Introduction to SymTFT and Condensation Defects via $\mathbb{Z}_2$ gauge theory.

Contents

§1 §2 → §3 → §4

Ref

§1. • 2205.05565

• "Topological Quantum"

§2.3. • "Lecture on anomalies and topological phases (2019)"

• Refs in 2601.09067

• 2107.13091

§4 • 2204.02407

§1 Quick review of 2+1 D TQFT

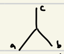
Fact

2+1 D TQFTs are described by Modular Tensor Cat (MTC).

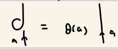
Basic data of MTC

Anyon : $a, b, c, \ldots \in \{\text{Anyon}\} =: \mathcal{C}$

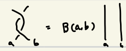
Fusion : $a \times b = \sum_c N_{ab}^c\, c$ w/ $N_{ab}^c \in \mathbb{Z}_{\geq 0}$



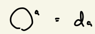
Top spin : $\theta(a) \in U(1)$ w/ $a \in \mathcal{C}$



Braiding : $B(a,b) \in U(1)$ w/ $a, b \in \mathcal{C}$

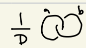


Quantum dim : $d_a \in \mathbb{R}_{\geq 1}$ w/ $a \in \mathcal{C}$



Tot. q dim : $\mathcal{D} := \sqrt{\sum_a d_a^2}$

S mat : $S_{ab} := \frac{1}{\mathcal{D}} \cdot \sum_c N_{a\bar{b}}^c \frac{\theta(c)}{\theta(a)\theta(b)} d_c$



chiral central charge : $\frac{1}{\mathcal{D}} \sum_a d_a^2 \theta(a) = e^{2\pi i c/8}$

Rmk

- $\forall a \in \mathcal{C},\ \exists!\ \bar{a} \in \mathcal{C}$ s.t. $a \times \bar{a} = \underbrace{1 + \cdots}_{\exists!\, 1 \in \mathcal{C} \text{ s.t. } \forall a \in \mathcal{C},\ 1 \times a = a}$
- TQFT is abelian iff $a \times b = c$ $\forall a, b \in \mathcal{C}$. $\Leftrightarrow d_a = 1\ \forall a \in \mathcal{C}$

Fact

$\mathcal{C}$: abelian ($\Leftrightarrow N_{ab}^c = \delta_{c, a\times b}$)

$$B(a,b) := \frac{\theta(a \times b)}{\theta(a)\theta(b)}$$

$$S_{ab} := \frac{1}{\sqrt{\sum 1}} \cdot \sum \delta_{c, a\times\bar{b}} \frac{\theta(c)}{\theta(a)\theta(b)} \cdot 1$$

$$= \frac{1}{|\mathcal{C}|^{1/2}} \cdot B(a, \bar{b})$$

Fact

$SL(2,\mathbb{Z}) \curvearrowright V(T^2)$

$$S_{ab} = \frac{1}{\mathcal{D}} \cdot \sum N_{a\bar{b}}^c \frac{\theta(c)}{\theta(a)\theta(b)} d_c \qquad T_{ab} = e^{-2\pi i c/24} \cdot \theta(a)\, \delta_{ab}.$$

E.g.

$\mathbb{Z}_2$ gauge theory = Toric Code (TC)

Anyon : $\mathcal{C} = \{1, e, m, f\}$

Fusion : $e^2 = m^2 = f^2 = 1 \quad e \times m = f \quad (\Rightarrow a = \bar{a} \ \forall a \in \mathcal{C})$

Top spin : $\theta(1) = \theta(e) = \theta(m) = 1 \quad \theta(f) = -1$

Braiding : $B(e,m) = \dfrac{\theta(e \times m)}{\theta(e)\theta(m)} = -1$

q. dim : $d_a = 1 \quad \forall a \in \mathcal{C}$

Tot. q. dim : $\mathcal{D} = 2$.

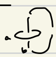
S mat : $S_{ab} = \dfrac{1}{2} \cdot \dfrac{\theta(a \times b)}{\theta(a)\theta(b)}$

ccc : $c = 0 \mod 8$

Fact

 $= \dfrac{S_{ab}}{S_{1b}} \Big|_b$ w/ $S_{1a} = \dfrac{1}{D} \cdot \underbrace{\sum N^c_{a1} \cdot \dfrac{\theta(c)}{\theta(a) \cdot 1} \cdot d_c}_{S_{ac}} = \dfrac{d_a}{D}$

Obs

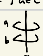
 $= \dfrac{S_{ab}}{S_{1b}}$ 

$= S_{ab} \cdot \left(\dfrac{D}{d_b}\right) \cdot d_b$

$= D \cdot S_{ab} \Leftrightarrow S_{ab} = \dfrac{1}{D} \cdot$ 

Cor

$\left\{\dfrac{S_{ax}}{S_{1x}}\right\}_{a \in \mathcal{C}}$ forms a fusion ring.

☺  $= \dfrac{S_{ax}}{S_{1x}} \dfrac{S_{bx}}{S_{1x}} \Big|_x$

$= \sum_c N^c_{ab}$ 

$= \sum_c N^c_{ab} \dfrac{S_{cx}}{S_{1x}} \Big|_x$

Prop

$N^c_{ab} = \sum_x \dfrac{S_{ax} S_{bx} \bar{S}_{cx}}{S_{1x}}$ (Verlinde formula)

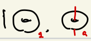
☺ $\dfrac{S_{ax}}{S_{1x}} \dfrac{S_{bx}}{S_{1x}} = \sum_c N^c_{ab} \dfrac{S_{cx}}{S_{1x}}$

$\Leftrightarrow \sum_x \dfrac{S_{ax} S_{bx} \bar{S}_{dx}}{S_{1x}} = \sum_{x,d} S_{cx} \bar{S}_{dx} N^c_{ab}$

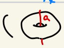
$= \sum_d (S S^\dagger)_{cd} N^c_{ab} = N^c_{ab}$

Fact

- $V(T^2) \cong \mathbb{C}^{\oplus |\mathcal{C}|} \cong \mathrm{span}_{\mathbb{C}}\{|a\rangle \mid a \in \mathcal{C}\} = \mathrm{span}_{\mathbb{C}}\{$  $, \dots \}$

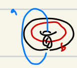
  $\cong \mathrm{span}_{\mathbb{C}}\{S|a\rangle \mid a \in \mathcal{C}\} = \mathrm{span}_{\mathbb{C}}\{$  $, \dots \}$

  w/ $|a\rangle := A(x)|0\rangle = Z($  $) = Z(D^2(a) \times S^1)$

  $S|a\rangle := A(y)(S|0\rangle) = Z($  $) = Z(S^1 \times D^2(a))$

- $A(x)|b\rangle = \sum_c N^c_{ab} |c\rangle$

Prop

$A(y)|b\rangle = \dfrac{S_{ab}}{S_{1b}} |b\rangle$  $= \dfrac{S_{ab}}{S_{1b}}$ 

☺ LHS $= S A(x) S^\dagger |b\rangle$

$= S\left(\sum_c A(x) (S^\dagger)_{bc} |c\rangle\right)$

$= S\left(\sum_{c,d} (S^\dagger)_{bc} N^d_{ac} |d\rangle\right)$

$= \sum_{c,d,e} \underbrace{S_{ed} N^d_{ac} \cdot (S^\dagger)_{bc}}_{} |e\rangle$ = RHS.

$= S_{eb} \dfrac{S_{ae}}{S_{1e}}$ ☺ Verlinde formula

E.g.

$E(x)|e\rangle = |1\rangle, \ E(x)|m\rangle = |f\rangle, \ E(y)|m\rangle = -|m\rangle, \dots$

Summary

2+1 D TQFT is described by anyon data, which is encoded by MTC.

§2. Duality web on 1+1 D QFT.

Claim

Duality Web on 1+1 D thry w/ $\mathbb{Z}_2$ sym :

$$\begin{array}{ccc} T_B & \overset{\mathrm{Fer}}{\underset{\mathrm{Fer}^{-1}}{\longleftrightarrow}} & T_F \\ \mathbb{Z}_2 \text{ gauging} \updownarrow & \circlearrowleft & \updownarrow \text{ Arf stacking.} \\ \mathcal{O}^B(T_B) & \longleftrightarrow & T_F \times \mathrm{Arf} \end{array}$$

← Toric code describe this duality via SymTFT !

Notation

- T_B : Bosonic thry coupled to $\mathbb{Z}_2$ back ground.
- T_F : Fermionic " dep on spin str.
- $a \in H^1(\Sigma, \mathbb{Z}_2)$: $\mathbb{Z}_2$ back ground.
- $\eta \in \mathrm{Spin}(\Sigma)$: spin str on Σ.
- $Z_{T_B}[a]$: part. fcn of T_B.

$\mathcal{O}^B$: Gauging

Recall

Gauging Lie grp G.

$G \to \mathfrak{g}$: $U(1) \ni e^{i\alpha}$, $\alpha \in \mathbb{R} = \mathfrak{g}$.

A : $\mathfrak{g}$ valued 1-form

① "Couple" back ground gauge fld A to the action.

$$Z_{T_B}[A] := \int \mathcal{D}\phi \, e^{iS[A,\phi]}$$

② Make it dynamical.

i.e. $Z_{T_B/G} := \int \mathcal{D}A \, Z[A]$ ← Very complicated

Fact

$G = \mathbb{Z}_2$,

$\exists$ Part fcn $Z_{T_B}[a]$ $\quad a \in H^1(\Sigma, \mathbb{Z}_2)$

Path int. $\int \mathcal{D}A \rightsquigarrow \frac{1}{|H^1(\Sigma,\mathbb{Z}_2)|^{1/2}} \cdot \sum_{a \in H^1(\Sigma,\mathbb{Z}_2)}$.

$$\therefore \; Z_{T_B/\mathbb{Z}_2} = \frac{1}{|H^1(\Sigma,\mathbb{Z}_2)|^{1/2}} \cdot \sum_a Z_{T_B}[a]$$

E.g.

$\Sigma = T^2$ $\therefore$ $H^1(T^2; \mathbb{Z}_2) \cong \{0, x, y, x+y\}$

$$Z_{T_B/\mathbb{Z}_2} = \frac{1}{2} \cdot \left(\square + \boxminus + \square\!\!| + \boxplus \right)$$

Rmk

We can refine it by considering dual sym.

Def

$$Z_{\mathcal{O}^B(T_B)}[a] := \frac{1}{|H^1(\Sigma,\mathbb{Z}_2)|^{1/2}} \cdot \sum_b (-1)^{\int a \cup b} Z_{T_B}[b]$$

a : b.g. fld of dual $\mathbb{Z}_2$ sym.

$$\begin{array}{c} T_B \cdots \\ \downarrow \\ \mathcal{O}^B(T_B) \cdots \end{array}$$

Obs

$$(\mathcal{O}^B)^2 = 1 \because Z_{\mathcal{O}^{B2}(T_B)}[a] = \frac{1}{|H^1(\Sigma,\mathbb{Z}_2)|^{1/2 \cdot 2}} \cdot \sum_{c,b} (-1)^{\int a \cup b} \cdot (-1)^{\int b \cup c} Z_{T_B}[c]$$

$$= \text{"} \sum_c Z_{T_B}[c] \underbrace{\frac{1}{|H^1(\Sigma,\mathbb{Z}_2)|} \cdot \sum_b (-1)^{\int b \cup (a+c)}}_{= \delta_{a,c}}$$

$$= Z_{T_B}[a].$$

Fer : Fermionization

$\mathrm{Fer}(T_B)$: thry which depends on the spin str η on the sptime.

Def

Spin str η on Σ

$\Leftrightarrow$ The choice of $\eta \in C^1(\Sigma, \mathbb{Z}_2)$ s.t. $w_2 = \delta\eta$.

Here w_2 : 2nd Stiefel-Whitney class. $\in H^2(\Sigma, \mathbb{Z}_2)$

obstruction to the existence of the spin str.

($\Leftrightarrow$ The choice of lift $P_{\mathrm{Spin}(n)} \to P_{SO(n)}$)

Prop

Spin(Σ) forms a torsor over $H^1(\Sigma,\mathbb{Z}_2)$

i.e. $\forall \eta,\eta' \in$ Spin(Σ), $\eta-\eta' \in H^1(\Sigma,\mathbb{Z}_2)$

☺ $\eta,\eta' \in$ Spin(Σ) $\Rightarrow$ $w_2 = \delta\eta$, $w_2 = \delta\eta'$

$\therefore$ $\delta(\eta-\eta') = 0$ i.e. $\eta-\eta' \in \mathrm{Im}\,\delta$

$\tilde{\eta} = \eta + \delta\lambda \Leftrightarrow \eta - \tilde{\eta} = \delta\lambda$ $\quad \delta(\eta-\tilde{\eta}) = 0$ trivially

$\therefore$ $\eta-\eta' \in \mathrm{Im}\,\delta / \mathrm{Ker}\,\delta = H^1(\Sigma,\mathbb{Z}_2)$

Fact

$\{\text{spin str } \eta \text{ on } \Sigma\} \overset{1:1}{\longleftrightarrow} \{q_\eta : H^1(\Sigma,\mathbb{Z}_2) \to \mathbb{Z}_2 \text{ s.t. } q_\eta(a+b) = q_\eta(a) + q_\eta(b) + \int a\cup b\}$

Prop

$q_{\eta+a}(b) = q_\eta(b) + \int a \cup b \quad \forall a,b \in H^1(\Sigma,\mathbb{Z}_2)$

☺ $r_a(x) := q_{\eta+a}(x) - q_\eta(x)$

then $r_a(x+y) = r_a(x) + r_a(y)$

w/ non-deg of $\int a\cup b$, $\exists z \in H^1(\Sigma,\mathbb{Z}_2)\setminus\{0\}$ s.t. $\int z \cup x = r_a(x) \quad \forall x \in H^1(\Sigma,\mathbb{Z}_2)$

$\cdots$.

Question

How to define Fermionization

- $Z_{T_B}[a]$: bos thy coupled to bkgd $a \in H^1(\Sigma,\mathbb{Z}_2)$
- $q_\eta : H^1(\Sigma,\mathbb{Z}_2) \to \mathbb{Z}_2$: quad fcn which is 1 to 1 to spin str η.

Def

$$Z_{\mathrm{Fer}(T_B)}[\eta] := \frac{1}{|H^1(\Sigma,\mathbb{Z}_2)|^{1/2}} \sum_a (-1)^{q_\eta(a)} Z_{T_B}[a]. \qquad T_B \longrightarrow T_F$$

$$Z_{\mathrm{Fer}^{-1}(T_F)}[a] := \text{〃} \sum_\eta (-1)^{q_\eta(a)} Z_{T_F}[\eta] \qquad T_B \longleftarrow T_F$$

Cor

$$Z_{\mathrm{Fer}^{-1}\circ \mathrm{Fer}(T_B)}[a] = Z_{T_B}[a]$$

Def

$$Z_{\mathrm{Arf}}[\eta] := \frac{1}{|H^1(\Sigma,\mathbb{Z}_2)|^{1/2}} \sum_a (-1)^{q_\eta(a)} \in \mathbb{Z}_2$$

Rmk

- $Z_{\mathrm{Arf}}[\eta]$ can be interpreted as a Fermionization of trivial thy.
- Physically, Arf inv is a low energy lim of the Kitaev chain.
- Mathematically, Arf inv gives a homomorphism $\Omega_2^{\mathrm{Spin}}(pt) \xrightarrow{\sim} \mathbb{Z}_2$

Prop

$$Z_{\mathrm{Fer}\circ O^B \circ \mathrm{Fer}^{-1}(T_F)}[\eta] = Z_{T_F}[\eta]\, Z_{\mathrm{Arf}}[\eta]$$

☺ Easy to show by defs.

$\hookrightarrow$ We obtain the duality web :

$$\begin{tikzcd}
T_B \arrow[r, shift left, "\mathrm{Fer}"] \arrow[d, leftrightarrow, "\text{gauge } \mathbb{Z}_2"'] \arrow[dr, phantom, "\circlearrowleft"] & T_F \arrow[l, shift left, "\mathrm{Fer}^{-1}"] \arrow[d, leftrightarrow, "\mathrm{Arf}"] \\
O^B(T_B) \arrow[r, shift left, "\mathrm{Fer}"] & T_F \times \mathrm{Arf} \arrow[l, shift left, "\mathrm{Fer}^{-1}"]
\end{tikzcd}$$

Rmk

$$(O^B)^2 = 1 \quad \longleftrightarrow \quad (Z_{\mathrm{Arf}}[\eta])^2 = 1$$

§3 SymTFT with $\mathbb{Z}_2$ gauge theory

Slogan

SymTFT : Sym in d-dim thry can be described by d+1 D TQFT

$\longrightarrow$ See "Module Category and Symmetry-TFT" for general discussions.

Fact

1+1D Duality Web in §2 $\leftrightarrow$ SymTFT : Toric code

i.e. $\mathcal{Z}(\mathrm{Vec}_{\mathbb{Z}_2}) =$ Toric code MTC.

Recall

- Fourier transf in QM

$$\mathcal{F}(\psi)(p) = \frac{1}{\sqrt{2\pi}} \int dx\, e^{-ipx} \psi(x)$$

- Gauging in 2+1 D

$$Z_{\mathcal{O}^B(T_B)}[a] = \frac{1}{|H^1(\Sigma,\mathbb{Z}_2)|^{\frac{1}{2}}} \cdot \sum_b (-1)^{\int a \cup b} Z_{T_B}[b]$$

- Fermionization in 1+1 D

$$Z_{\mathrm{Fer}(T_B)}[\eta] = \frac{1}{|H^1(\Sigma,\mathbb{Z}_2)|^{\frac{1}{2}}} \cdot \sum_a (-1)^{q_\eta(a)} Z_{T_B}[a]$$

Rmk

For even fcn $\psi(x) = \psi(-x)$

$$\mathcal{F}^2(\psi)(x) = \psi(x) \quad \text{i.e. } \mathcal{F}^2 = 1.$$

$\longleftrightarrow$ Compatible w/ $(\mathcal{O}^B)^2 = 1$.

Recall

Fourier transf = Change of basis.

i.e. $\mathcal{F}(\psi)(p) = \langle p | \psi \rangle$

$$= \int dx \langle p | x \rangle \langle x | \psi \rangle$$

$$= \int dx \frac{e^{-ipx}}{\sqrt{2\pi}} \psi(x)$$

Question

Can we interpret $\mathcal{O}^B$, Fer as a change of basis ?

Answer

$\langle e, a | T_B \rangle := Z_{T_B}[a]$

$$Z_{\mathcal{O}^B(T_B)}[a] = \langle m, a | T_B \rangle$$

$$= \frac{1}{|H^1(\Sigma,\mathbb{Z}_2)|^{\frac{1}{2}}} \cdot \sum_b \langle m, a | e, b \rangle \langle e, b | T_B \rangle$$

$$= \frac{1}{|H^1(\Sigma,\mathbb{Z}_2)|^{\frac{1}{2}}} \cdot \sum_b (-1)^{\int a \cup b} Z_{T_B}[b]$$

$$Z_{\mathrm{Fer}(T_B)}[\eta] = \langle \eta | T_B \rangle$$

$$= \frac{1}{|H^1(\Sigma,\mathbb{Z}_2)|^{\frac{1}{2}}} \cdot \sum_a \langle \eta | e, a \rangle \langle e, a | T_B \rangle$$

$$= \frac{1}{|H^1(\Sigma,\mathbb{Z}_2)|^{\frac{1}{2}}} \sum_a (-1)^{q_\eta(a)} Z_{T_B}[a]$$

w/ $V(\Sigma) \cong \mathrm{span}_{\mathbb{C}}\{ |e, a\rangle \mid a \in H^1(\Sigma, \mathbb{Z}_2) \}$

$\cong \mathrm{span}_{\mathbb{C}}\{ |m, a\rangle \mid a \in H^1(\Sigma, \mathbb{Z}_2) \}$

$\cong \mathrm{span}_{\mathbb{C}}\{ |\eta\rangle \mid \eta \in \mathrm{Spin}(\Sigma) \}$

Next let us construct these bases.

$V(\Sigma)$: Hilb sp assigned by 2+1 D TQFT w/ Toric code.

Claim

1+1 D w/ sym : fusion cat $\mathcal{C}$

$\leftrightarrow$ SymTFT ; MTC = $\mathcal{Z}(\mathcal{C})$ Drinfeld center.

Fact

Given MTC $\mathcal{C}$,

$\exists\, \mathcal{F}$: fusion cat s.t. $\mathcal{Z}(\mathcal{F}) \Leftrightarrow \exists\, \mathcal{A}$: lagrangian alg in $\mathcal{C}$.

$\downarrow$ $\mathcal{C}$: abelian

L: lagrangian subgrp

Def

For abelian TQFT $\mathcal{C}$, L is a lagrangian subgrp

$\iff$
- $\theta(a) = 1 \quad \forall a \in L$
- $B(a,b) = 1 \quad \forall a,b \in L$

indep of triangulation

- $|L|^2 = |\mathcal{C}|$

maximal.

- $\forall a \in \mathcal{C}\setminus L$, $\exists b \in L$ s.t. $B(a,b) \neq 1$.

Cor

$L = \{1,e\}, \{1,m\}$ ($\{1,f\}$: odd) are lagrangian

☺ For $\{1,e\}$,

- $\theta(1) = \theta(e) = 1$
- $B(e,e) = \dfrac{\theta(e\times e)}{\theta(e)\theta(e)} = \dfrac{1}{1\cdot 1} = 1$ etc...
- $|\{1,e\}|^2 = |\mathcal{C}| = 4$
- $\mathcal{C}\setminus L = \{m,f\}$, $B(e,m) = B(e,f) = -1$

Construction of gapped bdry

Input : Toric code TQFT $\leftarrow$ Realized as $U(1)^2$ Chern-Simons w/ $K = \begin{pmatrix} & 2 \\ 2 & \end{pmatrix}$

Wilson loop op : $\{1, E(a), M(a), F(a)\}$ satisfying

$$(*)\begin{cases} E(a+b) = E(a)E(b), \quad E(a)^\dagger = E(a) & \forall a,b \in H^1(\Sigma,\mathbb{Z}_2) \\ M(a+b) = M(a)M(b), \quad M(a)^\dagger = M(a) & \forall a,b \in H^1(\Sigma,\mathbb{Z}_2) \\ F(a) := E(a)M(a) & \\ E(a)M(b) = (-1)^{\int a\cup b} M(b)E(a) & \forall a,b \in H^1(\Sigma,\mathbb{Z}_2) \\ E(0) = M(0) = F(0) = 1 & \end{cases}$$

Output : Three kinds of bases corresponding to e, m, f

$$V(\Sigma) \cong \mathrm{span}_{\mathbb{C}}\{\, |e,a\rangle \mid a \in H^1(\Sigma,\mathbb{Z}_2)\}$$
$$\cong \;\;{}'' \;\; \{\, |m,a\rangle \mid \;\; {}'' \;\; \}$$
$$\cong \;\;{}'' \;\; \{\, |f,a\rangle \mid \;\; {}'' \;\; \}$$

Prop

- $E(a)^2 = M(a)^2 = F(a)^2 = 1$ ($\leftrightarrow$ fusion rule of $\{1,e,m,f\}$)
- $F(a+b) = (-1)^{\int a\cup b} F(a)F(b)$

☺ Easy to show w/ $(*)$

Basis w.r.t. $E(a)$

Define $|e,0\rangle$ by the following condition

$$E(a)|e,0\rangle = |e,0\rangle \quad \forall a \in H^1(\Sigma,\mathbb{Z}_2)$$

and define $|e,a\rangle = M(a)|e,0\rangle$

$\Rightarrow$ $|e,b\rangle$ is an eigenvector of $E(a)$

$$\begin{aligned} E(a)|e,b\rangle &= E(a)M(b)|e,0\rangle \\ &= (-1)^{\int a\cup b} M(b)E(a)|e,0\rangle \\ &= (-1)^{\int a\cup b} M(b)|e,0\rangle \\ &= (-1)^{\int a\cup b} |e,b\rangle \end{aligned}$$

Prop

w/ $\dim V(\Sigma) = |H^1(\Sigma,\mathbb{Z}_2)|$

- $|e,0\rangle$ always exists uniquely up to phase.
- $\{|e,a\rangle\}_{a\in H^1(\Sigma,\mathbb{Z}_2)}$ is an orthogonal basis of $V(\Sigma)$

☺ Using nondeg of $\int a\cup b$ many times.

$\rightsquigarrow$ $V(\Sigma) \cong \mathrm{span}_{\mathbb{C}}\{\, |e,a\rangle \mid a \in H^1(\Sigma,\mathbb{Z}_2)\}$

Basis w.r.t. $M(a)$

Define $|m,0\rangle$ by $M(a)|m,0\rangle = |m,0\rangle \quad \forall a \in H^1(\Sigma,\mathbb{Z}_2)$

$|m,a\rangle$ by $|m,a\rangle := E(a)|m,0\rangle$

$\rightsquigarrow$ $V(\Sigma) \cong \mathrm{span}_{\mathbb{C}}\{\, |m,a\rangle \mid a \in H^1(\Sigma,\mathbb{Z}_2)\}$

Normalization

$\langle e,0|m,0\rangle = 1$

$\langle e,a|e,b\rangle = |H^1(\Sigma,\mathbb{Z}_2)|^{\frac{1}{2}}\,\delta_{ab}$

$\langle m,a|m,b\rangle = |H^1(\Sigma,\mathbb{Z}_2)|^{\frac{1}{2}}\,\delta_{ab}$

$$\mathrm{id}_{V(\Sigma)} = \frac{1}{|H^1(\Sigma,\mathbb{Z}_2)|^{\frac{1}{2}}}\cdot\sum_a |e,a\rangle\langle e,a| = \frac{1}{|H^1(\Sigma,\mathbb{Z}_2)|^{\frac{1}{2}}}\cdot\sum_a |m,a\rangle\langle m,a|$$

Cor

$\langle e,a|m,b\rangle = (-1)^{\int a\cup b}$

$\because$ $\langle e,a|m,b\rangle = \langle e,0|M(a)E(b)|m,0\rangle$

$= (-1)^{\int a\cup b}\langle e,0|E(b)M(a)|m,0\rangle$

$= (-1)^{\int a\cup b}\langle e,0|m,0\rangle$

$= (-1)^{\int a\cup b}$

Then we find that

$$|m,a\rangle = \frac{1}{|H^1(\Sigma,\mathbb{Z}_2)|^{\frac{1}{2}}}\cdot\sum_b (-1)^{\int a\cup b}|e,b\rangle$$

Prop

$\langle m,a|T_B\rangle = Z_{\mathcal{O}^B(T_B)}[a]$

$\because$ $\mathrm{LHS} = \frac{1}{|H^1(\Sigma,\mathbb{Z}_2)|^{\frac{1}{2}}}\cdot\sum_b \langle m,a|e,b\rangle\langle e,b|T_B\rangle$

$= \quad '' \quad \sum_b (-1)^{\int a\cup b} Z_{T_B}[b]$

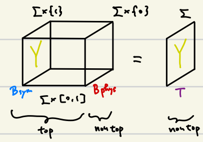
$= Z_{\mathcal{O}^B(T_B)}[a]$

$= \mathrm{RHS}$

$\to$ We can reinterpret the gauging as a change of basis,

or " of "bdry".

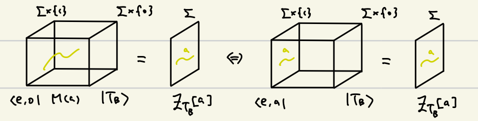
SymTFT picture



B_{phys}: phys, dynamical } bdry

B_{sym}: sym, top, gapped } bdry $\rightsquigarrow$ only capture sym.

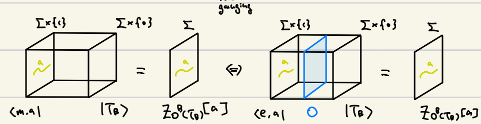
In our setup, let us define $|T_B\rangle$ s.t. $Z_{T_B}[a] = \langle e,a|T_B\rangle$



change of B_{sym}. $\Downarrow$ gauging

insertion of $\mathcal{O}$ domain wall $\Downarrow$ gauging

gauging



Basis w.r.t. $F(a)$

Define $|\eta\rangle$ by $F(a)|\eta\rangle = (-1)^{\int \eta(a)}|\eta\rangle \quad \forall a\in H^1(\Sigma,\mathbb{Z}_2)$

$|\eta+a\rangle$ by $|\eta+a\rangle := E(a)|\eta\rangle$

$\rightsquigarrow$ $V(\Sigma) \cong \mathrm{span}_{\mathbb{C}}\{|\eta+a\rangle \mid a\in H^1(\Sigma,\mathbb{Z}_2)\}$

Normalization

$\langle e,0|\eta\rangle = 1$

$\langle \eta|\eta'\rangle = |H^1(\Sigma,\mathbb{Z}_2)|^{\frac{1}{2}}\,\delta_{\eta\eta'}$ $\rightsquigarrow$ $\langle \eta|T_B\rangle = Z_{F_{er}(T_B)}[\eta]$

$$\mathrm{id}_{V(\Sigma)} = \frac{1}{|H^1(\Sigma,\mathbb{Z}_2)|^{\frac{1}{2}}}\cdot\sum_\eta |\eta\rangle\langle\eta|$$

Summary

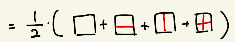
Sym part of d-dim thing is described by d+1-dim TQFT.

As an example, we saw Duality Web in 1+1D can be described by TC in 2+1D.

§4. Condensation defects with $\mathbb{Z}_2$ gauge theory

Recall

Gauging $\mathbb{Z}_2$ on T^2



We insert 1D defect on 2D spacetime mfd.

⤳ What if we do that for codim-$q+1$ defect on codim-p submfd?

Def

- Higher gauging

  $\Leftrightarrow$ Summing over insertions of codim-$(q+1)$ top defect on codim-p mfd. w/ $p \le q+1$.

  i.e. gauging a q-form sym on a codim-p mfd.

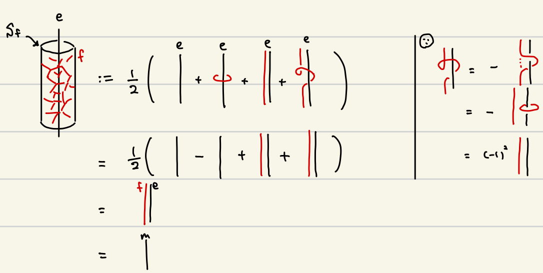
- Condensation defect

  $\Leftrightarrow$ Top defect resulting from higher gauging

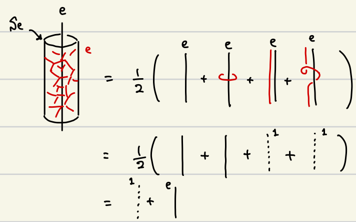
gauging a on Σ

E.g.

For TC: $\{1, e, m, f\}$, we can construct surface operators S_e, S_m, S_f by higher gauging.



→ invertible 0-form sym



⤳ Non-inv 0-form sym

Fact

$\forall$ 0-form sym in 2+1D TQFT arises from higher gauging of 1-form sym.

Fact

Condensation surfaces in TC have forms as

$S_e := |e,0\rangle\langle e,0|$, $S_m := |m,0\rangle\langle m,0|$, $S_{em} := |e,0\rangle\langle m,0|$ etc...

⤳ Let us check how they work.

Recall

- $V(T^2) \cong \mathrm{span}_{\mathbb{C}}\{|1\rangle, |e\rangle, |m\rangle, |f\rangle\}$ w/ $\langle a|b\rangle = \delta_{ab}$

  w/ $|e\rangle := E(x)|0\rangle$ etc...

- $E(a)|e,0\rangle = |e,0\rangle \quad \forall a \in H^1(\Sigma, \mathbb{Z}_2)$

⤳ How can we describe $|e,0\rangle$ wrt $\{|a\rangle\}_{a\in Z}$?

⤳ $|e,0\rangle = |1\rangle + |e\rangle$

☺ Set $|e,0\rangle = \sum C_a |a\rangle$ w/ $C_a \in \mathbb{C}$, $a \in \{1, e, m, f\}$

$E(x)|e,0\rangle = |e,0\rangle \Rightarrow \text{LHS} = C_1|e\rangle + C_e|1\rangle + C_m|f\rangle + C_f|m\rangle$

$\therefore C_1 = C_e,\ C_m = C_f$

$E(y)|e,0\rangle = |e,0\rangle \Rightarrow \text{LHS} = C_1|1\rangle + C_e|e\rangle - C_m|m\rangle - C_f|f\rangle$

$\therefore C_m = C_f = 0$ //

$\therefore$ We obtain $|e,0\rangle = |1\rangle + |e\rangle$

$$|e,x\rangle = |m\rangle + |f\rangle$$

$$|e,y\rangle = |1\rangle - |e\rangle$$

$$|e,x+y\rangle = |m\rangle - |f\rangle$$

$$\therefore \quad S_e|a\rangle = \begin{cases} |1\rangle + |e\rangle & a \in \{1, e\} \\ 0 & a \in \{m, f\} \end{cases}$$

$\rightsquigarrow$ One can compute S_m, S_{em}, etc...

Prop

$$S_f := \frac{1}{|H^1(\Sigma, \mathbb{Z}_2)|^{\frac{1}{2}}} \cdot \sum_q (-1)^{\mathrm{Arf}(q)} |q\rangle\langle q| \quad : \mathbb{Z}_2 \text{ 0-form sym op } e \leftrightarrow m$$

$$S_f E(a) = \frac{1}{|H^1(\Sigma, \mathbb{Z}_2)|^{\frac{1}{2}}} \cdot \sum_q (-1)^{\mathrm{Arf}(q)} |q\rangle\langle q+a|$$

$$= \frac{1}{|H^1(\Sigma, \mathbb{Z}_2)|^{\frac{1}{2}}} \cdot \sum_q (-1)^{\mathrm{Arf}(q+a)} |q+a\rangle\langle q|$$

$$= \frac{1}{|H^1(\Sigma, \mathbb{Z}_2)|^{\frac{1}{2}}} \cdot \sum_q (-1)^{\mathrm{Arf}(q)} E(a) \cdot \underbrace{(-1)^{q_y(a)} |q\rangle}_{= F(a)|q\rangle}\langle q|$$

$$= E(a) \cdot F(a) \frac{1}{|H^1(\Sigma, \mathbb{Z}_2)|^{\frac{1}{2}}} \cdot \sum_q (-1)^{\mathrm{Arf}(q)} |q\rangle\langle q|$$

$$= M(a)\, S_f$$