

Gradient Flow in Perturbation Theory

Robert Harlander

RWTH Aachen University

10 Sep 2026

Ricci flow in quantum gravity

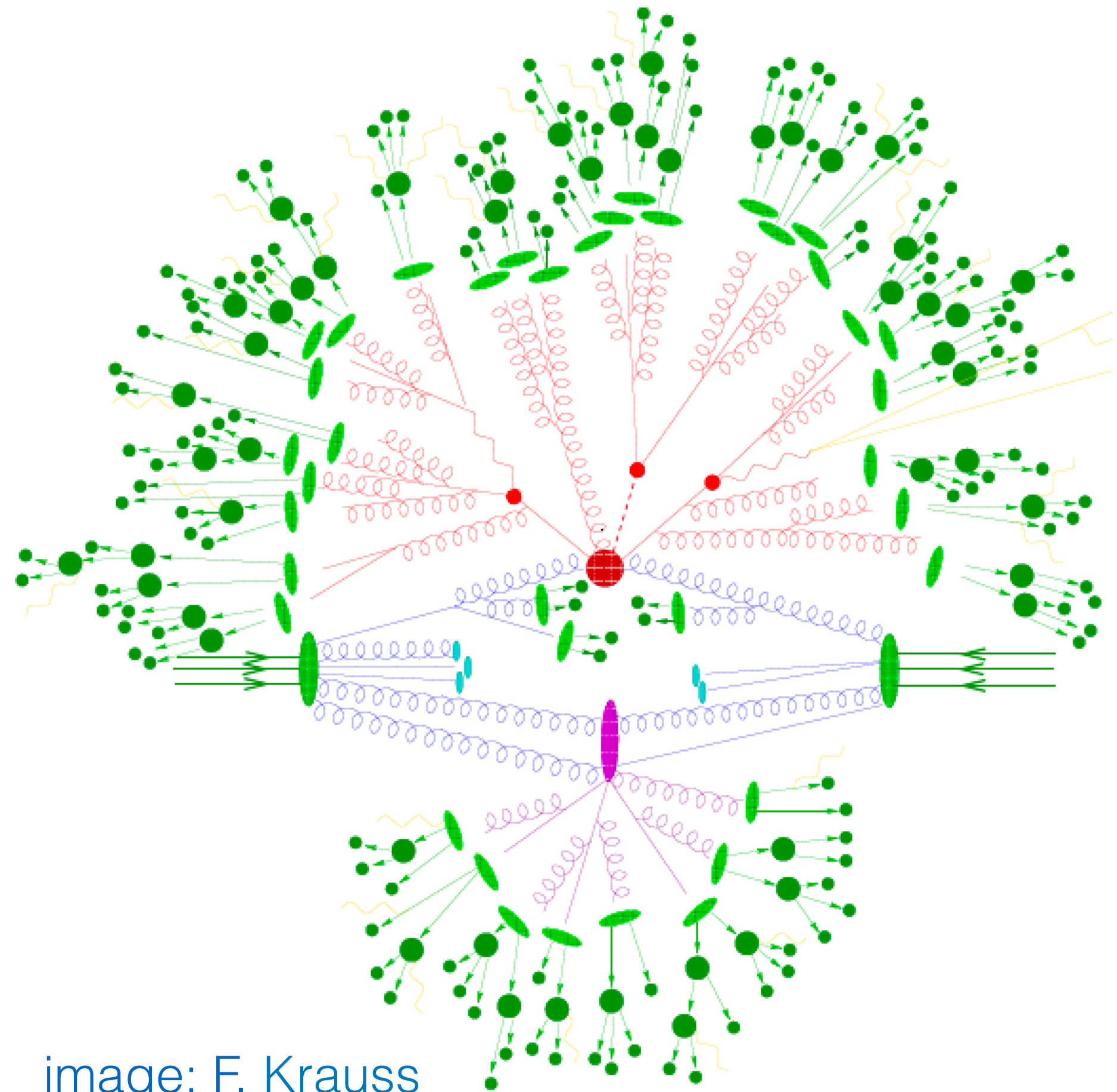
Aachen, 09-11 Sep 2026



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Motivation



Motivation

perturbative contributions:
first principles

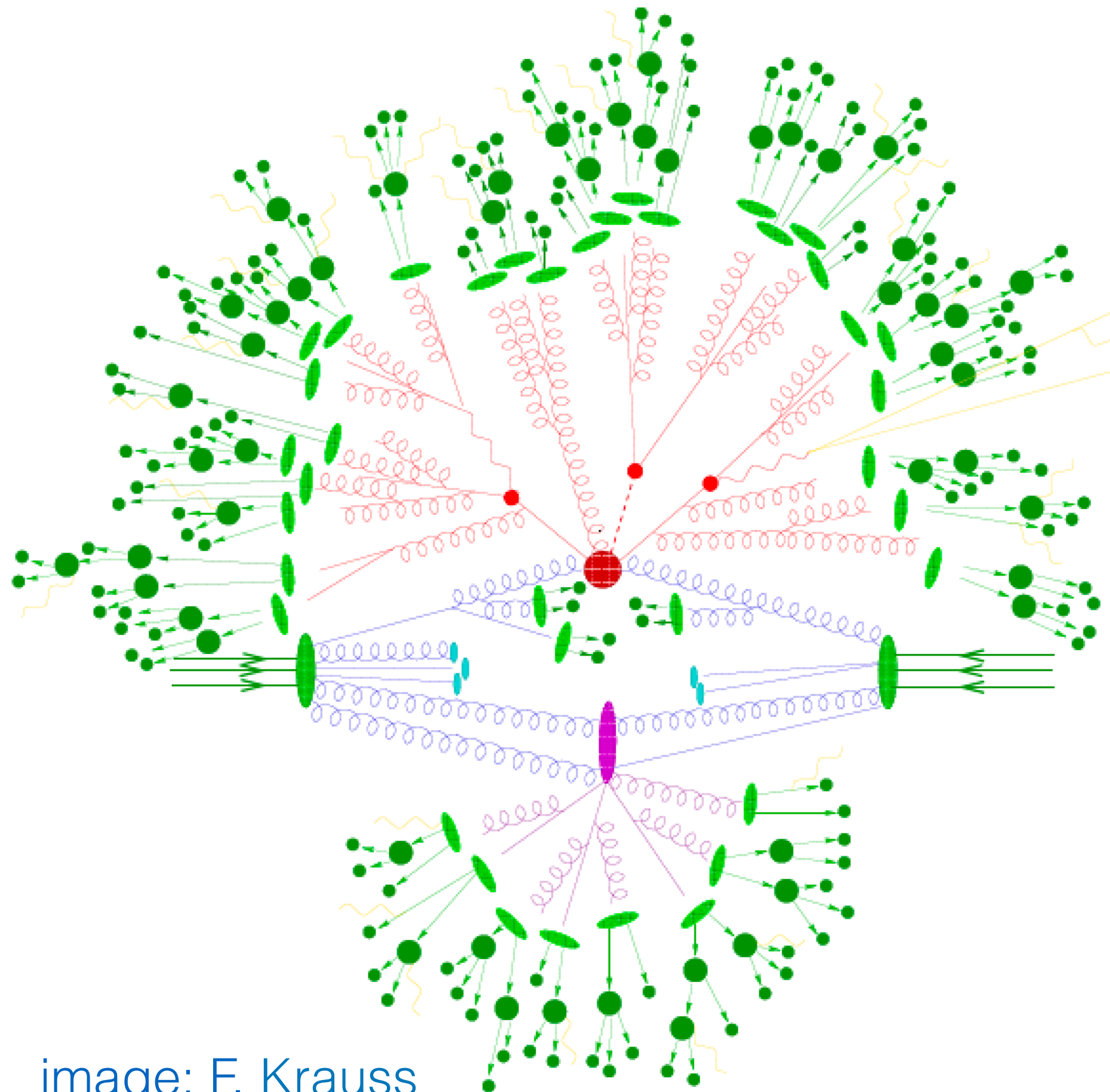


image: F. Krauss

Motivation

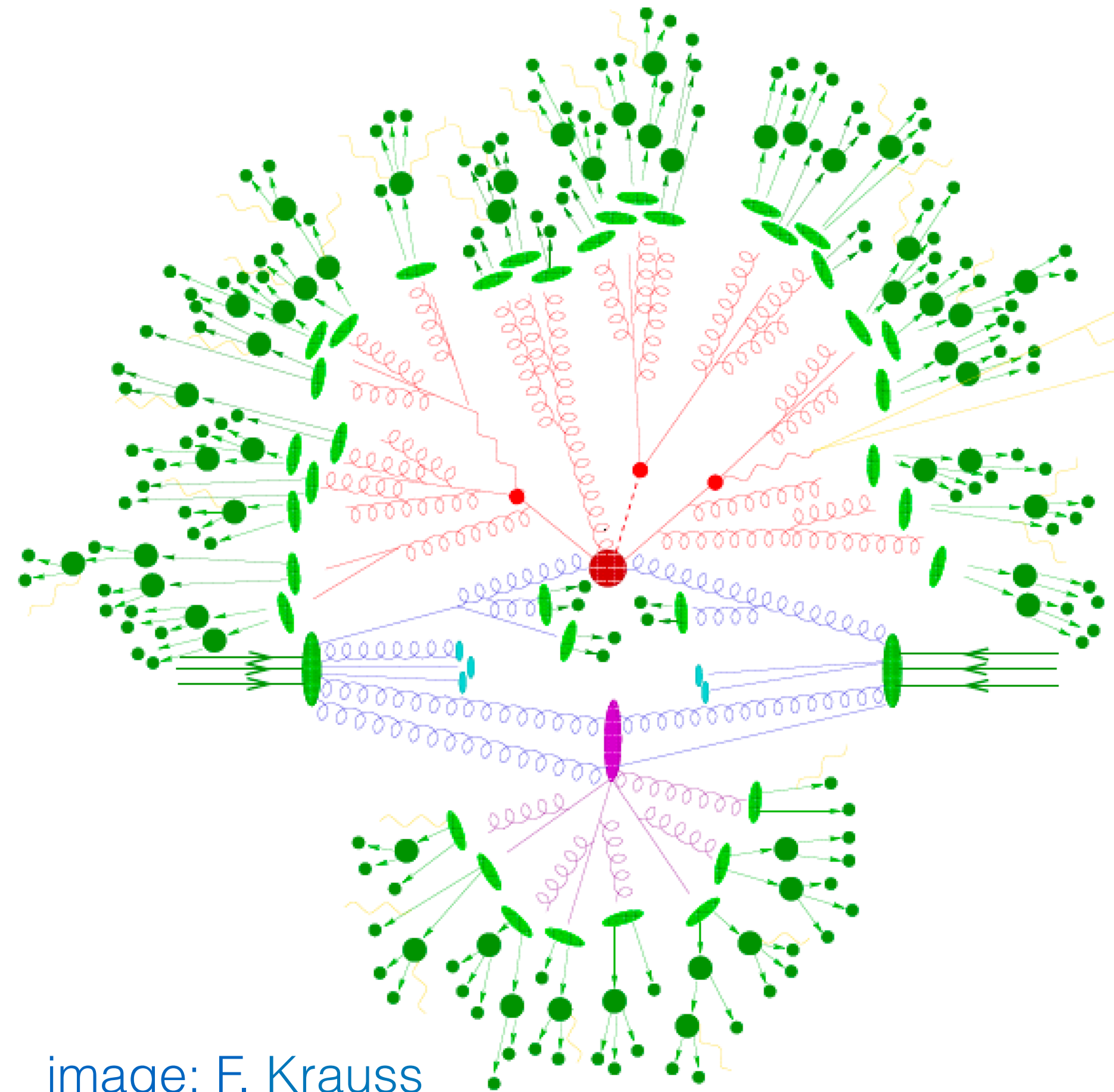


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perturbative contributions:
first principles

non-perturbative contributions:
phenomenological fits

Motivation

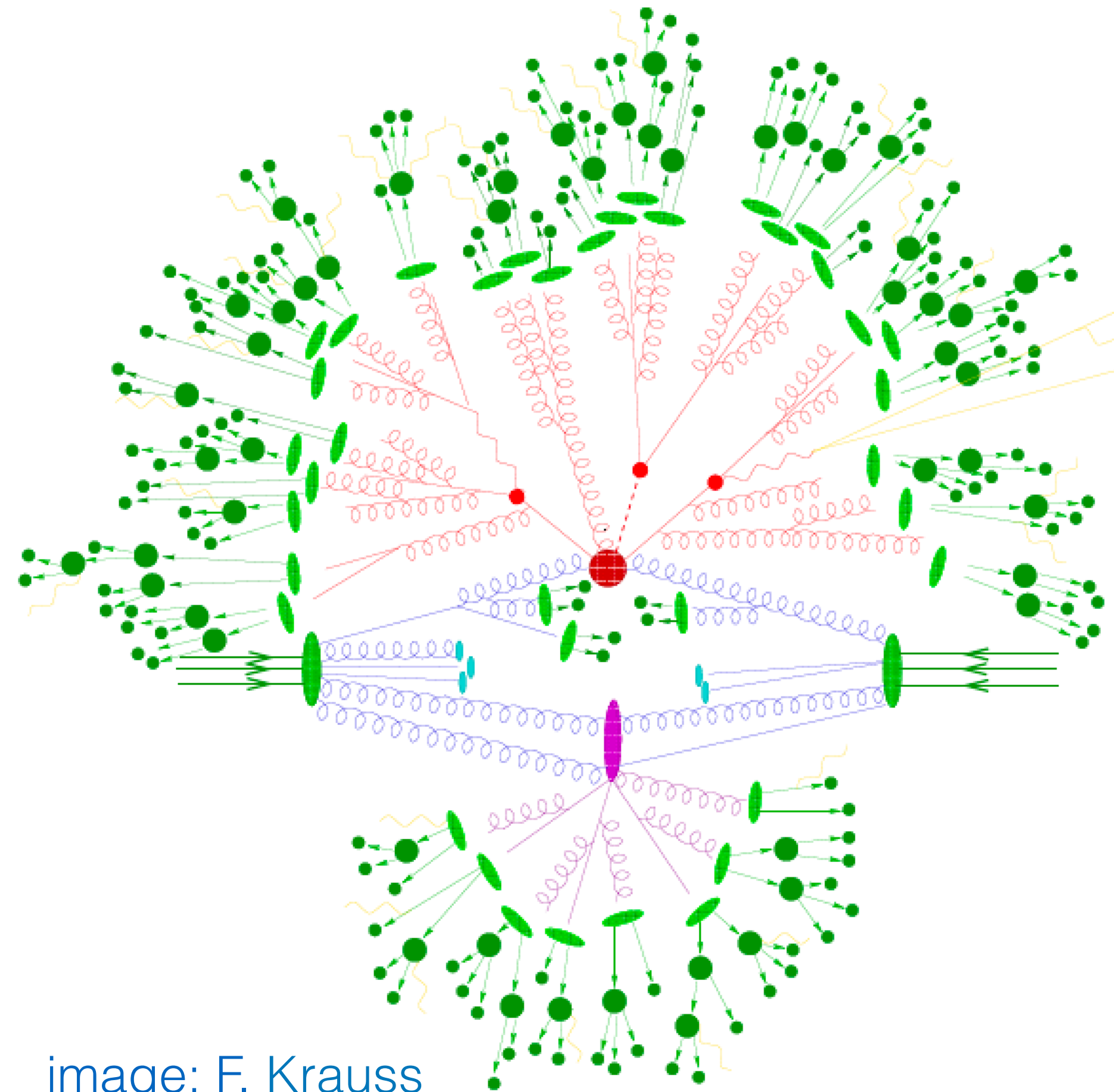


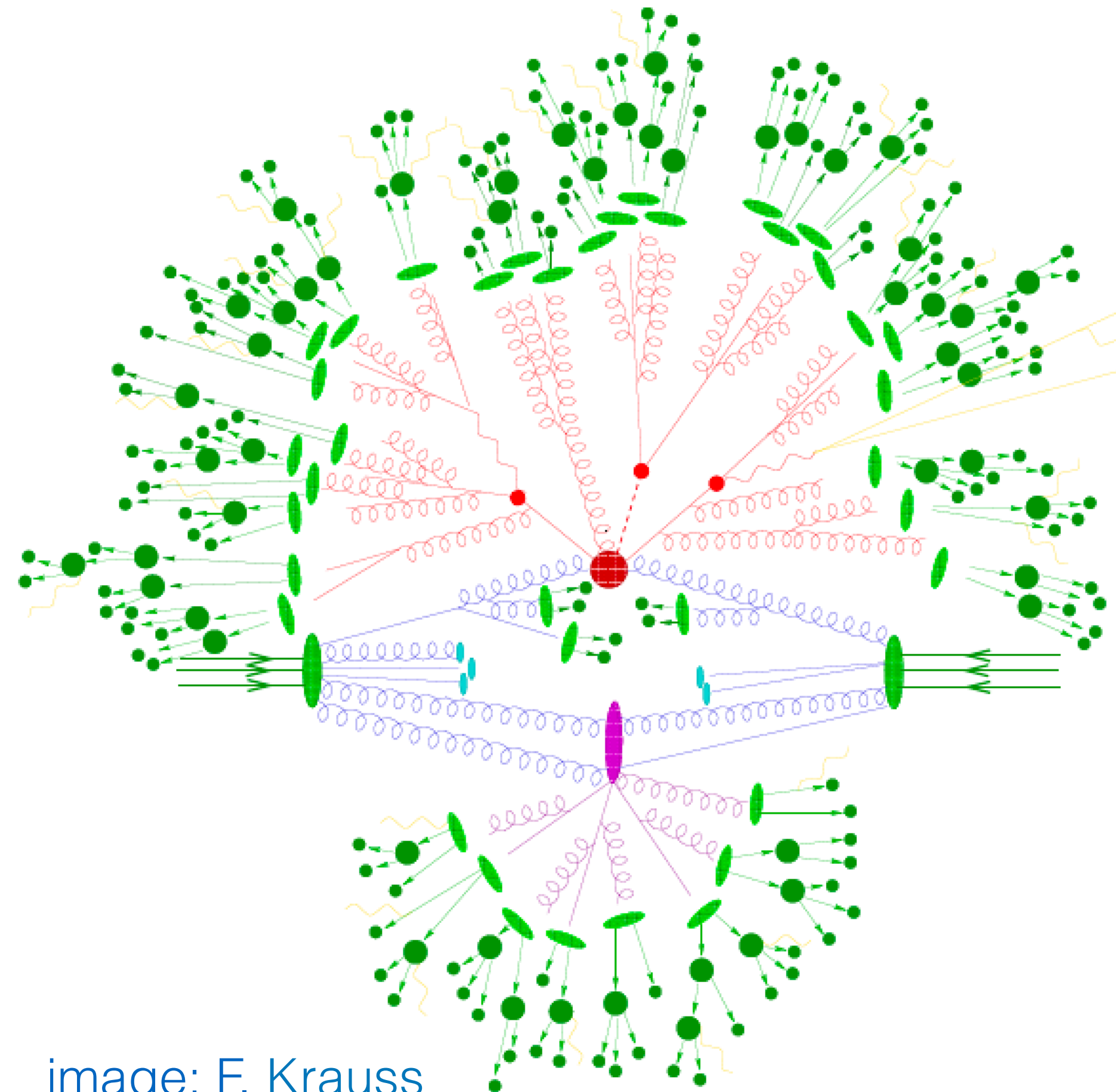
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phenomenological fits

lattice?

Motivation



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lattice?

PDFs defined via light-cone correlators,
lattice formulated in Euclidean space

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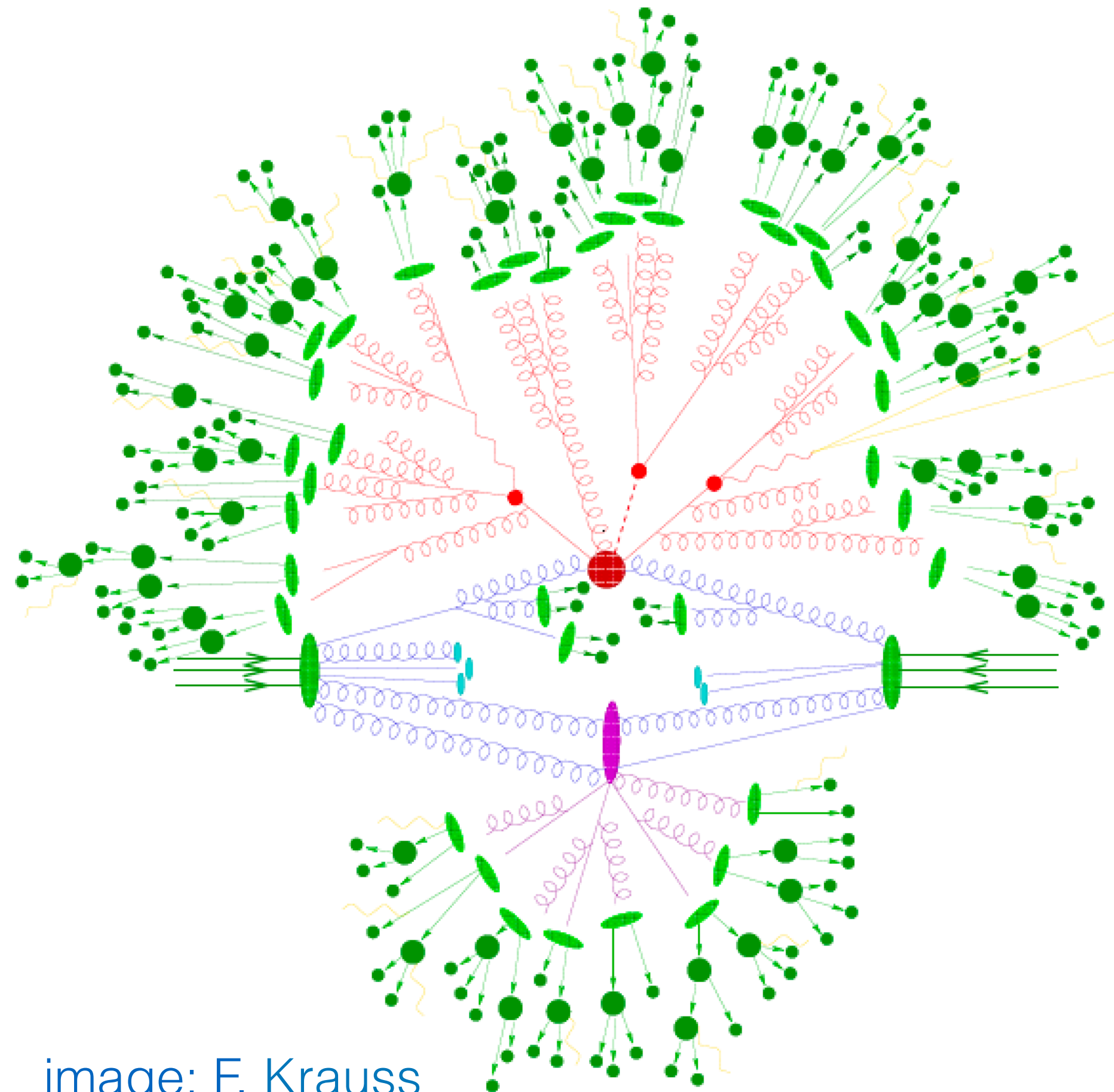


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perturbative contributions:
first principles

non-perturbative contributions:
phenomenological fits

lattice?

PDFs defined via light-cone correlators,
lattice formulated in Euclidean space

alternative: moments

$$\langle x^n \rangle_q = \int_0^1 dx x^n q(x)$$

Moments of parton densities

Moments of parton densities

$$\langle h(p) | \mathcal{O}_{\{\mu_1 \cdots \mu_n\}} | h(p) \rangle \sim p_{\mu_1} \cdots p_{\mu_n} \langle x^{n-1} \rangle$$

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$$\mathcal{O}_{\mu_1 \cdots \mu_n} \sim \int d^4x \, \bar{\psi}(x) \gamma_{\mu_1} \overleftrightarrow{D}_{\mu_2} \cdots \overleftrightarrow{D}_{\mu_n} \psi(x)$$

$$D_\mu = \partial_\mu + ig A_\mu$$

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→ $a \rightarrow 0$ limit challenging

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problem: Lattice breaks Lorentz symmetry

- mixing with lower-dimensional operators
- $a \rightarrow 0$ limit challenging
- only results up to $\langle x^3 \rangle$ available

Martinelli, Sachrajda '87, '88
Alexandrou et al. '20, '21

Gradient flow as cutoff

$$\langle h(p) | \mathcal{O}_{\{\mu_1 \dots \mu_n\}} | h(p) \rangle \sim p_{\mu_1} \dots p_{\mu_n} \langle x^{n-1} \rangle$$

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In $\mathcal{O}_{\mu_1 \dots \mu_n}$, remove high-momentum modes $p^2 \gtrsim 1/t$

$$\psi(x) \rightarrow \chi(x, t)$$

$$A_\mu(x) \rightarrow B_\mu(x, t)$$

in a gauge and Lorentz invariant way

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→ continuum limit can be taken, result depends on UV cutoff t

Gradient flow as cutoff

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$$B_\mu(x, t) = \int^{1/\sqrt{t}} d^4k e^{ikx} \tilde{A}_\mu(k)$$

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$$G_{\mu\nu}(x, t) \sim [D_\mu(t), D_\nu(t)]$$

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analogously:

$$\partial_t \chi(x, t) = D^2 \chi(x, t)$$

$$\chi(x, 0) = \psi(x)$$

$$D_\mu(t) = \partial_\mu + ig[B_\mu(x, t), \cdot]$$

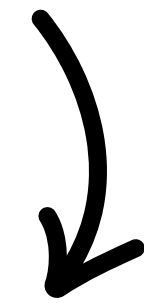
$$G_{\mu\nu}(x, t) \sim [D_\mu(t), D_\nu(t)]$$

Moments of parton densities

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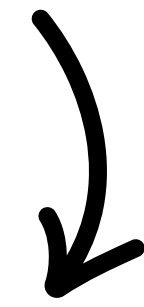


Shindler '24

$$\langle h(p) | \mathcal{O}_{\{\mu_1 \cdots \mu_n\}}(\boldsymbol{t}) | h(p) \rangle$$

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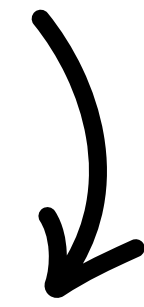
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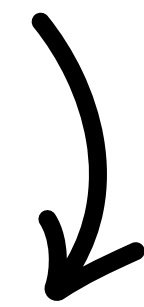
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Continuum limit exists.

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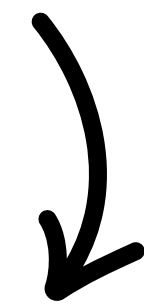
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Continuum limit exists.

Lorentz and gauge invariance preserved.

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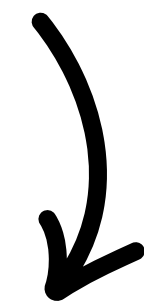
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Lorentz and gauge invariance preserved.

No mixing with other operators.

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$t \rightarrow 0$?

Short flow time expansion

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$$\mathcal{O}_n(t) = \sum_m \zeta_{nm}(t) \mathcal{O}_m + t (\cdots)$$

Lüscher '14
Suzuki '15

Short flow time expansion

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Lüscher '14

Suzuki '15

higher-dimensional operators $\rightarrow$ neglect

Short flow time expansion

$$\mathcal{O}_n(\textcolor{red}{t}) = \sum_m \zeta_{nm}(\textcolor{red}{t}) \mathcal{O}_m + \textcolor{red}{t} (\cdots)$$

Lüscher '14
Suzuki '15

higher-dimensional operators $\rightarrow$ neglect

$$\langle h(p) | \mathcal{O}_{\{\mu_1 \cdots \mu_n\}} | h(p) \rangle = \sum_m \zeta_{mn}^{-1}(\textcolor{red}{t}) \langle h(p) | \mathcal{O}_{\{\mu_1 \cdots \mu_n\}}(\textcolor{red}{t}) | h(p) \rangle$$

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lattice

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perturbation theory

lattice

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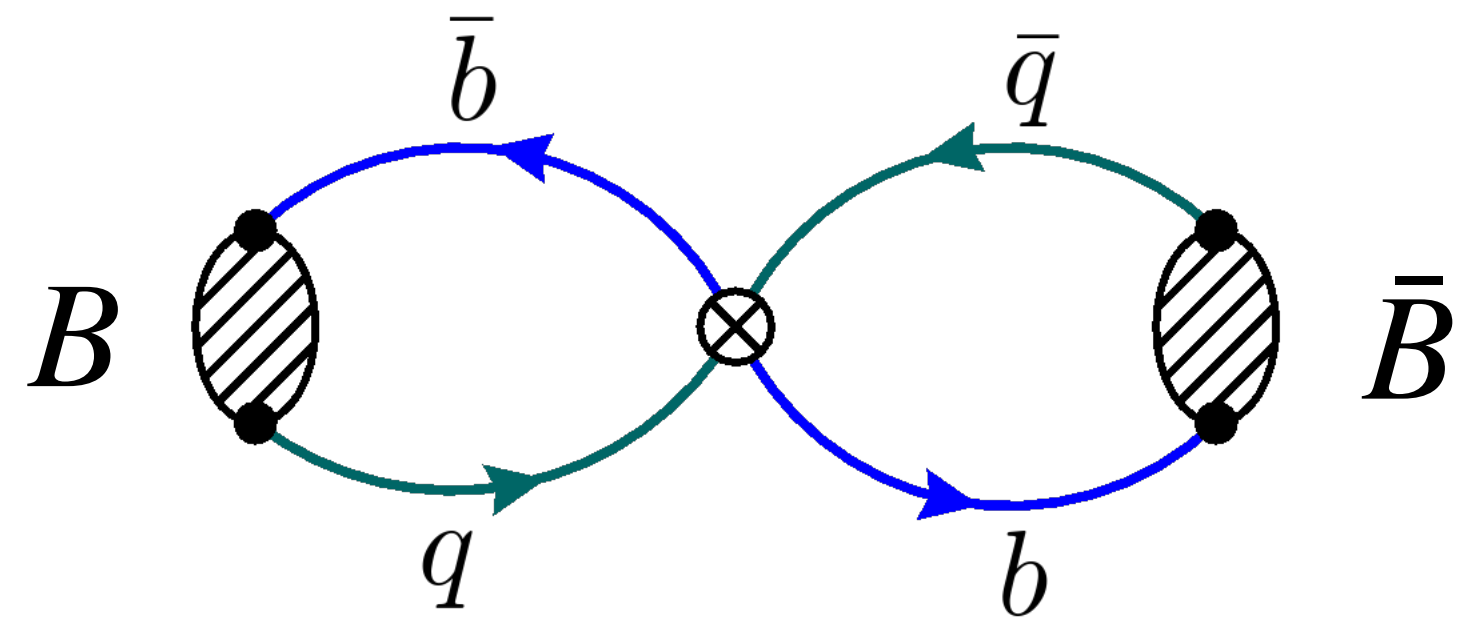
perturbation theory

lattice

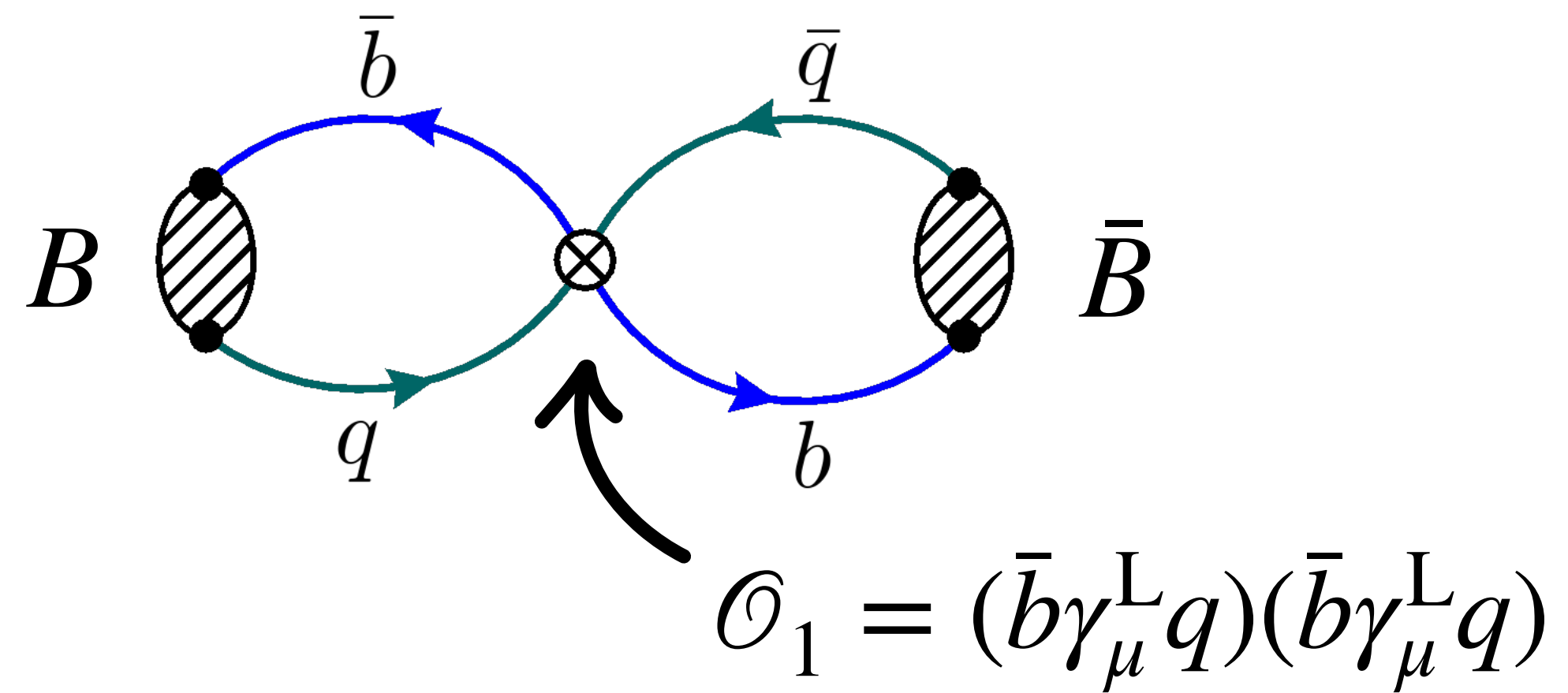
actually:

$$\zeta_{nm}^{\text{R}}(\textcolor{red}{t}, \mu) = \sum_k \zeta_{nk}(\textcolor{red}{t}) Z_{km}^{-1}$$

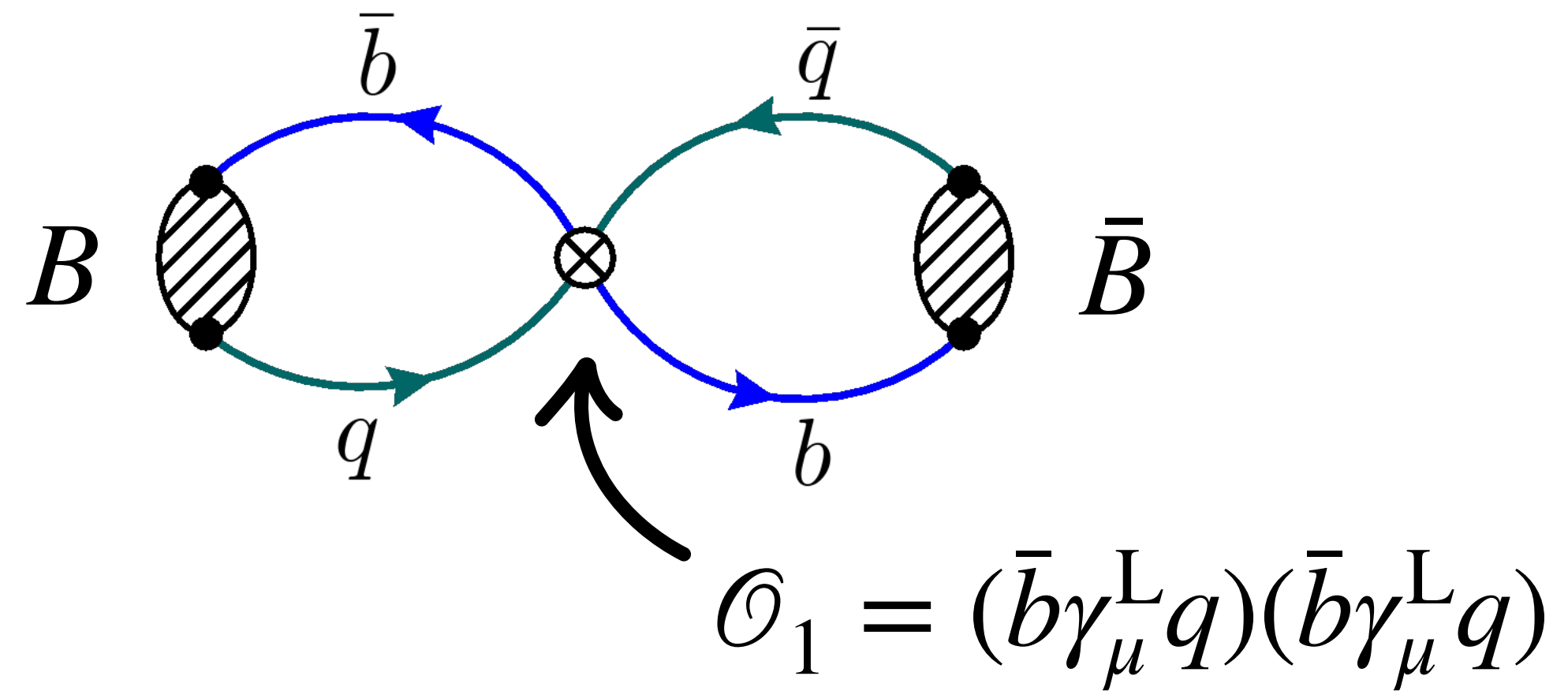
Meson mixing



Meson mixing

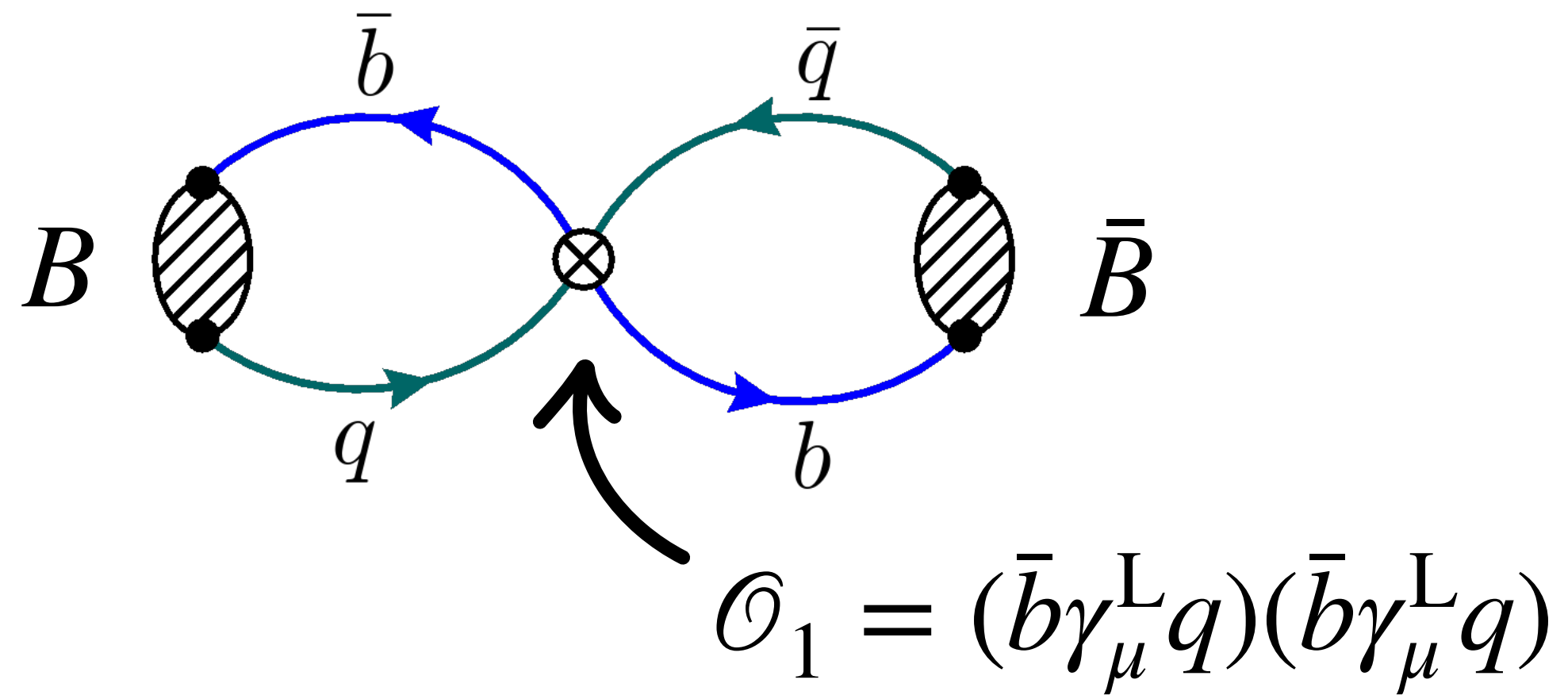


Meson mixing



$$\rightarrow B_1 \sim \langle B | \mathcal{O}_1 | B \rangle \quad \text{bag parameter}$$

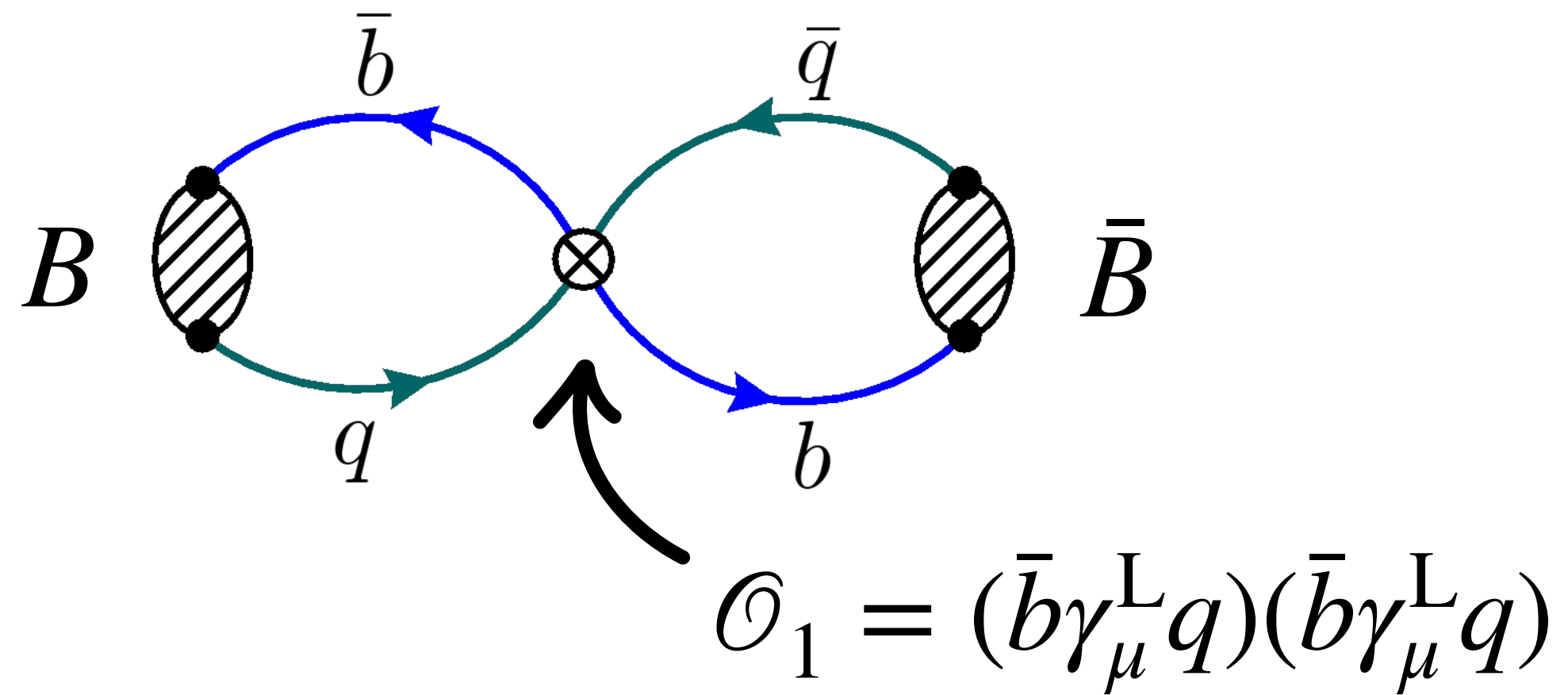
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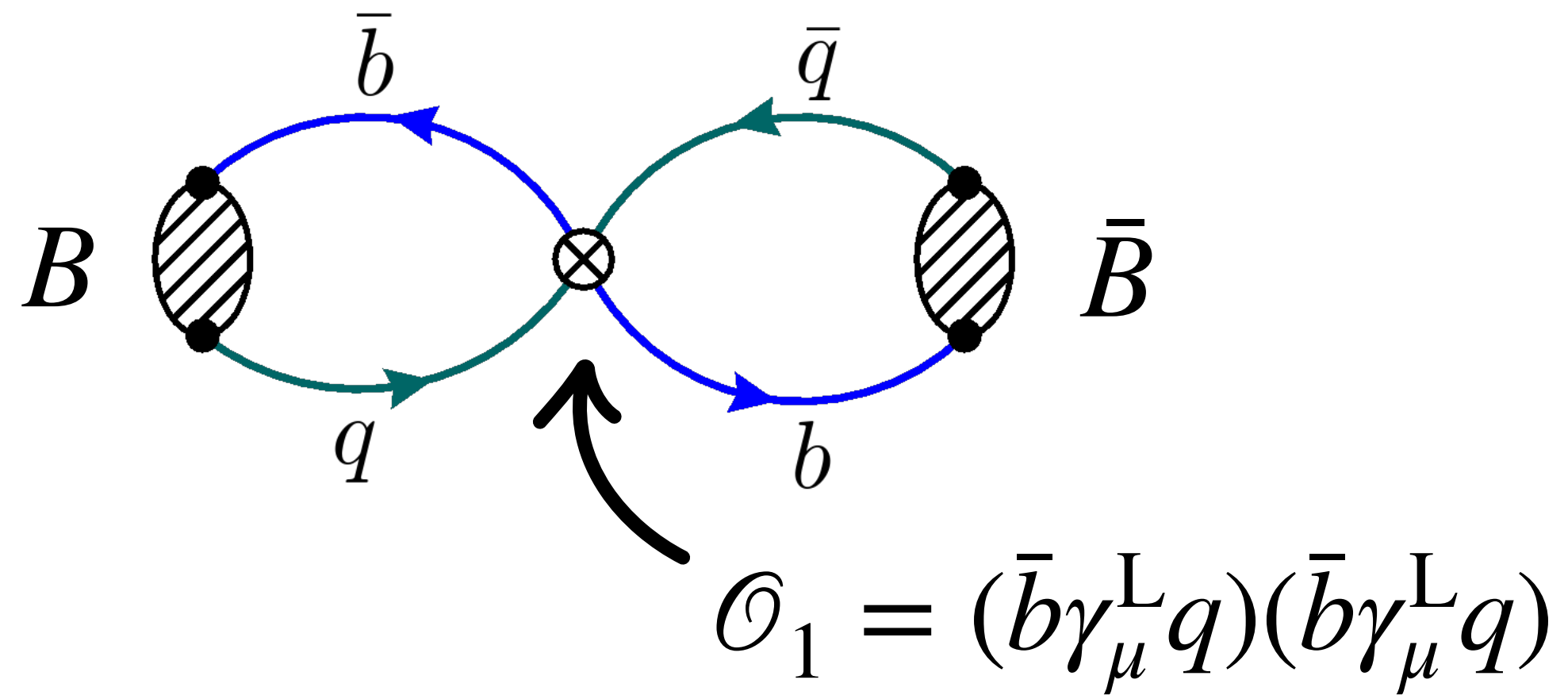


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perturbative $\nearrow$

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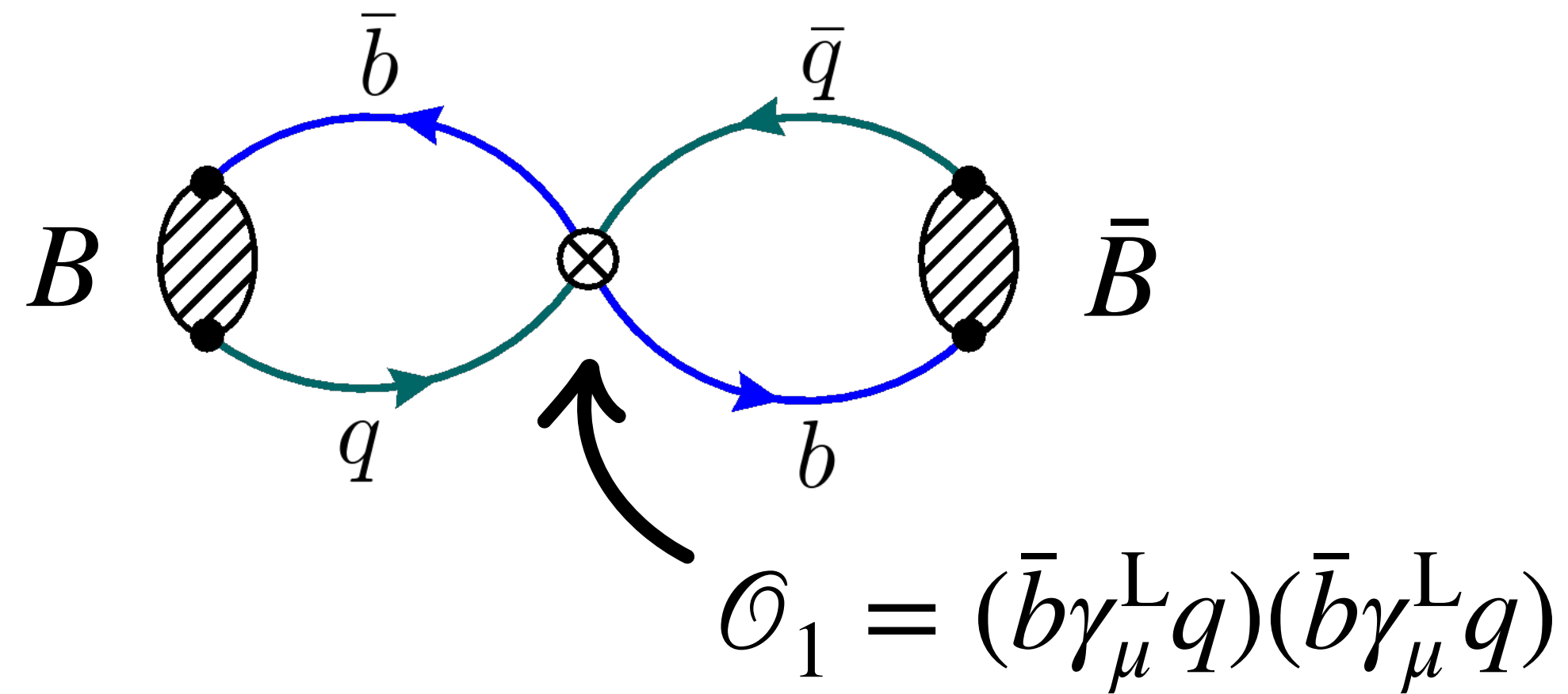


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perturbative $\nearrow$ $\nwarrow$ lattice

Meson mixing

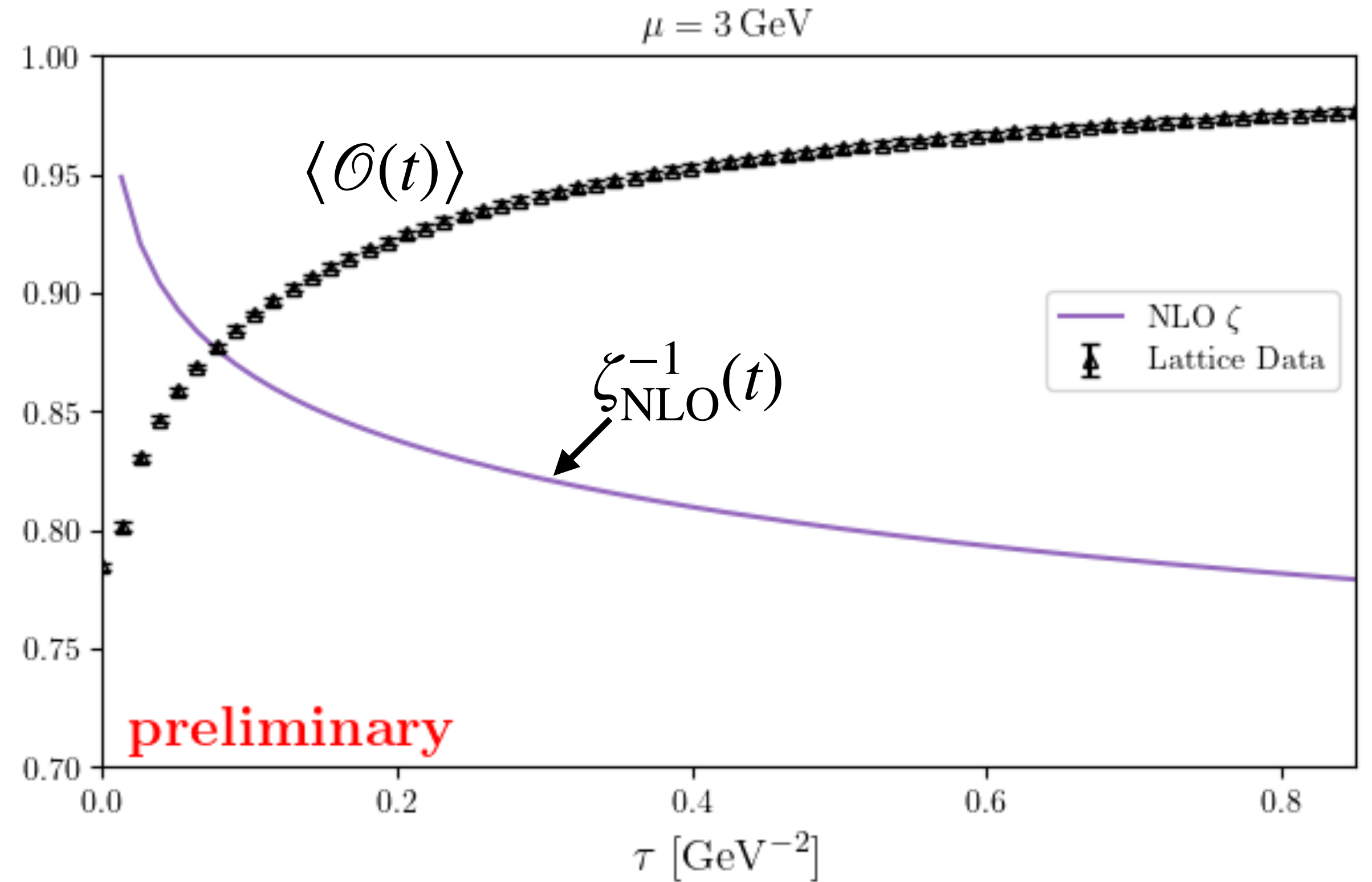


→ $B_1 \sim \langle B | \mathcal{O}_1 | B \rangle$ bag parameter

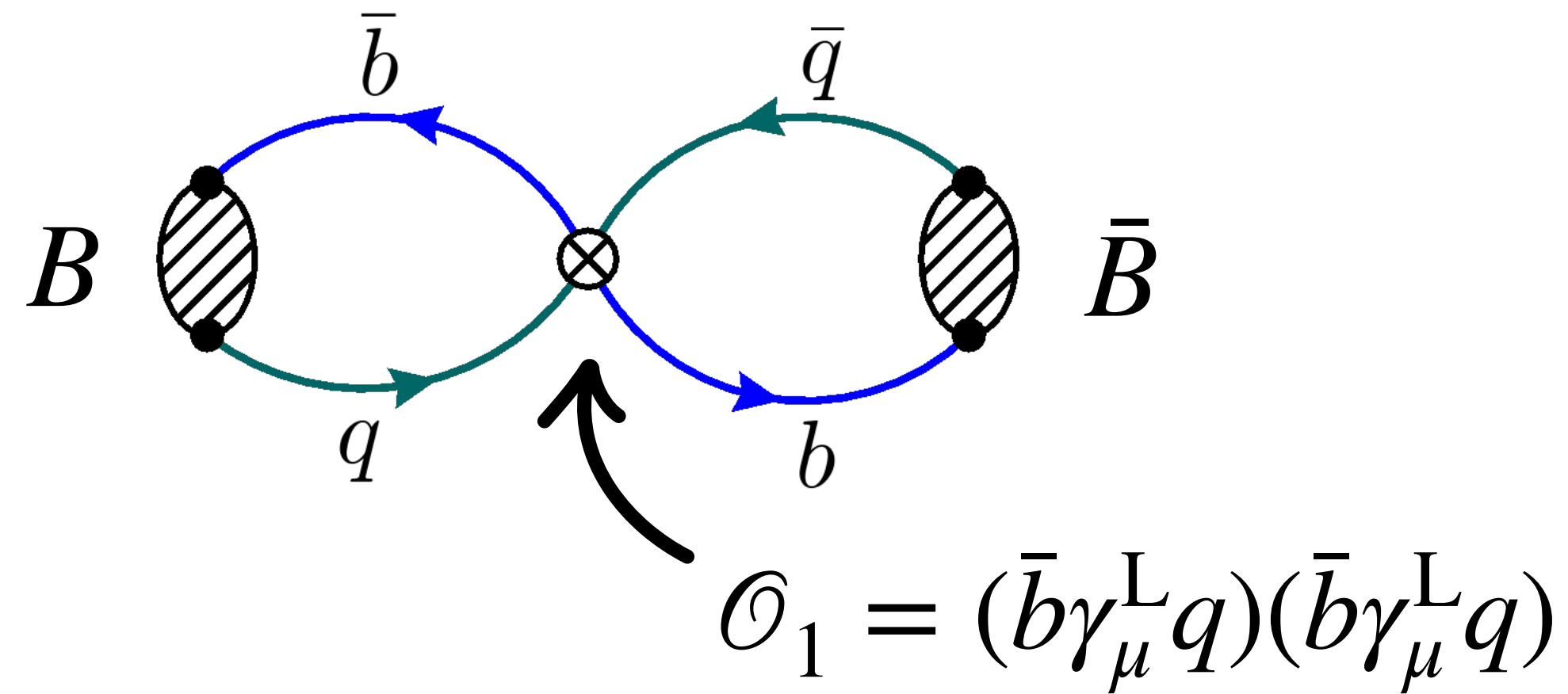
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perturbative

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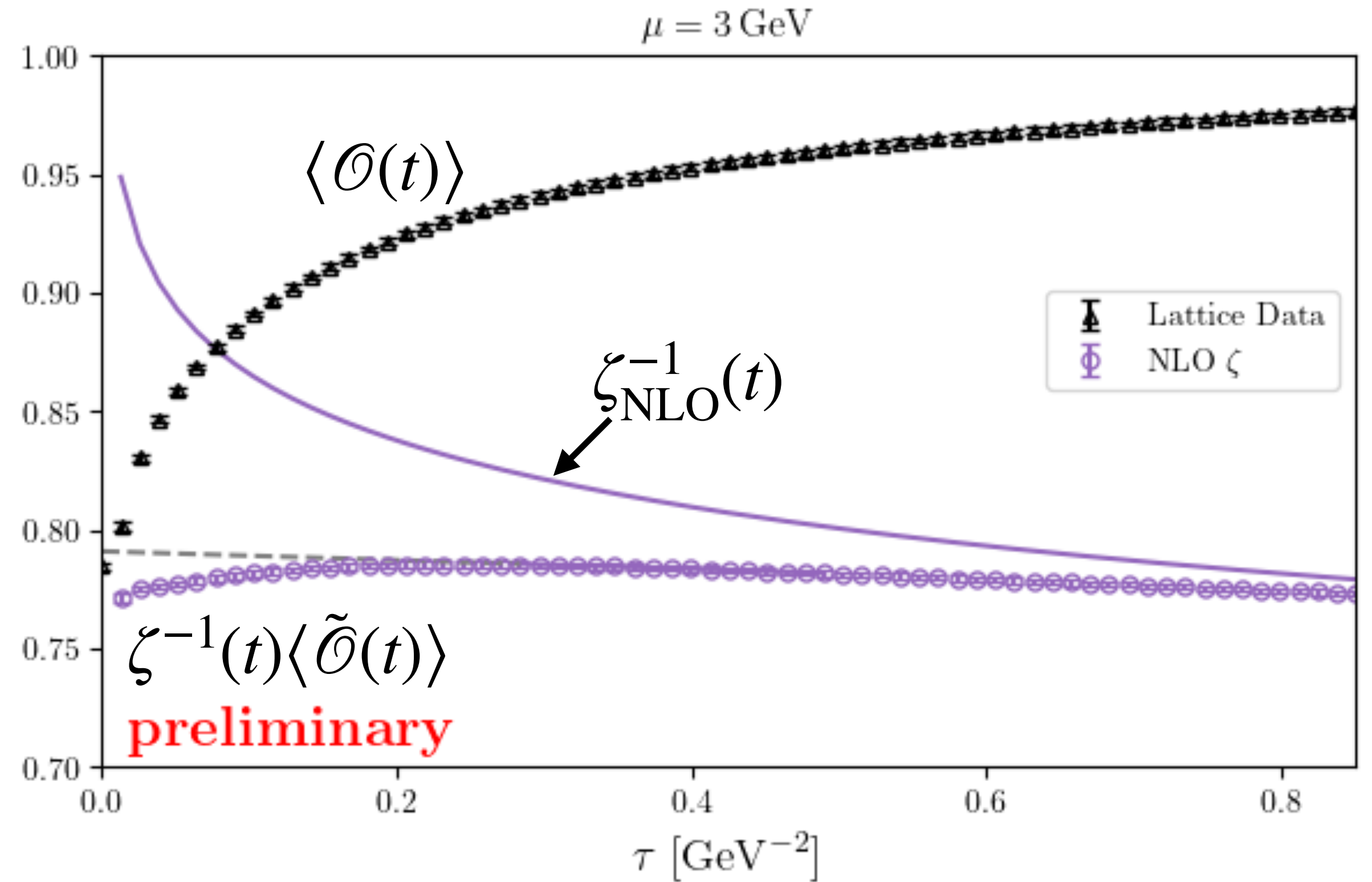


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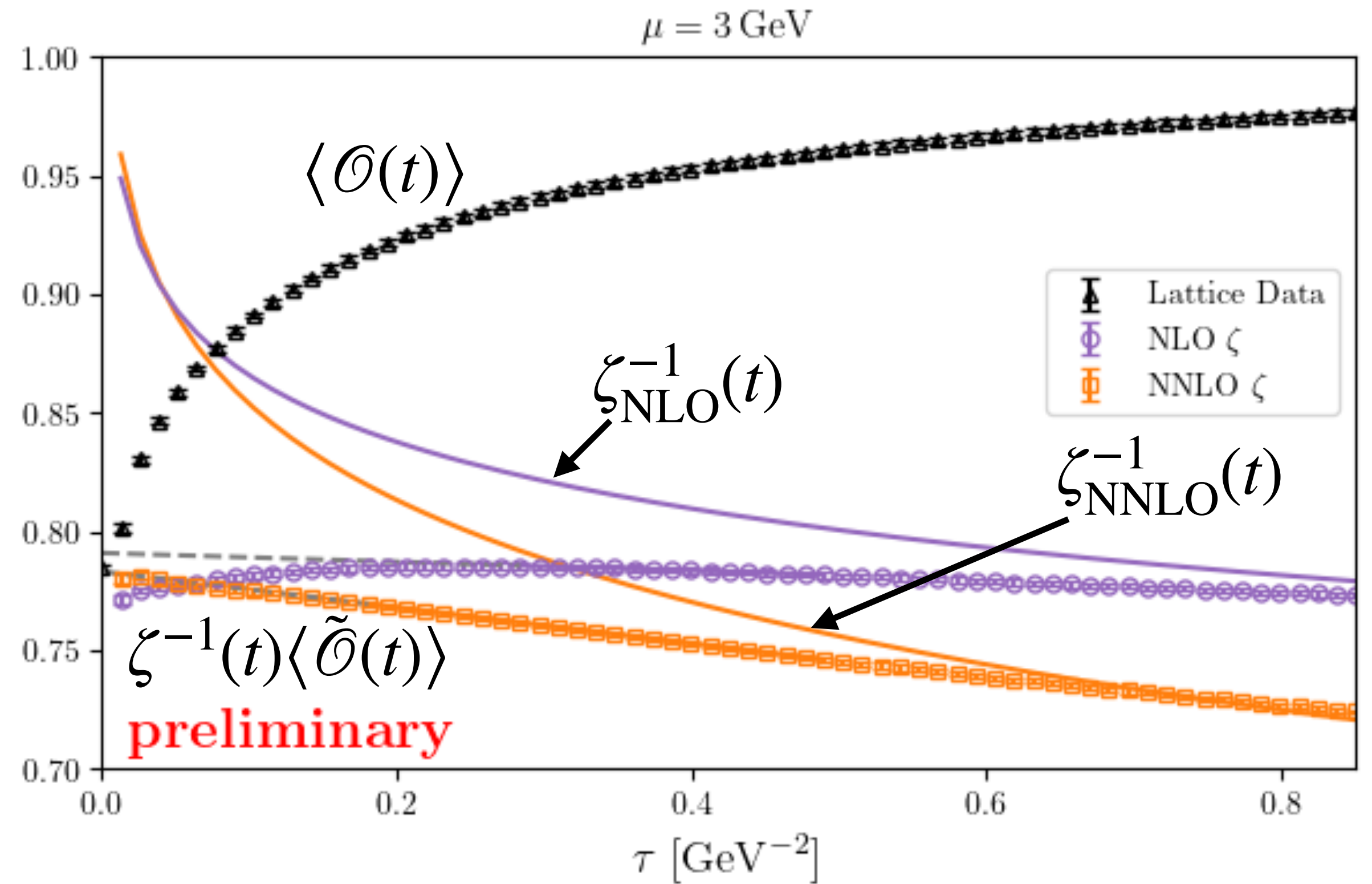
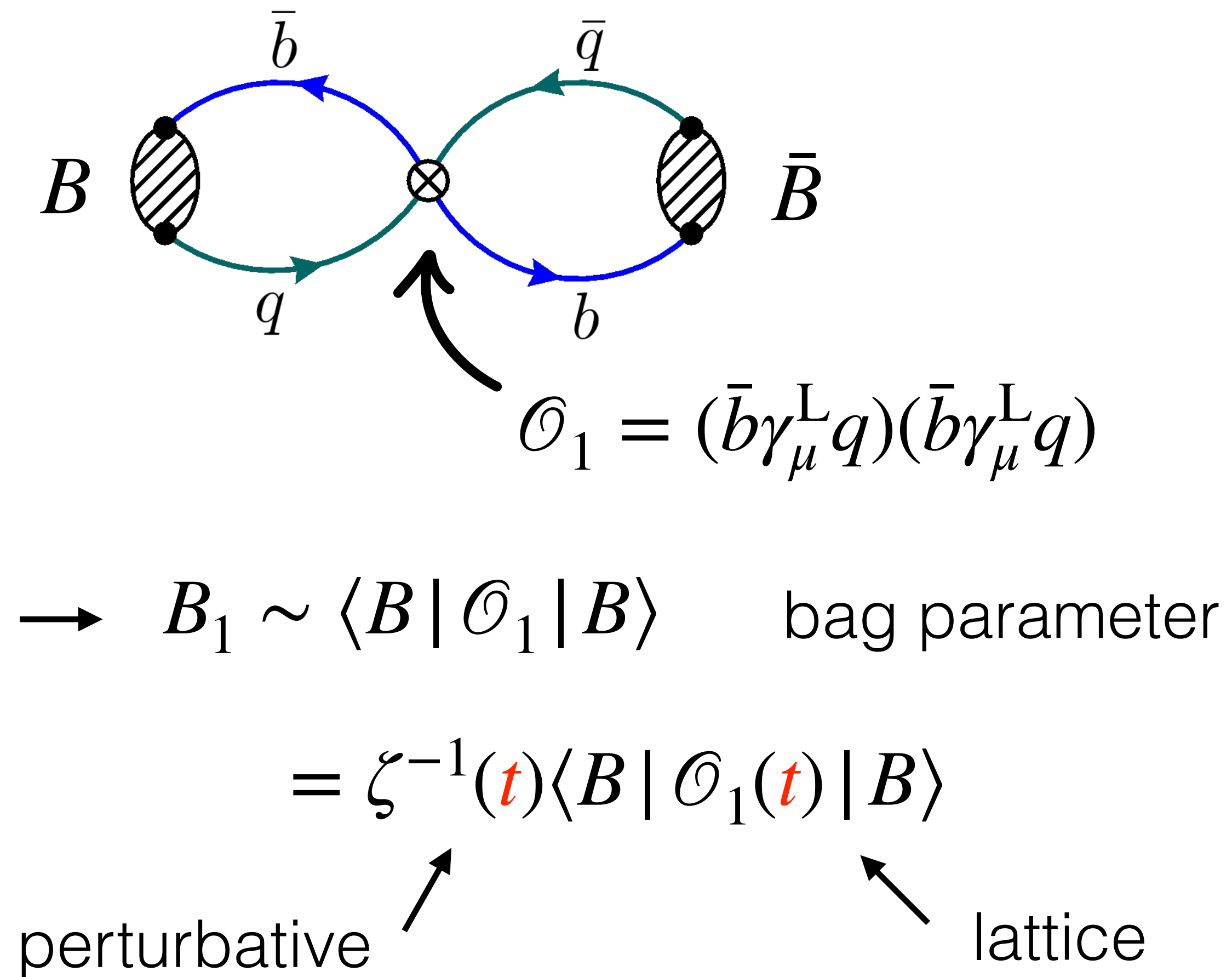
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perturbative

lattice



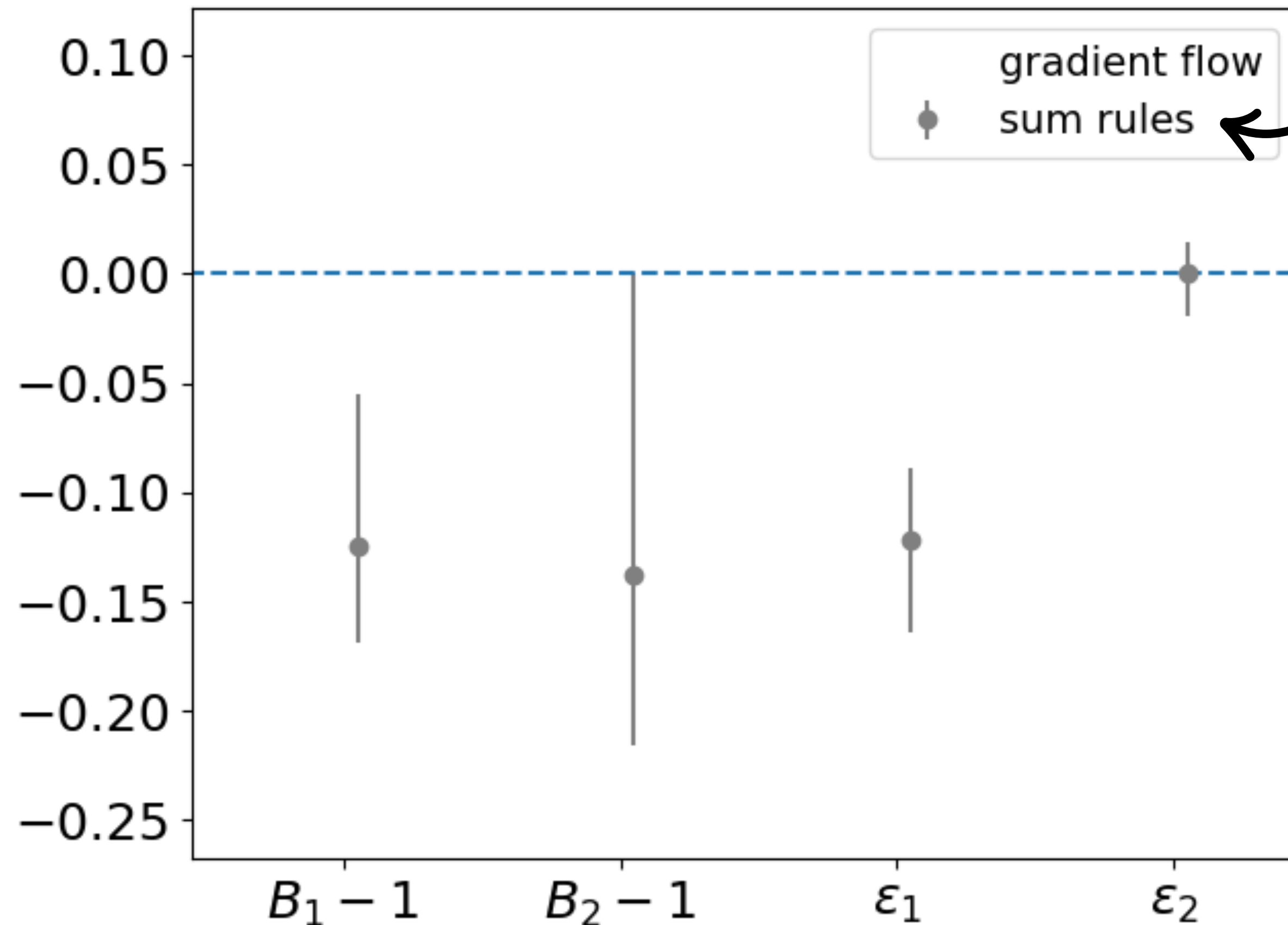
Meson mixing



Black, RH, Lange, Rago,
Shindler, Witzel '26

P³H
CRC TRR 257

D meson lifetime bag parameters



Black, Lang, Lenz, Wüthrich '25

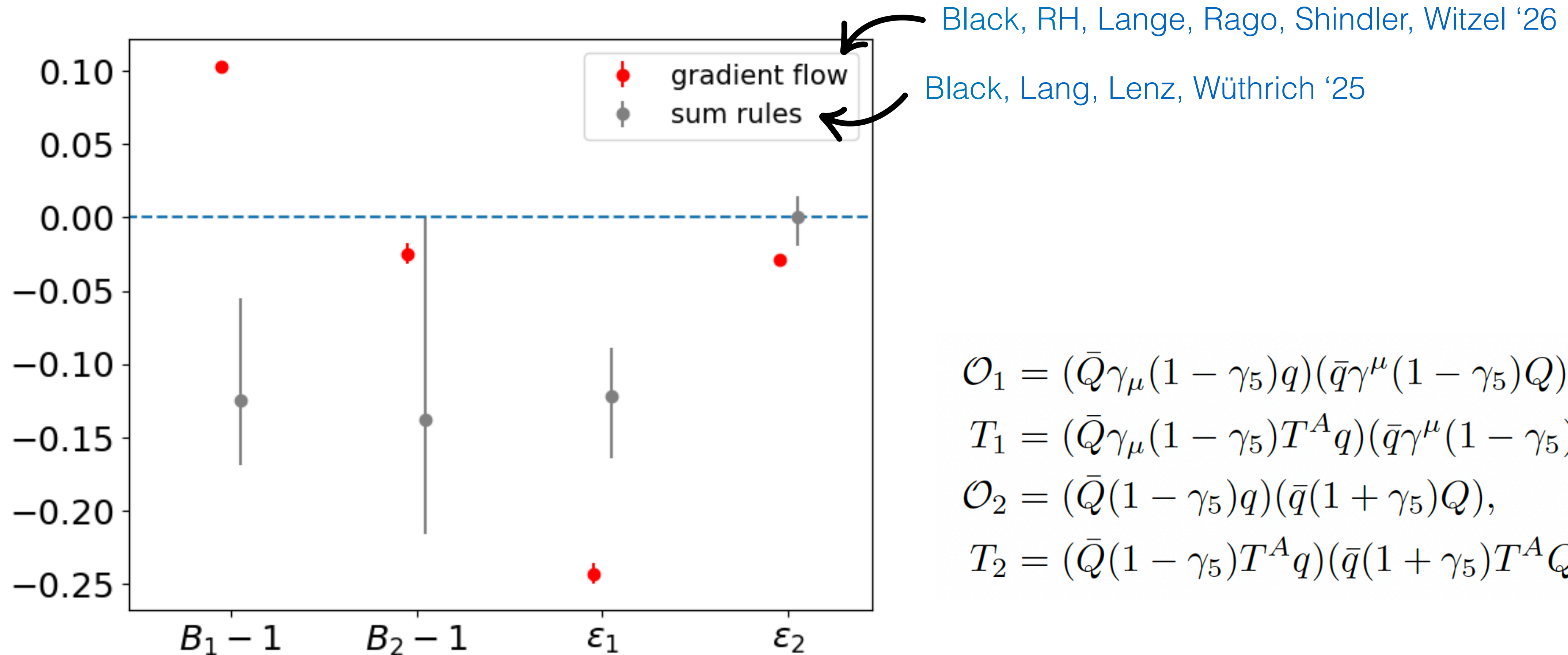
$$\mathcal{O}_1 = (\bar{Q}\gamma_\mu(1 - \gamma_5)q)(\bar{q}\gamma^\mu(1 - \gamma_5)Q),$$

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D meson lifetime bag parameters



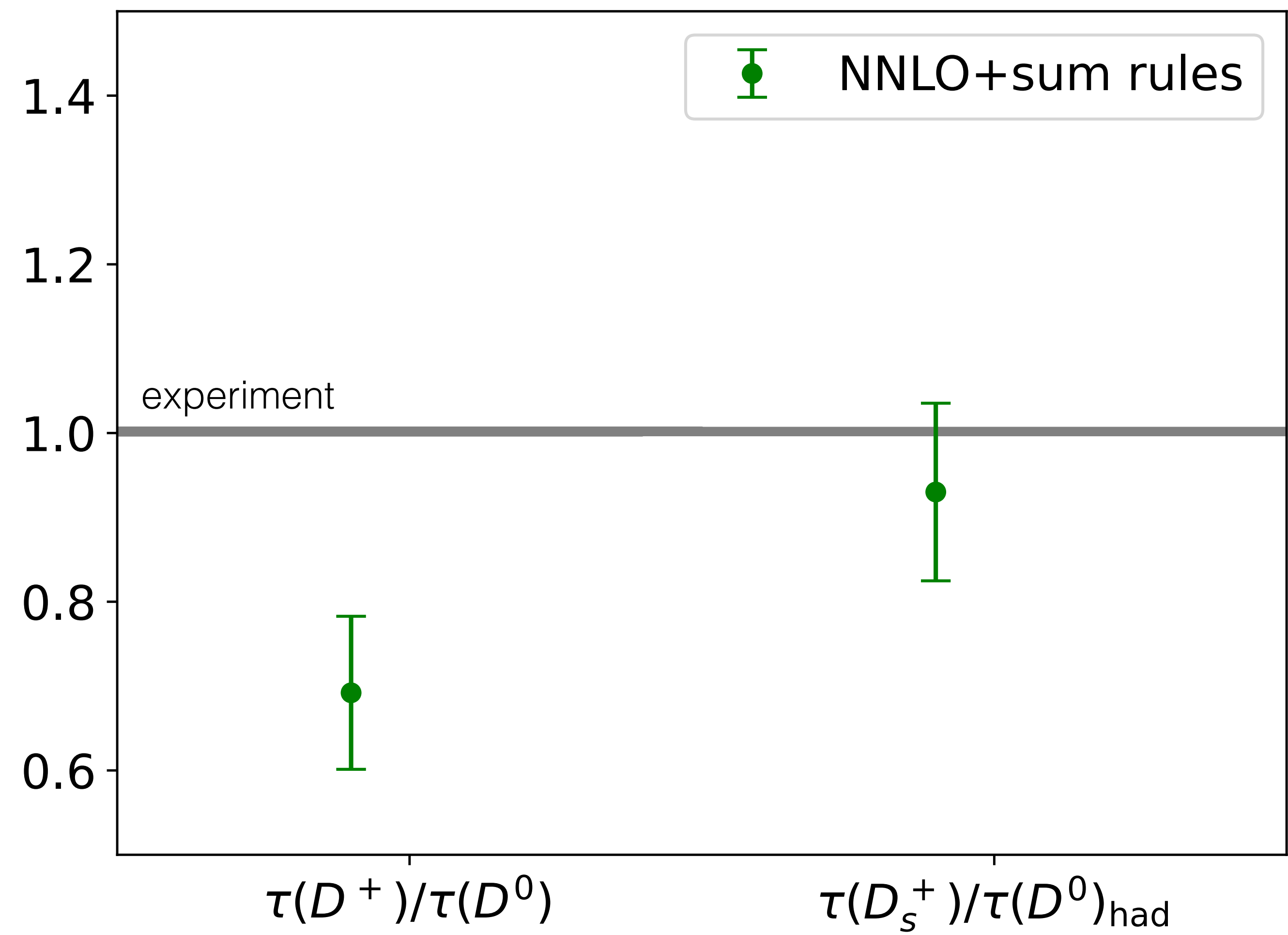
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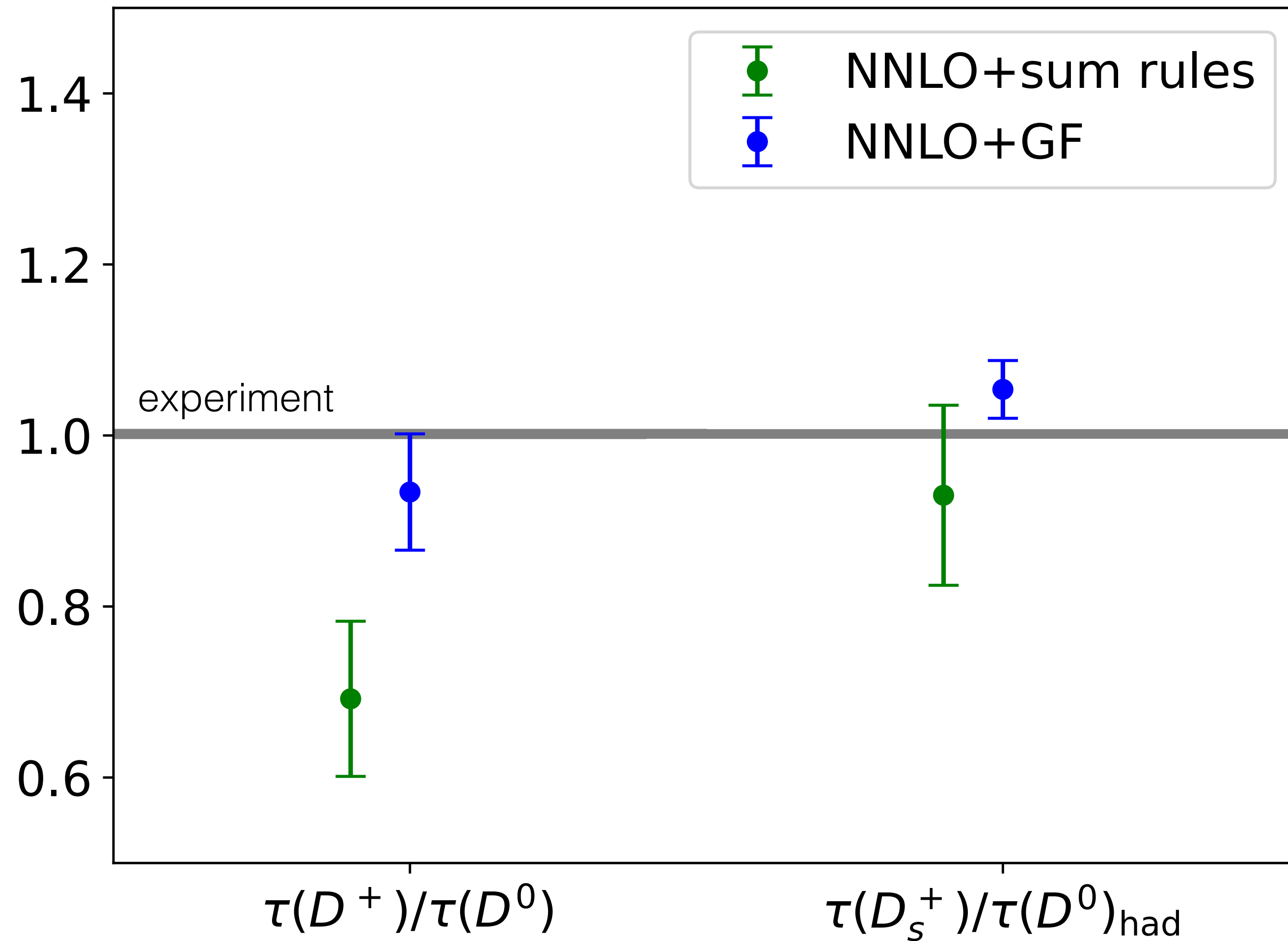
Lifetime ratios



Combining with NNLO Wilson coefficients

Moretti, Nierste, Reeck, Steinhauser '26

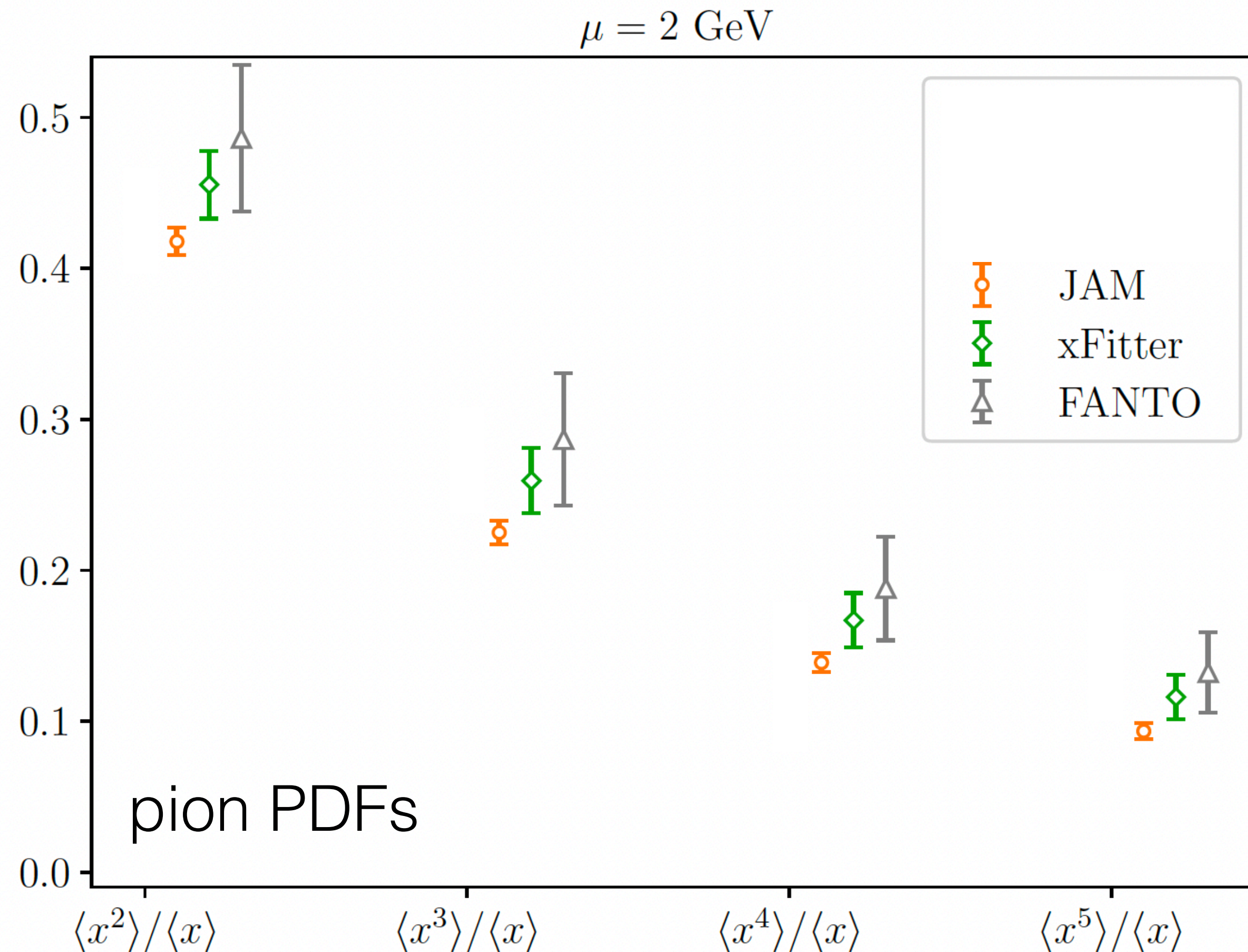
Lifetime ratios



Combining with NNLO Wilson coefficients

Moretti, Nierste, Reeck, Steinhauser '26

Moments of parton densities



$$\begin{aligned} \langle h(p) | \mathcal{O}_{\{\mu_1 \dots \mu_n\}} | h(p) \rangle &= \\ &= \sum_m \zeta_{mn}^{-1}(t) \langle h(p) | \mathcal{O}_{\{\mu_1 \dots \mu_n\}}(t) | h(p) \rangle \end{aligned}$$

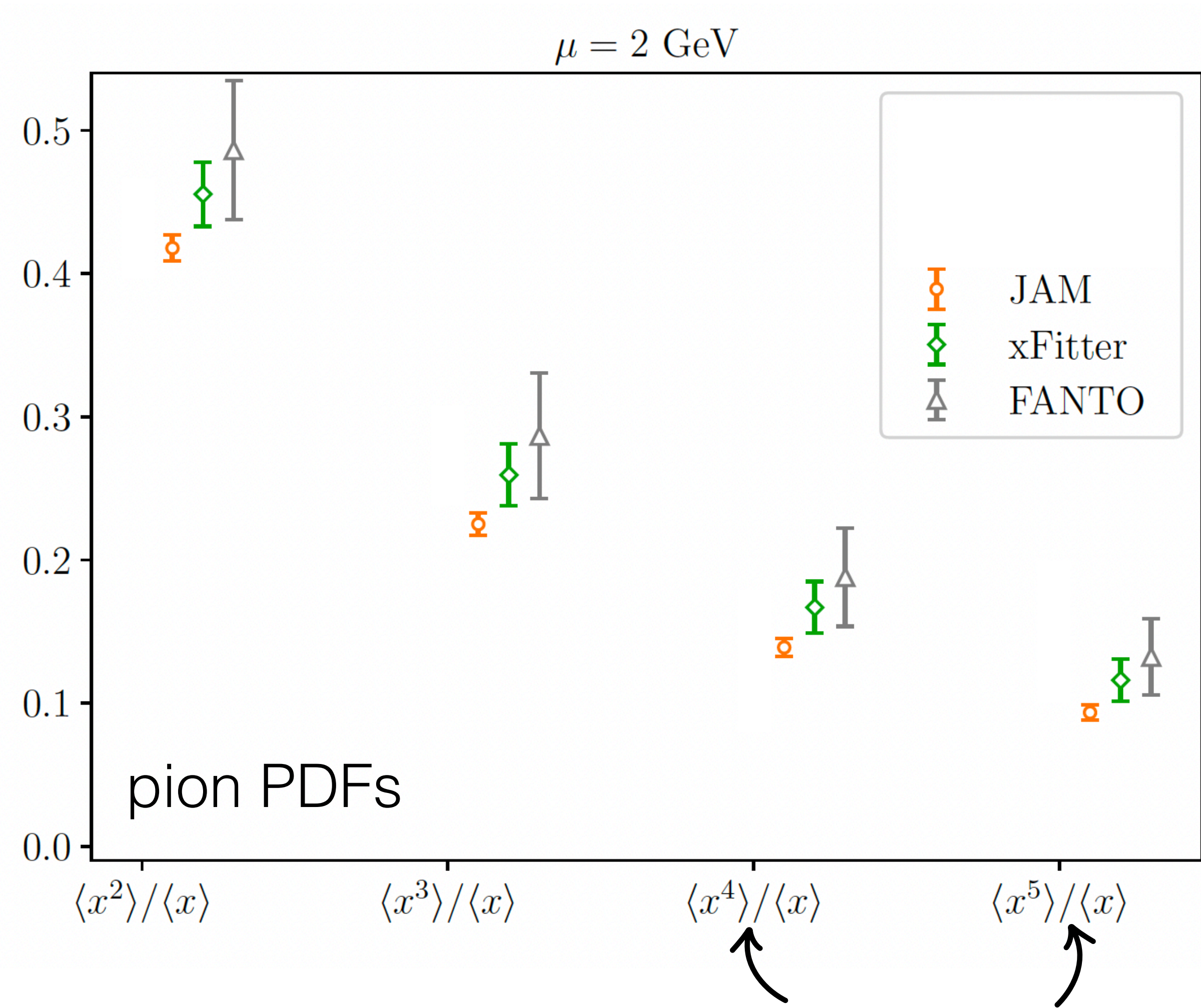
Francis, ..., Shindler, ... '25

Francis, ..., RH, Kohnen, Shindler, ... '25

RH, Kohnen, Shindler '25

see also Edwards et al. '26

Moments of parton densities



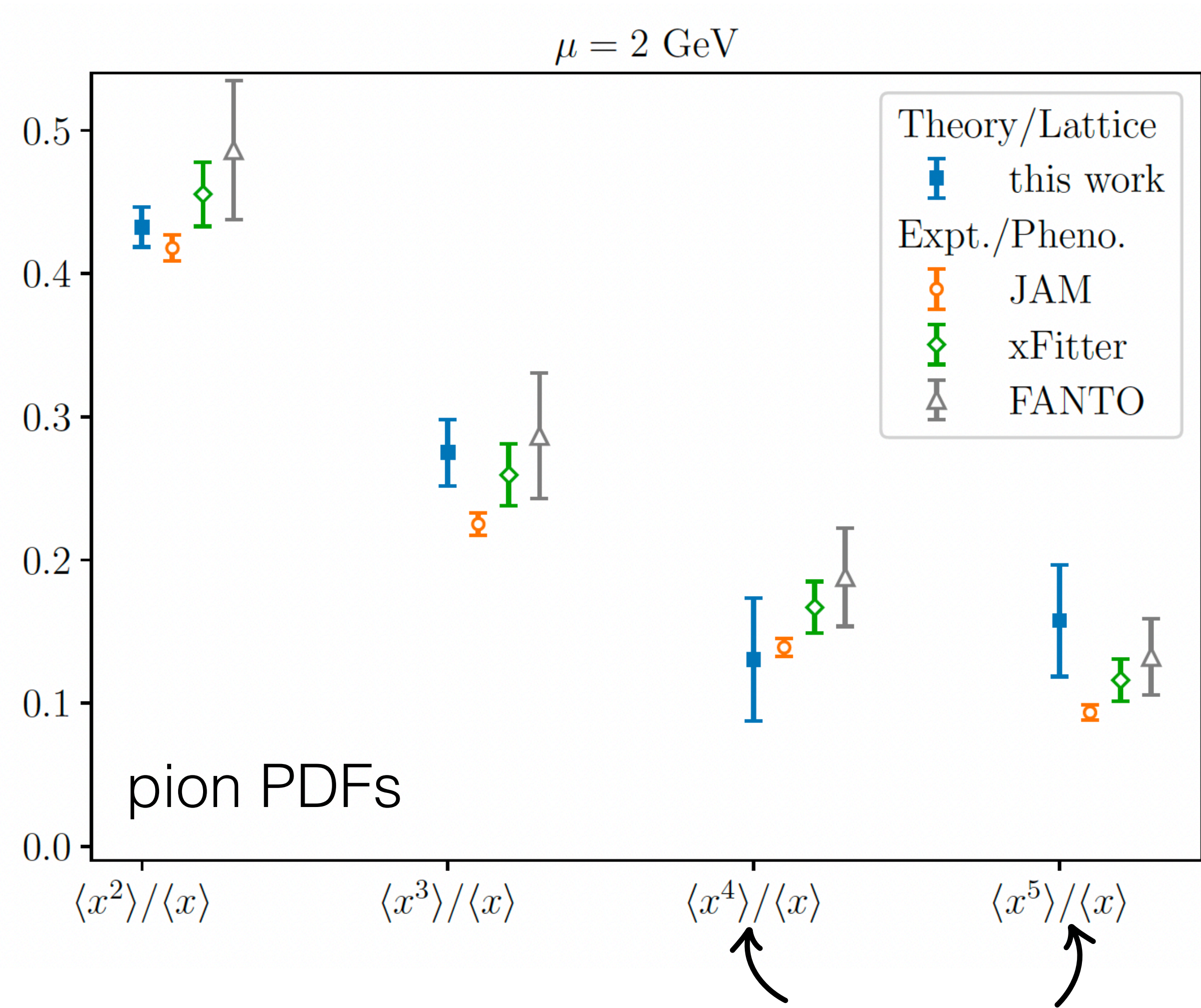
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unreachable w/o gradient flow

see also Edwards et al. '26

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2. Introduction

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~~**3. Main part**~~

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4. Conclusions

Gradient flow Lagrangian

flowed gauge field:

$$\begin{aligned}\frac{\partial}{\partial t} B_\mu(t, x) &= \mathcal{D}_\nu G_{\nu\mu}(t, x) \\ B_\mu(t=0, x) &= A_\mu(x)\end{aligned}$$

flowed quark field:

$$\begin{aligned}\frac{\partial}{\partial t} \chi(t, x) &= \mathcal{D}^2 \chi(t, x) \\ \chi(t=0, x) &= \psi(x)\end{aligned}$$

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$$\mathcal{L} = \mathcal{L}_{\text{QCD}} + \mathcal{L}_B + \mathcal{L}_\chi$$

$$\mathcal{L}_B \sim \int_0^\infty dt \, L_\mu \left(\partial_t B_\mu - \mathcal{D}_\nu G_{\nu\mu} \right)$$

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Lüscher, Weisz 2011

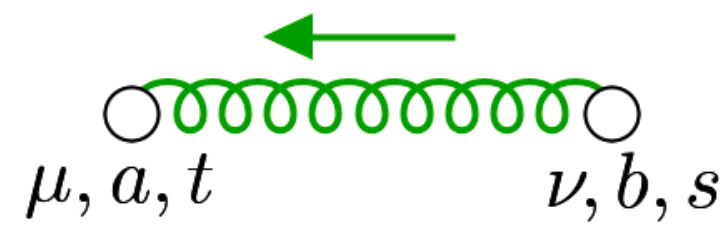
Lüscher 2013

Perturbative approach

$$\mathcal{L}_B \sim \int_0^\infty dt \, \mathbf{L}_\mu \left(\partial_t B_\mu - \mathcal{D}_\nu G_{\nu\mu} \right)$$

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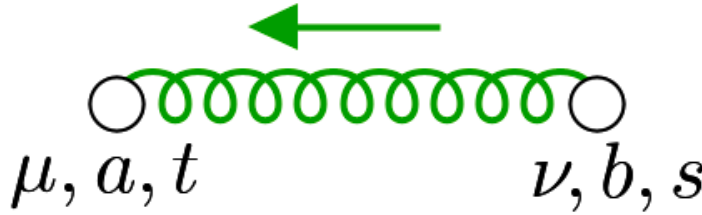
$$\delta_{ab} \delta_{\mu\nu} \theta(t-s) e^{-(t-s)p^2}$$

“gluon flow line”

$$\sim \langle 0 | T \mathbf{L}_\mu^a(t, x) B_\nu^b(s, 0) | 0 \rangle$$

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μ, a, t ν, b, s

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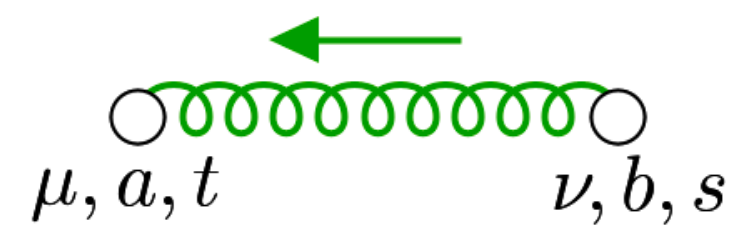
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$\frac{\delta^{ab}}{p^2} \left(\delta_{\mu\nu} - \xi \frac{p_\mu p_\nu}{p^2} \right) e^{-(t+s)p^2}$

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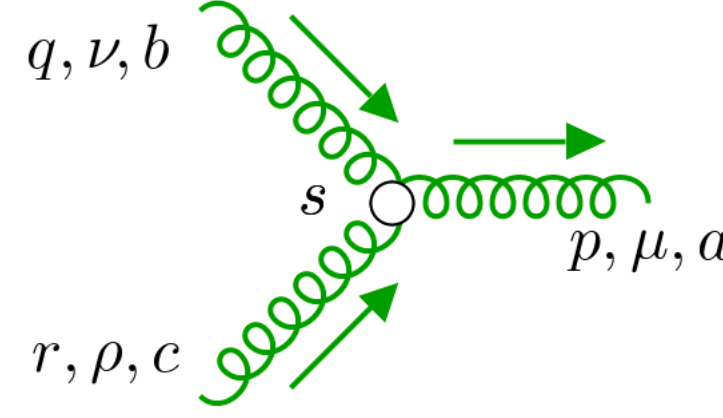
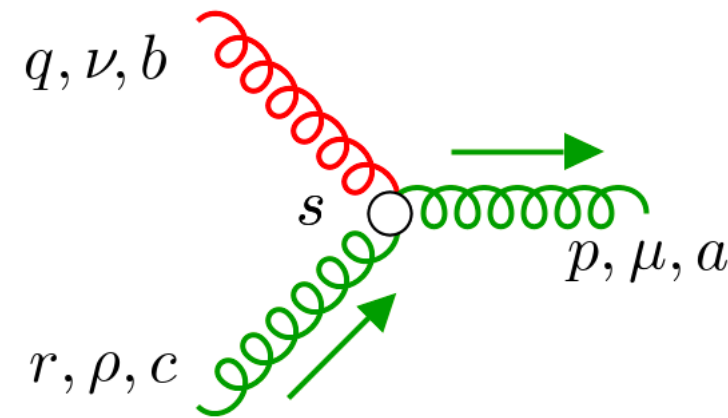
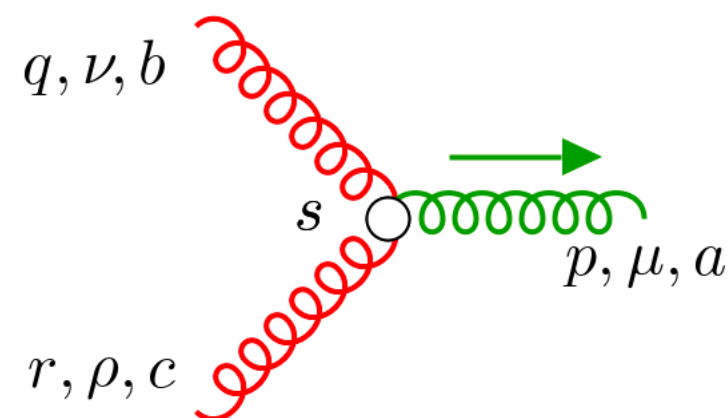
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$$-ig f^{abc} \int_0^\infty ds \left(\delta_{\nu\rho} (r-q)_\mu + 2\delta_{\mu\nu} q_\rho - 2\delta_{\mu\rho} r_\nu \right. \\ \left. + (\kappa - 1)(\delta_{\mu\rho} q_\nu - \delta_{\mu\nu} r_\rho) \right)$$

+ 4-gluon vertex

Quantum field theory

$$\mathcal{L} = \mathcal{L}_{\text{QCD}} + \mathcal{L}_B + \mathcal{L}_\chi$$

$$\mathcal{L}_B \sim \int_0^\infty dt \, \textcolor{green}{L}_\mu \left(\partial_t B_\mu - \mathcal{D}_\nu G_{\nu\mu} \right)$$

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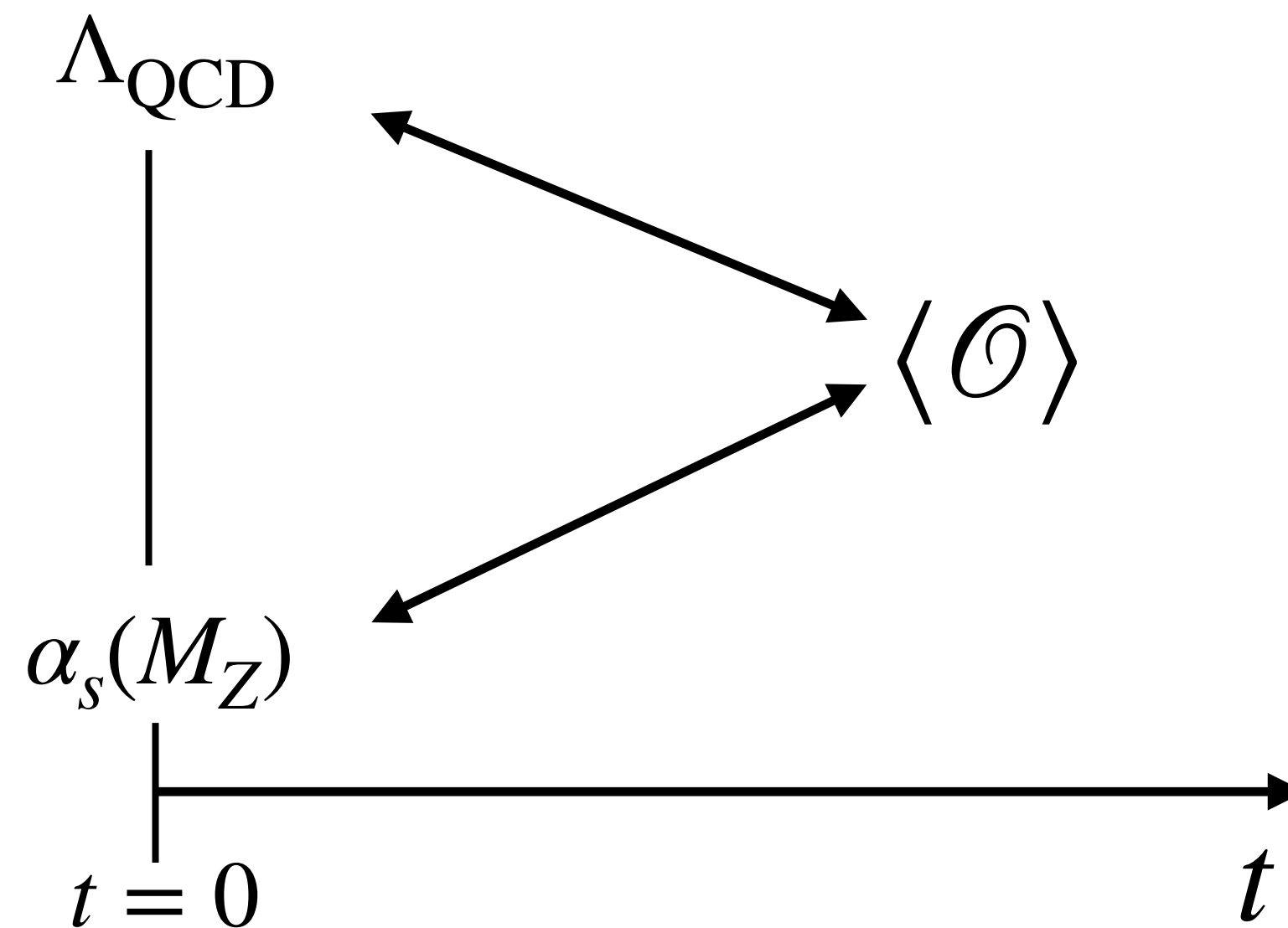
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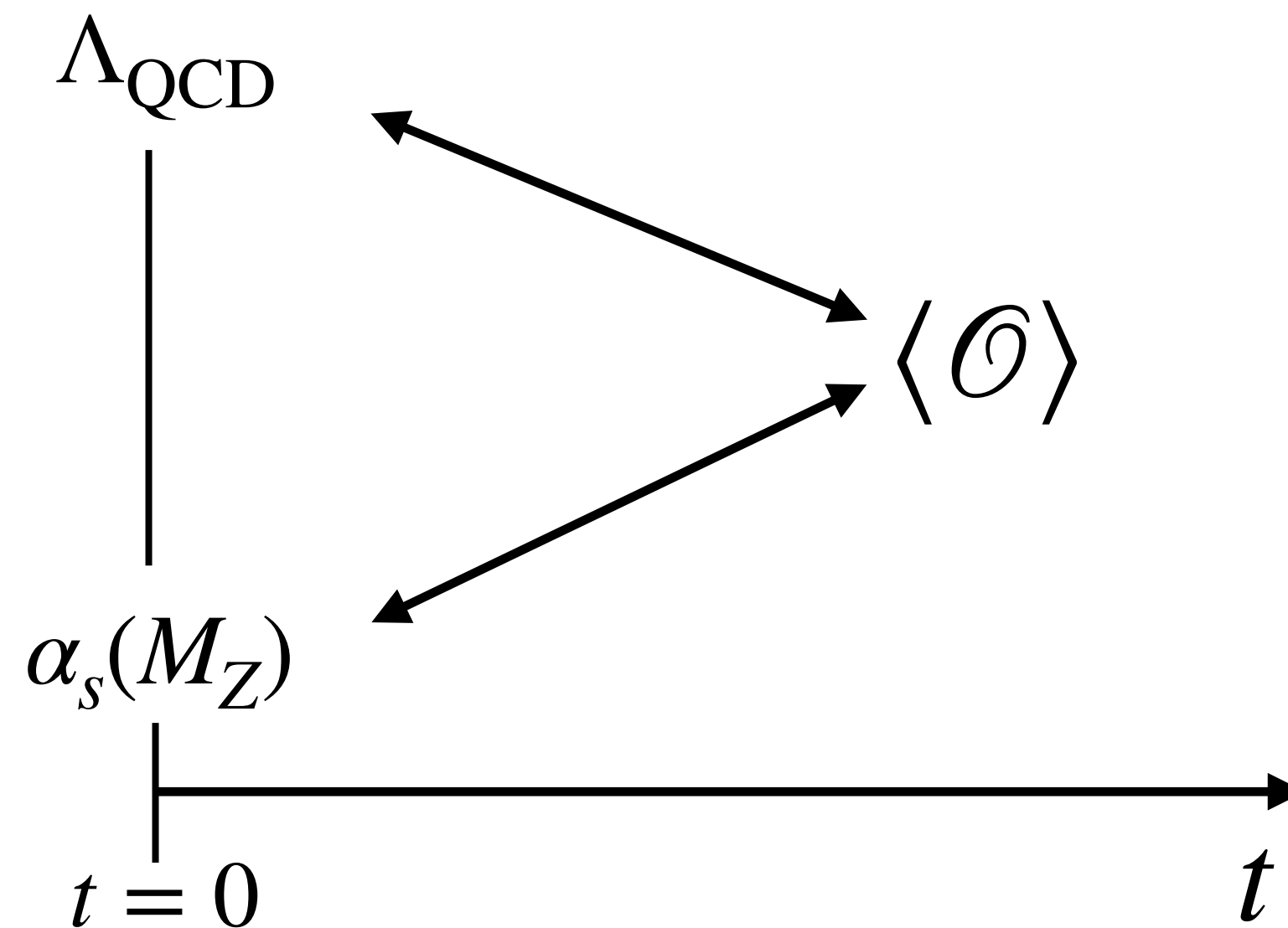
UV divergences can come
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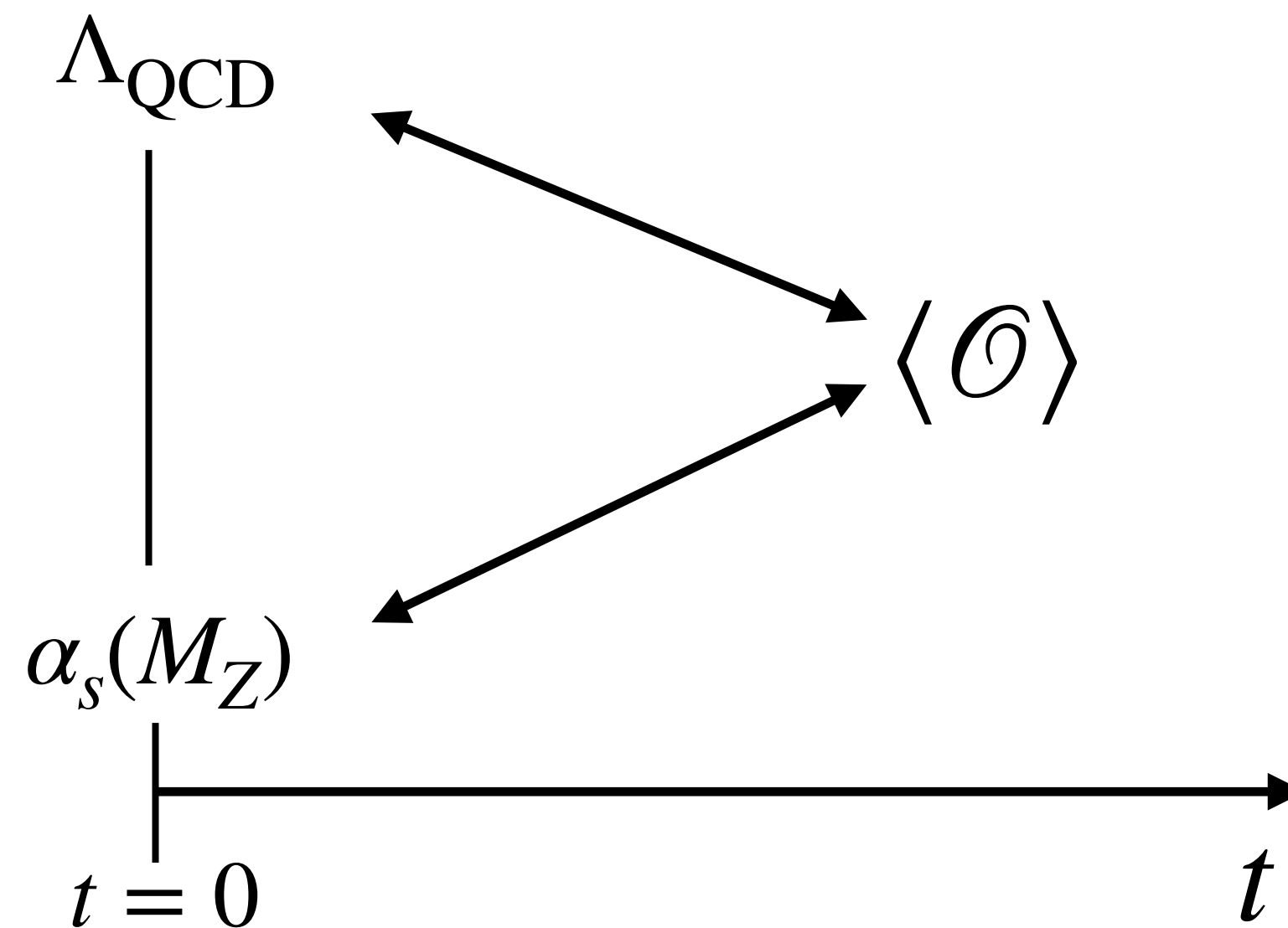
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Ringed scheme

$$\chi^{\text{R}}(t, x) = Z_{\chi}^{1/2} \chi(t, x)$$

$\overline{\text{MS}}$: specific to dimensional regularization
 μ dependent results



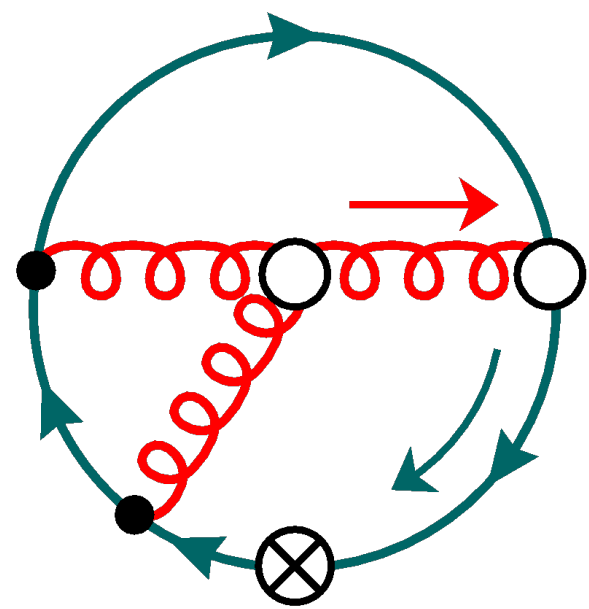
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Ringed scheme: [Suzuki, Makino 2014](#)



$$\dot{Z}_\chi(t) \langle \bar{\chi}(t) \mathbb{D} \chi(t) \rangle = \langle \bar{\chi}(t) \mathbb{D} \chi(t) \rangle \Big|_{\text{LO}} = - \frac{2n_c}{(4\pi t)^2}$$

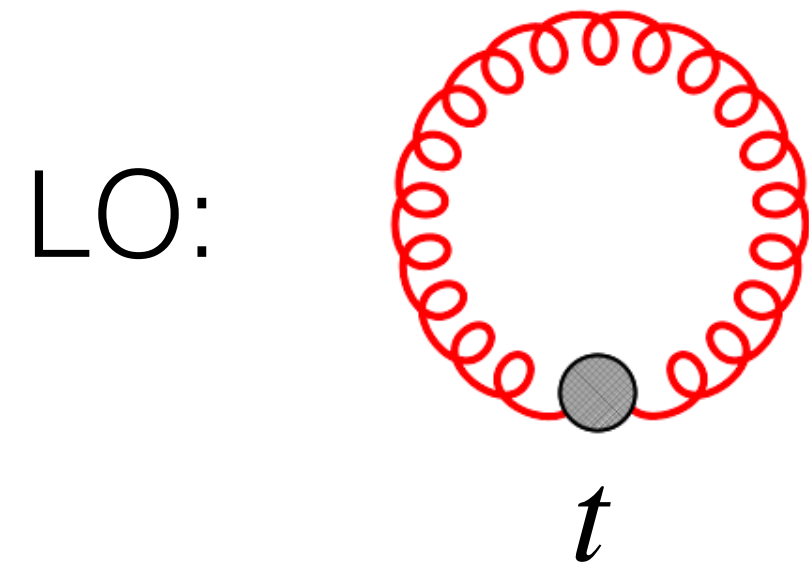
results become RG invariant

Let's calculate

$$E(t) \equiv \frac{1}{4} \langle G_{\mu\nu}^a(t) G^{a,\mu\nu}(t) \rangle$$

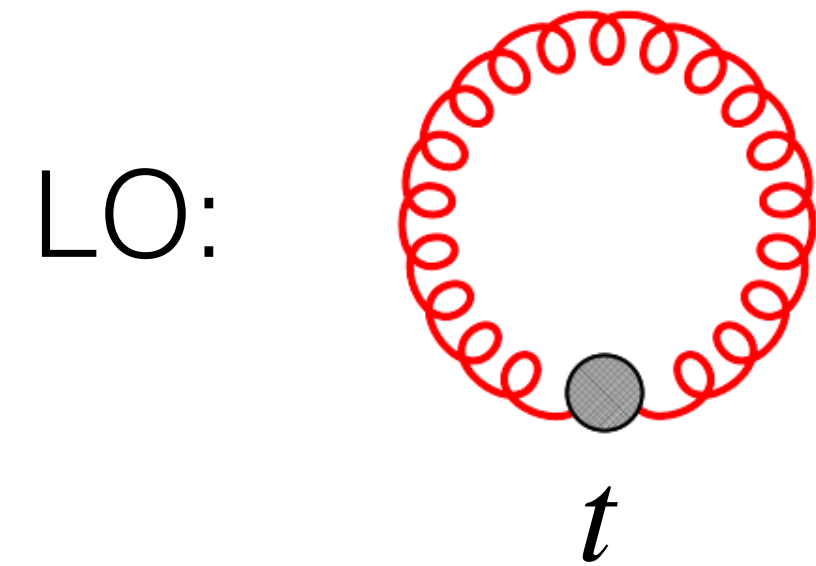
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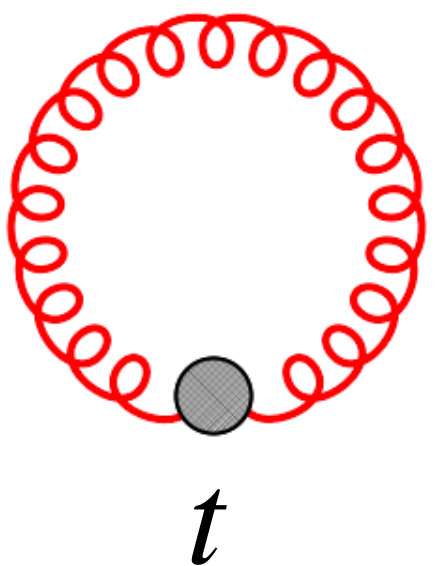
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$$\frac{\delta^{ab}}{p^2} \left(\delta_{\mu\nu} - \xi \frac{p_\mu p_\nu}{p^2} \right) e^{-(t+s)p^2}$$

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LO:  $\sim \int d^D p e^{-2tp^2} \sim t^{-2+\epsilon} \neq 0$

The diagram shows a red circular loop with a grey dot at the bottom labeled t .

 $\frac{\delta^{ab}}{p^2} \left(\delta_{\mu\nu} - \xi \frac{p_\mu p_\nu}{p^2} \right) e^{-(t+s)p^2}$

The diagram shows a red wavy line between two white circles. The left circle is labeled μ, a, t and the right circle is labeled ν, b, s .

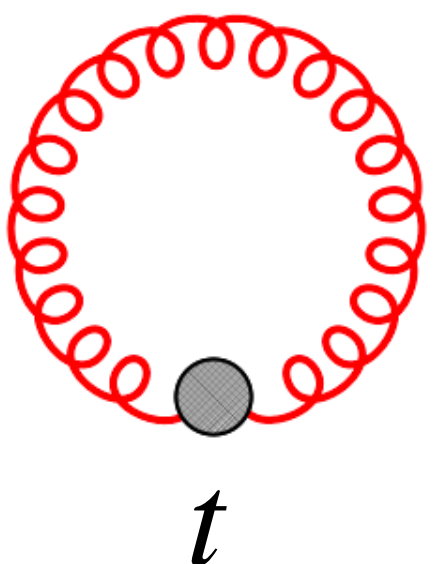
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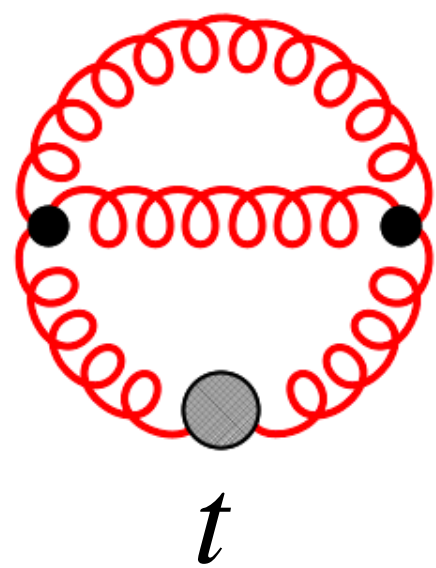
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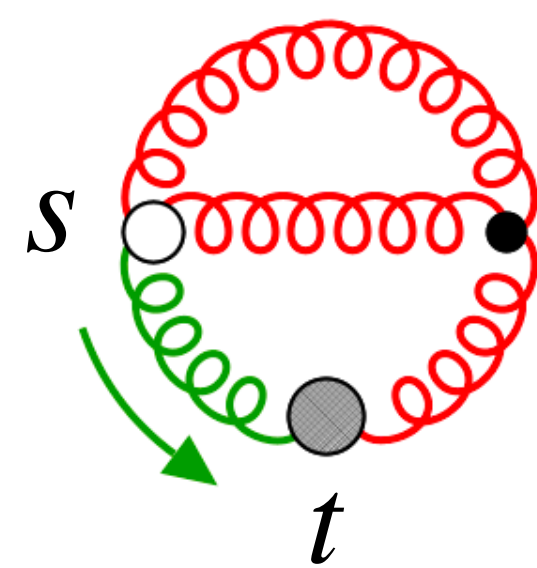
LO:



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NLO:



$$\sim \int_p \int_k \frac{e^{-2tp^2}}{p^4 k^2 (p-k)^2}$$


$$\int_0^t ds \int_p \int_k \frac{e^{-(2t-s)p^2}}{p^2 k^2 (p-k)^2}$$

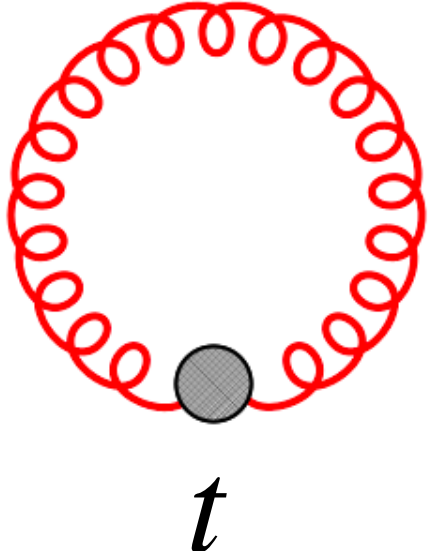
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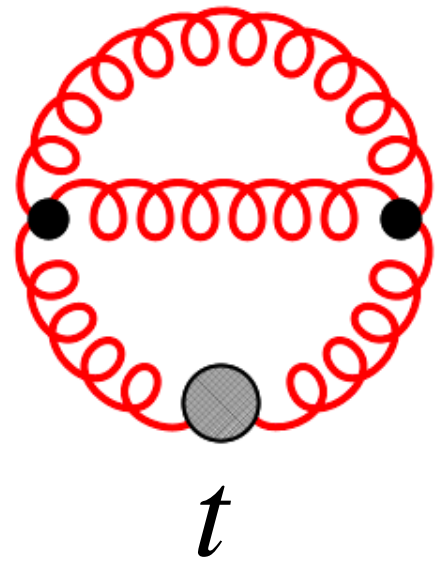
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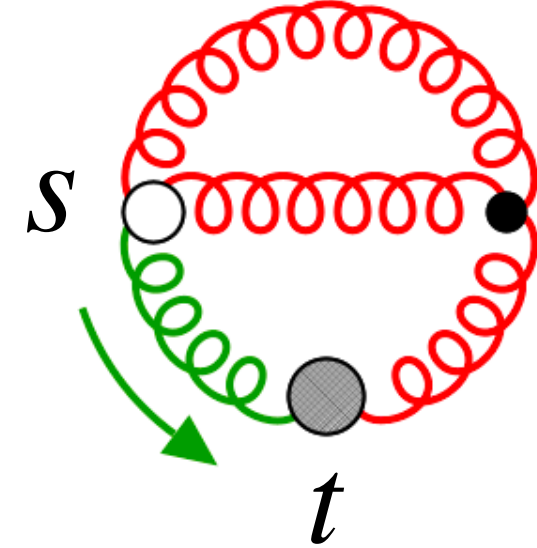


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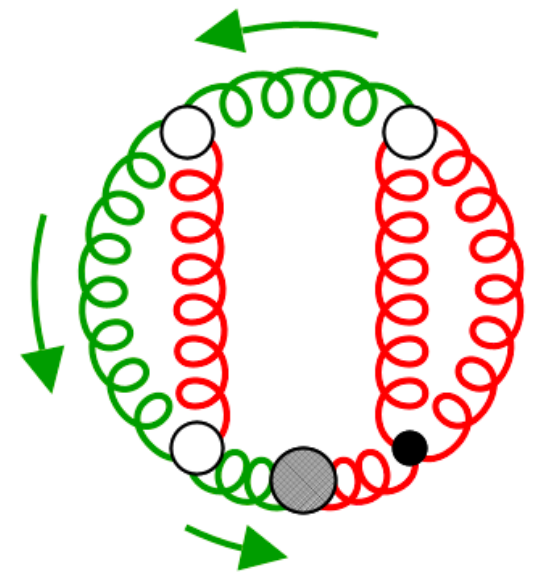


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NNLO:



$$\int_0^t ds_1 \int_0^{s_1} ds_2 \int_0^{s_2} ds_3 \int_p \int_k \int_l \dots$$

The perturbative toolbox

[For details, see: Artz, RH, Lange, Neumann, Prausa 2019]

diagram generation:

qgraf Nogueira 1993

diagram analyzation:

q2e/exp RH, Seidensticker, Steinhauser 1997

→ tapir/exp Gerlach, Herren, Lang 2022

algebraic manipulations:

FORM Vermaseren 2000, ...

reduction to masters:

Chetyrkin, Tkachov 1981

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$$\int d^D k \int d^D p \int_0^t ds \frac{e^{-tp^2 - s(k-p)^2}}{k^2 p^2 (k-p)^2} = \frac{A}{\epsilon^2} + \frac{B}{\epsilon} + C + \dots$$

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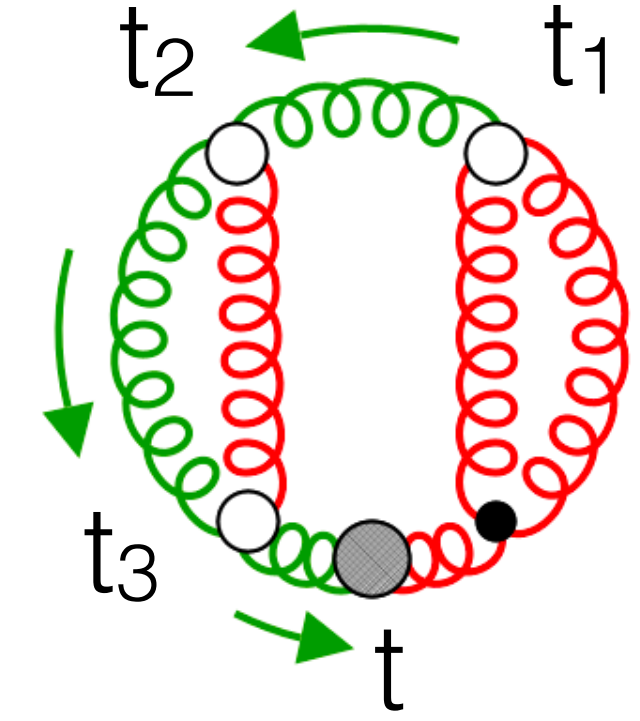
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→ ftint RH, Nellopoulos, Olsson, Wesle '25

Numerical evaluation

$$I(\{c_1, \dots, c_f\}, \{a_1(u), \dots, a_6(u)\}, \{b_1, \dots, b_6\}) =$$

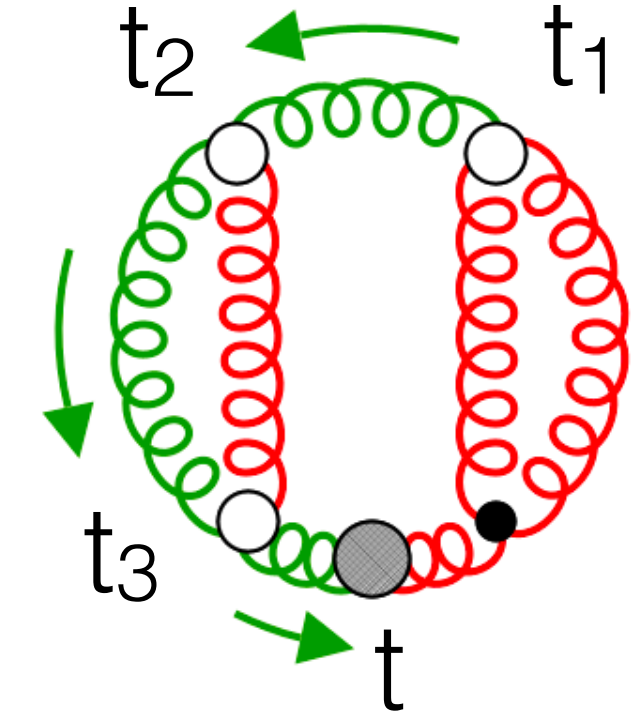
$$= \left(\prod_{i=1}^f \int_0^1 du_i u_i^{c_i} \right) \int d^D p_1 d^D p_2 d^D p_3 \frac{\exp \left[-t \left(a_1(u) p_1^2 + \dots + a_6(u) p_6^2 \right) \right]}{(p_1^2)^{b_1} \dots (p_6^2)^{b_6}}$$



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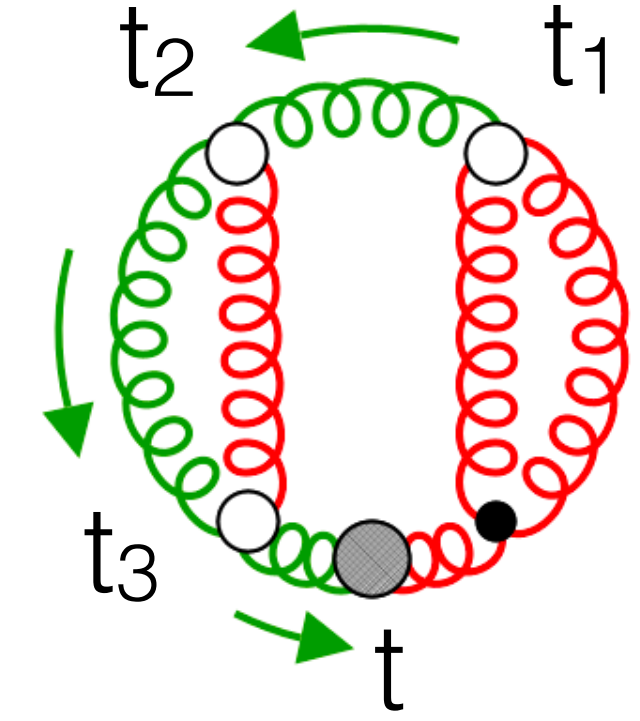
Schwinger parameters:

$$\frac{1}{(p^2)^b} \sim \int_0^\infty dx x^{b-1} e^{-xp^2}$$

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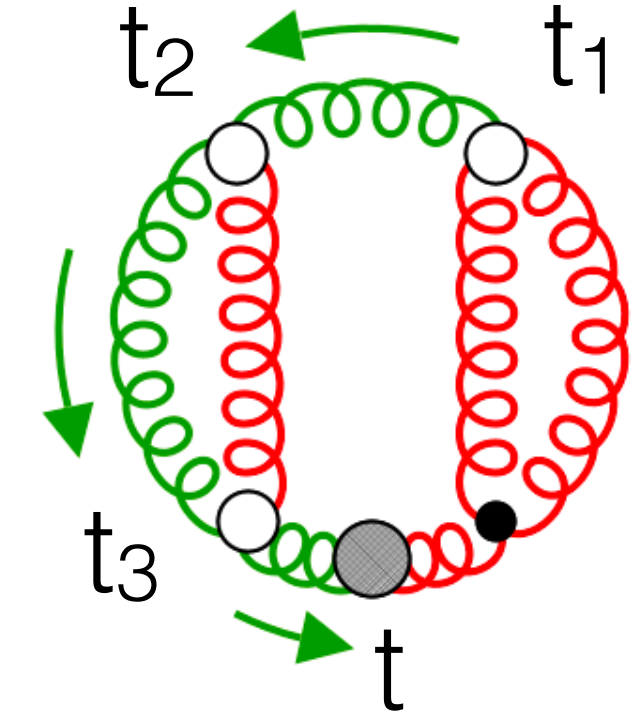
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$$\sim \left(\prod_{i=1}^f \int_0^1 du_i u_i^{c_i} \right) \left(\prod_{j=1}^6 \int_0^\infty dx_j x_j^{b_j-1} \right) \int d^D p_1 d^D p_2 d^D p_3 \exp \left[-t \mathbf{p}^T \mathbf{A}(x, u) \mathbf{p} \right]$$

Numerical evaluation

$$I(\{c_1, \dots, c_f\}, \{a_1(u), \dots, a_6(u)\}, \{b_1, \dots, b_6\}) =$$

$$= \left(\prod_{i=1}^f \int_0^1 du_i u_i^{c_i} \right) \int d^D p_1 d^D p_2 d^D p_3 \frac{\exp \left[-t \left(a_1(u) p_1^2 + \dots + a_6(u) p_6^2 \right) \right]}{(p_1^2)^{b_1} \dots (p_6^2)^{b_6}}$$



Schwinger parameters:

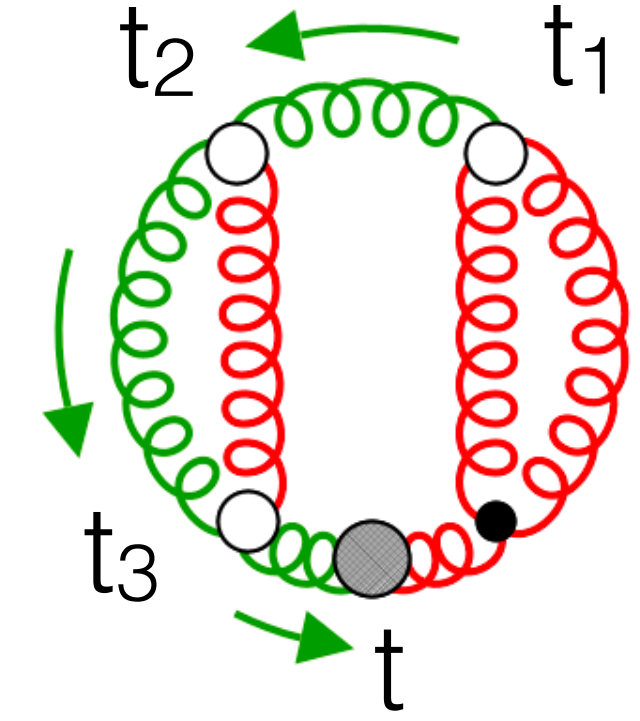
$$\frac{1}{(p^2)^b} \sim \int_0^\infty dx x^{b-1} e^{-xp^2} \quad \left(\xrightarrow{\text{map}} \int_0^1 dx \dots \right)$$

$$\sim \left(\prod_{i=1}^f \int_0^1 du_i u_i^{c_i} \right) \left(\prod_{j=1}^6 \int_0^\infty dx_j x_j^{b_j-1} \right) \int d^D p_1 d^D p_2 d^D p_3 \exp \left[-t \mathbf{p}^T \mathbf{A}(x, u) \mathbf{p} \right]$$

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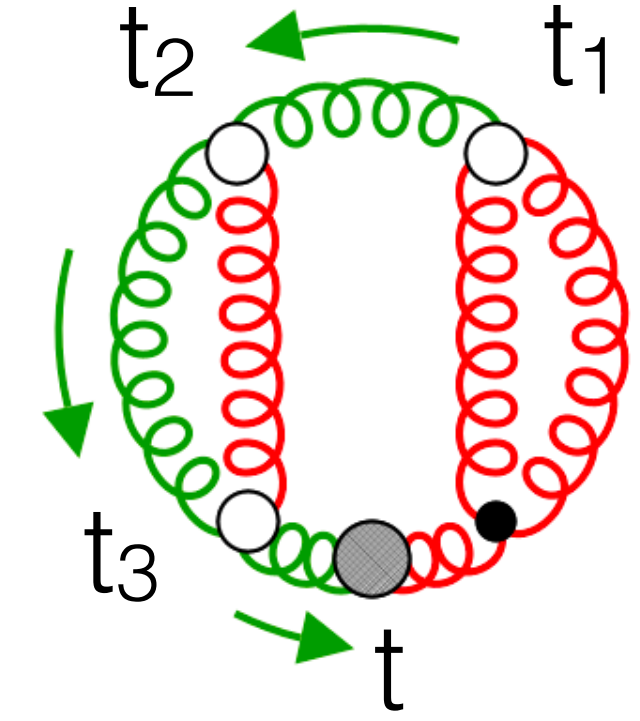
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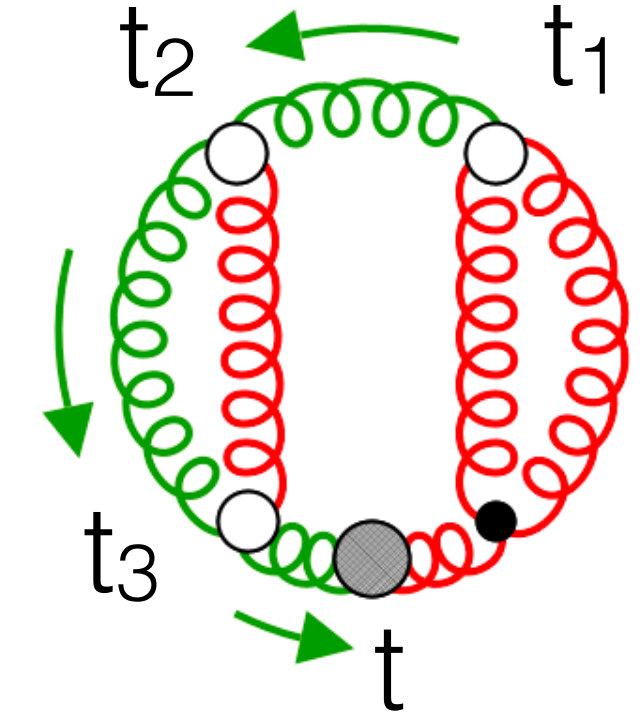
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Binoth, Heinrich (2000)

Implementation

$$I(\{c_1, \dots, c_f\}, \{a_1(u), \dots, a_6(u)\}, \{b_1, \dots, b_6\})$$

$$c_1 = c_2 = 0$$

$$a_1 = u_1 u_2, \quad a_2 = u_2, \quad a_3 = u_2 - u_1 u_2$$

$$a_4 = 1, \quad a_5 = 1 + u_1 u_2, \quad a_6 = 1 - u_2$$

$$b_1 = b_4 = 1$$

$$b_2 = b_3 = b_5 = b_6 = 0$$

Implementation

$$I(\{c_1, \dots, c_f\}, \{a_1(u), \dots, a_6(u)\}, \{b_1, \dots, b_6\})$$

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ftint RH, Nellopoulos, Olsson (in prep)

(based on pySecDec)

Heinrich, Magerya, Kerner, Jones, ...

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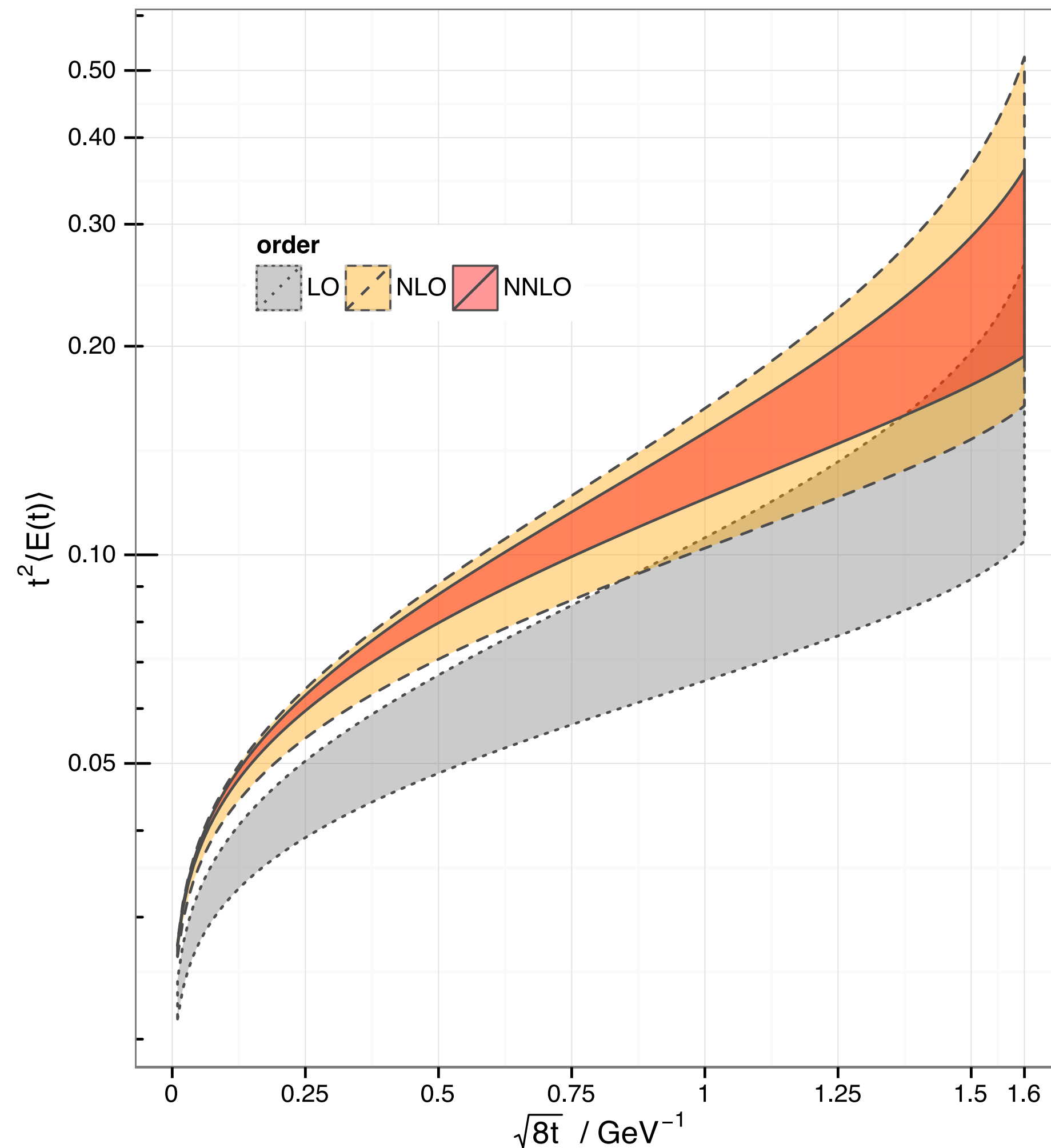
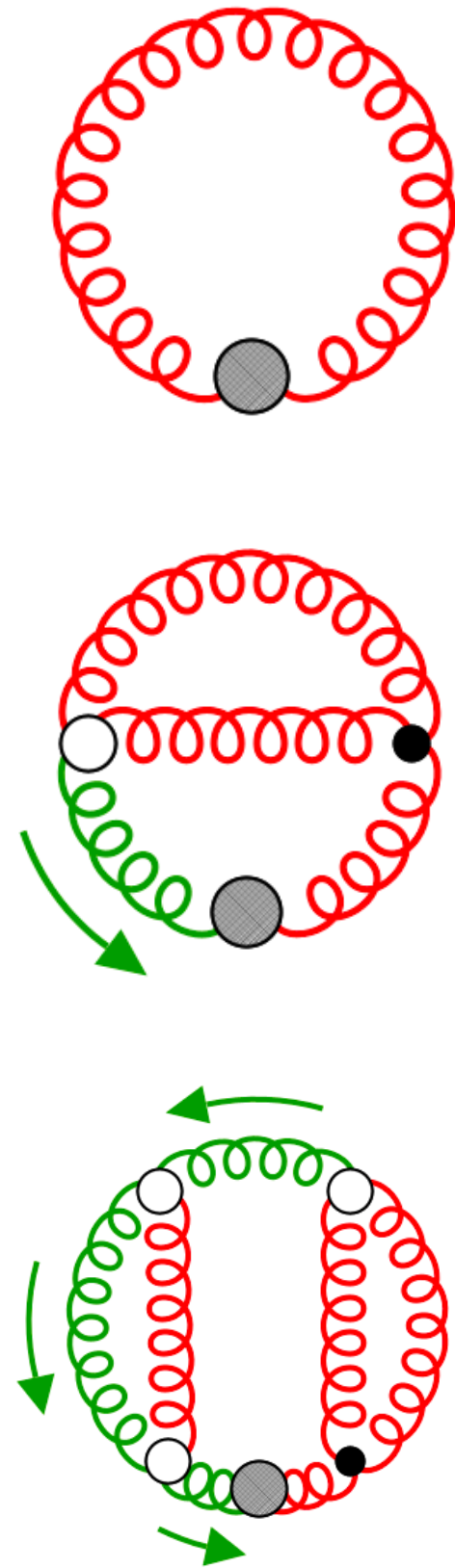
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Heinrich, Magerya, Kerner, Jones, ...

```
f[{{0,0},{u1*u2,u2,u2-u1*u2,1,1+u1*u2,1-u2}}, {1,0,0,1,0,0}] -> (
+eps^-1*(+8.33333333333333343*10^-02+0.000000000000000000*10^+00*I)
+eps^-1*(+1.4433895444086145*10^-15+0.000000000000000000*10^+00*I)*plusminus
+eps^0*(+3.0238270284562663*10^-01+0.000000000000000000*10^+00*I)
+eps^0*(+1.6918362746499228*10^-08+0.000000000000000000*10^+00*I)*plusminus
+eps^1*(+6.5531010458012129*10^-01+0.000000000000000000*10^+00*I)
+eps^1*(+3.7857260802916662*10^-08+0.000000000000000000*10^+00*I)*plusminus
),
```

$$t^2 E(t) = \frac{3\alpha_s(\mu)}{4\pi} \left[1 + k_1(t, \mu) \alpha_s(\mu) + k_2(t, \mu) \alpha_s^2(\mu) \right] \equiv \frac{3}{4\pi} \alpha_s^{\text{GF}}(t)$$

RH, Neumann 2016



Renormalization scheme for α_s
compatible with lattice and PT!

Three thoughts on the GF scheme

1

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perturbative contribution: renormalons!

Renormalon subtraction

Beneke, Takaura '25

$$\Pi(q^2) \sim \int d^4x e^{iqx} \langle 0 | T j(x) j(0) | 0 \rangle$$

Adler function:
$$D(Q^2) = -Q^2 \frac{d}{dQ^2} \Pi(Q^2) = C_1(Q^2) + \frac{1}{Q^4} C_F(Q^2) \langle F_{\mu\nu} F^{\mu\nu} \rangle + \dots$$

Baikov, Chetyrkin, Kühn '08   RH '98

perturbative

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RH '98

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RH, Neumann '16

RH, Lange, Neumann '20

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Baikov, Chetyrkin, Kühn '08 RH '98

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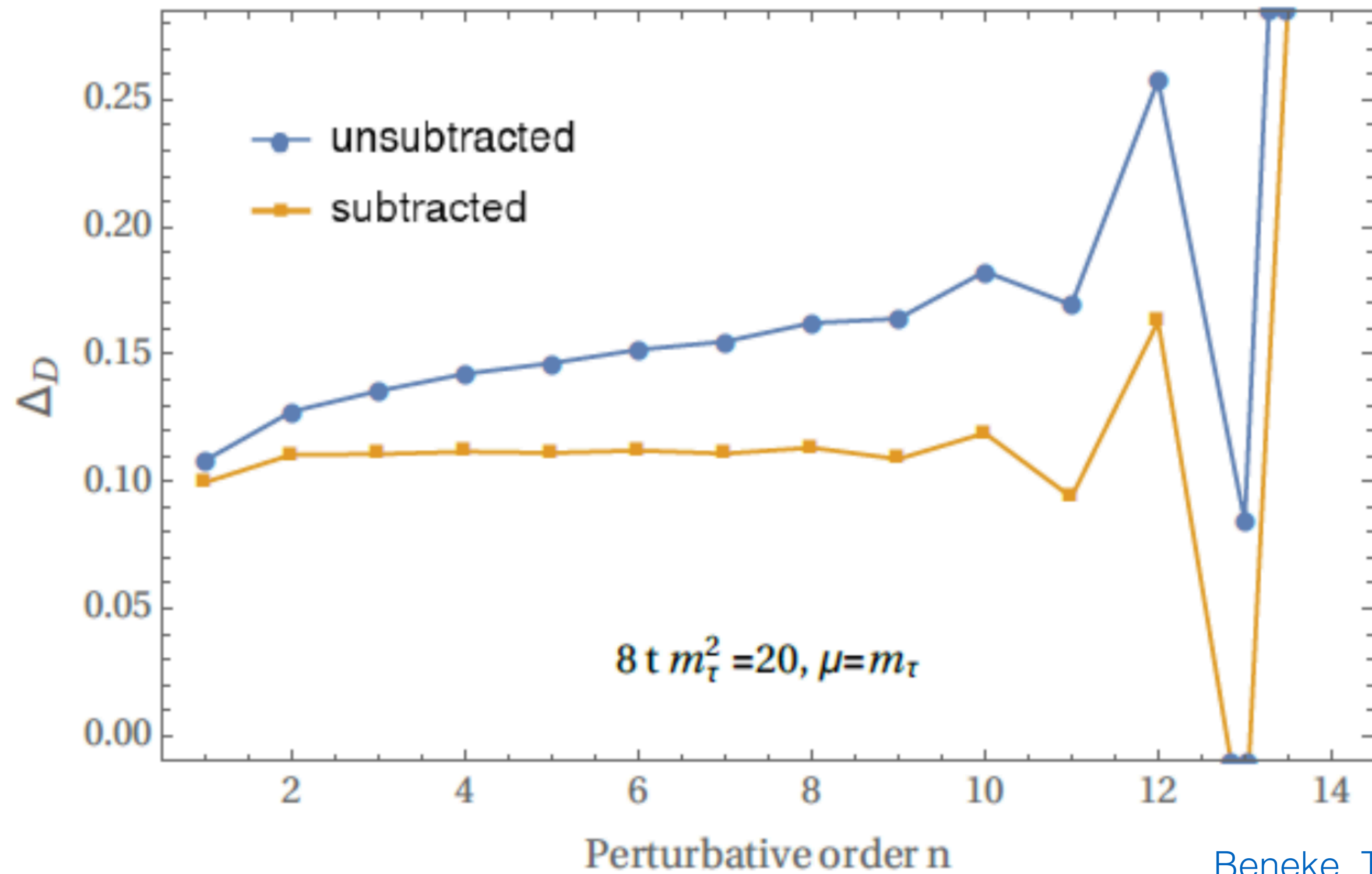
RH, Neumann '16

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renormalon cancels



Beneke, Takaura '25

Three thoughts on the GF scheme

2

Three thoughts on the GF scheme

2

Problems with $\overline{\text{MS}}$: mass independent

Three thoughts on the GF scheme

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decoupling must be implemented “by hand”

Three thoughts on the GF scheme

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e.g.: $\alpha_s^{(n_f=3)}(\mu)$ ok for $\mu \lesssim m_c$

Three thoughts on the GF scheme

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Three thoughts on the GF scheme

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e.g.: $\alpha_s^{(n_f=3)}(\mu)$ ok for $\mu \lesssim m_c$

$$\beta^{\overline{\text{MS}}}(\alpha_s) = -\frac{\alpha_s}{4\pi} \left(11 - \frac{2}{3}n_f \right) + \dots$$

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decoupling relation: $\alpha_s^{(4)}(\mu) = \alpha_s^{(3)}(\mu) \left[1 + c_1(m_c, \mu) \alpha_s^{(3)}(\mu) + \dots \right]$

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$\alpha_s(\mu)$ has a kink/step at some “decoupling scale” μ_{thr}

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$$k_1 = k_1 \Big|_{n_f=0} - \sum_q \frac{1}{6\pi} \left(1 + \Omega_{1q}(m_q, t) \right)$$

Lüscher 2010
RH, Neumann 2016

Three thoughts on the GF scheme

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NNLO: RH, Mason 2025

Three thoughts on the GF scheme

Kluth '24, '26

3

Three thoughts on the GF scheme

Kluth '24, '26

3

Problems with $\overline{\text{MS}}$: insensitive to power divergences

Three thoughts on the GF scheme

Kluth '24, '26

3

Problems with $\overline{\text{MS}}$: insensitive to power divergences

Example:

$$\int \frac{d^D p}{\pi^{D/2}} \frac{1}{p^2 + m^2} = m^{D-2} \Gamma(1 - D/2)$$

Three thoughts on the GF scheme

Kluth '24, '26

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$$D = 4 - 2\epsilon \quad = m^{2-2\epsilon} \Gamma(-1 + \epsilon) = \frac{m^{2-2\epsilon}}{\epsilon - 1} \Gamma(\epsilon) = \frac{m^2}{\epsilon}$$

Three thoughts on the GF scheme

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i.e., only logarithmic divergence contributes!

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→ Fixed point in gravity is hidden in $\overline{\text{MS}}$!

Three thoughts on the GF scheme

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Example:

$$\int \frac{d^D p}{\pi^{D/2}} \frac{1}{p^2 + m^2} = m^{D-2} \Gamma(1 - D/2)$$

$$D = 4 - 2\epsilon \quad = m^{2-2\epsilon} \Gamma(-1 + \epsilon) = \frac{m^{2-2\epsilon}}{\epsilon - 1} \Gamma(\epsilon) = \frac{m^2}{\epsilon}$$

$$\text{cf.} \quad \int \frac{d^D p}{\pi^{D/2}} \frac{1}{p^2 + m^2} = \int \frac{d^D p}{\pi^{D/2}} \left[\frac{1}{p^2} - \frac{m^2}{p^4} + \dots \right]$$

i.e., only logarithmic divergence contributes!

→ Fixed point in gravity is hidden in $\overline{\text{MS}}$!

Need mass dependent renormalization scheme

Application to gravity

RH, Kluth, Kohnen, Werthenbach '26

flowed gauge field:

$$\begin{aligned}\frac{\partial}{\partial t} B_\mu(t, x) &= \mathcal{D}_\nu G_{\nu\mu}(t, x) \\ B_\mu(t = 0, x) &= A_\mu(x)\end{aligned}$$

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Ricci flow

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flowed metric requires field renormalization (analogous to Z_χ in QCD)

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$$\langle \sqrt{g(t)} \rangle \equiv -\frac{3G_N}{4\pi t}$$

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$$\text{cf.: } \langle \dot{\tilde{\chi}}(t) \mathbb{D} \dot{\tilde{\chi}}(t) \rangle \equiv -\frac{2n_c}{(4\pi t)^2}$$

Application to gravity

RH, Kluth, Kohnen, Werthenbach '26

define Newton coupling in Ricci flow scheme:

$$g_{\text{RF}}(\mu) \equiv -\frac{t}{3\nu} \langle \sqrt{g(t)} R(t) \rangle \Big|_{t=c/\mu^2}$$

$$\text{cf. } \alpha_s^{\text{GF}}(\mu) \equiv \frac{4\pi t^2}{3} \langle G_{\mu\nu}(t) G^{\mu\nu}(t) \rangle \Big|_{t=c/\mu^2}$$

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$$= \frac{G_{\text{N}}}{4\pi t\nu} \left[1 - \frac{5G_{\text{N}}}{4\pi t} + \dots \right]$$

Application to gravity

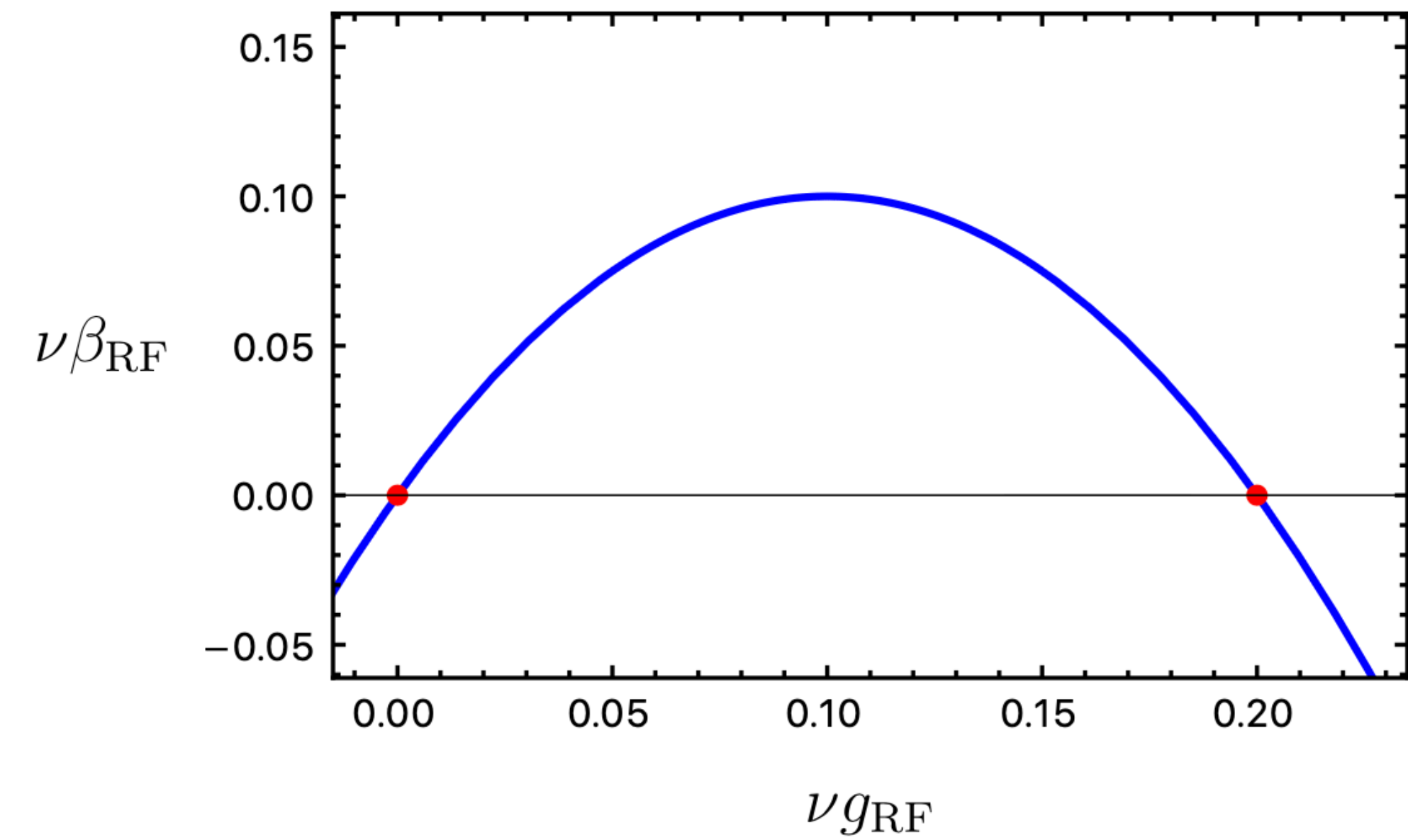
RH, Kluth, Kohnen, Werthenbach '26

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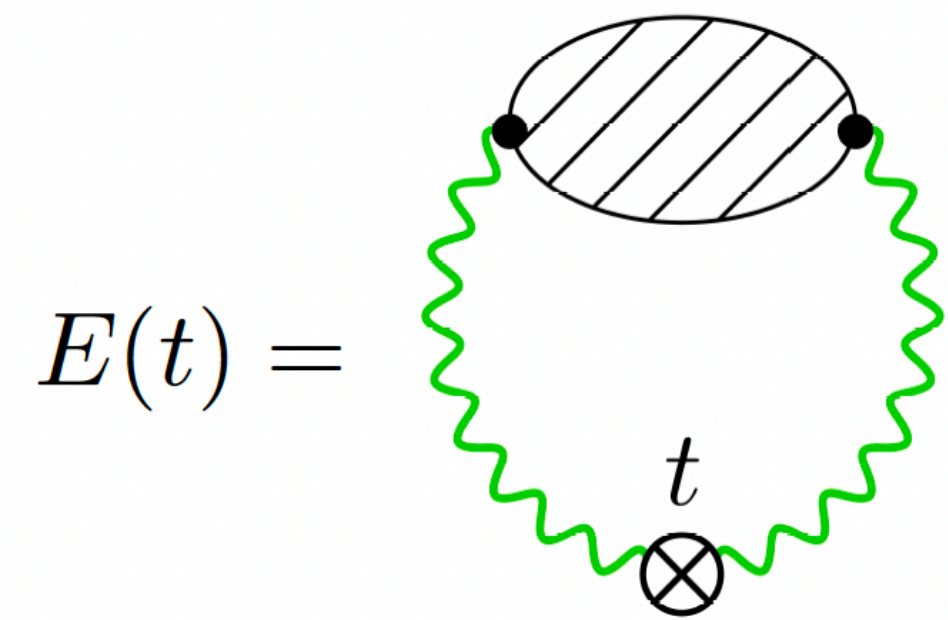
Application to QED₃

$$E(t) = \text{[Diagram]} = \langle F_{\mu\nu}(t) F_{\mu\nu}(t) \rangle$$

The diagram shows a green wavy loop with a shaded elliptical region at the top and a vertex at the bottom labeled t with a cross symbol.

Application to QED₃

in D dimensions:

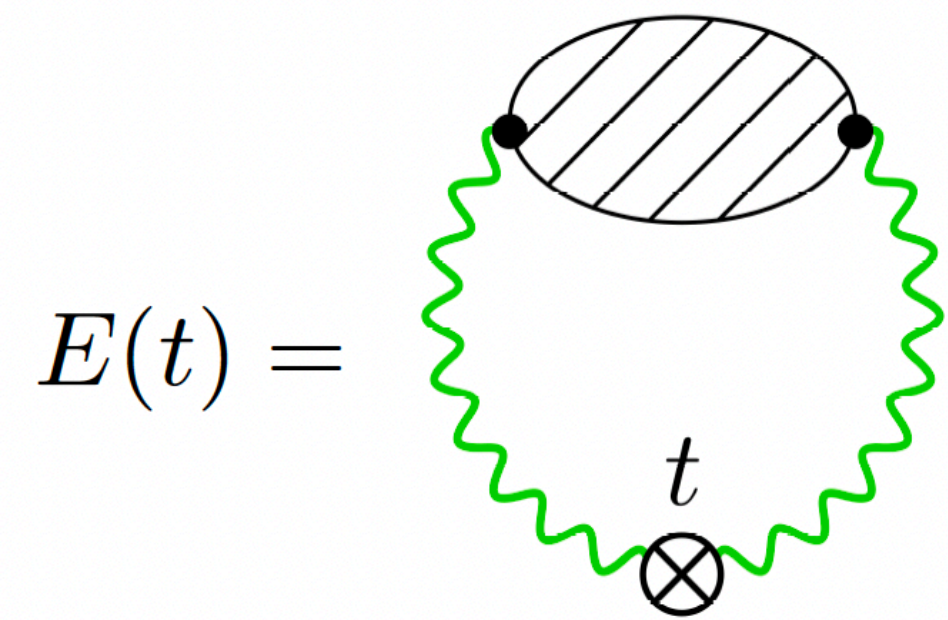


$$= \langle F_{\mu\nu}(t) F_{\mu\nu}(t) \rangle = \frac{e^2}{2} (D-1) \int \frac{d^D p}{(2\pi)^D} \frac{e^{-2tp^2}}{1 + \Pi_R(p)}$$

Georg, RH, Mason '26

Application to QED₃

in D dimensions:



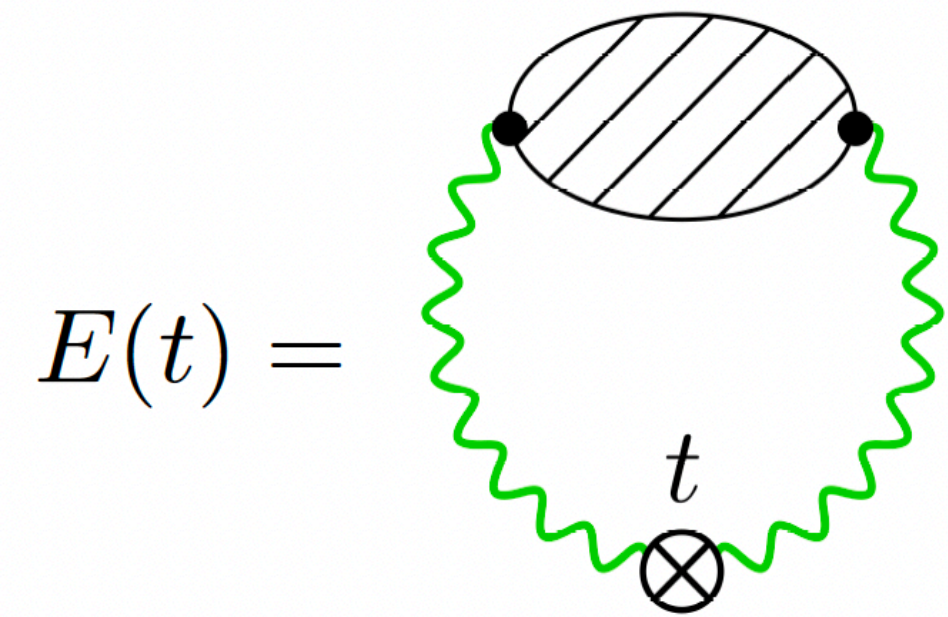
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Georg, RH, Mason '26

regular photon
polarization function

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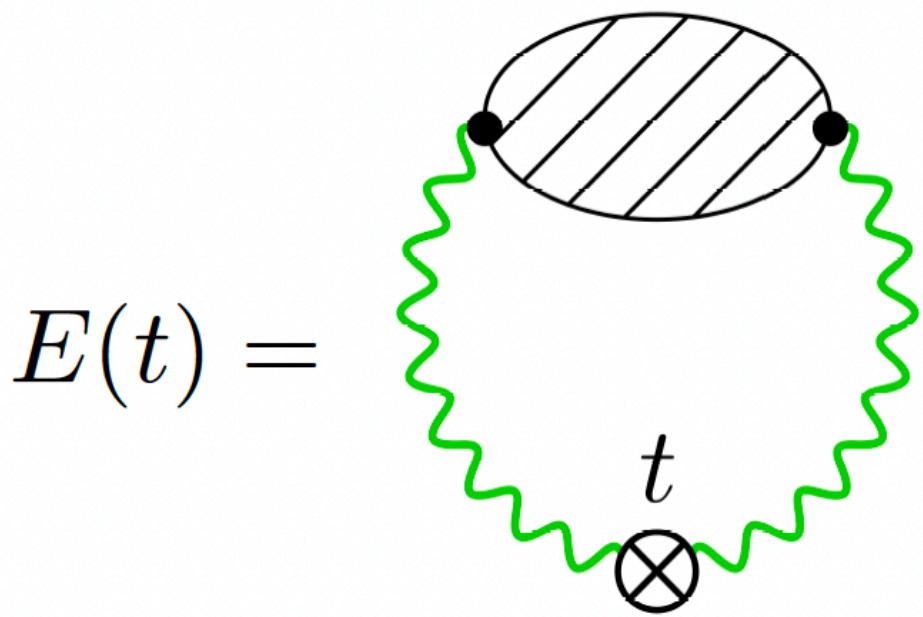
Georg, RH, Mason '26

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D=3:

Application to QED₃

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Georg, RH, Mason '26

regular photon polarization function

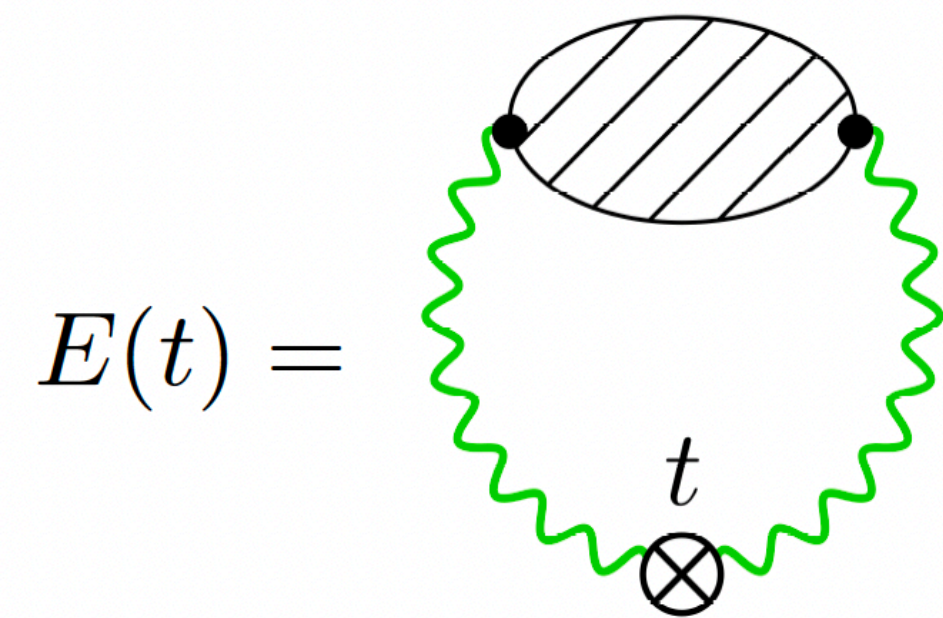
D=3: α mass dimension = 1

$$\hat{\alpha} \sim \alpha/p$$

Appelquist, Pisarski '81
Pisarski '84

Application to QED₃

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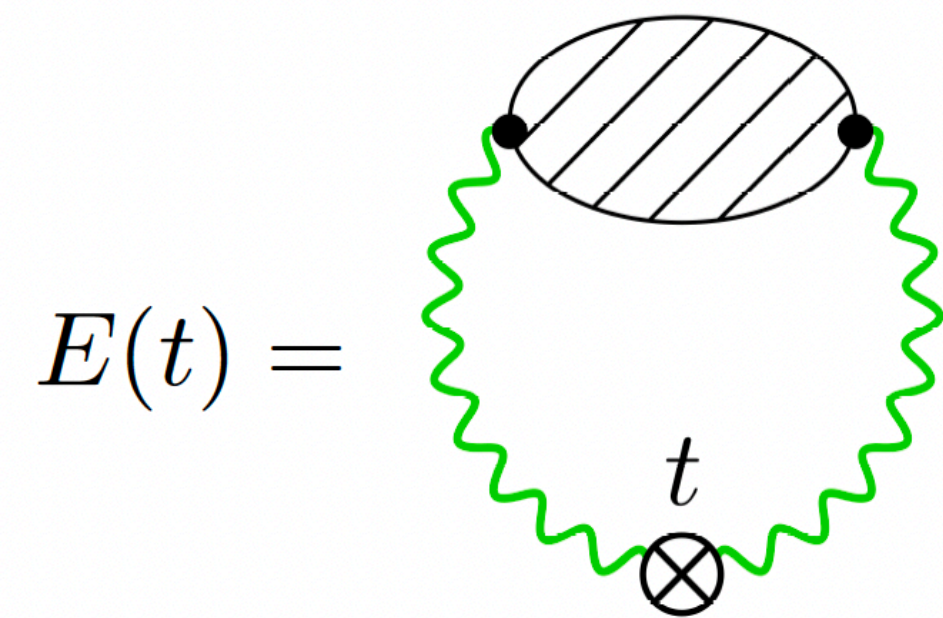
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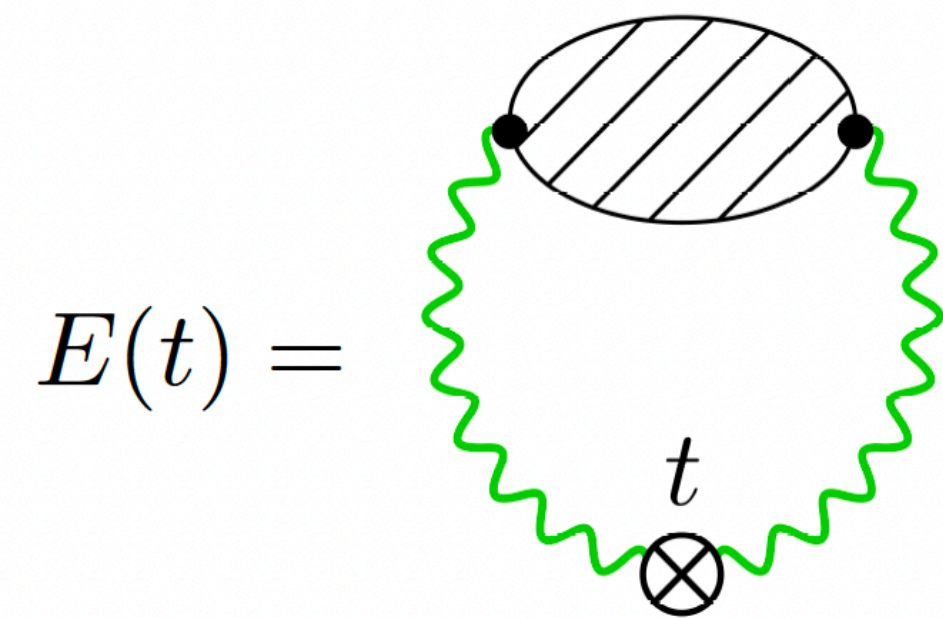
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Georg, RH, Mason '26

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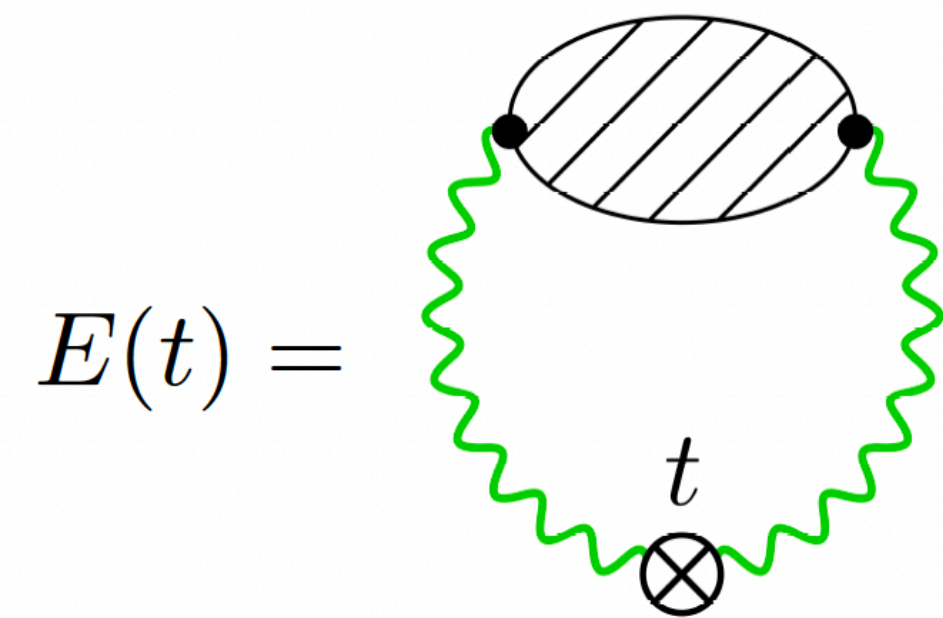
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Georg, RH, Mason '26

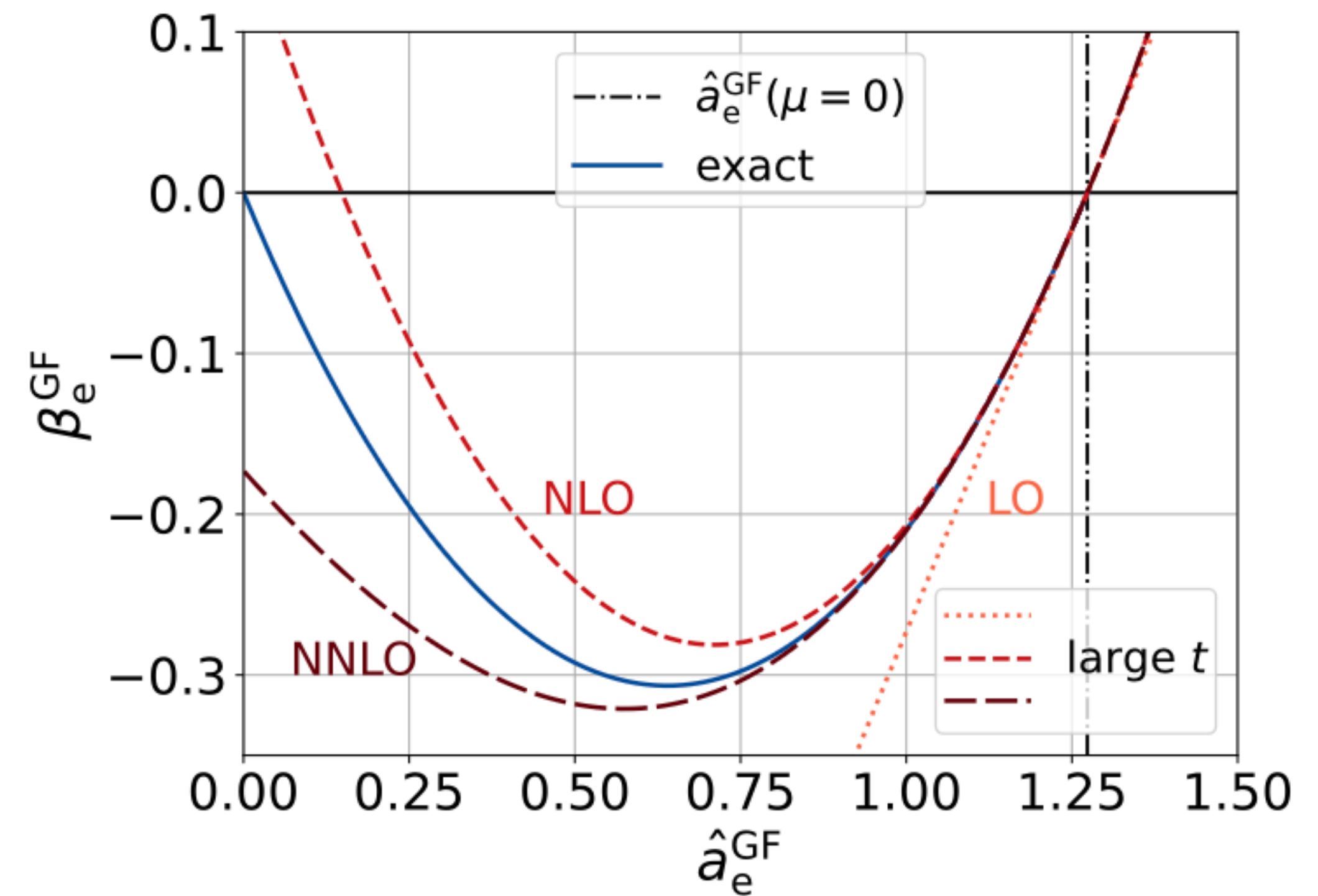
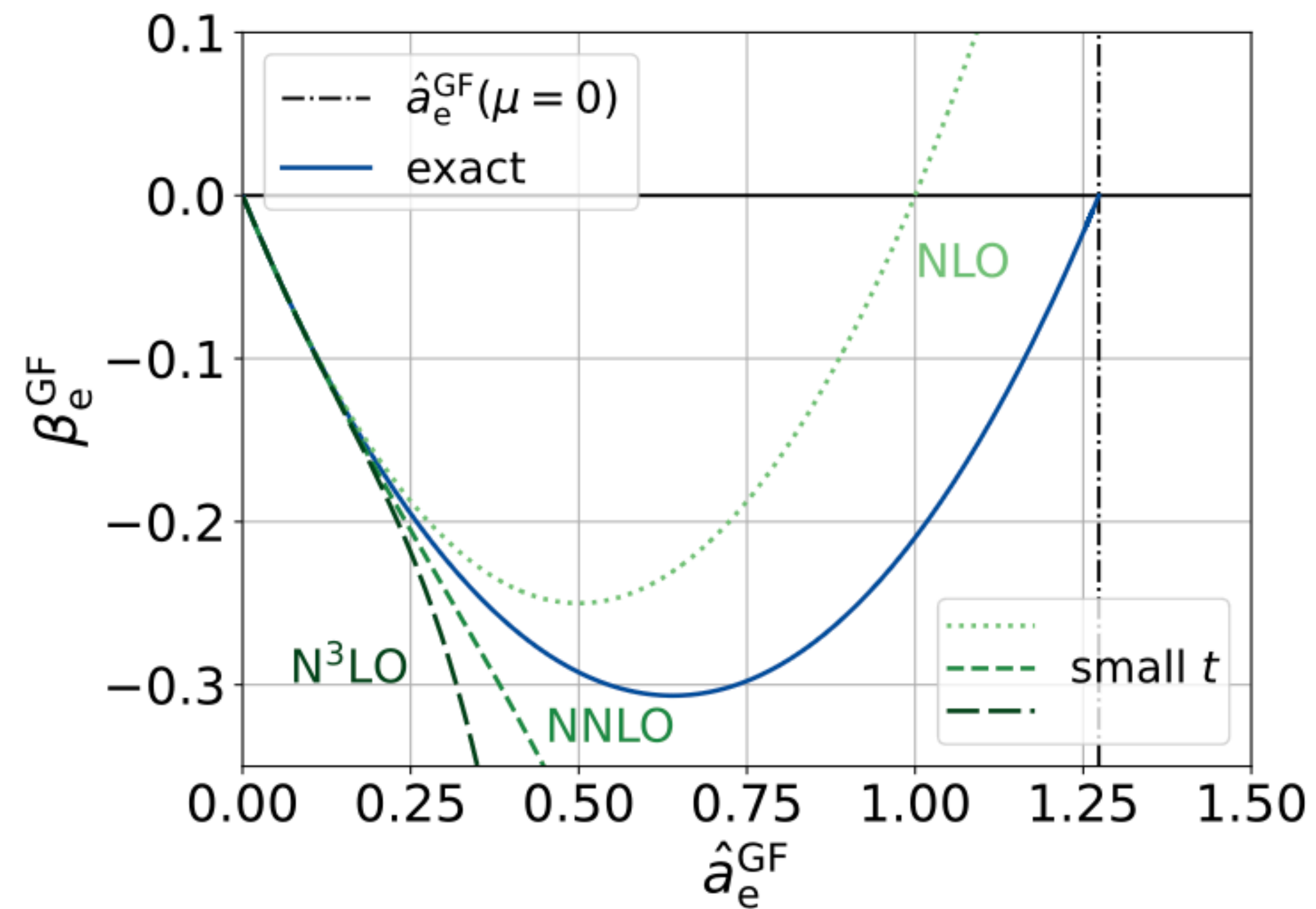
regular photon
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D=3: α mass dimension = 1
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 strongly interacting in IR
 IR fixed point at large n_f
 dynamical fermion mass generation

$$\hat{\alpha} \sim \alpha/p$$

Appelquist, Pisarski '81
 Pisarski '84

Application to QED₃



Georg, RH, Mason 2025

The GF scheme

$$\mathcal{L}_{\text{eff}} = \sum_n C_n \langle \mathcal{O}_n \rangle$$

GF

$\overline{\text{MS}}$

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$$= \sum_n (C \zeta^{-1}(t))_n \langle \zeta(t) \mathcal{O} \rangle_n$$

$$= \sum_n (C Z^{-1})_n \langle Z \mathcal{O} \rangle_n$$

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$$\gamma_{nm} = - \mu \frac{d}{d\mu} \ln Z$$

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RH, Lange, Neumann '20

Borgulat, Felten, RH, Kohnen '25

Conclusions

Gradient flow: rich field theoretical concept

Established tool in **lattice calculations**

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Lattice/PT interplay picks up momentum:

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Many things not even discussed:

Static QCD potential, EDMs, ...