,1,0,0,0](-s)^{2\epsilon}}{\epsilon^2},\\
I_6 &= -\frac{s e^{2\gamma\epsilon} G[0,2,0,1,0,1,1,0,0](-s)^{2\epsilon}}{\epsilon},\\
I_7 &= -\frac{s e^{2\gamma\epsilon} G[0,0,2,0,0,2,1,0,0](-s)^{2\epsilon}}{\epsilon^2},\\
I_8 &= \frac{s x e^{2\gamma\epsilon} G[0,2,0,0,2,0,1,0,0](-s)^{2\epsilon}}{\epsilon^2}
\end{aligned} \tag{177}$$

and the canonical DE is

$$A_x = \epsilon\left(\frac{a_{-1}}{x+1} + \frac{a_0}{x}\right) \tag{178}$$

with

$$a_{-1} = \begin{pmatrix} -1 & 1 & -18 & -4 & -1 & 3 & -3 & \frac{9}{2} \\ -2 & 2 & -12 & -4 & -2 & -6 & -6 & 3 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -3 & 0 & -\frac{3}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{179}$$

and

$$a_0 = \begin{pmatrix} 1 & -1 & 18 & 4 & -1 & -3 & 3 & -\frac{9}{2} \\ 0 & -2 & 12 & 4 & 0 & 0 & 3 & -3 \\ 0 & 0 & -2 & 0 & 0 & 0 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 0 & -2 & 0 & 0 & 0 & \frac{3}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 \end{pmatrix} \tag{180}$$

In practice, if we guess a MI I_k which may be proportional to a UT integral by a ϵ-*independent* factor, we can make a transformation ansatz

$$I_k \to f(\bar{x}) I_k \tag{181}$$

and then collect all ϵ^0 terms in the DE matrix. Cancel the ϵ^0 terms, and we are able to determine $f(\bar{x})$ analytically. This trial-and-error approach is convenient and saves time when finding the leading singularity.

Note that for a complicated integral, the leading singularities may be a list of rational functions that do not differ from one another by constant factors. In this case, it is impossible to construct a UT integral from this integral directly and we have to consider a linear combination.

B. Lee's algorithm

Another approach to finding UT integrals is to convert a noncanonical DE to a canonical DE. With a canonical DE, it is highly likely that the corresponding integrals are UT. However, we comment that a canonical DE does *not* guarantee that the integrals are UT. For example, if $\vec{I}$ is a UT basis which satisfies,

$$d\vec{I} = \epsilon(d\tilde{A})\vec{I} \tag{182}$$

Then a basis $\vec{J} = f(\epsilon)\vec{I}$ has the same DE. However, with a nontrivial function $f(\epsilon)$, J is in general not a UT basis. This subtlety can usually be fixed by evaluating the simplest integral in the basis analytically or computing the leading singularity of some integrals, and then remove possible $f(\epsilon)$ factor.

Here we introduce Roman Lee's algorithm [44] for finding the canonical DE. There are basically three steps. For simplicity, we consider the one-variable case,

1. Put the differential equation into Fuchsian form. This step is based on Moser's algorithm, which lowers the DE matrix's pole degree through a sequence of rational transformations.

   It is not guaranteed that every differential equation of Feynman integrals can always be transformed to a Fuchsian form by Moser's algorithm. When this is possible, extract the residue matrices as follows:

$$\partial_x I' = \big(\sum_i \frac{A_i(x,\epsilon)}{x-a_i}\big) I' \tag{183}$$

2. $A_i(x,\epsilon)$ are in general not proportional to ϵ. Find the eigenvalues of $A_i(x,\epsilon)$. (We also need to consider the residue matrix at infinity.) If the eigenvalues are all of the form $\mathbb{Z}+\mathbb{Q}\epsilon$, then apply the so-called *balance transformation*. For $i \neq j$, pick one eigenvalue λ_{ik} of A_i and one eigenvalue λ_{jl} of A_j. A balance transformation is

$$(\mathbb{I} - P) + \frac{x-x_i}{x-x_j} P \tag{184}$$

   with

$$P = \frac{1}{vu}\bigg(uv\bigg), \quad A_i u = \lambda_{ik} u, \quad v A_j = \lambda_{jl} v \tag{185}$$

   where u is a right eigenvector of A_i and v is a left eigenvector of A_j. A balance transformation increases A_i's eigenvalue λ_{ik} by one and decreases A_j's eigenvalue λ_{jl} by one.

   Repeat these steps until all eigenvalues are proportional to ϵ. Denote the total transformation by $I'' = T_1 I'$.

3. Now the differential equation reads

$$\partial_x I'' = \big(\sum_i \frac{A_i(x,\epsilon)}{x - a_i}\big) I'' \tag{186}$$

with all eigenvalues of the $A_i(x,\epsilon)$ proportional to ϵ. Note that A_i's in general do not commute, so a simultaneous diagonalization does not work. Lee's key insight is to make all entries of $A_i(x,\epsilon)$ proportional to ϵ.

Suppose there is a transformation matrix $T(x,\epsilon)$ such that

$$T(x,\epsilon)A_i(x,\epsilon)T^{-1}(x,\epsilon) = \epsilon B_i(x), \quad \forall i \tag{187}$$

then we introduce an auxiliary variable μ. It is clear,

$$T(x,\epsilon)A_i(x,\epsilon)T^{-1}(x,\epsilon)/\epsilon = T(x,\mu)A_i(x,\mu)T^{-1}(x,\mu)/\mu \tag{188}$$

Define $S(x,\epsilon,\mu) = T(x,\mu)^{-1}T(x,\epsilon)$, and

$$S(x,\epsilon,\mu)\frac{A_i(x,\epsilon)}{\epsilon} = \frac{A_i(x,\mu)}{\mu}S(x,\epsilon,\mu) \quad \forall i \tag{189}$$

This is a linear equation in $S(x,\epsilon,\mu)$ and in principle we can solve it for $S(x,\epsilon,\mu)$ with x,ϵ,μ treated as parameters. If for the solution, $S(x,\epsilon,\mu)$ is finite and invertible when $\mu \to \mu_0$ with some constant μ_0, then

$$S(x,\epsilon,\mu_0)A_i(x,\epsilon)S(x,\epsilon,\mu_0)^{-1} = \epsilon\frac{A_i(x,\mu_0)}{\mu_0} \tag{190}$$

Let $T = S(x,\epsilon,\mu_0)$; this makes all entries proportional to ϵ, yielding the canonical DE.

Lee's algorithm is implemented as an interactive Mathematica package *Libra* [45], or the automatic packages *epsilon* and *fuchsia* [46, 47].

We comment that sometimes in the second step, the eigenvalues do not have the form $\mathbb{Z} + \mathbb{Q}\epsilon$. This is usually caused by the variable choice x and implies the UT integrals have a square root factor over the original MIs. In this case, we need to rationalize the square root and then use Lee's algorithm.

Sometimes, we cannot find a nonsingular balance transformation that makes all eigenvalues proportional to ϵ. Usually, this means the UT basis does not exist and the Feynman integrals contain elliptic functions.

IX. INTEGRATION TO SYMBOLS

Before we consider solving the canonical DE analytically, we introduce the solution of the DE at the symbol level. The symbol is a powerful tool for loop-level scattering amplitudes that captures many features of analytic Feynman integrals.

We see that all polylogarithmic functions have the form of an integral of a simple rational function multiplied by a lower weight polylogarithm. To illustrate this structure, the symbol [30] is defined recursively as follows. If

$$dF = \sum_i F_i d\log R_i \tag{191}$$

with rational functions R_i, then the *symbol* of F is

$$\mathcal{S}(F) \equiv \sum_i \mathcal{S}(F_i) \otimes R_i \tag{192}$$

Here $\mathcal{S}$ is a $\mathbb{Q}$-linear map from the function space to the symbol space. $\otimes$ is the notation for noncommutative tensor product. Each entry of the tensor is called a symbol letter.

For the simplest function,

$$\mathcal{S}(\log(z)) = z \tag{193}$$

The original symbol definition treats

$$\mathcal{S}(\pi) = 0, \quad \mathcal{S}(\zeta_n) = 0\,, \tag{194}$$

to ignore these constants. Contrary to intuition, $\mathcal{S}(\pi\log(z)) = 0$.

Note that $d\log(R_1R_2) = d\log R_1 + d\log R_2$, $d\log(cR) = d\log(R)$, $d\log(R^{-1}) = -d\log(R)$, so we require the symbol to be distributive:

$$\begin{aligned}
\ldots \otimes (R_1R_2) \otimes \ldots &= \ldots \otimes R_1 \otimes \ldots + \ldots \otimes R_2 \otimes \ldots \\
\ldots \otimes (cR) \otimes \ldots &= \ldots \otimes R \otimes \ldots \\
\ldots \otimes (R^{-1}) \otimes \ldots &= - \ldots \otimes R \otimes \ldots
\end{aligned} \tag{195}$$

Example IX.1. *By definition,*

$$\begin{aligned}
&\mathcal{S}(\mathrm{Li}_2(z)) = -(1-z)\otimes z, \quad \mathcal{S}(\mathrm{Li}_4(z)) = -(1-z)\otimes z \otimes z \otimes z \\
&S\big(\mathrm{Li}_2(\frac{x}{y})\big) = -(1-\frac{x}{y})\otimes \frac{x}{y}, \quad \mathcal{S}(H(1,-1;z)) = -(z+1)\otimes(1-z)
\end{aligned} \tag{196}$$

For the second line, we do not need to decompose $d(x/y) = dx/y - xdy/y^2$; we simply treat $d(x/y)$ as one object.

Note that in $-(1-z) \otimes z$, the overall minus sign cannot be dropped. However, it is quite risky to confuse it with $(z-1) \otimes z$. Therefore, especially in implementations, we often write

$$\mathcal{S}(\mathrm{Li}_2(z)) = -S[1-z, z] \tag{197}$$

and the overall factor is emphasized.

The map $\mathcal{S}$ is a simple representation of polylogarithmic functions and can be used to check function identities. If a function combination is zero, then its symbol must be zero.

Example IX.2. *(Abel's identity)*

$$\mathrm{Li}_2\big(\frac{x}{1-y}\big) + \mathrm{Li}_2\big(\frac{y}{1-x}\big) - \mathrm{Li}_2\big(\frac{xy}{(1-x)(1-y)}\big) = \mathrm{Li}_2(x) + \mathrm{Li}_2(y) + \log(1-x)\log(1-y) \tag{198}$$

We use the symbol to check this identity.

$$\begin{aligned}\mathcal{S}[l.h.s] &= -S[1-\frac{x}{1-y}, \frac{x}{1-y}] - S[1-\frac{y}{1-x}, \frac{y}{1-x}] + S[1-\frac{xy}{(1-x)(1-y)}, \frac{xy}{(1-x)(1-y)}] \\ &= -S[1-x, x] + S[1-x, 1-y] + S[1-y, 1-x] - S[1-y, y]\end{aligned} \tag{199}$$

which is clearly the same as the letters on the right-hand side, since

$$d\big(\log(1-x)\log(1-y)\big) = \log(1-x)d\log(1-y) + \log(1-y)d\log(1-x) \tag{200}$$

However, even if the corresponding symbol is zero, we still cannot claim that the function is zero. The reason is that constants such as π or ζ_n are dropped in the symbol map $\mathcal{S}$. $\mathcal{S}$ is not an injective map. It is possible to make an ansatz proportional to π or ζ_n for the missing part and then use numerical evaluation to check a function relation completely.

Given a symbol $S[R_1, \ldots, R_n]$, can we find a function F such that $\mathcal{S}(F) = S[R_1, \ldots, R_n]$? Naively, we define the *iterated integral* as follows. Let M be the space (manifold) of variables and choose o as a base point. For any point $p \in M$, $p = (x_1, \ldots, x_m)$, we find a smooth map $\gamma : [0, 1] \to M$ such that

$$\gamma(0) = o, \quad \gamma(1) = p \tag{201}$$

With the abuse of notation $R_i(t) \equiv R_i(\gamma(t))$, we define

$$F(p) = \int_0^1 dt_n \frac{R_n'(t_n)}{R_n(t_n)} \bigg(\int_0^{t_n} dt_{n-1} \frac{R_{n-1}'(t_{n-1})}{R_{n-1}(t_{n-1})} \bigg(\int_0^{t_{n-1}} dt_{n-2} \frac{R_{n-2}'(t_{n-2})}{R_{n-2}(t_{n-2})} \Big(\ldots \Big) \bigg) \bigg) \tag{202}$$

This definition extends directly to linear combinations of symbols. For a connected region containing o, the function F appears to be defined. However, we need to check if the definition is homotopically invariant under an infinitesimal deformation of γ.

Example IX.3. *Consider the symbol* $(1+x)\otimes(1+y)$. *In the* x-y *plane, choose the base point* $o=(0,0)$. *For* $p=(1,1)$, *choose a path* $(x(t),y(t))$ *such that* $(x(0),y(0))=(0,0)$ *and* $(x(1),y(1))=(1,1)$. *The "function"* F *at* p *has the expression,*

$$F(p)=\int_0^1 dt_2 \frac{y'(t_2)}{1+y(t_2)}\int_0^{t_2} dt_1 \frac{x'(t_1)}{1+x(t_1)} \tag{203}$$

With $x(t)=t, y(t)=t$ *we get* $F(p)=\log(2)^2/2$, *while for* $x(t)=t, y(t)=t^2$ *we get* $F(p)=\log(2)^2/4+\pi^2/48$. *The two paths are homotopically equivalent, so this function is not well defined.*

The validity condition for the iterated integral is provided by Chen's theory of iterated integrals [48]. The integral (202) is well defined if and only if the symbol is *integrable*: Let A be a weight-m symbol in letters $\{R_1 \dots R_l\}$,

$$A=\sum_I c_I S[R_{i_1},\dots R_{i_m}] \tag{204}$$

where each $I=\{i_1,\dots,i_m\}$ is an m-tuple whose entries are integers in $[1,l]$. A is integrable if for any $1\le j\le m-1$

$$\sum_I c_I\left(d\log R_{i_j}\wedge d\log R_{i_{j+1}}\right)S[R_{i_1},\dots,\widehat{R_{i_j}},\widehat{R_{i_{j+1}}},\dots,R_{i_m}]=0 \tag{205}$$

Example IX.4. *The symbol*

$$-S[1-x,x]+S[1-x,1-y]+S[1-y,1-x]-S[1-y,y] \tag{206}$$

is integrable because

$$\begin{aligned}&-d\log(1-x)\wedge d\log(x)+d\log(1-x)\wedge d\log(1-y)\\&+d\log(1-y)\wedge d\log(1-x)-d\log(1-y)\wedge d\log(y)=0\end{aligned} \tag{207}$$

by the anticommutativity of the wedge product. This is the symbol of the well-defined function in Example IX.2.

The symbol

$$S[x+y,1-y,y]+S[1-y,x+y,y]+S[1-y,y,x+y] \tag{208}$$

is also integrable. To see this, with $j=1$ *for the condition in* (205), *we can check*

$$d\log(x+y)\wedge d\log(1-y)S[y]+d\log(1-y)\wedge d\log(x+y)S[y]+d\log(1-y)\wedge d\log(y)S[x+y]=0 \tag{209}$$

The check with $j=2$ *is similar.*

When a symbol A is integrable, the iterated integral (202) F is well-defined. By Newton-Leibniz theorem and the symbol definition

$$\mathcal{S}(F) = A. \tag{210}$$

Thus, we see that an integrable symbol corresponds to a function. Note that the iterated integral is homotopically invariant, but if the variable space is not simply connected (because of singular points), the iterated integral can have branch cuts.

Integrable symbols can be used to calculate an amplitude in the *bootstrap* approach; for examples, see refs. [49–51]. In practice, the integrable symbols with fixed weights can be found by the package [52].

Back to the canonical differential equation, with the background knowledge of Chen's theory, it is clear that the solution is an iterated integral. In many cases, we first want to know the symbol of the solution, instead of the analytic solution. By the definition, the symbol of the solution can be obtained without much effort.

Consider the canonical differential equation for UT integrals,

$$d\vec{I} = \epsilon(d\tilde{A})\vec{I} = \epsilon\left(\sum_{i=1}^{N} a_i d\log W_i\right)\vec{I} \tag{211}$$

with

$$\vec{I} = \epsilon^k \sum_{n=0}^{\infty} \vec{I}^{(n)} \epsilon^n \tag{212}$$

Recall that the canonical DE splits into different orders of ϵ in (155): each derivative of $I^{(n)}$ is the product of dlog's and $I^{(n-1)}$. This immediately indicates that

$$\mathcal{S}(I^{(m)}) = \left(\sum_{i_m=1}^{N}\sum_{i_{m-1}=1}^{N}\ldots\sum_{i_1=1}^{N} a_{i_m}a_{i_{m-1}}\ldots a_{i_1} I^{(0)}\right) S[W_{i_1},\ldots,W_{i_m}] \tag{213}$$

where $I^{(0)}$ is the leading order of the UT basis, which is a constant vector of rational numbers. The correctness of (213) follows directly from the definition of symbols.

Note that this formula requires only the leading order of the UT basis, $I^{(0)}$, which can be easily obtained from an infrared/ultraviolet structure analysis or a numerical computation. In practice, to evaluate the large sum in (213), we always compute $a_{i_1} I^{(0)}$ first, then $a_{i_2} a_{i_1} I^{(0)}$, and so on for the remaining multiplications, to speed up the computation. This computation should be simple.

An interesting consequence of (213) is that if the two matrices satisfy,

$$a_i a_j = 0, \tag{214}$$

then the symbol can never have W_j and W_i in adjacent positions.

We refer to [53] and [54] for the systematic way of determining letters from the Gram determinant in the Baikov representation and Landau singularity, respectively.

X. INTEGRATION TO FUNCTIONS

From the study of symbols, the idea of solving canonical differential equations in terms of iterated integrals becomes clear. However, in practice, we need the analytic function form of Feynman integrals. The new ingredient is the *boundary condition*.

The solution of a canonical differential equation is completely determined by the UT integrals at one boundary point o:

$$\vec{I}^{(n)}(o) \equiv B^{(n)} \tag{215}$$

Let γ be a smooth curve starting from o. Then, along the curve γ, the solution corresponding to the UT Feynman integral is

$$\begin{aligned}
I^{(0)} &= B^{(0)} \\
I^{(1)} &= B^{(1)} + \int_\gamma (d\tilde{A}) I^{(0)} \\
I^{(2)} &= B^{(2)} + \int_\gamma (d\tilde{A}) I^{(1)} \\
&\dots \\
I^{(n)} &= B^{(n)} + \int_\gamma (d\tilde{A}) I^{(n-1)}
\end{aligned} \tag{216}$$

Here the notation $\int_\gamma$ can be explicitly written as an integration over a t-parameterized curve from o to an arbitrary point p.

The main difference between (213) and (216) is that the higher-order boundary values $B^{(i)}$ are also taken into account. This means that the computation is significantly harder than the previous section.

For the integration in (216), note that $(d\tilde{A})$ contains only dlog's of the symbol letters. If all letters are rational functions, then $(d\tilde{A})$ contains only rational functions. Along the curve γ, the pullback of $(d\tilde{A})$ becomes a rational function of t. After factorization, each rational function has only linear denominators in t. The power of each linear factor would be at most one, because of the dlog form. Therefore, the integration, by the definition, produces GPLs only if all symbol letters are rational. Of course, in many cases the integration only gives HPLs or classical polylogarithms.

If the symbol letters contain square roots that cannot be rationalized simultaneously, then the integration may produce functions more complicated than GPLs.

Here the main problem is to find the higher order boundary values. This step is tricky: unfortunately, there is no automatic algorithm to do this. There are two types of strategies for determining the boundary values:

1. Direct evaluation of the boundary values at a special point. This may be done numerically with AMFlow [26, 28].

   If the special point has simple kinematics, one may directly evaluate UT integrals there with Cheng-Wu theorem, dimension recursion relation, expansion by regions or direct integration with HypInt.

2. Set an ansatz for the boundary values and determine them later from consistency conditions. There are many types of consistency conditions. For example, for the massless four-point kinematics, a *planar* UT integral should be finite in the limit $u \to 0$. More generically, a UT integral is likely to be finite when one of the symbol letters is zero and $\epsilon < 0$. These generic consistency conditions were systematically studied in []. Furthermore, it is also useful to consider the soft/collinear/Regge limit of the UT integrals.

When solving a canonical DE at the function level, one may need to use both strategies.

Example X.1. *(One-loop massless box) The UT basis and the canonical DE of the one-loop massless box integral family are given in* (170) *and* (171)*.*

The canonical DE has the symbol letter decomposition,

$$A_x = \epsilon\left(\frac{a_{-1}}{x+1} + \frac{a_0}{x}\right) \tag{217}$$

with

$$a_{-1} = \begin{pmatrix} 1 & -2 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad a_0 = \begin{pmatrix} -1 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \tag{218}$$

We choose the point $x = 1$. At this point, $I_2 = I_3$ and by a simple Feynman parametrization computation,

$$I_2|_{x=1} = I_3|_{x=1} = \epsilon^{-2}\left(1 - \frac{\pi^2\epsilon^2}{12} - \frac{7\zeta(3)\epsilon^3}{3} - \frac{47\pi^4\epsilon^4}{1440} + \dots\right) \tag{219}$$

The boundary value for I_1 can also be computed by Feynman parametrization, however the procedure is much more complicated. We leave its boundary value undetermined.

For convenience, we define

$$\vec{I} = \epsilon^{-2} \sum_{n=0}^{\infty} \vec{I}^{(n)}(x)\epsilon^n \,. \tag{220}$$

The boundary value is then

$$\sum_n \vec{B}^{(n)}\epsilon^n = \begin{pmatrix} c^{(0)} \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} c^{(1)} \\ 0 \\ 0 \end{pmatrix}\epsilon + \begin{pmatrix} c^{(2)} \\ -\frac{\pi^2}{12} \\ -\frac{\pi^2}{12} \end{pmatrix}\epsilon^2 + \begin{pmatrix} c^{(3)} \\ -\frac{7\zeta(3)}{3} \\ -\frac{7\zeta(3)}{3} \end{pmatrix}\epsilon^3 + \dots \tag{221}$$

c's are to be fixed by the condition that planar Feynman integral should not diverge at $x = -1$ ($u = 0$). Note that the fact that the DE itself has a pole at $x = -1$ does not mean that the solution must diverge there.

The condition $\lim_{x\to -1} \vec{I}^{(n+1)}(x) < \infty$ is translated as,

$$\lim_{x\to -1} a_{-1}\vec{I}^{(n)}(x) = \vec{0} \tag{222}$$

- *Weight-0.* (222) *determines that $c_0 = 4$ and* (216) *provides,*

$$\vec{I}^{(0)} = \begin{pmatrix} 4, & 1, & 1 \end{pmatrix}^T \tag{223}$$

- *Weight-1.* (222) *determines that $c_1 = 0$ and* (216) *provides,*

$$\vec{I}^{(1)} = \begin{pmatrix} -2\log(x), & 0, & -\log(x) \end{pmatrix}^T \tag{224}$$

  Note that in using the logarithm function, we impose the branch cut $(-\infty, 0)$ on the x-plane.

- *Weight-2. The condition* (222) *reads*

$$\lim_{x\to -1} c_2 - \log(x)^2 + \frac{\pi^2}{3} = 0 \tag{225}$$

  It appears subtle because $\log(x)$ has different values above and below the point $x = -1$. However, this subtlety is resolved by the square, and we get $c_2 = -4\pi^2/3$. By (216),

$$\vec{I}^{(2)} = \begin{pmatrix} -\frac{4\pi^2}{3}, & -\frac{\pi^2}{12}, & \frac{\log^2(x)}{2} - \frac{\pi^2}{12} \end{pmatrix}^T \tag{226}$$

- *Weight-3. The condition* (222) *determines that*

$$c_3 = \frac{1}{6}\left(-77\zeta(3) - 6\pi^2\log(2)\right) \tag{227}$$

And by (216)*, we get* $\vec{I}_3$*,*

$$\begin{pmatrix} \frac{1}{6}\left(12\text{Li}_3(-x)-12\text{Li}_2(-x)\log x+2\log^3 x-6\log(x+1)\log^2 x+7\pi^2\log x-6\pi^2\log(x+1)-68\zeta_3\right) \\ -\frac{7\zeta_3}{3} \\ \frac{1}{12}\left(-2\log^3 x+\pi^2\log x-28\zeta_3\right) \end{pmatrix} \tag{228}$$

Note that these functions also have the branch cut $(-\infty, 0)$*.*

Higher orders can be computed analytically in a similar way. After the analytic computation, it is important to check the result against numerical computations. With $s=-1$ *and* $t=-1$*, the package pySecdec we have*

$$I_1|_{x\to 1} = 4.00000\epsilon^{-2} - \left(1.74986\times 10^{-7}\right)\epsilon^{-1} - 13.1595 - 22.2675\epsilon \tag{229}$$

which is consistent with our analytic boundary condition. Furthermore, we can check the integrated result on a generic physical point. For example, consider the point $s=-1$ *and* $t=+3$*, pySecdec gives*

$$I_1|_{x\to -3} = \frac{4.00000}{\epsilon^2} - \frac{2.19725-6.28323i}{\epsilon} - (13.15975-0.00021i) + (-9.8244-10.2675i)\epsilon \tag{230}$$

which is consistent with our analytic computation. Note that in the physical region, we have to shift the value as,

$$s\to -1, \quad t\to 3+i\delta, \quad \delta>0 \tag{231}$$

in order to get the correct value for functions like $\log x$ *and* $\text{Li}_2(-z)$*. Otherwise, we are using the wrong branch.*

Example X.2. *(Two-loop massless box) The UT basis and the canonical DE were given in* (177) *and* (178)*. Again the UT basis is dimensionless and we consider* $x=1$ *as our boundary. Here we see that this example has new features compared with the one-loop box family.*

The last four UT integrals in (177) *can be easily computed from Feynman parameterization. The UT integrals* I_3 *and* I_4 *can also be computed from Feynman parametrization; however, the computation is more expensive. We leave the boundary values for the first four UT integrals undetermined.*

We use the convention

$$\vec{I} = \epsilon^{-4}\sum_{n=0}^{\infty}\vec{I}^{(n)}(x)\epsilon^n\,. \tag{232}$$

The boundary condition for the 8 UT integrals is

$$\sum_{n=0}^{\infty} \vec{B}^{(n)}\epsilon^n = \begin{pmatrix} c_1^{(0)} + \epsilon c_1^{(1)} + \epsilon^2 c_1^{(2)} + \epsilon^3 c_1^{(3)} + \epsilon^4 c_1^{(4)} + O\left(\epsilon^5\right) \\ c_2^{(0)} + \epsilon c_2^{(1)} + \epsilon^2 c_2^{(2)} + \epsilon^3 c_2^{(3)} + \epsilon^4 c_2^{(4)} + O\left(\epsilon^5\right) \\ c_3^{(0)} + \epsilon c_3^{(1)} + \epsilon^2 c_3^{(2)} + \epsilon^3 c_3^{(3)} + \epsilon^4 c_3^{(4)} + O\left(\epsilon^5\right) \\ c_4^{(0)} + \epsilon c_4^{(1)} + \epsilon^2 c_4^{(2)} + \epsilon^3 c_4^{(3)} + \epsilon^4 c_4^{(4)} + O\left(\epsilon^5\right) \\ 1 - \frac{\pi^2\epsilon^2}{6} - \frac{14\zeta(3)\epsilon^3}{3} - \frac{7\pi^4\epsilon^4}{120} + O\left(\epsilon^5\right) \\ \frac{1}{4} + \frac{\pi^2\epsilon^2}{24} - \frac{13\zeta(3)\epsilon^3}{6} - \frac{41\pi^4\epsilon^4}{1440} + O\left(\epsilon^5\right) \\ -1 + \frac{\pi^2\epsilon^2}{6} + \frac{32\zeta(3)\epsilon^3}{3} + \frac{19\pi^4\epsilon^4}{120} + O\left(\epsilon^5\right) \\ 1 - \frac{\pi^2\epsilon^2}{6} - \frac{32\zeta(3)\epsilon^3}{3} - \frac{19\pi^4\epsilon^4}{120} + O\left(\epsilon^5\right) \end{pmatrix} \tag{233}$$

We may use the planar consistency condition $\lim_{x\to -1} \vec{I}^{n+1}(x) < \infty$ *or*

$$\lim_{x\to -1} a_{-1}\vec{I}^{n}(x) = 0 \tag{234}$$

- *Weight 0. In this case,* (234) *means,*

$$c_2^{(0)} = \frac{7}{4} + c_1^{(0)}, \quad c_3^{(0)} = 0, \quad c_4^{(0)} = \frac{9}{4} \tag{235}$$

  It does not look good because $c_1^{(0)}$ *is still undetermined. We leave* $c_1^{(0)}$ *as a free parameter and turn to the next order.*

- *Weight 1. We integrate* $\vec{I}^{(0)}$ *to get* $\vec{I}^{(1)}$. *Then* (234) *reads,*

$$c_2^{(1)} = c_1^{(1)} + \lim_{x\to -1}(2c_1^{(0)} - \frac{9}{2})\log x, \quad c_3^{(1)} = 0, \quad c_4^{(1)} = 0 \tag{236}$$

  Whether we approach the point $x = -1$ *from above or below,* $\log x$ *would be complex. However* $c_2^{(1)}$ *and* $c_1^{(1)}$ *should be real, since the boundary point is in the Euclidean region. Hence,*

$$c_1^{(0)} = \frac{9}{4} \tag{237}$$

  At weight 1, we have fixed the weight-0 boundary condition. Here the value of $c_1^{(1)}$ *is undetermined yet.*

- *Weight 2. We integrate* $\vec{I}^{(1)}$ *to get* $\vec{I}^{(2)}$. *Then* (234) *reads,*

$$c_2^{(2)} \to \frac{1}{24}\left(48c_1^{(1)} \lim_{x\to -1}(\log(x)) + 24c_1^{(2)} - 13\pi^2\right) \quad c_3^{(2)} \to \frac{\pi^2}{2} \quad c_4^{(2)} = -\frac{13\pi^2}{8} \tag{238}$$

  Again by the Euclidean region argument,

$$c_1^{(1)} = 0 \tag{239}$$

So we completely obtained the UT basis up to weight 1*, and the result is,*

$$\vec{I}^{(0)} + \epsilon \vec{I}^{(1)} = \begin{pmatrix} \frac{9}{4} - 2\epsilon \log(x) \\ 4 - 5\epsilon \log(x) \\ 0 \\ \frac{9}{4} - 3\epsilon \log(x) \\ 1 \\ \frac{1}{4} \\ -1 \\ 1 - 2\epsilon \log(x) \end{pmatrix} \tag{240}$$

Boundary values at all orders can be fixed by (1) the planar-integral consistency condition and (2) the Euclidean-region condition. The rest of the computation is left as homework.

We remark that when solving a canonical DE, it may be cumbersome to use the "Integrate" command in Mathematica, even if a result contains only classical polylogarithm. It is more convenient to define the iterated-integration rules and formal functions such as HPLs or GPLs in the computer algebra system and then call a package like HPL to simplify the result.

To simplify combinations of polylogarithms, the main tool is the *shuffle identity.*

Example X.3. *To understand the shuffle identity, it is useful to start with a simple example. Consider* $f_1(z) = H(1, -1; z)$ *and* $f_2(z) = H(-1, 1; z)$*. Assume that* $0 < z < 1$*, and they both have the iterated-integral form,*

$$\begin{aligned} f_1(z) &= \int_0^z dt_2 \frac{1}{1-t_2} \int_0^{t_2} dt_1 \frac{1}{1+t_1} \\ f_2(z) &= \int_0^z dt_2 \frac{1}{1+t_2} \int_0^{t_2} dt_1 \frac{1}{1-t_1} \end{aligned} \tag{241}$$

In the second integral, we rename the variables $t_1 \leftrightarrow t_2$*, and we get*

$$H(1, -1; z) + H(-1, 1; z) = \int_0^z dt_1 \frac{1}{1+t_1} \int_0^z dt_2 \frac{1}{1-t_2} = -\log(1+z)\log(1-z) \tag{242}$$

We see that

$$\begin{aligned} &\mathcal{S}(\log(1+z)) = S[1+z], \quad \mathcal{S}(-\log(1-z)) = -S[1-z] \\ &\mathcal{S}(H(1, -1; z) + H(-1, 1; z)) = -S[1+z, 1-z] - S[1-z, 1+z] \end{aligned} \tag{243}$$

Thus, the symbol of the product of $\log(1+z)$ *and* $\log(1-z)$ *is the sum of the shuffle products of the two letters.*

We also comment that all terms in (242) *have the same branch cuts* $(-\infty, -1)$ *and* $(1, \infty)$*, and by analytic continuation, the identity holds everywhere on the complex plane except for* $z = \pm 1$*.*

For the product of two polylogarithms, the shuffle identity can be derived by a simplex decomposition. For GPLs,

$$G(a_1 \dots a_n; z) G(a_{n+1}, \dots a_{n+m}; z) = \sum_{\sigma \in \Sigma_{n,m}} G(a_{\sigma(1)} \dots a_{\sigma(n+m)}; z) \tag{244}$$

where $\Sigma_{n,m}$ is the subset of the permutation group Σ_{n+m} consisting of all permutations such that

$$\sigma^{-1}(1) < \dots < \sigma^{-1}(n), \quad \text{and,} \quad \sigma^{-1}(n+1) < \dots < \sigma^{-1}(n+m), \tag{245}$$

For HPLs, the shuffle identity is the same, with $G \to H$.

The shuffle identity means that GPLs (HPLs) of the same weights in z are not independent. Thus, we can use the shuffle identity to simplify polylogarithms, and extract the divergence.

Example X.4. *Consider the function* $f(z) = H(1, 0, -1, 0; z)$*. This function is divergent when* $z \to 1$*. To find the leading divergence, we use the shuffle identity:*

$$H(1; z) H(0, -1, 0; z) = H(1, 0, -1, 0; z) + H(0, 1, -1, 0; z) + H(0, -1, 1, 0; z) + H(0, -1, 0, 1; z) \tag{246}$$

The last three functions are not divergent at $z \to 1$*. Hence we see that,*

$$H(1, 0, -1, 0; z)|_{z \to 0} \to -H(0, -1, 0; z) \log(1 - z) \sim \frac{3}{2} \zeta(3) \log(1 - z) \tag{247}$$

XI. SIMPLIFIED CANONICAL DIFFERENTIAL EQUATION

Sometimes, for a real-world problem, we do not need to solve *all* the integrals in a family. Instead, we are interested only in a subset of integrals to a certain ϵ order. In this case, a canonical DE can be simplified [55].

We illustrate the idea with a concrete example from ref. [55]:

Example XI.1. *(One-loop box with internal mass).*

$$D_1 = l_1^2 - m^2, \quad D_2 = (l_1 - p_1)^2 - m^2, \quad D_3 = (l_1 - p_1 - p_2)^2 - m^2, \quad D_4 = (l_1 + p_4)^2 - m^2 \tag{248}$$

with $p_i^2 = 0,\ i = 1,2,3,4$. $(p_1+p_2)^2 = s$, $(p_1+p_4)^2 = t$ *and* $(p_2+p_4)^2 = -s-t$. *There are* 6 *master integrals in this family. With the leading singularity analysis and some reasonable guessing, the UT basis is,*

$$\begin{aligned}
I_1 &= e^{\epsilon\gamma}\left(m^2\right)^{\epsilon}\sqrt{st\left(st-4m^2(s+t)\right)}G[1,1,1,1]\\
I_2 &= e^{\epsilon\gamma}s\left(m^2\right)^{\epsilon}G[1,0,1,1]\\
I_3 &= e^{\epsilon\gamma}t\left(m^2\right)^{\epsilon}G[0,1,1,1]\\
I_4 &= e^{\epsilon\gamma}\frac{\sqrt{s(s-4m^2)}\left(m^2\right)^{\epsilon}}{\epsilon}G[1,0,2,0]\\
I_5 &= e^{\epsilon\gamma}\frac{\sqrt{t(t-4m^2)}\left(m^2\right)^{\epsilon}}{\epsilon}G[0,1,0,2]\\
I_6 &= e^{\epsilon\gamma}\frac{\left(m^2\right)^{\epsilon+1}}{\epsilon^2}G[0,0,0,3]
\end{aligned} \tag{249}$$

It is convenient to define

$$u = -\frac{4m^2}{s}, \quad v = -\frac{4m^2}{t} \tag{250}$$

and,

$$\beta_u = \sqrt{1+u}, \quad \beta_v = \sqrt{1+v}, \quad \beta_{uv} = \sqrt{1+u+v}, \tag{251}$$

The alphabet $\{W_1,\ldots,W_8\}$ *is*

$$\{\frac{u}{1+u}, \frac{v}{1+v}, \frac{u+v}{1+u+v}, \frac{\beta_u-1}{\beta_u+1}, \frac{\beta_v-1}{\beta_v+1}, \frac{\beta_{uv}-\beta_u}{\beta_{uv}+\beta_u}, \frac{\beta_{uv}-\beta_v}{\beta_{uv}+\beta_v}, \frac{\beta_{uv}-1}{\beta_{uv}+1}\} \tag{252}$$

By a numerical fitting, the canonical differential equation has the $\tilde{A}$ *matrix (homework):*

$$\tilde{A} = \begin{pmatrix}
\log(W_3) & -2\log(W_8) & -2\log(W_8) & -2\log(W_6) & -2\log(W_7) & 0\\
0 & 0 & 0 & -\log(W_4) & 0 & 0\\
0 & 0 & 0 & 0 & -\log(W_5) & 0\\
0 & 0 & 0 & \log(W_1) & 0 & 2\log(W_4)\\
0 & 0 & 0 & 0 & \log(W_2) & 2\log(W_5)\\
0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix} \tag{253}$$

It is easy to find the boundary value at $m \to \infty$

$$u \to \infty, \quad v \to \infty \tag{254}$$

and solve the canonical DE. However, if we only need the I_1*'s* ϵ^0 *order, there is a shortcut.*

Note that from the UV/IR analysis, in the UT list

$$I_1, I_2, I_3 \sim O(\epsilon^0), \quad I_4, I_5 \sim O(\epsilon^{-1}), \quad I_6 \sim O(\epsilon^{-2}) \tag{255}$$

Then we define

$$J_1 = I_1, \quad J_2 = I_2, \quad J_3 = I_3, \quad J_4 = \epsilon I_4, \qquad J_5 = \epsilon I_5, \qquad J_6 = \epsilon^2 I_5, \tag{256}$$

So J_i's are finite as $\epsilon \to 0$. The DE for J's is,

$$d\vec{J} = d\begin{pmatrix} \epsilon\log(W_3) & -2\epsilon\log(W_8) & -2\epsilon\log(W_8) & -2\log(W_6) & -2\log(W_7) & 0 \\ 0 & 0 & 0 & -\log(W_4) & 0 & 0 \\ 0 & 0 & 0 & 0 & -\log(W_5) & 0 \\ 0 & 0 & 0 & \epsilon\log(W_1) & 0 & 2\log(W_4) \\ 0 & 0 & 0 & 0 & \epsilon\log(W_2) & 2\log(W_5) \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}\vec{J} \tag{257}$$

If we are only interested in the ϵ^0 order of J, then

$$d\vec{J}^{(0)} = d\begin{pmatrix} 0 & 0 & 0 & -2\log(W_6) & -2\log(W_7) & 0 \\ 0 & 0 & 0 & -\log(W_4) & 0 & 0 \\ 0 & 0 & 0 & 0 & -\log(W_5) & 0 \\ 0 & 0 & 0 & 0 & 0 & 2\log(W_4) \\ 0 & 0 & 0 & 0 & 0 & 2\log(W_5) \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}\vec{J}^{(0)} \tag{258}$$

The DE is significantly simplified. Especially,

$$d\begin{pmatrix} J_1^{(0)} \\ J_4^{(0)} \\ J_5^{(0)} \\ J_6^{(0)} \end{pmatrix} = d\begin{pmatrix} 0 & -2\log(W_6) & -2\log(W_7) & 0 \\ 0 & 0 & 0 & 2\log(W_4) \\ 0 & 0 & 0 & 2\log(W_5) \\ 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} J_1^{(0)} \\ J_4^{(0)} \\ J_5^{(0)} \\ J_6^{(0)} \end{pmatrix} \tag{259}$$

That means $J_1^{(0)}$ is an integration of $J_4^{(0)}$ and $J_5^{(0)}$, where the latter two are the integration of $J_6^{(0)}$. We then have a simple iterative integration, unlike the standard canonical DE, here the iterative structure is over different integrals.

From Feynman parameterization,

$$J_6^{(0)} = -\frac{1}{2} \tag{260}$$

and then

$$J_4^{(0)} = -\log(W_4) + c_4 = -\log(\frac{\beta_u - 1}{\beta_u + 1}) + c_4, \quad J_5^{(0)} = -\log(W_5) + c_5 = -\log(\frac{\beta_v - 1}{\beta_v + 1}) + c_5 \tag{261}$$

From the boundary condition at $s \to 0$ and $t \to 0$, we see that $c_4 = c_5 = 0$. Then, the symbol of $J_1^{(0)}$ is clear,

$$\mathcal{S}(J_1^{(0)}) = 2S[W_4, W_6] + 2S[W_5, W_7] = 2\left(S[\frac{\beta_u - 1}{\beta_u + 1}, \frac{\beta_{uv} - \beta_u}{\beta_{uv} + \beta_u}] + S[\frac{\beta_v - 1}{\beta_v + 1}, \frac{\beta_{uv} - \beta_v}{\beta_{uv} + \beta_v}]\right). \tag{262}$$

If we pick up a boundary point where $J_1^{(0)} \to 0$, then $J_1^{(0)}$ has a compact expression in terms of iterative integral

$$J_1^{(0)} = 2\int_\gamma \log\left(\frac{\beta_u - 1}{\beta_u + 1}\right) d\log\left(\frac{\beta_{uv} - \beta_u}{\beta_{uv} + \beta_u}\right) + \log\left(\frac{\beta_v - 1}{\beta_v + 1}\right) d\log\left(\frac{\beta_{uv} - \beta_v}{\beta_{uv} + \beta_v}\right) \tag{263}$$

This is a perfect answer.

However, in practice, we may want the expression in terms of polylogarithms. In general, Chen's iterative integral may not be a combination of polylogarithms, but the integral (263) *is, because all three square roots can be rationalized simultaneously:*

$$u = \frac{(1 - w^2)(1 - z^2)}{(w - z)^2}, \quad v = \frac{4wz}{(w - z)^2} \tag{264}$$

For the simplicity, we choose the Euclidean region,

$$u = -\frac{4m^2}{s} > 0, \quad v = -\frac{4m^2}{t} > 0 \tag{265}$$

and correspondingly $0 < w < z < 1$. In this region,

$$\beta_u = \frac{1 - wz}{z - w}, \quad \beta_v = \frac{z + w}{z - w}, \quad \beta_{uv} = \frac{1 + wz}{z - w} \tag{266}$$

Then

$$W_4 = \frac{(w + 1)(1 - z)}{(1 - w)(z + 1)}, \quad W_5 = \frac{w}{z}, \quad W_6 = wz, \quad W_7 = \frac{(1 - w)(1 - z)}{(w + 1)(z + 1)} \tag{267}$$

Here we see that in the new coordinates, the relevant symbol letters become $\{z, w, 1 \pm z, 1 \pm w\}$. For a generic point (z, w), $0 < w < z < 1$, we pick the boundary point as (w, w) and γ to be the straight line from (w, w) to (z, w). At (w, w), $J_1^{(0)}$ is zero. Then the computation of (263) *is straightforward, and the result is*

$$\begin{aligned} J_1^{(0)} &= -2\mathrm{Li}_2(1 - w) - 4\mathrm{Li}_2(-w) + 2\mathrm{Li}_2(w) + 2\mathrm{Li}_2(1 - z) + 4\mathrm{Li}_2(-z) - 2\mathrm{Li}_2(z) + 2\log(w + 1)\log(z) \\ &+ 2\log(w)\log(1 - z) - 2\log(1 - w)\log(z) - 2\log(w)\log(z + 1) - 2\log(w)\log(w + 1) + 2\log(z)\log(z + 1)\,. \end{aligned} \tag{268}$$

This result is consistent with pySecDec.

The lesson here is that we do not need to compute the two triangles; instead, we obtain the box directly by integrating the bubbles. Equation (259) is not just upper triangular, but has all diagonal elements zero. Then the iterative integration structure over UT integrals is manifest.

This kind of treatment certainly simplified the canonical DE a lot. However, it would be even better if one could completely skip the full canonical DE and directly get a simple DE like (259). This is possible and the key idea is to find integral relations valid only to a fixed order in ϵ.

XII. UT DETERMINATION, II

Determining UT multiloop Feynman integrals, especially those with multiple legs, is not an easy task. In this section, we give an overview of some more recent methods.

A. Integrand dlog construction

Roughly speaking, integrand dlog construction can be understood as an advanced version of the leading singularity analysis. Again, it was developed in the study of $N = 4$ super-Yang–Mills theory [56, 57]. For details of the relation between the dlog integrand and UT integrals in ref. [58].

A $4D$ dlog integrand has the form

$$\int d\log f_1 \wedge \ldots d\log f_{4L} \tag{269}$$

We can consider $f_1, \ldots, f_{4L}$ as variables and thus any residue would be ± 1. Note that this argument is actually stronger than a leading singularity based on cuts because no cut is taken here. A Feynman integral with a $4D$ dlog integrand is likely to be UT, while a Feynman integral with constant leading singularity tends to be UT up to some lower-sector integrals.

Example XII.1. *(Dlog form of the one-loop massless box). The inverse propagators are*

$$D_1 = l_1^2, \quad D_2 = (l_1 - p_1)^2, \quad D_3 = (l_1 - p_1 - p_2)^2, \quad D_4 = (l_4 + p_4)^2 \tag{270}$$

We define a "magic factor"

$$F = (l - l_*)^2 \tag{271}$$

where l_* is one of the maximal-cut solutions for l. It is then easy to check

$$\begin{aligned} &\int d\log\left(\frac{F}{D_1}\right) \wedge d\log\left(\frac{F}{D_2}\right) \wedge d\log\left(\frac{F}{D_3}\right) \wedge d\log\left(\frac{F}{D_4}\right) \\ &\propto st \int d^4 l \frac{1}{D_1 D_2 D_3 D_4} \end{aligned} \tag{272}$$

Therefore, $stG[1,1,1,1]$ is a dlog integral. We already know that this is a UT integral.

In practice, it is not easy to find an elegant dlog integrand like (XII.1). However, for the purpose of UT determination, a linear combination of several dlog integrands is equally good. Such a linear combination can be found with Wasser's package DLOG [59].

B. Baikov leading singularity

The Baikov representation is convenient for computing leading singularities and has recently been used to determine UT integrals [60–63]. In Baikov representation, the ϵ parameter is preserved while the integration is over an integer number of variables.

Example XII.2. *(two-loop massless slashed box). With the notation in the previous examples, the integral $G[0,1,1,0,1,1,1,0,0]$ is the so-called slashed box integral.*

$$G[0,1,1,0,1,1,1,0,0] = \int \frac{d^d l_1}{i\pi^{d/2}} \frac{1}{(l-p_1)^2(l-p_1-p_2)^2} \int \frac{d^d l_2}{i\pi^{d/2}} \frac{1}{(l_2)^2(l_2-p_4)^2(l_2+l_1)^2} \tag{273}$$

We can do a right-to-left Baikov analysis. For the loop with l_2, the external lines are l_1 and p_4, so the Baikov representation is

$$G\begin{pmatrix} l_1 & p_4 \\ l_1 & p_4 \end{pmatrix}^{\frac{3-d}{2}} \int dz_1 dz_2 dz_3 \, G\begin{pmatrix} l_2 & l_1 & p_4 \\ l_2 & l_1 & p_4 \end{pmatrix}^{\frac{d-4}{2}} \frac{1}{z_1 z_2 z_3} \tag{274}$$

with the Baikov-variable definition

$$l_2^2 = z_1, \quad l_1 \cdot l_2 = \frac{1}{2}(-l_1^2 - z_1 + z_3), \quad l_2 \cdot p_4 = \frac{1}{2}(z_1 - z_2). \tag{275}$$

Taking the residue of (274) at $\{z_1, z_2, z_3\} \to \{0,0,0\}$ with $d \to 4$, we obtain

$$\frac{1}{l_1 \cdot p_4} \tag{276}$$

For the left loop with external lines p_2, the Baikov representation is

$$G\begin{pmatrix} p_2 & p_4 \\ p_2 & p_4 \end{pmatrix}^{\frac{3-d}{2}} \int dw_1 dw_2 dw_3 \, G\begin{pmatrix} l_1-p_1 & p_2 & p_4 \\ l_1-p_1 & p_2 & p_4 \end{pmatrix}^{\frac{d-4}{2}} \frac{1}{w_1 w_2 w_3} \tag{277}$$

with the Baikov-variable definition

$$(l_1-p_1)^2 = w_1, \quad (l_1-p_1-p_2)^2 = \frac{1}{2}(w_1 - w_2), \quad l_1 \cdot p_4 = w_3 - \frac{t}{2} \tag{278}$$

Note that from the right loop analysis we have a new propagator $(l_1 \cdot p_4)^{-1}$. *We have to add* p_4 *as an external line. The residue of* (277) *is then,*

$$\frac{1}{s+t} \tag{279}$$

From the loop-by-loop Baikov representation, we see that the leading singularity of $G[0,1,1,0,1,1,1,0,0]$ *is* $1/(s+t)$.

More importantly, in multileg cases, an *integrand* may be explicitly zero in the $4D$ limit, but the integral does not vanish. In such cases, the $4D$ UT-determination method fails completely, but the Baikov method works.

Example XII.3. *(Two-loop five-point nonplanar UT integral). The propagators of this integral family are*

$$\begin{aligned}
&D_1 = l_1^2, \quad D_2 = (l_1 - p_1)^2, \quad D_3 = (l_1 - p_{12})^2, \quad D_4 = l_2^2,\\
&D_5 = (l_2 - p_{123})^2, \quad D_6 = (l_2 - p_{1234})^2, \quad D_7 = (l_1 - l_2)^2,\\
&D_8 = (l_1 - l_2 + p_3)^2, \quad D_9 = (l_1 - p_{1234})^2, \quad D_{10} = (l_2 - p_1)^2,\\
&D_{11} = (l_2 - p_{12})^2.
\end{aligned} \tag{280}$$

Consider the integral

$$I[\mu_{12}] \equiv \int \frac{d^d l_1 d^d l_2}{(i\pi^{d/2})^2} \frac{G\begin{pmatrix} l_1 & p_1 & p_2 & p_4 & p_5 \\ l_2 & p_1 & p_2 & p_4 & p_5 \end{pmatrix}}{D_1 \ldots D_8}. \tag{281}$$

This one has no 4D leading singularity, since in the 4D limit the numerator vanishes. However, the integral itself is not zero. A Baikov leading singularity analysis finds nonzero residues of $I[\mu_{12}]$, *and we can predict that*

$$\frac{s_{12} - s_{45}}{\epsilon_{1245}} I[\mu_{12}]\,. \tag{282}$$

would be a UT integral [60]. This is confirmed by the differential equation.

C. Module lift method

Sometimes the search for UT integrals has a flavor of algebraic geometry: we need to combine several integrals to satisfy some conditions (leading singularity, dlog form, etc.). However, the coefficients must be "nice," without certain poles. These dual constraints tend to form an algebraic-geometry problem.

Mathematically, we are dealing with a system of linear equations:

$$Mv = b \tag{283}$$

where the $m \times n$ matrix M and the m-dimensional vector b are known. The vector v is to be solved for. Sometimes, we want all entries of v to be polynomials. This is a classical problem in algebraic geometry. Each column of M is a generator in the module R^m so M defines a submodule $\mathcal{S}$ in R^m. To find a polynomial vector solution v is equivalent to lifting $\mathcal{S}$ to b [60]. This can be done by a Groebner basis computation for $\mathcal{S}$.

Example XII.4. *(Module lift for finding UT integrals) Again, consider the integral family in Example XII.3. The sector* $(1,1,1,1,1,1,1,1,0,0,0)$ *contains* 9 *master integrals. It is then challenging to construct* 9 *UT integrals for this sector. To simplify the discussion, we consider only the maximal-cut level. The building blocks are the integrals,*

$$G[1,1,1,1,1,1,1,1,a_9,a_{10},a_{11}] \tag{284}$$

with

$$-2 \leq a_9 + a_{10} + a_{11} \leq 0, \quad a_j \leq 0, \quad j = 9, 10, 11 \tag{285}$$

There are 10 *such integrals, denoted by* $I_1, \ldots, I_{10}$. *A UT ansatz at the maximal-cut level is given by*

$$\sum_{j=1}^{10} c_j I_j \tag{286}$$

Here we simply consider the 4D *leading singularities. There are* 8 *solutions for the maximal cut, say* $p_1, \ldots, p_8$. *Define the residues by*

$$Res_{p_i}[I_j] \equiv M_{ij} \tag{287}$$

We then have the 4D *leading-singularity requirement*

$$\sum_{j=1}^{10} M_{ij} c_j = b_i \tag{288}$$

where b_i *must be rational numbers. The* (8×10) *matrix* M *contains large expressions, so it is not listed explicitly here.*

In this example, it is clear that from the viewpoint of linear algebra, for a fixed vector b, the solution for c is not unique even if a solution exists. If we naively choose a solution for c for the UT ansatz, we unfortunately do not obtain a canonical DE. The problem is that most solutions for c contain complicated unphysical poles, which are unlikely to yield a UT integral even if the 4D leading singularities are all constants.

Here we impose a strong condition: for even integrals, the entries of c must be polynomials in the Mandelstam variables, while for odd integrals, the entries of c must be polynomials in the Mandelstam variables divided by ϵ_{1234}.

In the even case, the entries of c are polynomials, but note that M itself contains fractions. This subtlety can be easily solved by a rescaling $\tilde{M} = FM$, where F is a known polynomial of Mandelstam variables to ensure that $\tilde{M}$ contains polynomials only. Then,

$$\tilde{M}c = Fb \tag{289}$$

is a module lift problem in the module R^8, with $R = \mathbb{Q}[s_{12}, s_{23}, s_{34}, s_{45}, s_{15}]$. For a given b, an algebraic geometry software package such as SINGULAR *can find the lifted solution c within a second.*

For example, with $b = (1, 0, 0, -1, 0, 0, 0, 0)^T$, SINGULAR *provides the solution for c,*

$$c = \begin{pmatrix} 0 \\ 4s_{12}s_{23}(s_{12} - 2s_{45}) \\ 4s_{12}s_{34}s_{45} \\ 4s_{12}s_{15}(s_{12} - s_{45}) \\ 0 \\ 0 \\ -4s_{12}s_{15} \\ -4s_{12}(s_{12} - s_{45}) \\ -4s_{12}(s_{23} + s_{45}) \\ -4s_{12}(s_{12} - s_{34}) \end{pmatrix} \tag{290}$$

The solution is surprisingly simple, and such solutions can be used to build a 9×9 canonical DE on the maximal cut [60]. After fitting the lower subsector contributions, this solution becomes a UT integral.

---

[1] J. M. Henn, J. Phys. A **48**, 153001 (2015), 1412.2296.

[2] V. A. Smirnov, *Analytic tools for Feynman integrals*, vol. 250 of *Springer Tracts in Modern Physics* (Springer, 2012), ISBN 978-3-642-34885-3.

[3] S. Weinzierl, *Feynman Integrals. A Comprehensive Treatment for Students and Researchers*, UNITEXT for Physics (Springer, 2022), ISBN 978-3-030-99557-7, 978-3-030-99560-7, 978-3-030-99558-4, 2201.03593.

[4] P. Nogueira, J. Comput. Phys. **105**, 279 (1993).

[5] P. Nogueira, Nucl. Instrum. Meth. A **559**, 220 (2006).

[6] T. Hahn, Comput. Phys. Commun. **140**, 418 (2001), hep-ph/0012260.

[7] V. Shtabovenko, R. Mertig, and F. Orellana, Comput. Phys. Commun. **306**, 109357 (2025), 2312.14089.

[8] Y.-Q. Ma, *CalcLoop*, `https://gitlab.com/multiloop-pku/calcloop`, a tool for automated multiloop calculations in quantum field theory.

[9] J. M. Henn, Phys. Rev. Lett. **110**, 251601 (2013), 1304.1806.

[10] G. Heinrich, Int. J. Mod. Phys. A **23**, 1457 (2008), 0803.4177.

[11] R. N. Lee and A. A. Pomeransky, JHEP **11**, 165 (2013), 1308.6676.

[12] R. N. Lee, in *49th Rencontres de Moriond on QCD and High Energy Interactions* (2014), pp. 297–300, 1405.5616.

[13] P. A. Baikov, Phys. Lett. B **385**, 404 (1996), hep-ph/9603267.

[14] S. Laporta, Int.J.Mod.Phys. **A15**, 5087 (2000), hep-ph/0102033.

[15] A. V. Smirnov, Comput. Phys. Commun. **189**, 182 (2015), 1408.2372.

[16] F. Lange, J. Usovitsch, and Z. Wu, Comput. Phys. Commun. **322**, 109999 (2026), 2505.20197.

[17] A. von Manteuffel and C. Studerus (2012), 1201.4330.

[18] X. Guan, X. Liu, Y.-Q. Ma, and W.-H. Wu, Comput. Phys. Commun. **310**, 109538 (2025), 2405.14621.

[19] Z. Wu, J. Boehm, R. Ma, H. Xu, and Y. Zhang, Comput. Phys. Commun. **295**, 108999 (2024), 2305.08783.

[20] A. V. Smirnov and A. V. Petukhov, Lett. Math. Phys. **97**, 37 (2011), 1004.4199.

[21] R. N. Lee (2012), 1212.2685.

[22] R. N. Lee, J. Phys. Conf. Ser. **523**, 012059 (2014), 1310.1145.

[23] A. Pak, J. Phys. Conf. Ser. **368**, 012049 (2012), 1111.0868.

[24] Z. Wu and Y. Zhang, Comput. Phys. Commun. **314**, 109681 (2025), 2406.20016.

[25] X. Liu, Y.-Q. Ma, and C.-Y. Wang, Phys. Lett. B **779**, 353 (2018), 1711.09572.

[26] Z.-F. Liu and Y.-Q. Ma, Phys. Rev. Lett. **129**, 222001 (2022), 2201.11637.

[27] X. Liu and Y.-Q. Ma, Phys. Rev. D **105**, L051503 (2022), 2107.01864.

[28] R.-J. Huang, X. Liu, and Y.-Q. Ma (2026), 2607.08477.

[29] J. Bosma, K. J. Larsen, and Y. Zhang, Phys. Rev. D **97**, 105014 (2018), 1712.03760.

[30] A. B. Goncharov, M. Spradlin, C. Vergu, and A. Volovich, Phys. Rev. Lett. **105**, 151605 (2010), 1006.5703.

[31] C. Duhr, JHEP **08**, 043 (2012), 1203.0454.

[32] D. Maitre, Comput. Phys. Commun. **174**, 222 (2006), hep-ph/0507152.

[33] E. Panzer, Comput. Phys. Commun. **188**, 148 (2015), 1403.3385.

[34] C. W. Bauer, A. Frink, and R. Kreckel, J. Symb. Comput. **33**, 1 (2002), cs/0004015.

[35] C. Duhr and F. Dulat, JHEP **08**, 135 (2019), 1904.07279.

[36] J. Broedel, C. Duhr, F. Dulat, B. Penante, and L. Tancredi, JHEP **08**, 014 (2018), 1803.10256.

[37] C. Duhr, F. Porkert, and S. F. Stawinski, JHEP **02**, 014 (2025), 2412.02300.

[38] S. Pögel, X. Wang, and S. Weinzierl, Phys. Rev. Lett. **130**, 101601 (2023), 2211.04292.

[39] I. Bree et al. (ε-collaboration), Phys. Rev. Lett. **136**, 241602 (2026), 2506.09124.

[40] J. Boehm, M. Wittmann, Z. Wu, Y. Xu, and Y. Zhang, JHEP **12**, 054 (2020), 2008.13194.

[41] C. Anastasiou, R. Britto, B. Feng, Z. Kunszt, and P. Mastrolia, JHEP **03**, 111 (2007), hep-ph/0612277.

[42] D. A. Kosower and K. J. Larsen, Phys. Rev. D **85**, 045017 (2012), 1108.1180.

[43] M. Søgaard and Y. Zhang, JHEP **12**, 008 (2013), 1310.6006.

[44] R. N. Lee, JHEP **04**, 108 (2015), 1411.0911.

[45] R. N. Lee, Comput. Phys. Commun. **267**, 108058 (2021), 2012.00279.

[46] M. Prausa, Comput. Phys. Commun. **219**, 361 (2017), 1701.00725.

[47] O. Gituliar and V. Magerya, Comput. Phys. Commun. **219**, 329 (2017), 1701.04269.

[48] K.-T. Chen, Bulletin of the American Mathematical Society **83**, 831 (1977).

[49] L. J. Dixon, J. M. Drummond, and J. M. Henn, JHEP **11**, 023 (2011), 1108.4461.

[50] L. J. Dixon, M. von Hippel, and A. J. McLeod, JHEP **01**, 053 (2016), 1509.08127.

[51] S. Caron-Huot, L. J. Dixon, A. McLeod, and M. von Hippel, Phys. Rev. Lett. **117**, 241601 (2016), 1609.00669.

[52] V. Mitev and Y. Zhang (2018), 1809.05101.

[53] X. Jiang, J. Liu, X. Xu, and L. L. Yang, Phys. Lett. B **864**, 139443 (2025), 2401.07632.

[54] M. Correia, M. Giroux, and S. Mizera, Comput. Phys. Commun. **320**, 109970 (2026), 2503.16601.

[55] S. Caron-Huot and J. M. Henn, JHEP **06**, 114 (2014), 1404.2922.

[56] N. Arkani-Hamed, J. L. Bourjaily, F. Cachazo, S. Caron-Huot, and J. Trnka, JHEP **01**, 041 (2011), 1008.2958.

[57] N. Arkani-Hamed, J. L. Bourjaily, F. Cachazo, and J. Trnka, JHEP **06**, 125 (2012), 1012.6032.

[58] E. Herrmann and J. Parra-Martinez, JHEP **02**, 099 (2020), 1909.04777.

[59] J. Henn, B. Mistlberger, V. A. Smirnov, and P. Wasser, JHEP **04**, 167 (2020), 2002.09492.

[60] D. Chicherin, T. Gehrmann, J. M. Henn, P. Wasser, Y. Zhang, and S. Zoia, Phys. Rev. Lett. **123**, 041603 (2019), 1812.11160.

[61] H. Frellesvig, F. Gasparotto, S. Laporta, M. K. Mandal, P. Mastrolia, L. Mattiazzi, and S. Mizera, JHEP **03**, 027 (2021), 2008.04823.

[62] J. Chen, X. Jiang, X. Xu, and L. L. Yang, Phys. Lett. B **814**, 136085 (2021), 2008.03045.

[63] C. Dlapa, X. Li, and Y. Zhang, JHEP **07**, 227 (2021), 2103.04638.