

The perturbative Ricci Flow in Quantum Gravity

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Mini-Workshop on the Ricci Flow

in collaboration with Harlander, Kluth, Kohnen

TTK, RWTH Aachen University

A short CV

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1982 introduced by Hamilton



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9 Sep Miniworkshop



Motivation

Why a flow?

Asymptotic safety

- power divergences
- $\overline{MS}$ is blind to them
- want: symmetry-preserving scheme with power divergencies

Gradient flow

- finite flowed operators
- lattice $\leftrightarrow$ perturbation theory
- CDT / EDT

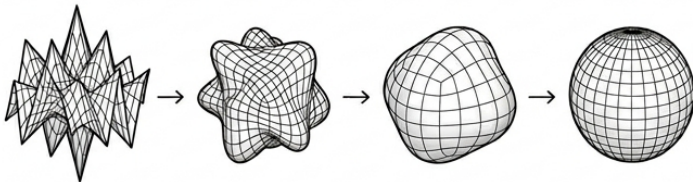
Ricci flow: the diffeomorphism invariant candidate.

First step: **perturbation theory**.

Intuition

What the Ricci flow does

$$\partial_t \hat{g}_{\mu\nu} = -2\hat{R}_{\mu\nu}, \quad \hat{g}_{\mu\nu}(0, x) = g_{\mu\nu}(x)$$



From geometry to a quantum field

$$S_{\text{EH}} = -\frac{2}{\kappa^2} \int_x \sqrt{g} R, \quad \Lambda = 0$$

classical

$$\bar{R}_{\mu\nu} = 0 \Rightarrow \partial_t \bar{g}_{\mu\nu} = 0$$

stationary solutions of the flow

quantum $g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}$

$$\partial_t \hat{h}_{\mu\nu} = \partial^2 \hat{h}_{\mu\nu} \Rightarrow \hat{h}_{\mu\nu}(t, p) = e^{-tp^2} h_{\mu\nu}(p)$$

damped: $p^2 \gg 1/t$

Perturbative solution

Solving flow and QFT together

Expand the flow equation in powers of $\hat{h}_{\mu\nu}$:

$$\partial_t \hat{h}_{\mu\nu} = \partial^2 \hat{h}_{\mu\nu} + \hat{E}_{\mu\nu}^{(2)}[\hat{h}, \hat{h}] + \mathcal{O}(\hat{h}^3)$$

Iterate, with $K(t, p) = e^{-tp^2}$:

$$\hat{h}_{\mu\nu}(t, p) = K(t, p) h_{\mu\nu}(p) + \int_0^t ds K(t-s, p) \hat{E}_{\mu\nu}^{(2)}[\hat{h}(s), \hat{h}(s)](p) + \dots$$

$\Rightarrow$ ordinary perturbation theory for $h_{\mu\nu}$ at $t = 0$

Equivalently: a $(d + 1)$ -dimensional field theory

$$S = S_{EH} + \int_0^\infty dt \int_x \hat{L}^{\mu\nu} \left[\partial_t \hat{h}_{\mu\nu} + 2\hat{R}_{\mu\nu} - 2\alpha_0 \hat{\nabla}_{(\mu} \hat{F}_{\nu)} \right]$$

- t : extra, artificial dimension
- $\hat{L}^{\mu\nu}$: Lagrange multiplier, tensor **density**
- flow gauge fixing $\hat{F}_\mu = \partial^\nu \hat{h}_{\mu\nu} + \beta_0 \partial_\mu \hat{h}^\rho{}_\rho$
- invariant objects: independent of α_0, β_0

Feynman rules

Propagator

$$\langle \hat{h} \hat{h} \rangle \sim \frac{e^{-(t+s)p^2}}{p^2}$$



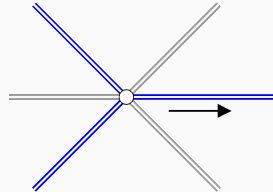
Flow line

$$\langle \hat{L} \hat{h} \rangle \sim \theta(t-s) e^{-(t-s)p^2}$$



Flow vertices

$$\hat{L} \hat{h} \hat{h}, \hat{L} \hat{h} \hat{h} \hat{h}, \dots$$



$$\begin{aligned}\mathfrak{s}(\hat{h}_{\mu\nu}) &= \hat{\nabla}_\mu \hat{C}_\nu + \hat{\nabla}_\nu \hat{C}_\mu \\ \mathfrak{s}(\hat{C}_\mu) &= -\hat{C}^\rho \hat{\nabla}_\mu \hat{C}_\rho\end{aligned}$$

$$\begin{aligned}\mathfrak{s}(\hat{L}^{\mu\nu}) &= (\partial_\rho \hat{C}^\rho) \hat{L}^{\mu\nu} + \hat{C}^\rho \hat{\nabla}_\rho \hat{L}^{\mu\nu} \\ &\quad - 2(\hat{\nabla}^\rho \hat{C}^{(\mu} \hat{L}^{\nu)}_\rho\end{aligned}$$

- gauge fixing $\rightarrow$ BRST, as in pure gravity
- flowed ghosts $\hat{C}^\mu$ needed — but never contribute
- density $\Rightarrow$ the $(\partial_\rho \hat{C}^\rho)$ term

Renormalization

QCD:

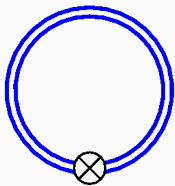
- flowed gauge field observables: **no** operator renormalization
- flowed fermions: multiplicative only

Ingredients:

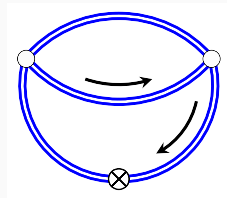
- power counting in $d + 1$ dimensions
- BRST symmetry

Let us do the same for gravity.

Bulk vs. boundary



lines carry e^{-tp^2} : **finite**



only through a one-loop **subdivergence**

No bulk counterterms. Counterterms only at $t = 0$.

What symmetry allows

- power counting
- counterterms must be BRST invariant

One loop:

gravity sector

$$R^2, \quad R_{\mu\nu} R^{\mu\nu}$$

absorbed by

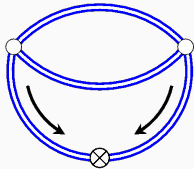
$$g_{\mu\nu} \rightarrow g_{\mu\nu} + \frac{G_{\mathbf{N}}}{4\pi} [c_1 R g_{\mu\nu} + c_2 R_{\mu\nu}]$$

flow sector (at $t = 0$)

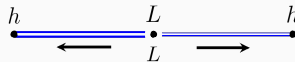
$$\hat{L}^{\mu\nu} R_{\mu\nu}, \quad \hat{L}^\mu{}_\mu R$$

$$\frac{1}{\sqrt{g}} \hat{L}^{\mu\nu} \hat{L}_{\mu\nu}, \quad \frac{1}{\sqrt{g}} \hat{L}^\mu{}_\mu \hat{L}^\nu{}_\nu$$

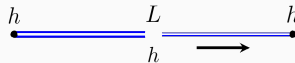
Mapping $\hat{L}$ onto $\hat{h}$ counterterms



$\hat{L}\hat{L}$ divergence

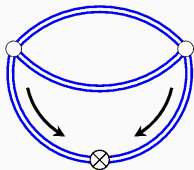


$\hat{L}\hat{L}$ counterterm

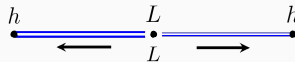


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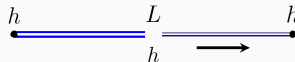
Mapping $\hat{L}$ onto $\hat{h}$ counterterms



$\hat{L}\hat{L}$ divergence



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$\hat{L}\hat{h}$ counterterm

$$\frac{1}{\sqrt{g}}\hat{L}^{\mu\nu}\hat{L}_{\mu\nu}, \frac{1}{\sqrt{g}}\hat{L}^{\mu}_{\mu}\hat{L}^{\nu}_{\nu} \longrightarrow \hat{L}^{\mu\nu}R_{\mu\nu}, \hat{L}^{\mu}_{\mu}R$$

Everything is an initial-condition renormalization

$$\partial_t \hat{g}_{\mu\nu} = -2\hat{R}_{\mu\nu} + (\text{g.f.}) + \delta(t) [\hat{z}_1 R g_{\mu\nu} + \hat{z}_2 R_{\mu\nu}]$$

$$\Longleftrightarrow \quad \hat{g}_{\mu\nu}(0, x) = g_{\mu\nu} + \hat{z}_1 R g_{\mu\nu} + \hat{z}_2 R_{\mu\nu}, \quad g_{\mu\nu} \rightarrow g_{\mu\nu} + \frac{G_N}{4\pi} [c_1 R g_{\mu\nu} + c_2 R_{\mu\nu}]$$

$$\Rightarrow \quad \delta \hat{g}_{\mu\nu} = \frac{G_N}{4\pi} [\hat{c}_1 R g_{\mu\nu} + \hat{c}_2 R_{\mu\nu}]$$

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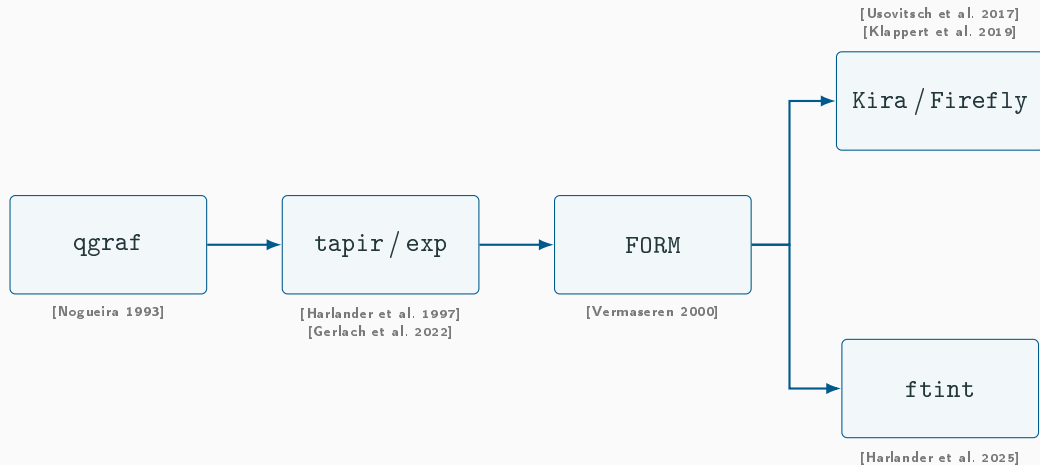
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✓ all flowed observables at two loops: renormalized by $\hat{c}_1, \hat{c}_2$ — **redundant** operators only.

How we computed

Toolchain



What we computed

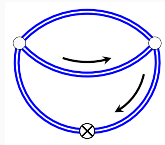
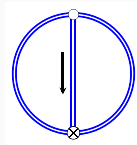
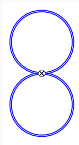
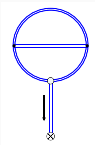
Flowed observables

$$\mathcal{I}_n(t) \equiv \frac{1}{V} \left\langle 0 \left| \int_x \sqrt{\hat{g}} \hat{\mathcal{O}}_n \right| 0 \right\rangle, \quad \hat{\mathcal{O}}_1 = 1, \quad \hat{\mathcal{O}}_{\hat{R}} = \hat{R}$$

$\mathcal{I}_1$: quantum **volume**

$\mathcal{I}_{\hat{R}}$: quantum **curvature**

$\partial_t \mathcal{I}_1 = -\mathcal{I}_{\hat{R}}$ (check)



Two-loop results

$$\mathcal{I}_1^{\overline{\text{MS}}} = 1 - \frac{3G_N}{4\pi t} \left[1 + \frac{G_N}{4\pi t} \left(5 L_{\mu t} + 9 \ln \frac{4}{3} + \frac{1231}{120} \right) \right]$$

$$t \mathcal{I}_{\hat{R}}^{\overline{\text{MS}}} = -\frac{3G_N}{4\pi t} \left[1 + \frac{G_N}{2\pi t} \left(5 L_{\mu t} + 9 \ln \frac{4}{3} + \frac{931}{120} \right) \right]$$

$$L_{\mu t} = \ln \mu^2 t + \gamma_E$$

also: $\hat{R}_{\mu\nu\rho\sigma}^2$, $\hat{R}_{\mu\nu}^2$, $\hat{R}^2$, $\hat{R}\square\hat{R}$ — same counterterms, gauge parameters cancel

A scheme for Newton's coupling

Flowed-volume scheme, to all orders:

$$\mathcal{I}_1^{\text{FVS}} \equiv \mathcal{I}_1^{\text{LO}}$$

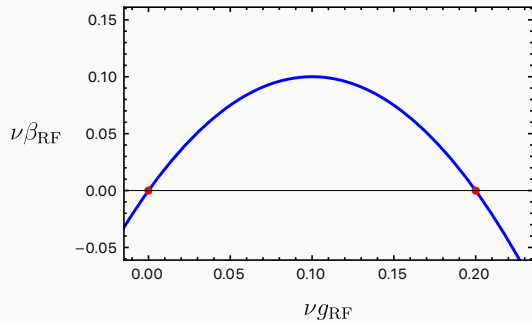
$$\implies \mathcal{I}_{\hat{R}}^{\text{FVS}} = -\frac{3G_{\text{N}}}{4\pi t^2} \left[1 - \frac{5G_{\text{N}}}{4\pi t} \right] \quad (\mu\text{-independent})$$

$$g_{\text{RF}}(\mu) \equiv -\frac{4t}{3} \mathcal{I}_{\hat{R}}^{\text{FVS}} \Big|_{t=\rho/\mu^2}$$

A non-Gaussian fixed point

$$\beta_{\text{RF}} = 2g_{\text{RF}} \left(1 - \frac{5}{4} g_{\text{RF}}\right)$$

- $g_{\text{RF}}^* = 4/5$
- perturbative regime
- $\theta = 2$



Summary

Summary

- Ricci flow: diffeomorphism invariant
UV regulator
- flowed Feynman rules, BRST invariant
- Renormalisation
- g_{RF} with a NGFP

Outlook

- higher orders
- $\Lambda \neq 0$
- coupling to matter

Discussion

1. Which flow? $\Lambda \neq 0$, normalized, ...
2. Counterterms on the lattice — automatic or tuned?
3. Which coupling? what else can we extract?

What else?

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classical

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stationary solutions of the flow

quantum $g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}$

$$\partial_t \hat{h}_{\mu\nu} = \partial^2 \hat{h}_{\mu\nu} \Rightarrow \hat{h}_{\mu\nu}(t, p) = e^{-tp^2} h_{\mu\nu}(p)$$

damped: $p^2 \gg 1/t$

Discussion #1

$\Lambda \neq 0$: normalized flow? Which flow goes on the lattice?

Everything is an initial-condition renormalization

$$\partial_t \hat{g}_{\mu\nu} = -2\hat{R}_{\mu\nu} + (\text{g.f.}) + \delta(t) [\hat{z}_1 R g_{\mu\nu} + \hat{z}_2 R_{\mu\nu}]$$

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$$\Rightarrow \quad \delta \hat{g}_{\mu\nu} = \frac{G_N}{4\pi} [\hat{c}_1 R g_{\mu\nu} + \hat{c}_2 R_{\mu\nu}]$$

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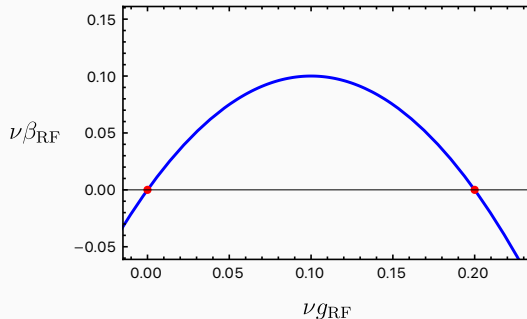
Discussion #2

What would that mean for the lattice?

A non-Gaußian fixed point

$$\beta_{\text{RF}} = 2g_{\text{RF}} \left(1 - \frac{5}{4} g_{\text{RF}}\right)$$

- $g_{\text{RF}}^* = 4/5$
- perturbative regime
- $\theta = 2$



Discussion #3

Other coupling definitions? What other physics can the Ricci flow give us?

Backup: conventions

$$S_{\text{EH}} = -\frac{2}{\kappa^2} \int_x \sqrt{g} R, \quad \kappa^2 = 32\pi G_{\text{N}}, \quad \Lambda = 0,$$

$$g_{\mu\nu} = \delta_{\mu\nu} + h_{\mu\nu}, \quad \hat{g}_{\mu\nu} = \delta_{\mu\nu} + \hat{h}_{\mu\nu},$$

$$F_\mu = \partial^\nu h_{\mu\nu} + \beta \delta^{\rho\sigma} \partial_\mu h_{\rho\sigma}, \quad S_{\text{gf}} = \int_x \frac{1}{\alpha \kappa^2} \delta^{\mu\nu} F_\mu F_\nu,$$

$$\mathcal{P}_{\mu\nu\rho\sigma}(p) = \frac{\kappa^2}{p^2} \left(\delta_{\mu(\rho} \delta_{\sigma)\nu} - \frac{1}{d-2} \delta_{\mu\nu} \delta_{\rho\sigma} \right) \quad \text{for } \alpha = -2\beta = 1.$$

Backup: counterterms and scheme

$$\begin{aligned}\hat{c}_1^{\overline{\text{MS}}} &= -\frac{107}{30} \frac{1}{\varepsilon}, & \hat{c}_2^{\overline{\text{MS}}} &= \frac{407}{30} \frac{1}{\varepsilon} \\ \hat{c}_i &= \hat{c}_i^{\overline{\text{MS}}} + \hat{c}_i^{\text{fin}}, & \hat{c}_2^{\text{fin}} &= 10 L_{\mu t} + 18 \ln \frac{4}{3} + \frac{1231}{60} - \hat{c}_1^{\text{fin}}, \\ & & \hat{c}_1^{\text{fin}} &\text{arbitrary}\end{aligned}$$

Backup: general ν

$$g_{\text{RF}}(\mu) \equiv -\frac{t}{3\nu} \mathcal{I}_{\hat{R}}^{\text{FVS}} \Big|_{t=\rho/\mu^2}, \quad \beta_{\text{RF}} = 2g_{\text{RF}}(1 - 5\nu g_{\text{RF}})$$

$$g_{\text{RF}}^* = \frac{1}{5\nu}, \quad \frac{G_{\text{N}}}{4\pi t} = 3\nu g_{\text{RF}} + 15\nu^2 g_{\text{RF}}^2 + \dots$$

perturbative fixed point: $\nu \gtrsim 1/5$ — perturbative conversion: $\nu \lesssim 1/4$ — $\theta = 2$ for any ν