# Study of the heavy ion collisions dynamics through an hybrid approach: IP Glasma + Kinetic theory at fixed $\eta/s$

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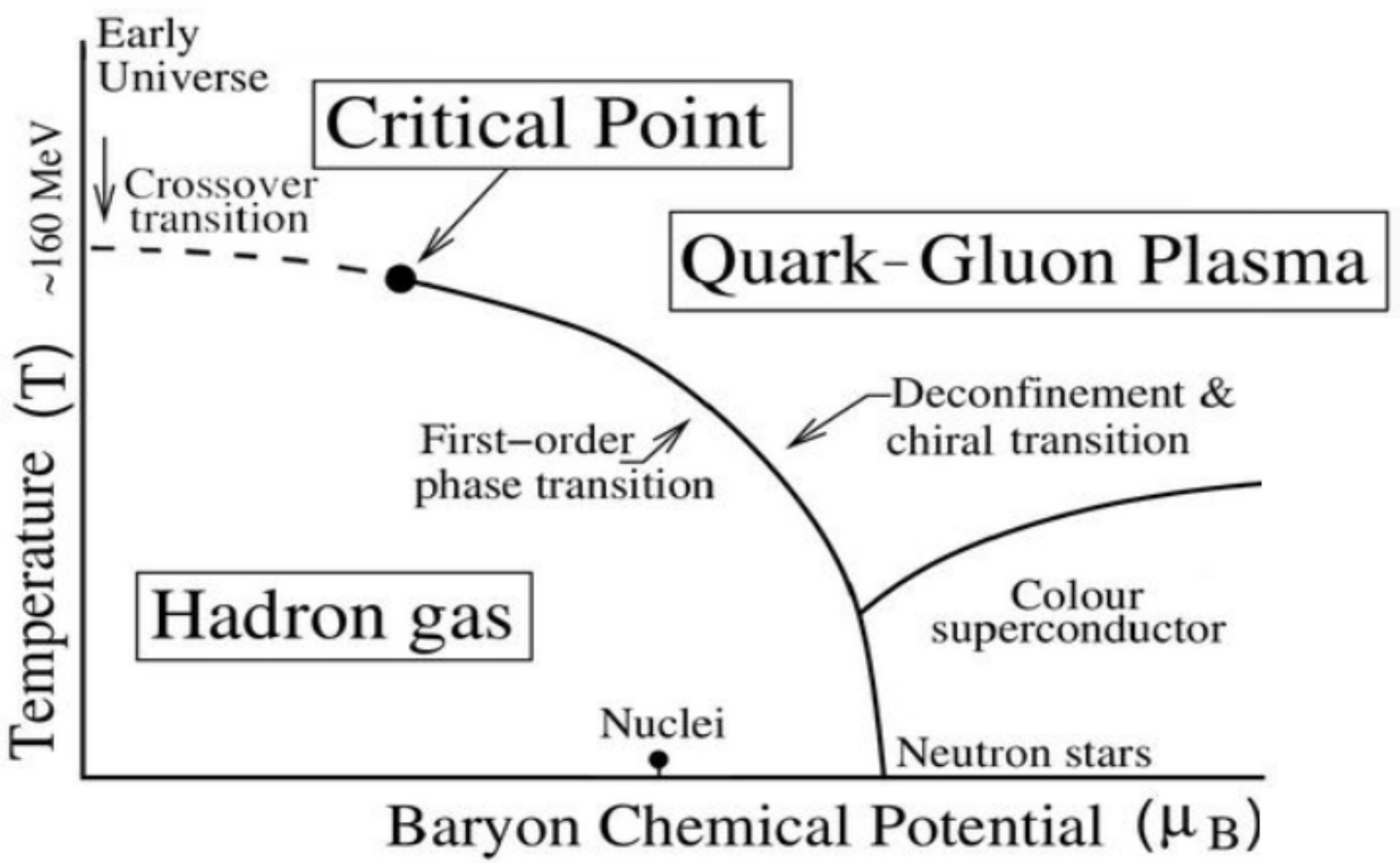
University of Jyväskylä, 5th June, 2026

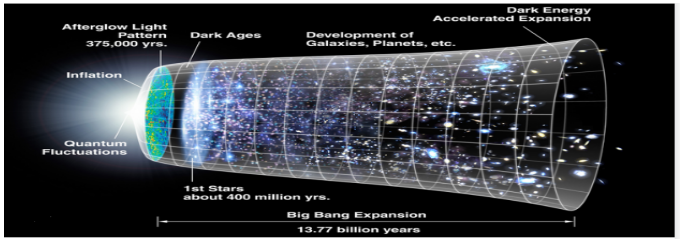


# Phase Diagram of QCD

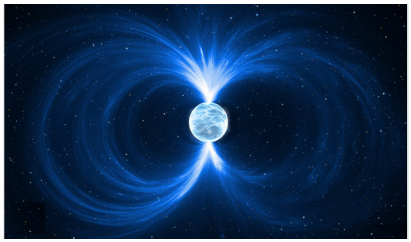
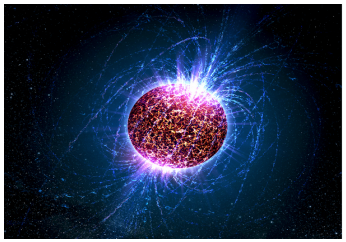


# Where can we find QGP?

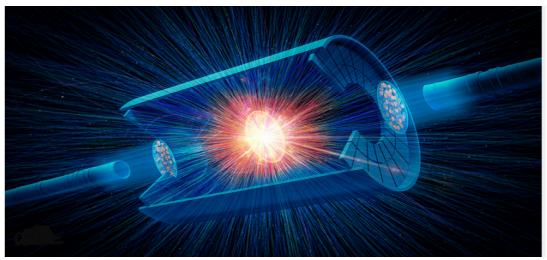
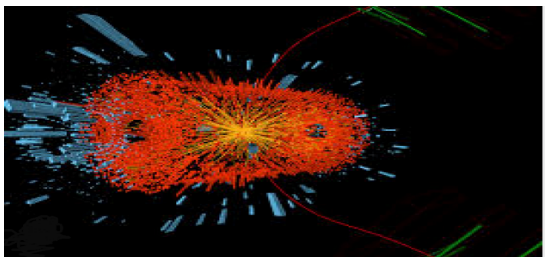
- Early universe, about $10^{-5}\,\mathrm{s}$ after the Big Bang



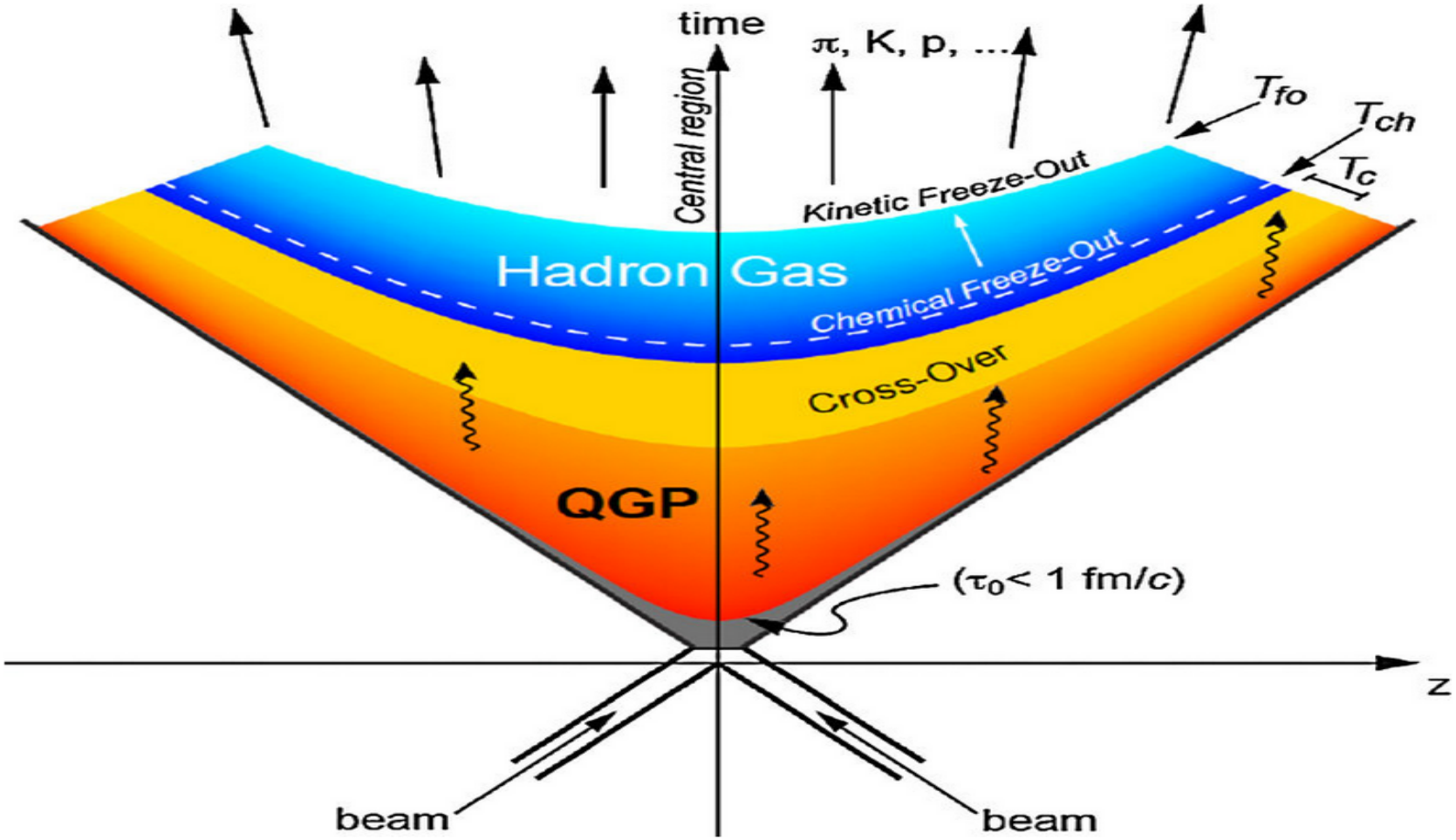

- Core of superdense stars: neutron stars and quark stars


- Ultrarelativistic heavy ion collisions (Little Bang)
  LHC, RHIC; energy of the order of $10^2$–$10^4\,\mathrm{GeV}$

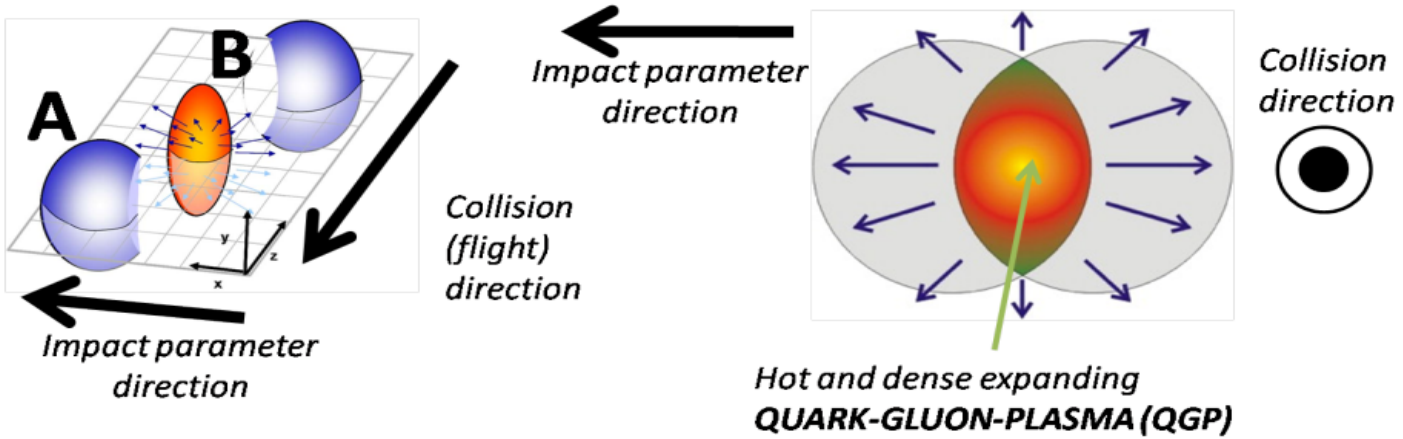
# Time evolution of the QGP



# QGP as a fluid

After the initial stage of collision the system evolves to a thermalized QGP within $\tau \simeq 1\ fm/c$.

Looking at the geometry of collision, the overlap region between two projectiles has the famous "almond shape".




The almond shape indicates an initial anisotropic configuration in space coordinates, translating into a non-zero eccentricity

$$\epsilon = \left\langle \frac{x^2 - y^2}{x^2 + y^2} \right\rangle \neq 0$$

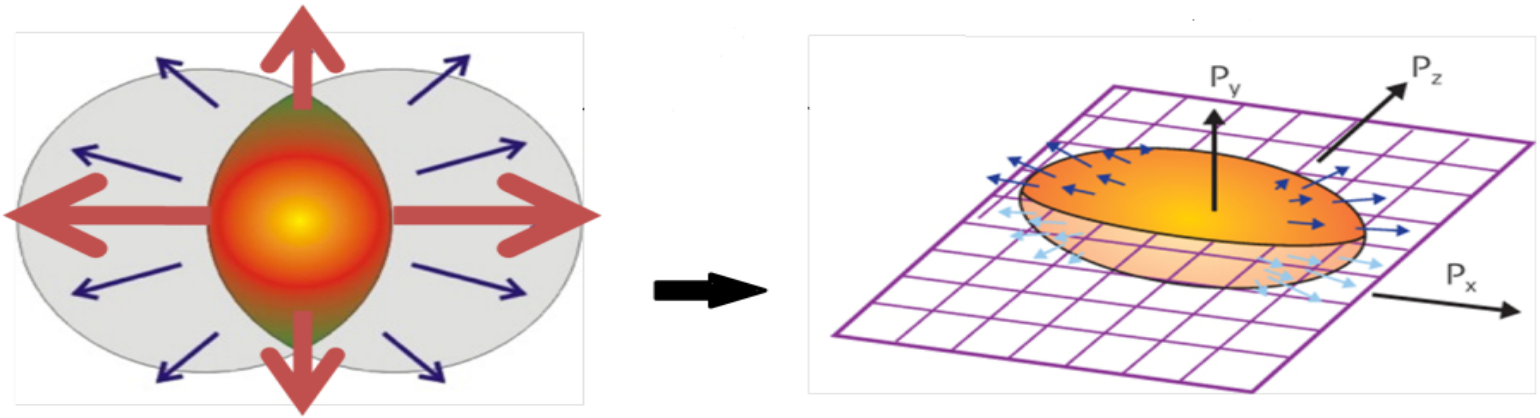
The final angular distribution of hadrons produced from QGP can be expanded in Fourier series as

$$\frac{dN}{d\phi} = \frac{N}{2\pi}\left[1 + 2v_2\cos(2\phi) + ....\right]$$

being $\phi$ the azimuthal angle and $v_2$ the elliptic flow coefficient defined as

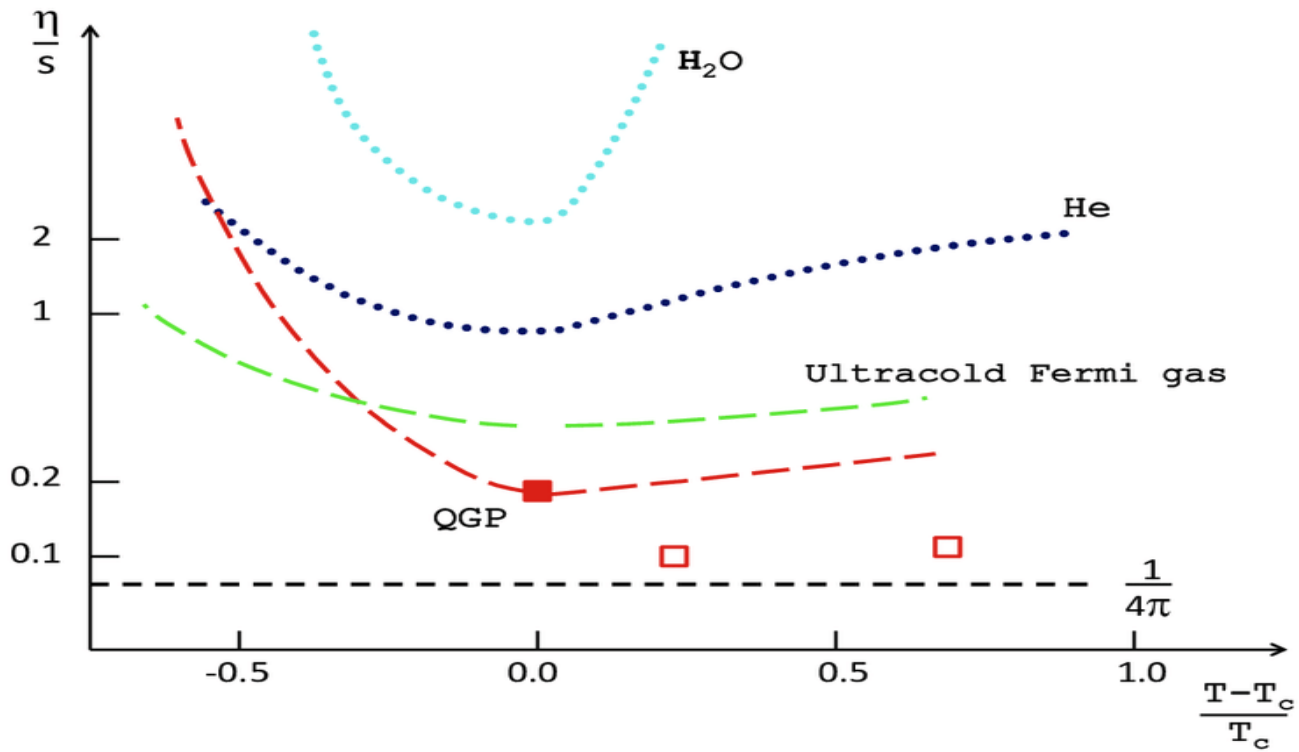
$$v_2 = \left\langle \frac{p_x^2 - p_y^2}{p_x^2 + p_y^2} \right\rangle$$

The initial anisotropy in space coordinate described by $\epsilon$ is transferred into an anisotropy in momentum coordinates described by $v_2$. This indicates the presence of a **flow**.



The experimental measure of $v_2 \neq 0$ is the most important evidence that QGP behaves like a fluid.

QGP has the smallest $\eta/s$ ratio ever measured, near to the lower limit $\eta/s = 1/4\pi$ conjectured for strongly interacting systems [0405231v2].




S. Cremonini, U. Gursoy, and P. Szepietowski, JHEP, 2012. arXiv:1206.3581

From these observations QGP is considered an almost **perfect fluid**.

# The approach

To properly study the QGP evolution, simulations in literature have in common three elements:

**Initial conditions**

A model describing the initial state, including event-by-event geometric fluctuations.

**Dynamical evolution**

A set of equations describing the time evolution of the created QGP.

**Hadronization mechanism**

A model describing the transition from QGP to hadrons as the temperature drops.

The common used approach is:

IP Glasma + Hydrodynamics + Cooper-Frye formula

My approach:

IP Glasma + Relativistic Transport Theory at fixed $\eta/s$

The hadronization mechanism unfortunately at the moment is lacking so in replacement the **parton-hadron duality** is used.

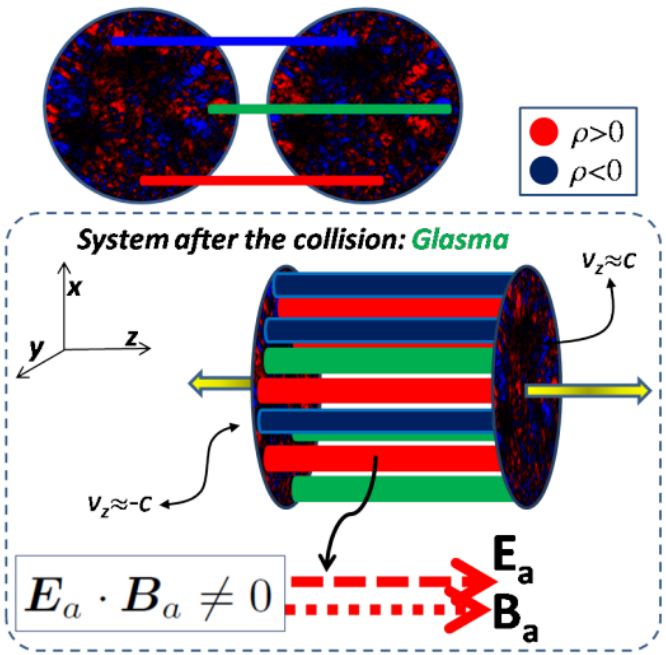
The approach is general, either for nucleus-nucleus (A-A) collisions that for small systems: proton-nucleus (p-A) and proton-proton (p-p) collisions.

# IP Glasma model

The earliest phase of the collision is dominated by highly occupied gluonic degrees of freedom, forming a medium known as the **Glasma**.

The Glasma is effectively described within the **Colour Glass Condensate (CGC)** framework.

The central idea of CGC is to describe the small Bjorken-$x$ degrees of freedom as classical colour fields generated by the large-$x$ degrees of freedom.



The large $x$ colour sources are randomly distributed on $N_s$ thin colour sheets stacked on top of one another. The distribution of these charges is gaussian, characterized by zero average

$$\langle \rho^a(x_T) \rangle = 0$$

and a variance given by

$$\langle \rho^a_{n,x} \ \rho^b_{m,y} \rangle = g^2 \mu^2 \frac{\delta_{n,m}}{N_s} \delta_{a,b} \frac{\delta_{x,y}}{a_T^2}$$

For the proton is introduced the thickness function

$$T_P(x_T) = \frac{1}{3} \sum_{i=1}^{3} \frac{1}{2\pi B_q} \ exp\left[ -\frac{(x_T - x_i)^2}{2B_q} \right]$$

$x_i$ denotes the constituent quarks,which are randomly extracted from

$$T_{cq}(x_T) = \frac{1}{2\pi B_{cq}} \ exp\left( -\frac{x_T^2}{2B_{cq}} \right)$$

For a the nucleus there are $N_{part}$ participating nucleons, each of them has 3 quarks hence the total number of quarks to distribute is $N_q = 3N_{part}$
The nucleus thickness function is

$$T_P(x_T) = \frac{N_{part}}{N_q} \sum_{i=1}^{N_q} \frac{1}{2\pi B_q} \; exp\left[-\frac{(x_T - x_i)^2}{2B_q}\right]$$

which is correctly normalized as

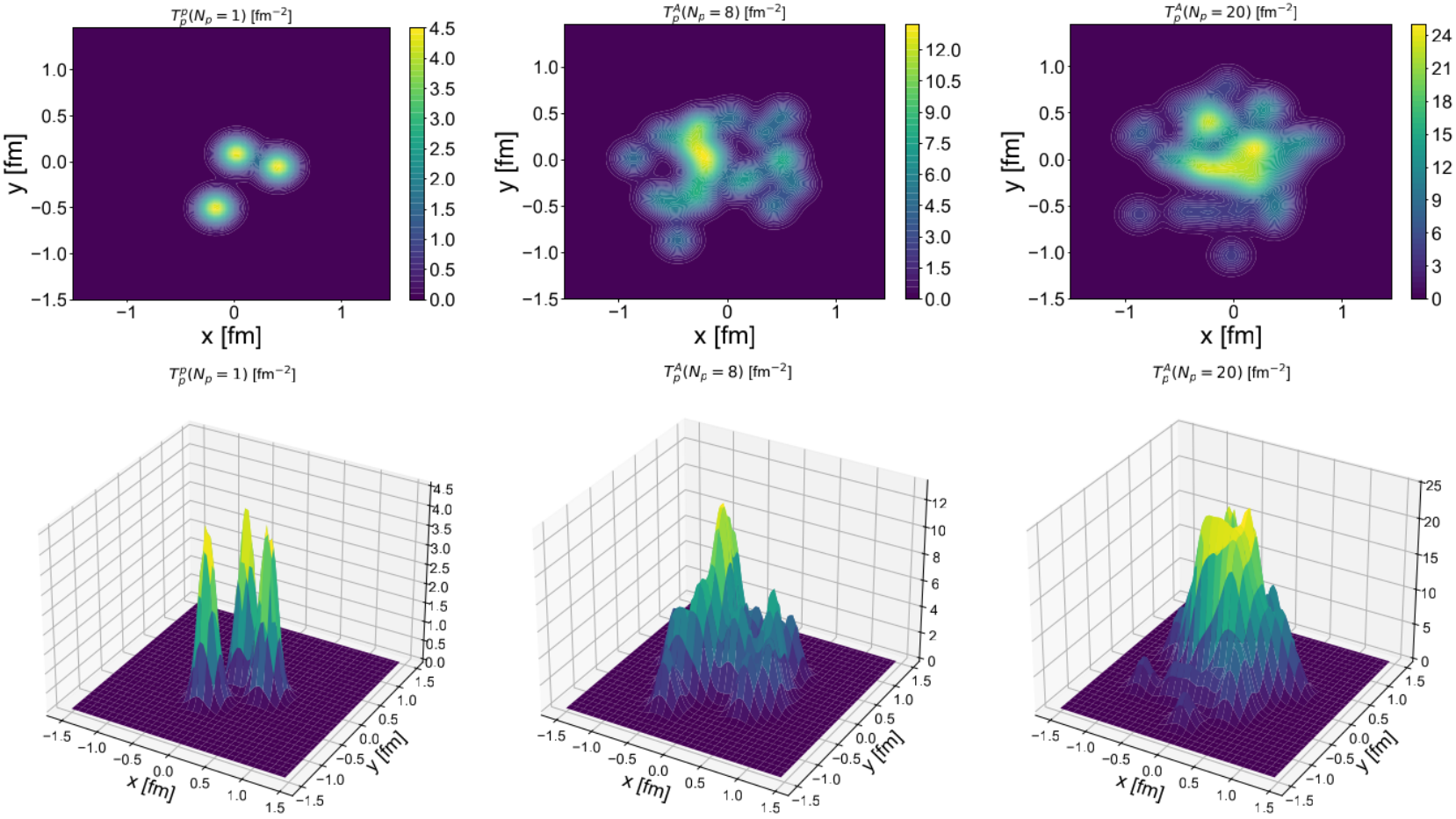
$$\int d^2x_T \; T_P(x_T) = N_{part}$$

Obtained $T_p$, it is possible to get saturation scale $Q_S(x, x_T)$

$$Q_s^2(x, x_T) = \frac{2\pi^2}{N_c}\alpha_s \; x \; g(x, Q_0^2) T_{\;P}(x_T)$$

From $Q_s(x, x_T)$ it is possible to obtain $\mu(x, x_T)$, needed to distribute the colour charges, through the equation

$$g^2 \; \mu(x, x_T) = c \; Q_s(x, x_T)$$




G.Parisi, F.Murgana, V.Greco, M.Ruggieri,Physical Review D, 2026. arXiv:2601.11123v2

The small $x$ degrees of freedom are classical fields which obey to Yang-Mills equation

$$D_\mu F^{\mu\nu} = J^\nu$$

where $D_\mu = \partial_\mu - ig[A_\mu, \cdot\,]$ and $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu + ig\,[A^\mu, A^\nu]$
Putting ourself in the gauge where $A^- = A^i = 0$ with $i = x, y$ and $A^+ = A$ the equation becomes a 2-D Poisson equation

$$\nabla_T^2\ A = -\rho\ (x_T)$$

this equation is solved by using the Fast Fourier Transform (FFT). Achieved the solution, it is possible to evaluate the Wilson line at position $\mathbf{x}$ in transverse plane for each projectile

$$V_{\mathbf{x}} = \prod_{n=1}^{N_s} exp\left\{i\ g\ A^a_{\mathbf{x},n}\ t^a\right\}$$

from which the gauge link for both projectiles A and B are obtained as

$$U^{A,B}_{\mathbf{x},i} = V^{A,B}_{\mathbf{x}}\, V^{\dagger\,A,B}_{\mathbf{x}+\hat{i}}$$

Since the QCD is a non-Abelian theory, the total gauge link $U_{\mathbf{x},i}$ for the combined system of the two projectiles can be obtained by solving the set of equation

$$Tr\left[T_a\left(U^A_{\mathbf{x},i}+U^B_{\mathbf{x},i}\right)\left(\mathbb{1}+U_{\mathbf{x},i}\right)-h.c.\right]=0$$

The components of colour-electric field at initial time after the collision can be expressed in terms of gauge links

$$E^x = E^y = 0$$

$$E^\eta = -\frac{i}{4ga_T^2}\sum_{i=x,y}\Big[\left(U_i(x_T)-\mathbb{1}\right)\left(U^{\dagger B}_i(x_T)-U^{\dagger A}_i(x_T)\right) + \left(U_i(x_T-\hat{i})-\mathbb{1}\right)\left(U^{\dagger B}_i(x_T-\hat{i})-U^{\dagger A}_i(x_T-\hat{i})\right)-h.c.\Big]$$

With this approach, the time evolutions of colour-fields is described by the equations

$$\partial_\tau U_i(\mathbf{x}) = \frac{-i\ g\ a_T}{\tau}\ E^i(\mathbf{x})\ U_i(\mathbf{x})$$

$$\partial_\tau U_\eta(\mathbf{x}) = -i\ g\ a_\eta\ \tau\ E^\eta(\mathbf{x})\ U_\eta(\mathbf{x})$$

Solving the time evolution equations and defining the plaquette variable

$$U_{\mu\nu}(\mathbf{x}) = U_{\mu}(\mathbf{x})U_{\nu}(\mathbf{x}+\hat{\mu})U^{\dagger}{}_{\mu}(\mathbf{x}+\hat{\nu})U^{\dagger}{}_{\nu}(\mathbf{x})$$

it is possible to obtain the longitudinal and transverse component of colour-electric field at time $\tau$

$$E_L^2(\tau) = E^{\eta}(\tau)E^{\eta}(\tau)$$
$$E_T^2(\tau) = \frac{1}{\tau^2}\left[E^{x}(\tau)E^{x}(\tau) + E^{y}(\tau)E^{y}(\tau)\right]$$

and the components of colour-magnetic field

$$B_L^2(\tau) = \frac{2}{g^2 a_T^4}\,\mathrm{Tr}\,(\mathbb{1} - U_{xy})$$
$$B_T^2(\tau) = \frac{2}{(g^2 a_\eta a_T \tau)^2}\sum_{i=x,y}\mathrm{Tr}\,(\mathbb{1} - U_{\eta i})$$

from which we can get physical observables, such as energy density $\epsilon$

$$\epsilon = \mathrm{Tr}\left[E_L^2 + B_L^2 + E_T^2 + B_T^2\right]$$

# Relativistic Boltzmann Transport Theory

The transport approach to heavy ion collisions involves to study the evolution of parton distribution function $f(\mathbf{x}, \mathbf{p}, t)$ through Boltzmann equation

$$p_\mu \partial^\mu f = C[f]$$

with $C[f]$ the collision integral

$$C[f] = \int d\Gamma_2 d\Gamma_{1'} d\Gamma_{2'} \left(f_{1'} f_{2'} - f_1 f_2\right) |\mathcal{M}|^2 \delta^4 \left(p_1 + p_2 - p_{1'} - p_{2'}\right)$$

We consider only $2 \leftrightarrow 2$ collisions.

The effect due to other types of collisions are negligible since the microscopic details doesn't matter so much in this approach.

# Test particles method

The simulation works with a discrete number of particles so the distribution function can be seen as a sum of delta functions

$$f(\mathbf{x},\mathbf{p}) = \sum_{i=1}^{N} \delta^3(\mathbf{x}-\mathbf{x}_i(t))(2\pi)^3\delta^3(\mathbf{p}-\mathbf{p}_i(t))$$

$N$ is the total number of particles, it has to be

$$\int d^3\mathbf{x} \int \frac{d^3\mathbf{p}}{(2\pi)^3 p_0}\ p_0\ f(\mathbf{x},\mathbf{p}) = N$$

To ensure a proper statistics the number of particles is incremented by adding $N_{test}$ fictitious particles.

The simulation is performed over a total number of particles

$$N_{tot} = N \times N_{test}$$

To be physically coherent after the introduction of test particles, all physical observables have to be properly "scaled".

The distribution function changes as

$$f(\mathbf{x},\mathbf{p}) = A\sum_{i=1}^{N_{tot}} \delta^3(\mathbf{x}-\mathbf{x}_i(t))(2\pi)^3\delta^3(\mathbf{p}-\mathbf{p}_i(t))$$

As consequence

$$\int d^3\mathbf{x}\int \frac{d^3\mathbf{p}}{(2\pi)^3 p_0}\; p_0\; f(\mathbf{x},\mathbf{p}) = A\; N_{tot} = N \quad \rightarrow \quad A = \frac{1}{N_{test}}$$

From this the energy-momentum tensor has expression

$$T^{\mu\nu} = \int \frac{d^3p}{(2\pi^3)p^0}\; p^\mu\; p^\nu\; f(x,p) \rightarrow \frac{1}{V\; N_{test}}\sum_{i=1}^{N_{tot}} \frac{p_i^\mu\, p_i^\nu}{p_i^0}$$

and other thermodynamic quantities such as energy density

$$\epsilon = T^{00} = \frac{1}{V\; N_{test}}\sum_{i=1}^{N_{tot}} p_i^0$$

The essential components of the simulation are:

**Particles array**: an array of length $N_{\mathrm{tot}}$ in which particles are stored. Each particle is a structure containing spatial and momentum coordinates, mass, and other physical quantities.

**Lattice**: space is discretized into cells forming a lattice. Both Milne and Cartesian coordinates can be used.
The lattice is initialized with finite dimensions and allowed to expand during the simulation.

The cell's dimensions cannot be freely chosen but three conditions must be satisfied

- **Causality condition**: $l_{max} < \Delta t$; with $l_{max}$ the largest distance within a cell and $\Delta t$ the time step. This condition preserves causality
- **Puglisi condition**: $\Delta x << \lambda_{mpf}$, with $\Delta x$ distance between two particles $\lambda_{mpf}$ mean free path. This condition preserves the locality of collisions
- **Greiner condition**: the diffusion of particles within a time step should not destroy to much the homogeneity of the cells

$$\Delta t < \Delta z_{min} = t\ (tanh^{-1}(\eta_m) - tanh^{-1}(\eta_m - \Delta\eta))$$

The simulation consists in repeating three actions for each time step:

- **Cell Filling**: particles are placed in the cells of the lattice basing on their space coordinates
- **Collision**: particles in the same cell interact and their momentum coordinates are transformed
- **Propagation**: particles move to other cells, propagating in the lattice

Initial and final time instants are set manually before the compilation

# Collision

Basing on the stochastic method to solve numerically Boltzmann equation, the probability that the particles in a cell can interact is given by

$$P_{22} = v_{rel}\ \sigma_{22} \frac{\Delta t}{\Delta^3 x}$$

with $v_{rel}$ relative velocity between two particles and $\sigma_{22}$ the cross section.

A random number $r$ in $[0, 1]$ is extracted, if $P_{22} > r$ the collision happens.

Test particles method is employed so the cross section has to be rescaled $\sigma \to \sigma / N_{test}$
The expression for the collision probability turns into:

$$P_{22} = v_{rel}\ \frac{\sigma_{22}}{N_{test}} \frac{\Delta t}{\Delta^3 x}$$

# Cross section calculation

The cross section is obtained by fixing the ratio $\eta/s$ in order to simulate the dynamical evolution of a fluid. To this purpose, second order Chapman-Enskog expansion is chosen.
For massless particles it imposes

$$\sigma = 1.2558 \, \frac{T}{\eta} = 1.2558 \, \frac{T}{\eta/s \; s}$$

with $T$ temperature, $s$ entropy density.
In general it has expression

$$\sigma = f(z) \, \frac{T}{\eta/s \; s}$$

where $f(z)$ is a certain function of $z = m/T$, depending on the order the approximation is performed.

In a paper of 2015 [1208.0481v2] the numerical results for shear viscosity $\eta$ obtained from the Green-Kubo correlator were compared with the ones obtained by Chapman-Enskog approach (CE).
For $\eta$ Green-Kubo formula assumes the expression

$$\eta = \frac{1}{T}\int_0^\infty dt \int_V d^3x \, \langle \pi^{xy}(x,t)\pi^{xy}(0,t)\rangle$$

In CE approach shear viscosity $\eta$ is given by

$$\eta = \frac{1}{10}(T)^2 \sum_{m=0}^{\infty} c_m \gamma_m$$

The first and second order rare

$$\eta_I = \frac{1}{10} T \frac{\gamma_0^2}{c_{00}}$$

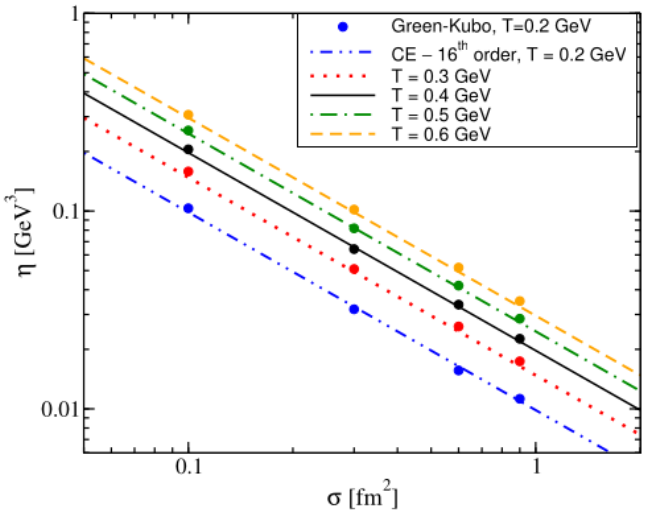
$$\eta_{II} = \frac{1}{10} T \frac{\gamma_0^2 c_{00} - 2\gamma_0\gamma_1 c_{01} + \gamma_1^2 c_{00}}{c_{00}\ c_{11} - c_{01}^2}$$

In the case of massless particles and isotropic cross section, the CE first order approximation is simply

$$\eta^{I}_{CE} = 1.2\frac{T}{\sigma_{tot}}$$

in this particular case the calculation was extended up to $16^{th}$ order

$$\eta^{16^{th}}_{CE} = 1.267\frac{T}{\sigma_{tot}}$$




S. Plumari, A. Puglisi, F. Scardina and V. Greco 2015. arXiv:1208.0481v2

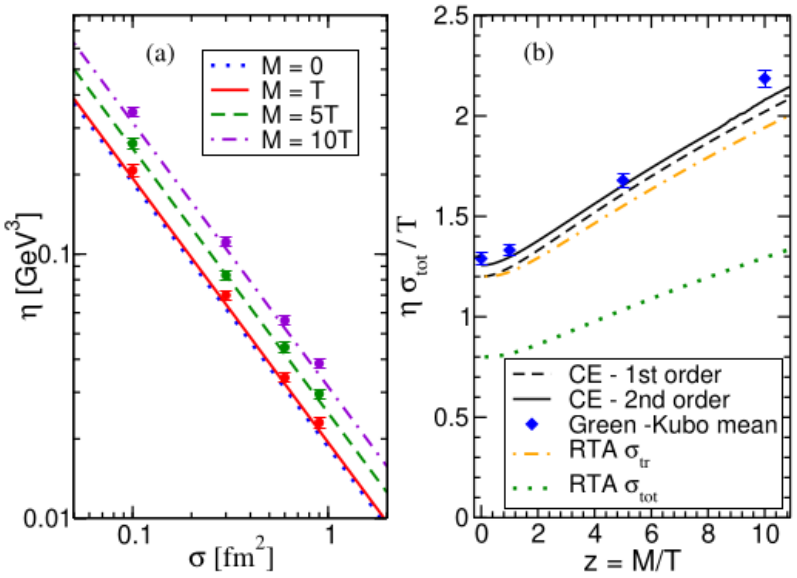
There is a good agreement in all the cross section and temperature range examined.

Remaining in the case of isotropic cross section but passing to massive particles, CE at first order becomes

$$\eta = f(z)\frac{T}{\sigma_{tot}}$$

being $f(z)$ a function of $z = m/T$ which has expression

$$f(z) = \frac{15}{16}\frac{z^4 K_3^2(z)}{(15z^2+2)K_2(2z)+(3z^3+49z)K_3(2z)}$$




S. Plumari, A. Puglisi, F. Scardina and V. Greco 2015. arXiv:1208.0481v2

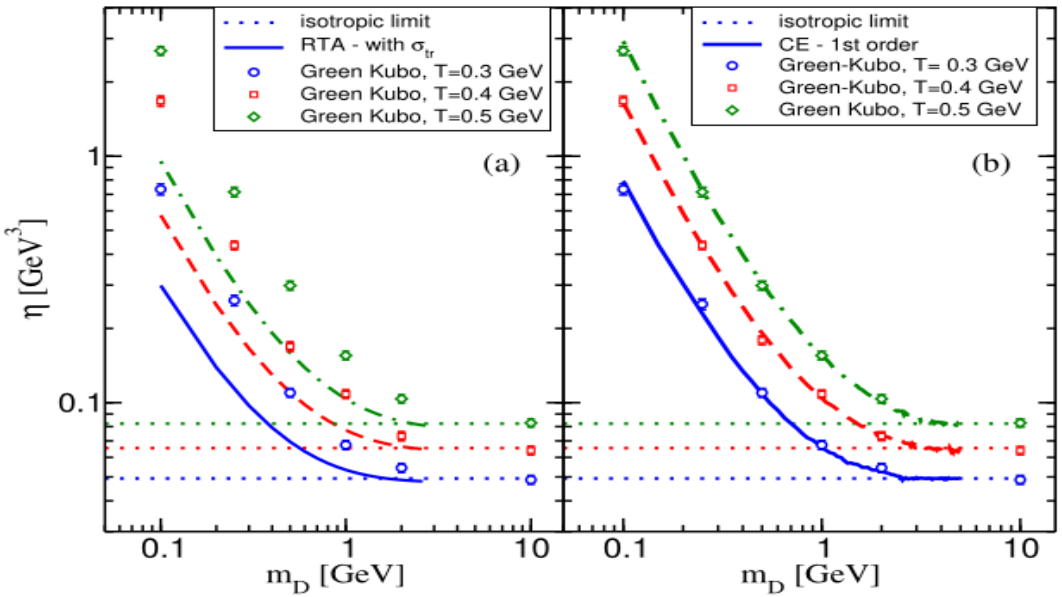
First order CE underestimates $\eta$ with respect to the Green–Kubo formula, while second order CE is in good agreement.

In the case of massless particles and anisotropic cross section first order CE for $\eta$ is

$$\eta^{I}_{CE} = 0.8\,\frac{1}{g(z, T/m_D)}\,\frac{T}{\sigma_{tot}}$$

with

$$g(z, T/m_D) = \frac{32}{25}\,\frac{z}{K_3^2(z)}\int_1^\infty dy\,(y^2-1)^3 h(2zyT/m_D)\left[(z^2y^2+1/3)K_3(2zy) - zyK_2(2zy)\right]$$




S. Plumari, A. Puglisi, F. Scardina and V. Greco 2015. arXiv:1208.0481v2

# Thermodynamic issue

For an ideal fluid, in local rest frame (LRF) energy-momentum tensor is

$$T^{\mu\nu} = \begin{pmatrix} \epsilon & 0 & 0 & 0 \\ 0 & P & 0 & 0 \\ 0 & 0 & P & 0 \\ 0 & 0 & 0 & P \end{pmatrix}$$

this is valid only in the hypothesis of local thermal equilibrium.
In a realistic case this hypothesis is not valid, we need to assume Landau matching condition

$$\epsilon = u_\mu u_\nu T^{\mu\nu}$$

$$n = u_\mu N^\mu$$

and to correctly define the four velocity $u^\mu$ we choose the Landau frame

$$u_\nu T^{\mu\nu} = \epsilon \; u^\nu$$

In each cell of the lattice it is possible to obtain the energy-momentum tensor according to

$$T^{\mu\nu} = \frac{1}{V\ N_{test}} \sum_{i=1}^{N_{tot}} \frac{p_i^\mu\, p_i^\nu}{p_i^0}$$

Eigenvalues and eigenvectors of $T^{\mu\nu}$ are calculated through the **GNU Scientific library** (GLS)*.

Thanks to matching condition and being in the Landau frame, $\epsilon$ and $n$ are calculated. From them temperature is obtained by inverting the relation

$$\epsilon = n\left(3T + m\frac{K_1(m/T)}{K_2(m/T)}\right)$$

Again, thanks to matching conditions, entropy density is given by

$$s = \frac{\epsilon + P}{T}$$

* https://www.gnu.org/software/gsl/

# Propagation

After the collision particles propagate in the lattice. For each particle the simualtion solves the system of equations

$$\begin{cases} \dot{x}_i(t) = \frac{p_i(t)}{p_i^0} \\ \dot{p}_i(t) = F\left(x_i(t), p_i(t)\right) \end{cases}$$

$F = 0$, assuming no external fields are present. In this way the propagation between a collision and another one is free.

**Numerical implementation**: Runge-Kutta method, the order can be chosen depending on computational resources

To describe correctly the propagation of particles, boundary conditions are needed. There are two possible options:

- **Periodic boundary condition** If a particle exits from the borders of domain it re-enters with the same instant but from the opposite directions
- **Dynamic boundary condition** When a particle overcomes the boundary the code simply arrests itself or a warning advice is printed. This conditions are used when the domain is sufficient to contain all the evolution or the lattice is set to expand in the properly way.

# How do we link Glasma with Transport?

Glasma and Transport are different approaches, involving different objects

## Glasma

- Description in terms of chromo-electric $E(\tau)$ and chromo-magnetic $B(\tau)$ fields
- Dynamics governed by Yang–Mills equations

## Transport

- Description in terms of particles (partons)
- Dynamics governed by the relativistic Boltzmann equation

**Bridge between the two: thermodynamic quantities**

Stopping the Glasma evolution at certain time instant $\tau$, energy density is given by

$$\epsilon(\tau) = \mathrm{Tr}\left[E_L^2(\tau) + B_L^2(\tau) + E_T^2(\tau) + B_T^2(\tau)\right]$$

Using thermodynamic relations we get temperature $T$ and particles number density $n$

**Massless**

$$T = \sqrt[4]{\frac{\epsilon\ \pi^2}{3\ d}}$$

$$n = d\ \frac{T^3}{\pi^2}$$

**Massive**

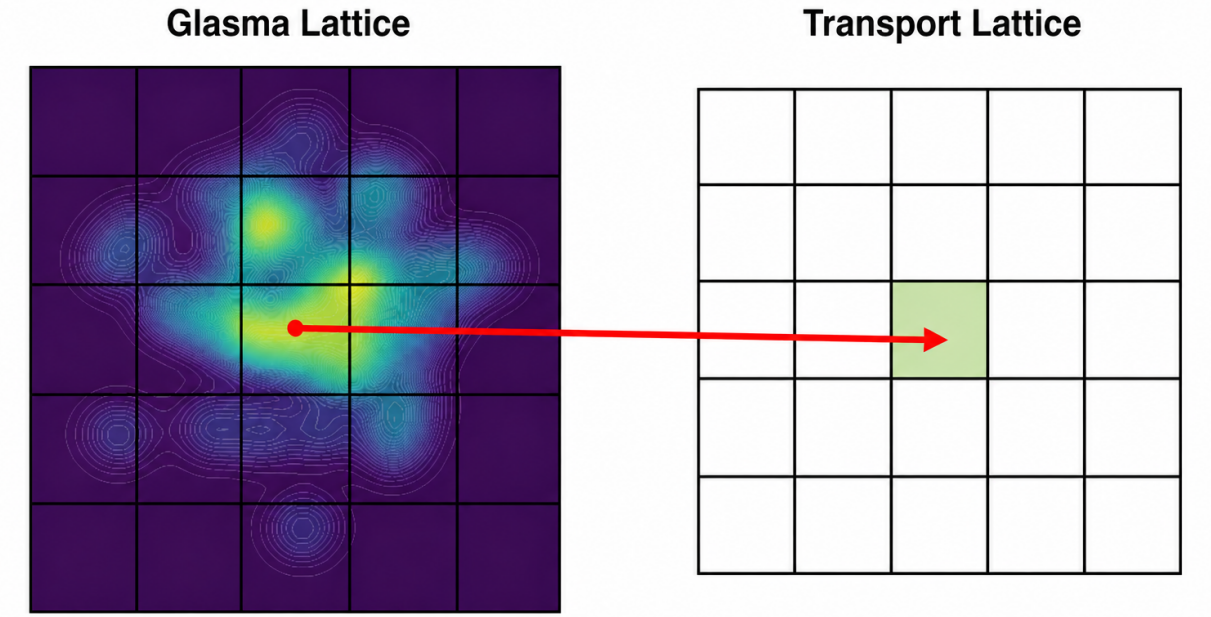
$$\epsilon = \frac{T^4}{2\pi}\left[3\left(\frac{m}{T}\right)^2 K_2\left(\frac{m}{T}\right) + \left(\frac{m}{T}\right)^3 K_1\left(\frac{m}{T}\right)\right]$$

$$n = d\ \frac{m^2\, T}{2\pi^2}\ K_2(m/T)$$

$d$ are the degrees of freedom, this approach has 3 flavours $(u, d, s)$ and 3 colours

$$d = \underbrace{2_{spin}\ (N_c^2 - 1)}_{gluons} + \underbrace{2_{spin}\ 2_{q\bar{q}}\ N_f\ N_c}_{quarks}$$

In each cell of the lattice the energy density $\epsilon$ is converted in temperature $T$ and number particle density $n$




Space coordinates on transversal plane are assigned to particles by generating random numbers

$$x \in [-x_{max}; x_{max}] \qquad y \in [-y_{max}; y_{max}]$$

following the distribution of the density profile through the **acceptance-rejection method**.

In p-A collisions the measured particle distribution in $\eta$ is asymmetric.

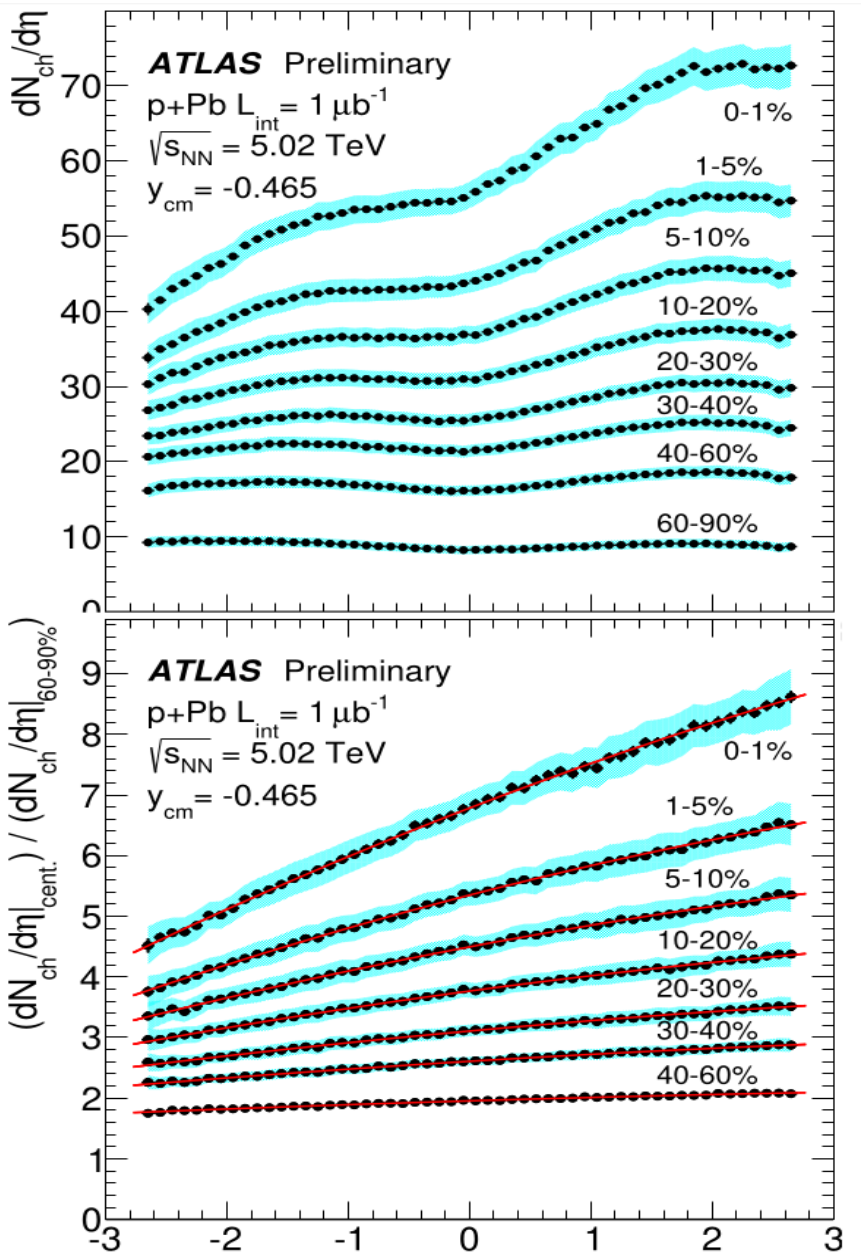
The pseudorapidity coordinate is generated according to

$$\rho_\eta = (a\eta + b)\exp\left[-\frac{(|\eta|-\eta_0)^2}{2\sigma_\eta^2}\Theta(|\eta| - \eta_0)\right]$$

with

$$\eta_0 = 2.5, \qquad \sigma_\eta = 2.5$$

The parameters $a$ and $b$ are extracted from fits to experimental data.




ATLAS collaboration, 2014, arXiv:1403.5738v1

Knowing the temperature of the $i_{th}$ cell, momentum coordinates are assigned using **acceptance-rejection method**.

A random number is generated in the range $[0, p_{max}]$ following the Boltzmann distribution at the temperature of the cell

$$f(p) = exp\left(-\frac{\sqrt{p^2 + m^2}}{T_i}\right)$$

using polar coordinates we get momentum coordinates

$$p_x = p\cos\theta \qquad p_y = p\sin\theta$$

Assuming $Y \simeq \eta$ the third momentum component is given by

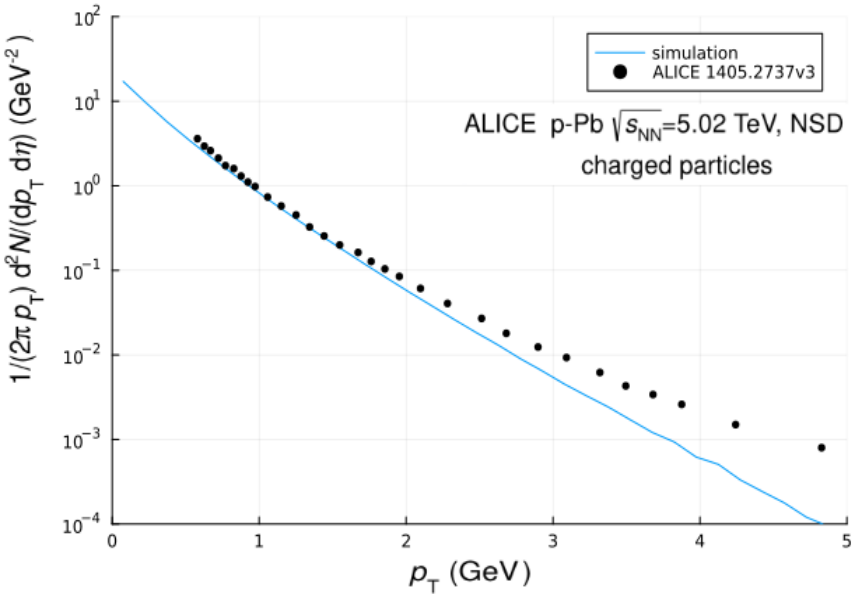
$$p_z = \sinh\eta \ \sqrt{p^2 + m^2}$$

# Results

The first physical observable to compare with experimental data is the spectrum of charged particles in transverse momentum $p_T$.

The low $p_T$ region (until 2 $\mathrm{GeV}$) is in agreement with experimental data.

The high $p_T$ region is underestimated.



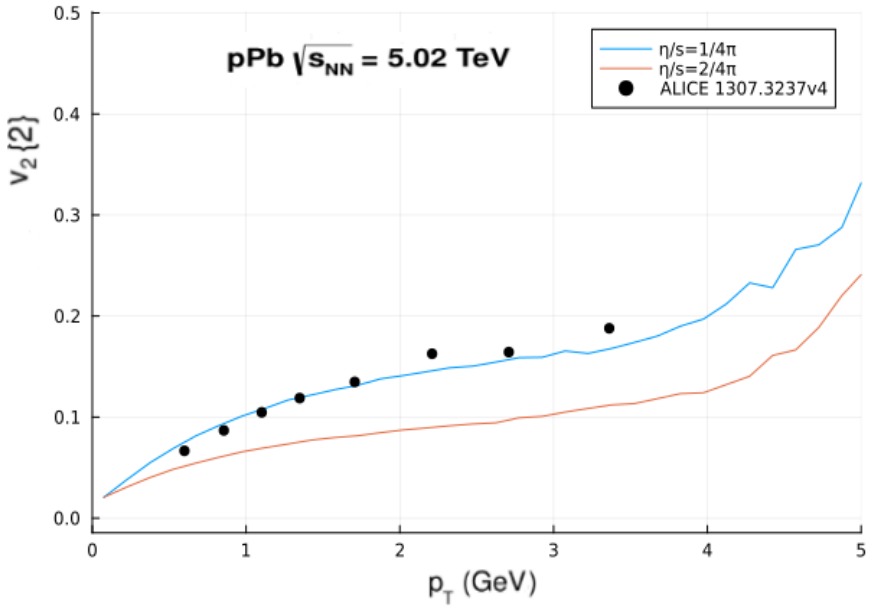

*Preliminary

The simulated spectrum is purely thermal; it is necessary to include mini-jets production or modify the momentum distribution approach.

The $v_2$ elliptical coefficient is simulated for two values of $\eta/s$.

For $\eta/s = 1/4\pi$ the simulation is in agreement with experimental data.

With $\eta/s = 2/\pi$ the functional shape is the same but the curve is lowered.



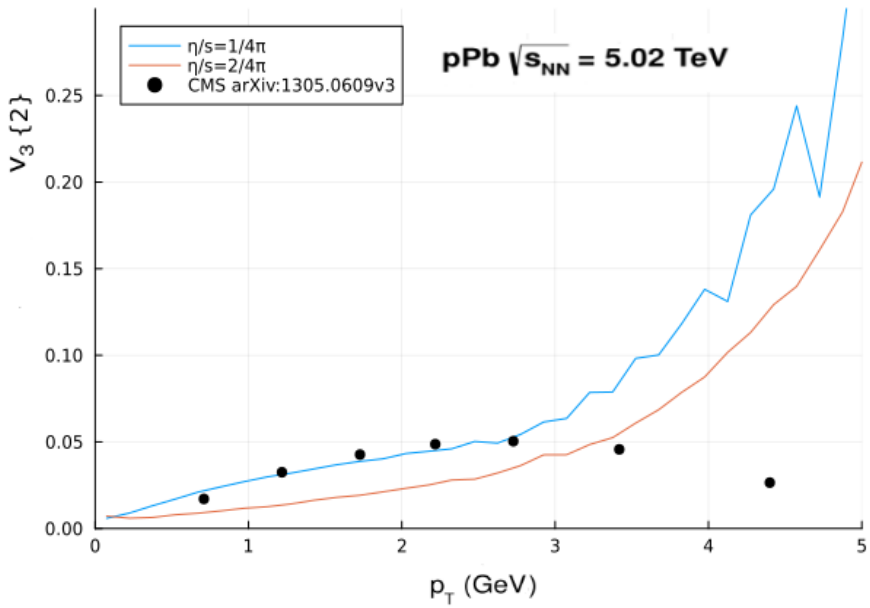

*Preliminary

The behaviour of the two curves is consistent with hydrodynamic simulations, showing the appropriate value to choose for $\eta/s$.

Also $v_3$ triangular coefficient is simulated for two values of $\eta/s$.

For $\eta/s = 1/4\pi$ the simulation is in agreement with experimental data.

With $\eta/s = 2/\pi$ the functional shape is the same but the curve is lowered.




*Preliminary

The trend of $v_2$ is maintained, giving a further confirmation that $\eta/s = 1/4\pi$ is the appropriate value to choose.

# Future perspectives

- Make simulation with massive particles
- Improve the connection method between Glasma and Transport
- Study the correlation between different coefficient $v_n$ and $v_m$
- Include an hadronization mechanism

Thanks for your attention!