Lattice fermions on anisotropic lattices

Johannes H. Weber (they/them)

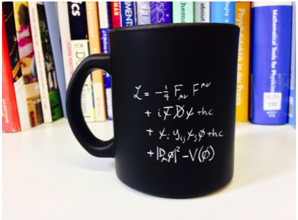
[Bazavov, Trimis, Weber, arXiv:2026.XXXXX (to appear today?)]

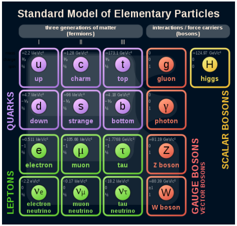
[Technische Universität Darmstadt] [Werner-Heisenberg-Gymnasium Bad Dürkheim]

Università degli Studi di Milano-Bicocca, Milano, Italy, 29.06.2026

QCD in the Standard Model

- **Standard Model of Particle Physics**: most elementary, **precisely tested** theory of nature; inconsistencies would provide hints of **New Physics**.





- **Interactions** between fields of related *charges* $\Rightarrow$ **non-linear equations**.
- **Particles: local excitations** of the quantum fields permeating space-time.

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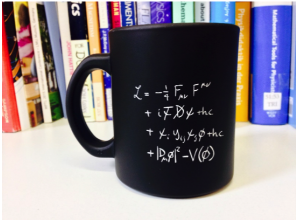

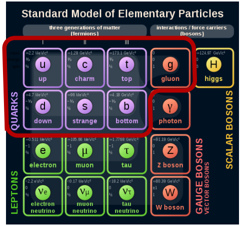

Matter Mandala

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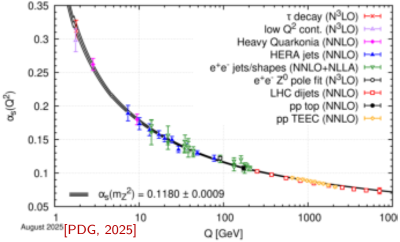
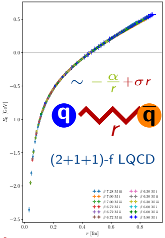
- **Interactions** between fields of related *charges* ⇒ **non-linear equations**.
- **Particles: local excitations** of the quantum fields permeating space-time.
- **Nuclear matter** (quarks, gluons): **Quantum Chromodynamics (QCD)**.
- **Flavors**: u, d, s: **light**; c: **intermediate**; b: **heavy**; t: (irrelevant in QCD).

A particle physics perspective on nuclear matter

- No direct observation of **elementary nuclear matter**



[PDG, 2025]



[TUMQCD, PRD 107, 2022]

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- **Color charges cannot be separated**:

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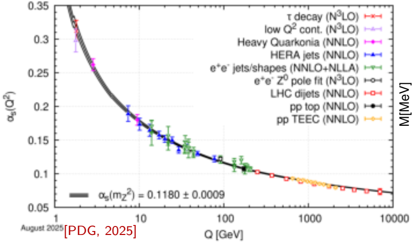

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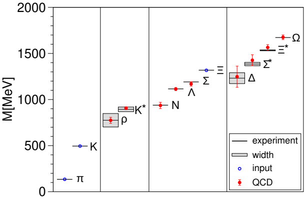
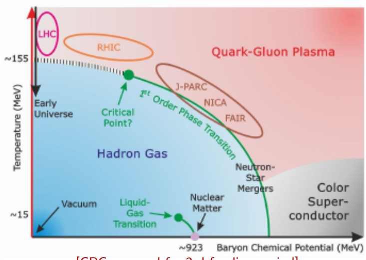


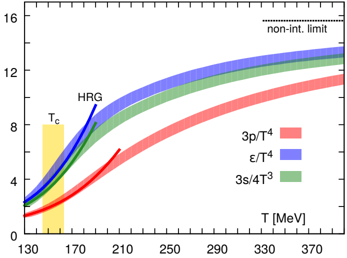
[BMW, Science 322, 2008]

- **QCD coupling** α_s **grows** at smaller E: diverges at $E \sim \Lambda_{\mathrm{QCD}}$.
- **Color charges cannot be separated**: instead, we only observe **hadrons**, e.g. $q\bar{q}$ pairs: mesons; heavy $q\bar{q}$: **quarkonia**; qqq: baryons; and exotica. . .
- Structuring this cornucopia of hadrons (**particle zoo**) gave rise to QCD, whose stochastic solution using supercomputers is called **Lattice QCD**.

Strong-Interaction matter under Extreme Conditions



[CRC proposal for 3rd funding period]



[HotQCD, PRD 90, 2014]

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- Light hadrons cease to be relevant degrees of freedom at higher T or μ_B.

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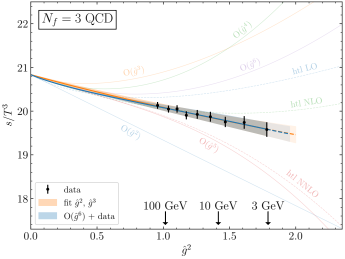
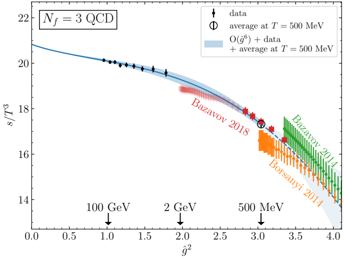


[Bresciani et al., PRL 134, 2025]



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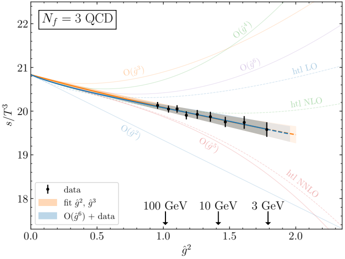

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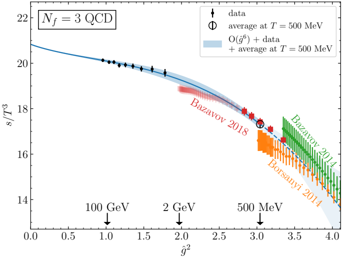

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- **Strongly-coupled dynamics** stays relevant even at the electroweak scale.
- LQCD as a first-principles approach is a reliable baseline at $\mu_{\mathrm{B}} = 0$.
- Problems at $\mu_{\mathrm{B}} \neq 0$: a **complex fermion determinant**
  1. gives rise to the **sign problem** and breakdown of importance sampling,
  2. makes naively **rooting staggered fermions** fail in reweighting step.

[Borsanyi et al.: PRD 105, 2021; PRD 107, 2023; PRD 109,2024].

- Anisotropic **Karsten-Wilczek fermions** avoid rooting! [Borsanyi et al., PRD 111, 2025]

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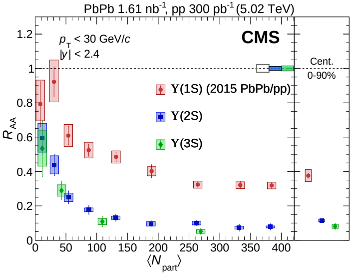
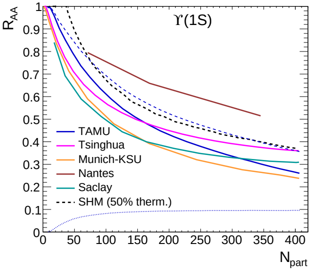
In-medium heavy hadrons



[CMS, PRL 133, 2024]



[Andronic et al., EPJ A60, 2024]

- **Quarkonia suppression** is signature of QGP formation: [Matsui, Satz, PLB 178, 1986].
- Heavy hadron rates are nontrivially **modified in heavy-ion collisions.**
- EFT suggests **medium-modified heavy hadrons** may persist above T_{pc}:
  [Laine et al., JHEP 09, 2007]; [Brambilla et al., PRD 78, 2008]; [Laine, NPA 820, 2009]; [Rothkopf, PR 858, 2020].
- Models built on **different assumptions yield similar** R_{AA} for $\Upsilon(1S)$, etc.
- LQCD can help discriminate, but **real-time dynamics** accessed in imag. time correlation functions via spectral reconstruction: **ill-posed problem**.
- **Imaginary time resolution** of lattice correlators is improved by **anisotropy**.

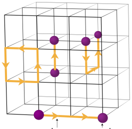
Isotropic lattice gauge theory in a nutshell

- **Non-perturbative lattice regularization** of QCD path integral: IR or UV via finite volume or lattice spacing $\Rightarrow$ infinite volume and **continuum limit**.
- No Leibniz rule! **Gauge invariance** via **change of variables** $A_{x,\mu} \Rightarrow U_{x,\mu}$

$$
\begin{aligned}
U_{x,\mu} &= \mathrm{e}^{\mathrm{i}aA_{x,\mu}} && \text{gauge link} \\
P_{x,\mu\nu} &= U_{x,\mu} U_{x+\hat{\mu},\nu} U^{\dagger}_{x+\hat{\nu},\mu} U^{\dagger}_{x,\nu} && \text{plaquette} \\
R_{x,\mu\nu} &= U_{x,\mu} U_{x+\hat{\mu},\mu} U_{x+2\hat{\mu},\nu} U^{\dagger}_{x+\hat{\mu}+\hat{\nu},\mu} U^{\dagger}_{x+\hat{\nu},\mu} U^{\dagger}_{x,\nu} && \text{rectangle}
\end{aligned}
$$

$$
S_{\mathrm{QCD}}[U,\bar{\psi},\psi] = -\sum_{x\in V_4} \beta \sum_{\mu\neq\nu} \Big(c_P \,\mathrm{Re}\,\mathrm{Tr}\left[1 - P_{x,\mu\nu}\right] + c_R \,\mathrm{Re}\,\mathrm{Tr}\left[1 - R_{x,\mu\nu}\right] \Big) + a^4 \sum_{x,y\in V_4} \sum_f \bar{\psi}^f_x M^f_{xy}[U] \psi^f_y
$$

- $M^f_{xy}[U]$ is a discretized pendant of the fermion matrix $\not{D}[A](x) + m_f$.
- Any lattice regularizations **break various symmetries** of the continuum theory, e.g. $O(4) \to W_4$, but these are always **systematically improvable**.



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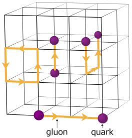
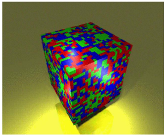
- **MCMC evaluation** of path integral ⇒ quantum nature as **fluctuations**.



$\xrightarrow{\text{HPC}}$



Overview

1 Introduction

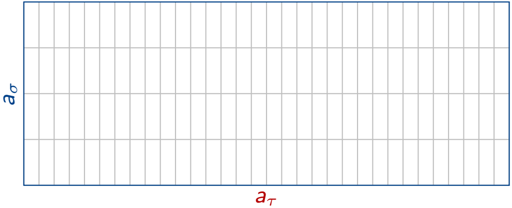
2 Anisotropic LQCD
- Anisotropic lattice gauge theory
- Anisotropic gradient flow

3 Anisotropic fermions
- Staggered fermions

4 Taste structure
- Staggered fermions
- Independent link fluctuations
- Staggered chiral perturbation theory

5 Summary

Anisotropic lattice gauge theory



- **Anisotropy** $\xi = \frac{a_\sigma}{a_\tau}$: different spatial or **temporal** lattice spacings $a_\sigma \neq a_\tau$.
- $\xi > 1$: spatial links $U_{x,\sigma} = \mathrm{e}^{\mathrm{i}a_\sigma A_{x,\sigma}}$ (**temporal** $U_{x,\tau} = \mathrm{e}^{\mathrm{i}a_\tau A_{x,\tau}}$) drawn from rougher (**smoother**) distributions linked by the **bare anisotropy** ξ_0.

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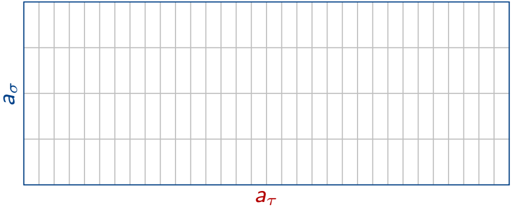

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- **Two** bare parameters ($\beta_\sigma = \frac{\beta}{\xi_0}$, $\beta_\tau = \beta\xi_0$) for gauge action [Karsch, NPB 205, 1982], are **simultaneously** mapped to (a_σ, a_τ) or (a_σ, ξ) by **scale setting**.
- For each fermion flavor **two further** bare parameters $(a_\sigma m_f, \xi_0^f)$ are **simultaneously** tuned to the **spectrum**, e.g. $M_{f\bar{f}}^{\mathrm{LCP}}$, and to $\xi^q = \xi$.
- Light flavors via **improved chiral fermions**: $\frac{\xi_0^f}{\xi}$ is fairly independent of m_f, and mass ratios $\frac{m_{f'}}{m_f}$ are regularization independent $\Rightarrow$ **only one** ξ_0^f.

Anisotropic fermions for hot or dense QCD

- **Finite temperature** $T = \frac{1}{a_\tau N_\tau}$ limits the time resolution for given a_τ.
- **NRQCD**: NR hierarchy needs $p \sim \frac{1}{a_\sigma} \ll m_h \ll \frac{1}{a_\tau}$.
- Similar approach: **anisotropic Wilson clover fermion** ensembles used for spectroscopy [HadSpec coll.: PRD 78, 2008; PRD 79, 2009]. Tuning is hard!
- **HadSpec lines of constant physics** used for lattice NRQCD and thermodynamics by **FASTSUM**, e.g. [FASTSUM coll., 2026].
  - Fixed scale approach, temperatures of interest reach $N_\tau = 36$.
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- **Anisotropic staggered** has not been pursued after initial attempts: on quenched [Nomura et al., PTP 111, 2004]; $N_\mathrm{f} = 2$ dynamical [Levkova et al., PRD 73, 2006].
- **Anisotropic HISQ** might permit $N_\tau \sim 200$ with current resources...
- Target **spatially coarse** lattices ($a_\sigma \gtrsim 0.15\,\mathrm{fm}$), **high anisotropy** $\xi \sim 4-8$. Reduce lattice artifacts via improvement: **anisotropic HISQ.** Currently limited to **only one lattice spacing** $a_\sigma = 0.1665$ fm. [Bazavov et al., arXiv].

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- **Finite density** ($\mu_B = 0$): avoid rooting a complex determinant to $N_f = 2$ via **Karsten-Wilczek fermions** [Karsten, PLB 104, 1981]; [Wilczek, PRL 59, 1987]. Reduce lattice artifacts via **Naik improvement** and **anisotropy** $\xi \sim 2-4$. Recent preparatory work: [Borsanyi et al., PRD 111, 2025]; [Weber, PRD 112, 2025].

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Tuning strategy

- Plan: reach $\xi \sim 4 - 8$ iteratively via intermediate anisotropies $\xi \sim 1.5 - 4$, e.g. in sequence $\xi: \; 1 \to 1.5 \to 2 \to 4 \to 6 \to 8$ to understand the RG flow.
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- Proposed QCD tuning strategy:
  1. General **trial ensembles** with $(\beta, \xi_0; \hat{m}_s, \xi_0^f)$.
  2. **Simultaneous scale setting** to obtain (a_σ, ξ).
  3. Identify $(\{M_{s\bar{s}}\}, \xi^f)$ for the fermions.
  4. Change parameter set. Rinse and repeat.
  5. Identify (linear?) **scaling window**.
  6. **Interpolate** in $(\beta, \xi_0; \hat{m}_q, \xi_0^f)$ to **target** $(a_\sigma, \xi; M_{s\bar{s}}^{\mathrm{phys}}, \xi^f = \xi)$.

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- Unknown lattice artifacts from taste-symmetry breaking of the fermions complicate the **simultaneous tuning: simplify by sequestration.**
  1. Generate pure gauge **trial ensembles** with (β, ξ_0).
  2. **Simultaneous scale setting** to obtain (a_σ, ξ).
  3. Identify $(\{M_{s\bar{s}}\}, \xi^f)$ for trial $(\hat{m}_s, \xi_0^f)$.
  4. **Tune** $(\hat{m}_s, \xi_0^f)$ to $(M_{s\bar{s}}^{\mathrm{phys}}, \xi^f = \xi)$.
  5. Study **taste patterns** and lattice artifacts.

Pure gauge theory on anisotropic lattices

$$S_g = \beta\frac{1}{\xi_0}\left(c_P\sum_{P_{\sigma\sigma}}\left[1-\frac{1}{3}\mathrm{ReTr}(P_{\sigma\sigma})\right]+c_R\sum_{R_{\sigma\sigma}}\left[1-\frac{1}{3}\mathrm{ReTr}(R_{\sigma\sigma})\right]\right)$$
$$+\beta\xi_0\left(c_P\sum_{P_{\sigma\tau}}\left[1-\frac{1}{3}\mathrm{ReTr}(P_{\sigma\tau})\right]+c_R\sum_{R_{\sigma\tau}}\left[1-\frac{1}{3}\mathrm{ReTr}(R_{\sigma\tau})\right]\right),$$

- P or R represent of 1×1 or 1×2 Wilson loops in all possible orientations.
- Indices $\sigma\sigma$ ($\sigma\tau$) distinguish **spatial-spatial** **(spatial-temporal)** orientations.
- **LW action** [Weisz, NPB 212, 1982];[Lüscher, CMP 98, 1985]: $c_P = 1$, $c_R = -\frac{1}{20}$ and $\beta = \frac{10}{g^2}$.
- Traditionally: simultaneous scale setting via **Wilson loops** [Klassen, NPB 533, 1998].

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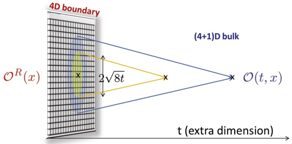
Anisotropic gradient flow

- **Gradient flow** is a gauge-invariant smoothing via evolution in a fictitious fifth direction (**flow time** $t \sim \mathrm{length}^2$)

  [Lüscher, CMP 293, 2010]

- GF automatically **renormalizes composite operators** at $t \gtrsim a_\sigma^2$.



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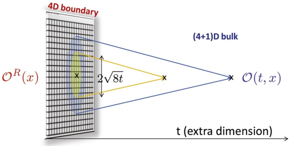

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$$\frac{\mathrm{d}U_\mu}{\mathrm{d}t} = Z_\mu U_\mu, \qquad Z_\mu = -\sum_{\nu\neq\mu} \rho_{\mu\nu} \mathcal{P}_A \left[U_\mu S^{\mathrm{fl}\,\dagger}_{\mu\nu}[U] \right]$$

- Antihermitean projection $\mathcal{P}_A$, staple $S^{\mathrm{fl}\,\dagger}_{\mu\nu}$ from the **flow action** S^{fl}.
- GF made **anisotropic** via staple weights $\rho_{i4} = (\xi_0^{gf})^2$, $\rho_{ij} = \rho_{4i} = 1$.

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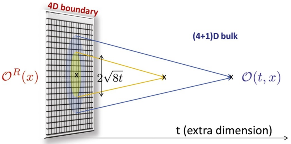

- GF automatically **renormalizes composite operators** at $t \gtrsim a_\sigma^2$.

$$\frac{\mathrm{d}U_\mu}{\mathrm{d}t} = Z_\mu U_\mu, \qquad Z_\mu = -\sum_{\nu\neq\mu} \rho_{\mu\nu} \mathcal{P}_A \left[U_\mu S^{\mathrm{fl}\,\dagger}_{\mu\nu}[U] \right]$$

- Antihermitean projection $\mathcal{P}_A$, staple $S^{\mathrm{fl}\,\dagger}_{\mu\nu}$ from the **flow action** S^{fl}.
- GF made **anisotropic** via staple weights $\rho_{i4} = (\xi_0^{gf})^2$, $\rho_{ij} = \rho_{4i} = 1$.
- GF allows **scale setting via action density** S^{obs} [Lüscher JHEP 08, 2010; JHEP 03, 2014]:

$$S^{\mathrm{obs}}_{\sigma\sigma}(t) = \frac{1}{4}\sum_{x,i\neq j} F_{ij}^2(x,t)\,, \quad S^{\mathrm{obs}}_{\sigma\tau}(t) = \frac{1}{2}\sum_{x,i} F_{i4}^2(x,t)$$

- **Gauge action** S^{g}, **flow action** S^{fl} or **observable** S^{obs} may differ.
- Simultaneous scale setting via gradient flow [Borsanyi et al.: unpub. 2012; PRD 98, 2018].

Small flow-time lattice artifacts

- GF scales $t_{0,\sigma}$, $t_{0,\tau}$ are obtained from $t^2 S^{\mathrm{obs}}$ or scales $w_{0,\sigma}$, $w_{0,\tau}$ with reduced cutoff effects from the flow derivative of $t^2 S^{\mathrm{obs}}$ [BMW, JHEP 09, 2012]

$$\left[t\frac{\mathrm{d}}{\mathrm{d}t}t^2\langle S_{\sigma\sigma}(t)\rangle\right]_{t=w_{0,\sigma}^2} = 0.15\,, \quad (\xi_0^{gf})^2\left[t\frac{\mathrm{d}}{\mathrm{d}t}t^2\langle S_{\sigma\tau}(t)\rangle\right]_{t=w_{0,\tau}^2} = 0.15\,.$$

- Setting $\xi_0^{gf} = \xi$ (target renormalized anisotropy): adjust ξ_0 until $\frac{w_{0,\sigma}}{w_{0,\tau}} = 1$. Better than [Borsanyi, unpub. 2012], only one flow per trial ξ_0 [Borsanyi et al., PRD 98, 2018]
- Spatial lattice spacing a_σ controls the **small flow-time** lattice artifacts. Is $\frac{w_{0,\sigma}}{w_{0,\tau}}$ safe for coarse spatial lattice spacing $a_\sigma \gtrsim 0.15$?
- Ratio of **spatial-spatial or spatial-temporal** derivatives:

$$R(t) = t\frac{\mathrm{d}}{\mathrm{d}t}t^2\langle S_{\sigma\sigma}(t)\rangle \Big/ \, t\frac{\mathrm{d}}{\mathrm{d}t}t^2\langle S_{\sigma\tau}(t)\rangle \xrightarrow{t\gtrsim a_\sigma^2} \xi^2.$$

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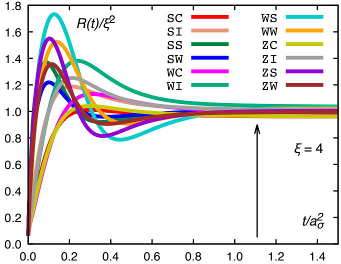
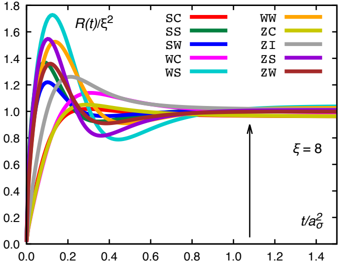
WERNER-HEISENBERG-GYMNASIUM

Small flow-time lattice artifacts





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- We have $S^{\mathrm{g}} = S^{\mathrm{LW}}$ and consider $S^{\mathrm{fl}} \to W, S, Z$ and $S^{\mathrm{obs}} \to W, S, C, I$.
  - **Zeuthen flow**: $\mathcal{O}(a_\sigma^2)$-improvement of flow evolution [Ramos, Sint, EPJ C76, 2016]
  - Clover operator [Sheikholeslami, NPB 259, 1985]
  - **Improved clover** operator [Bilson-Thompson et al., AP 304, 2003]
- **Nonmonotonic** for W, S observables, yet decreasing after peak for C, I.
- **Peak height** increasing for SC, ZC, WC, SI, ZI, WI. Suggests C instead of I!

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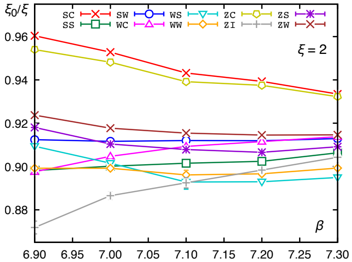
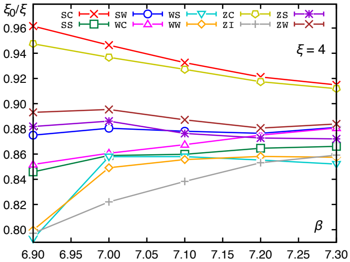
Scheme choices and lines of constant renormalized anisotropy





- Study systematics at $(\xi = 2, 4)$ with 5×9 resp. 5×10 ensembles.
- $\frac{\xi_0}{\xi}$ for each β with various schemes. Continuum $(\beta \to \infty)$ requires $\frac{\xi_0}{\xi} \to 1$.

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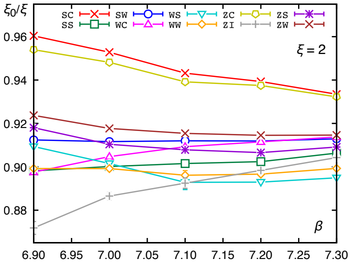
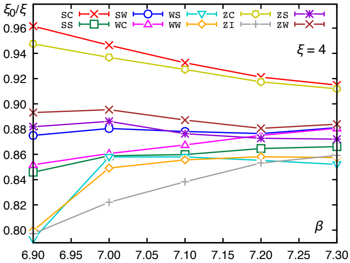
Scheme choices and lines of constant renormalized anisotropy





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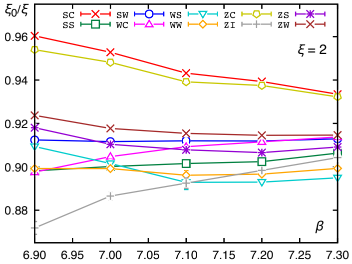
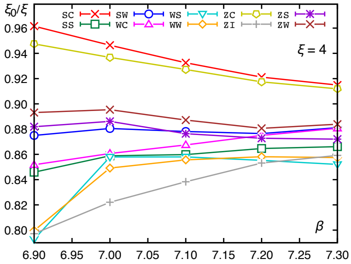
- Only WC, ZI are increasing $\Rightarrow$ WC scheme, same as [Borsanyi et al., PRD 98, 2018]!

Scheme choices and lines of constant renormalized anisotropy





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- $\frac{\xi_0}{\xi}$ for each β with various schemes. Continuum ($\beta \to \infty$) requires $\frac{\xi_0}{\xi} \to 1$.
- SC, ZC smallest renormalization, but decreasing $\Rightarrow$ must turn around!
- Only WC, ZI are increasing $\Rightarrow$ WC scheme, same as [Borsanyi et al., PRD 98, 2018]!
- Interpolate LCRA to desired $\frac{a_\sigma}{w_0}(\beta)$ with Allton-type Ansatz [Allton, NPB PS 53, 1997], then determine $\xi_0(\beta)$ with rational function [Borsanyi et al., PRD 98, 2018].

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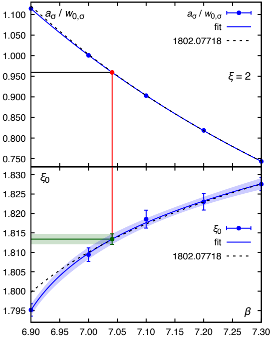
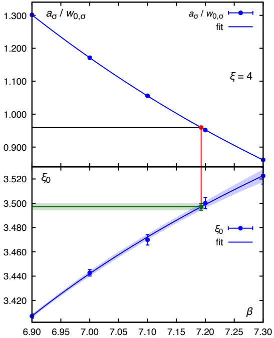
Lines of constant renormalized anisotropy



$$\frac{a^*_\sigma}{w_0} = \frac{0.1665}{0.17355} \approx 0.959378$$



$$\frac{a_\sigma}{w_0}(\beta) = c_0 f(\beta) \frac{1 + c_2 \frac{10}{\beta} f^2(\beta)}{1 + d_2 \frac{10}{\beta} f^2(\beta)}, \quad f(\beta) = \left(\frac{10 b_0}{\beta}\right)^{-\frac{b_1}{2b_0^2}} \mathrm{e}^{-\frac{\beta}{20 b_0}},$$

$$\xi_0(\beta) = \xi \left[1 + \frac{10}{\beta} \frac{u_2 + u_4 \frac{10}{\beta}}{1 + v_2 \frac{10}{\beta}}\right]$$

Expect significant changes to coefficients in QCD with dynamical fermions.

Anisotropic lattice fermions

- Generic anisotropic lattice fermion action

$$S_f = a_\sigma^3 a_\tau \sum_{x,y} \bar{\tilde{\psi}}_x \tilde{M}_{xy} \tilde{\psi}_y \ , \quad \tilde{M}_{xy} = \tilde{D}_{xy} + m\delta_{xy} \ .$$

- Lattice derivatives: $\tilde{\nabla}_{xy,\sigma} \sim \frac{1}{a_\sigma}$, $\tilde{\nabla}_{xy,\tau} \sim \frac{1}{a_\tau} = \frac{\xi^f}{a_\sigma}$ do not scale uniformly!
- Introduce dimensionless lattice quantities with uniform scaling:

$$\psi = a_\sigma \sqrt{a_\tau}\, \tilde{\psi} \ , \ D_{xy,\sigma} = a_\sigma \tilde{D}_{xy,\sigma} \ , \ \xi^f D_{xy,\tau} = a_\sigma \tilde{D}_{xy,\tau} \ .$$

- **Bare fermion anisotropy** ξ_0^f is renormalized, differs from ξ_0 (gauge fields)!
- Dimensionless lattice fermion action

$$S_f = \sum_{x,y} \bar{\psi}_x M_{xy} \psi_y = \sum_{x,y} \bar{\psi}_x \left[D_{xy,\sigma} + \xi_0^f D_{xy,\tau} + a_\sigma m \delta_{xy} \right] \psi_y \ .$$

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- Find the **renormalized fermion anisotropy** ξ^f, then tune $\xi^f \to \xi$:

(temporal) dispersion relation: $a_\tau^2 E_\tau^2(p^2) = a_\tau^2 E_\tau^2(0) + \frac{a_\sigma^2 p^2}{(\xi^f)^2}$,

(spatial-temporal) mass ratio: $\xi^f = \frac{E_\sigma}{E_\tau} = \frac{a_\sigma M}{a_\tau M} = \frac{a_\sigma}{a_\tau}$,

spatial dispersion relation: $a_\sigma^2 E_\sigma^2(p^2) = a_\sigma^2 E_\sigma^2(0) + (\xi^f)^2 (a_\tau^2 p_\tau^2)$.

Staggered fermion operators

- **Anisotropic** (naive) staggered [Nomura et al., PTP 111, 2004]; [Levkova et al., PRD 73, 2006]:

$$2D^{naive}_{xy}[U] = \sum_{\mu=1}^{3} \eta_{x,\mu}\left[U_{x,\mu}\delta_{x+\hat{\mu},y} - U^{\dagger}_{x-\hat{\mu},\mu}\delta_{x-\hat{\mu},y}\right] + \xi_0^f \eta_{x,4}\left[U_{x,4}\delta_{x+\hat{4},y} - U^{\dagger}_{x-\hat{4},4}\delta_{x-\hat{4},y}\right].$$

- **Temporal** staples in **smearing** w. **anisotropy factors** $\rho_{i4} = \xi^2$. [Borsanyi, 2012]

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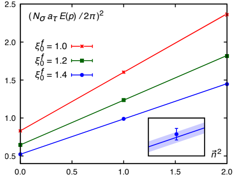
- **Temporal** staples in **smearing** w. **anisotropy factors** $\rho_{i4} = \xi^2$. [Borsanyi, 2012]
- **Anisotropic highly improved staggered quarks** [Follana et al., PRD 75, 2007]:

$$\begin{aligned} 2D^{aHISQ}_{xy}[U] = \sum_{\mu=1}^{3} \eta_{x,\mu} \Big[& X^F_{x,\mu}\delta_{x+\hat{\mu},y} - X^{F\dagger}_{x-\hat{\mu},\mu}\delta_{x-\hat{\mu},y} \\ & + (1+\epsilon(a_\sigma m)) \left[X^L_{x,\mu}\delta_{x+3\hat{\mu},y} - X^{L\dagger}_{x-3\hat{\mu},\mu}\delta_{x-3\hat{\mu},y} \right] \Big] \\ & + \xi_0^f \eta_{x,4} \Big[X^F_{x,4}\delta_{x+\hat{4},y} - X^{F\dagger}_{x-\hat{4},4}\delta_{x-\hat{4},y} \\ & + (1+\epsilon(a_\sigma m)) \left[X^L_{x,4}\delta_{x+3\hat{4},y} - X^{L\dagger}_{x-3\hat{4},\mu}\delta_{x-3\hat{4},y} \right] \Big]. \end{aligned}$$

- **Fat7 smeared links** for **(a)HISQ** [Blum et al., PRD 55, 1997]; [Orginos et al., PRD 60, 1999]:
  $U_{x,\mu} \xrightarrow{Fat7} V_{x,\mu}$, $V_{x,\mu} \xrightarrow{proj\ U(3)} W_{x,\mu}$, $W_{x,\mu} \xrightarrow{Fat7} X^F_{x,\mu}$, and $W_{x,\mu} \xrightarrow{Naik} X^L_{x,\mu}$.
- **Anisotropy factors** ξ_0^f just multiply $X^F_{x,4}, X^L_{x,4}$ due to **Fat7 smearing**.

Anisotropy tuning with (naive) staggered

Temporal **dispersion relation**
for unsmeared naive staggered
3-4 m_s bracket m_s^{phys}
3-4 ξ_0^f bracket expected ξ_0^f
for each anisotropy $\xi^f = \xi$



- Tuning scheme for **unsmeared** (aka naive) staggered (**quenched**)
    1. Dispersion relation @ fixed $(a_\sigma m_s, \xi_0^f) \Rightarrow \xi^f(a_\sigma m_s, \xi_0^f)$.

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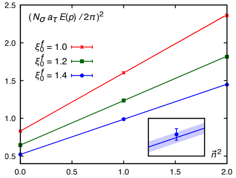
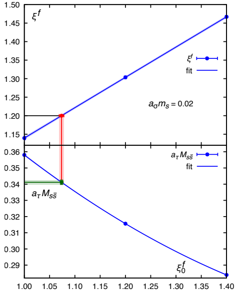
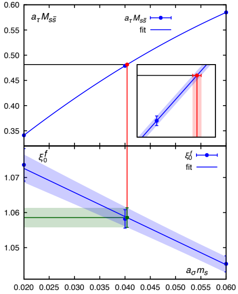


Tuning ξ_0^f at fixed m_s



Tuning $a_\sigma m_s$ @ tuned ξ^f



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  3. Interpolate $a_\tau M_{s\bar{s}}(a_\sigma m_s, \xi^f) \to a_\tau M_{s\bar{s}}^{phys}(\xi^f)$ @ fixed $\xi^f \Rightarrow a_\sigma m_s^*(\xi^f)$.
  4. Finally **readjust** $\xi_0^f(a_\sigma m_s, \xi^f) \to \xi_0^f(a_\sigma m_s^*, \xi^f)$ @ fixed ξ^f.

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Anisotropy tuning with (naive) staggered

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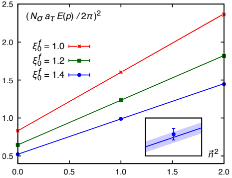

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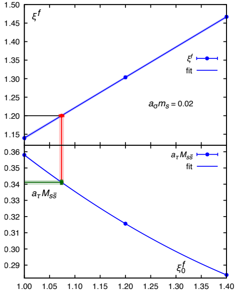

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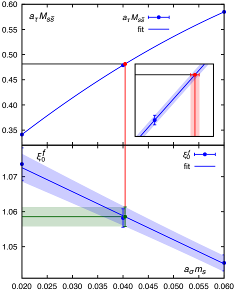

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monotonic: $\frac{\xi_0^f}{\xi^f} = 0.882(2),\ 0.826(3),\ 0.779(1)$ @ $\xi^f = 1.2,\ 1.5,\ 2$.

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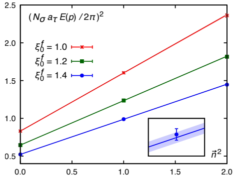
for each anisotropy $\xi^f = \xi$



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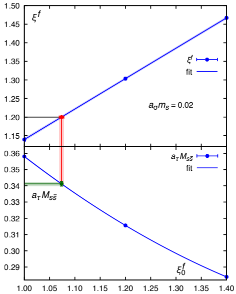

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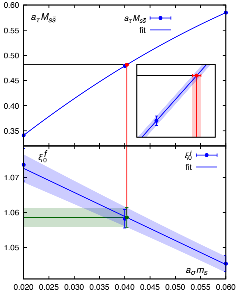

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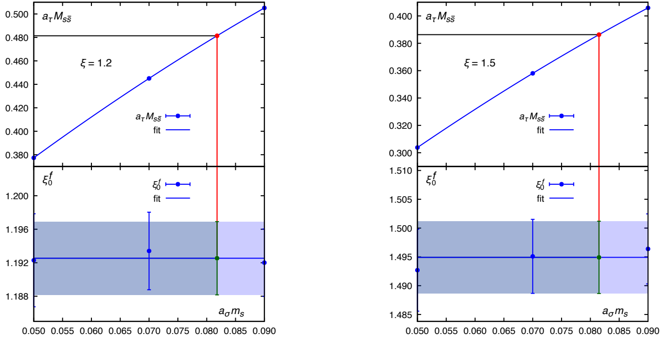
  monotonic: $\frac{\xi_0^f}{\xi^f} = 0.882(2),\ 0.826(3),\ 0.779(1)$ @ $\xi^f = 1.2,\ 1.5,\ 2.$
- **No tuning** for **anisotropic 4-stout** staggered ($N_f = 3$ **dynamical**): $\xi_0^f \simeq \xi^f$ at $\mathrm{SU}(3)$ symmetric point, confirmed from **mass ratios** up to $\xi = 4$.

Anisotropy tuning with aHISQ



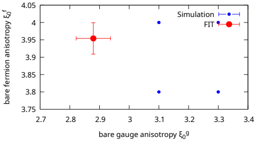
- Tuning scheme for aHISQ (**quenched**): steps 1 - 3 as for naive, no step 4.
- **Simplified tuning scheme**: 1) ξ_0^f @ fixed $a_\sigma m_s$. 2) $a_\sigma m_s$ @ fixed ξ_0^f.

ξ	1.1	1.2	1.5	2	4	8
ξ_0^f/ξ	0.998(8)	0.994(4)	0.997(4)	1.005(6)	1.011(9)	0.998(10)

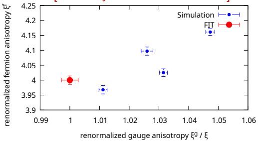
- Apparently, **no tuning needed for light fermions** w. aHISQ!
- **Beware**: this does not hold for **heavy fermions** w. aHISQ!

Simultaneous anisotropy tuning with dynamical staggered

1. Choose **anisotropy**, i.e. $\xi = 4$. For each β: Pick a **reasonable quark mass**.
2. **Bracket expected** (ξ_0, ξ_0^f) with square in **bare anisotropy plane** (upper).
3. **Determine** (ξ, ξ^f) (lower). Assuming **linear dependence** find **Jacobian**.
4. **Invert Jacobian** to find expected (ξ_0, ξ_0^f) corresponding to target (red).



[Courtesy of Alexander Adam]



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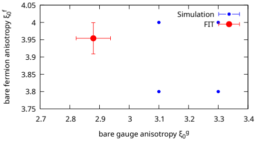
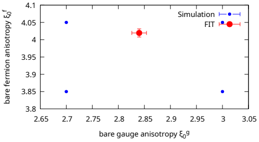
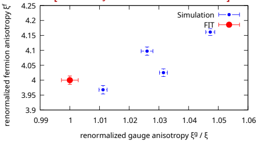
5. **Readjust to bracket new** (ξ_0, ξ_0^f)(middle). **Repeat until bracketed**.

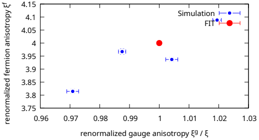




[Courtesy of Alexander Adam]

VERY PRELIMINARY!





Simultaneous anisotropy tuning with dynamical staggered

1. Choose **anisotropy**, i.e. $\xi = 4$. For each β: Pick a **reasonable quark mass**.
2. **Bracket expected** (ξ_0, ξ_0^f) with square in **bare anisotropy plane** (upper).
3. **Determine** (ξ, ξ^f) (lower). Assuming **linear dependence** find **Jacobian**.
4. **Invert Jacobian** to find expected (ξ_0, ξ_0^f) corresponding to target (red).
5. **Readjust to bracket new** (ξ_0, ξ_0^f)(middle). **Repeat until bracketed**.
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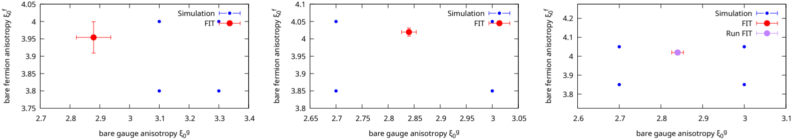
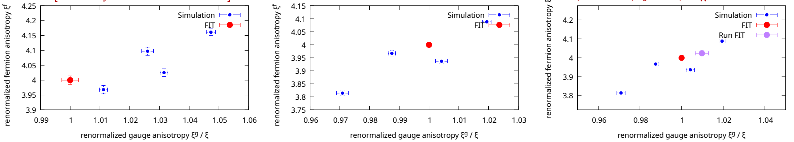


[Courtesy of Alexander Adam] **VERY PRELIMINARY!** $\beta = 4.0,\ \xi = 4,\ m_\pi = 411$ MeV



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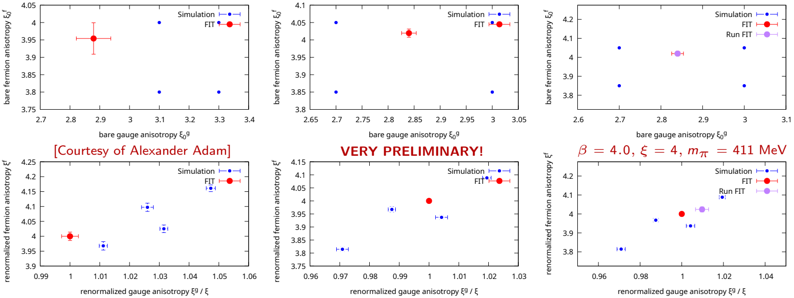
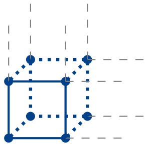


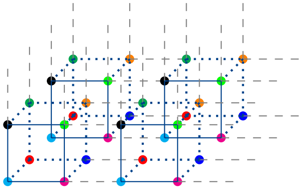
7. Finally **tune the quark masses**. Then **reconfirm actual** (ξ, ξ^f).

Doubling problem and staggered fermions

Lattice for bosons



(Staggered) fermion lattice



- Staggered fermions: 1-**component field** represents 4×4 spin-taste d.o.f., spread out over a **fermion hypersite** of 16 boson sites, i.e. one hypercube.
- Boson sites have different local gauge links: **parallel transporters** $X_{x,\mu}[U]$.

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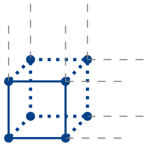

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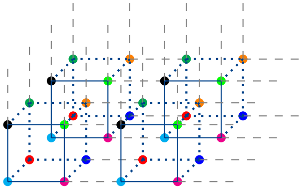

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- Staggered representation of **spin matrices**:

$$S^{\gamma}_{\mu,xy} \equiv \eta_{x,\mu} X_{x,\mu}[U] \delta_{x \pm \hat{\mu},y}, \quad \eta_{x,\mu} \equiv (-1)^{\sum\limits_{\nu<\mu} x_\nu} .$$

- Staggered representation of **taste matrices**:

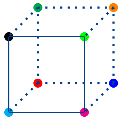
$$S^{\xi}_{\mu,xy} \equiv \zeta_{x,\mu} X_{x,\mu}[U] \delta_{x \pm \hat{\mu},y}, \quad \zeta_{x,\mu} \equiv (-1)^{\sum\limits_{\nu>\mu} x_\nu} .$$

- $S^{\gamma}_{\mu,xy}$ or $S^{\xi}_{\mu,xy}$ satisfy **Clifford algebras** in the free theory (broken in QCD).

Staggered spin-taste operators

- **Spin-taste operators** via chaining multiple $S^{\gamma}_{\mu,xy}$ and $S^{\xi}_{\nu,xy}$.

Staggered hypersite



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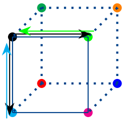

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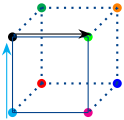

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- Spin-pseudoscalar **taste-"sth.else"** needs uncancelled $X_{x,\mu}[U]$, that are the origin of **taste-symmetry breaking**: $0 \to P$, $1 \to A$, $2 \to T$, $3 \to V$, $4 \to S$.
- **UV fluctuations** of uncancelled $X_{x,\mu}[U]$ are independent at leading order, each contributes $(a\Lambda^2)$ to squared energy: large effect for pions $M_\pi^2 \ll \Lambda^2$.
- Taste **hierarchy** of $P < A < T < V < S$ with $1, 4, 6, 4, 1$-fold **degeneracies**. Well-established and -explained by SChPT [Luo, PRD 57, 1998]; [Lee, Sharpe, PRD 60, 1999].

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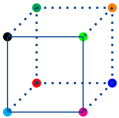

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- **Anisotropy**: temporal links fluctuate less than spatial links, hierarchy breaks up to $P \lesssim A_\tau < A_\sigma \lesssim T_{\sigma\sigma} < T_{\sigma\tau} \lesssim V_\sigma < V_\tau \lesssim S$ with **anisotropic** $1, 1, 3, 3, 3, 3, 1, 1$-fold **degeneracies**. $\xi \to \infty$: $2, 6, 6, 2$-fold **degeneracies**.

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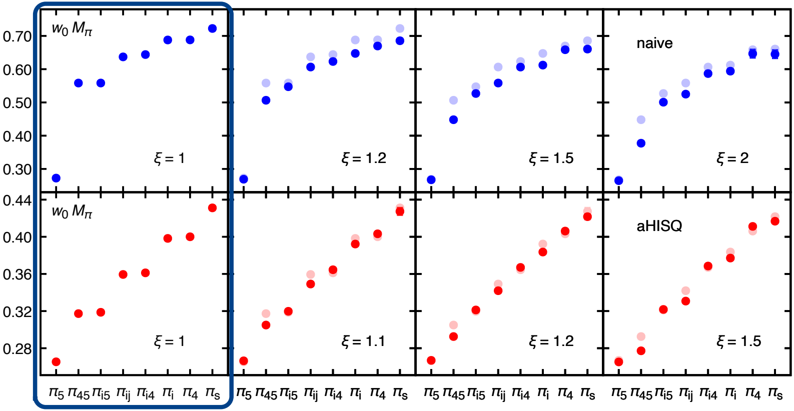
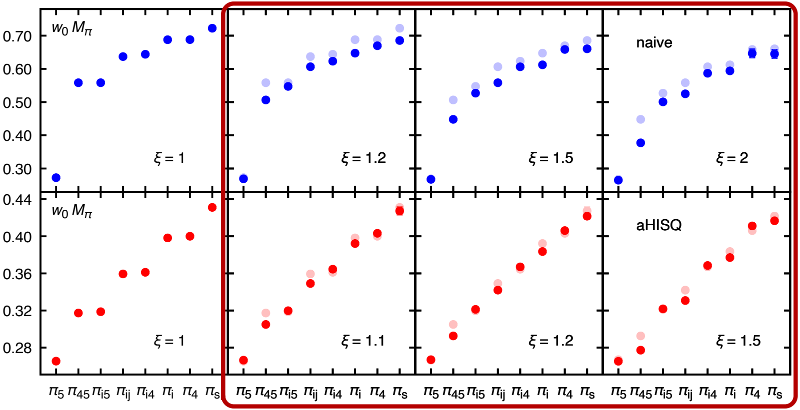
Anisotropic staggered pion tastes



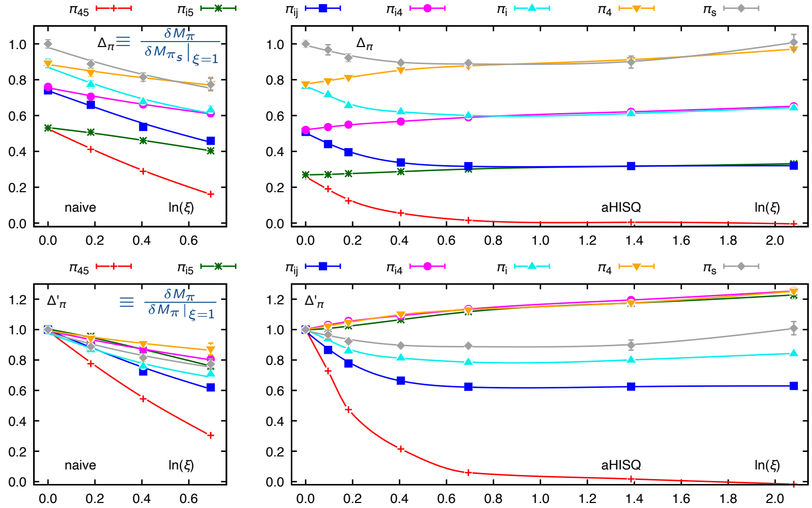
- **Tastes** $\pi_\mu \sim \xi_\mu \sim S_5^\gamma S_\mu^\xi$, $\pi_s \sim 1 \sim S_5^\gamma$ and **splittings** $\delta M_\pi \equiv M_\pi^2 - M_{\pi_5}^2$.
- **Isotropic**: five multiplets $P < A < T < V < S$, approximately equidistant splittings for HISQ, but not for naive! Well established and explained by SχPT [Lee, Sharpe, PRD 60, 1999] and lattice perturbation theory [Follana et al., PRD 75, 2007].
- **Isotropic**: Lowest, highest, and mean splitting all much larger for naive!

Anisotropic staggered pion tastes



- **Anisotropic**: The $A < T < V \Rightarrow A_\tau < A_\sigma \lesssim T_{\sigma\sigma} < T_{\sigma\tau} \lesssim V_\sigma < V_\tau$.
- **Naive**: odd tastes $\pi_{45}, \pi_{ij}, \pi_i, \pi_s$ decrease faster than even tastes $\pi_{i5}, \pi_{i4}, \pi_4$. **The lower the tastes** (among odd or even) **the faster** the decrease.
- **aHISQ**: odd tastes $\pi_{45}, \pi_{ij}, \pi_i, \pi_s$ fall, yet even tastes $\pi_{i5}, \pi_{i4}, \pi_4$ rise. **The lower** the odd/even tastes **the faster/slower** the decrease/increase.

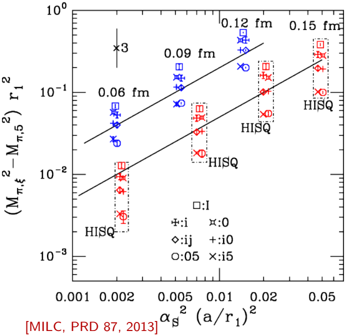
Anisotropic staggered pion tastes



- **New degeneracy pattern** of $\xi \to \infty$ limit emerges much faster for **aHISQ**.
- aHISQ: hints of **universal behavior** of Δ'_π for the rising even tastes.

Independent link fluctuations

- Staggered TSB is due to UV fluctuations of links.
- **Naive** staggered TSB is $\alpha_s^1 a^2$ + h.o. [Golterman, NPB 245, 1984].
- **Improved/link-smeared** staggered TSB is $\alpha_s^2 a^2$ + h.o. [MILC, PRD 87, 2013].
- QCD coupling α_s at the lattice scale, e.g. α_V in the potential scheme.
- Powers of α_s ⇒ **perturbative** picture?



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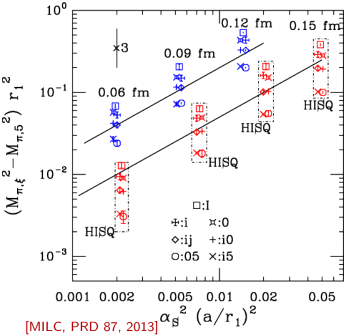

[MILC, PRD 87, 2013]

- In a spin-pseudoscalar **taste-"sth.else"** operator $S_{5,xy}^{\gamma} S_{\#,yz}^{\xi}$, each **uncancelled link** is in a different direction.
- The tree-level gluon propagator is **diagonal** in Euclidean and color spaces.
- To leading order, each link in a given direction **fluctuates independently**.
- Define $\delta E(\xi = 1) = \delta M_{\mu 5}(\xi = 1)$, and thus, $\delta M_\pi(\xi = 1) = (l_\sigma + l_\tau)\delta E(\xi = 1)$.
- **Independent link fluctuations** explain the equidistant HISQ splittings.

Anisotropic link fluctuations

- At larger ξ, **temporal** links get **smoother**, while **spatial** links get **rougher**.
- We generalize the tadpole factor $u_0 = \sqrt[4]{P_{\mu\nu}}$ as a measure of roughness:

$$Q_{\mu\nu} = \frac{P_{\mu\nu}(\xi)}{P_{\mu\nu}(\xi = 1)} \quad \Rightarrow \quad u_\tau(\xi) = \sqrt[4]{\frac{Q^2_{\sigma\tau}}{Q_{\sigma\sigma}}} \, , \; u_\sigma(\xi) = \sqrt[4]{Q_{\sigma\sigma}} \; .$$

- Thus, **fluctuations** get redistributed from **temporal** to **spatial** links:

$$u_\tau(\xi > 1) > 1 \; , \; u_\sigma(\xi > 1) < 1 \quad \Rightarrow \quad \frac{u_\tau(\xi > 1) - 1}{1 - u_\sigma(\xi > 1)} > 1$$

- The decrease of **temporal link fluctuations** is balanced by a **weaker** increase of **spatial link fluctuations** (for each direction).
- We observe $\frac{u_\tau(\xi>1)-1}{1-u_\sigma(\xi>1)} \approx 3.7$ for $\xi \lesssim 2$ and logarithmic rise thereafter.

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Anisotropy as a symmetry-breaking perturbation

- $\delta M_s(\xi) = 4\delta E(\xi)$ differs by at most 17% from isotropic value $4\delta E(\xi=1)$.
- Grotrian diagrams reveal in atomic physics or in the eightfold way that **multiplets break up unevenly s.t. the weighted mean is unchanged**.
- **Universal breakup function** $f(\xi)$: limits $f(\xi \to 1) \to 0$ and $f(\xi \to \infty) \to 1$.
- **Units of fluctuations** distributed $3:1$ between temporal or spatial links.
- **Mass formula** ξ dependence via $f(\xi)$ and $\delta E(\xi)$

$$\delta M_\pi(\xi) = \left(I_\sigma \left(1+\frac{1}{3}f(\xi)\right) + I_\tau \left(1-f(\xi)\right)\right) \delta E(\xi).$$

- Taking ratios eliminating $\delta E(\xi)$ to isolate $f(\xi)$, we find **data collapse:**

$$f(\xi) = 3\,\frac{\delta M_{i5}(\xi)-\delta M_{45}(\xi)}{3\delta M_{i5}(\xi)+\delta M_{45}(\xi)} = 3\,\frac{\delta M_{i4}(\xi)-\delta M_{ij}(\xi)}{\delta M_{i4}(\xi)+\delta M_{ij}(\xi)} = 9\,\frac{\delta M_4(\xi)-\delta M_i(\xi)}{\delta M_4(\xi)+3\delta M_i(\xi)}.$$

- Then we eliminate $f(\xi)$ to isolate $\delta E(\xi)$, e.g.

$$\delta E(\xi) = \frac{3\delta M_{i5}(\xi)+\delta M_{45}(\xi)}{4} = \frac{\delta M_{i4}(\xi)+\delta M_{ij}(\xi)}{4} = \frac{\delta M_4(\xi)+3\delta M_i(\xi)}{12} = \frac{\delta M_s(\xi)}{4},$$

which also exhibit approximate **data collapse.**

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Staggered chiral perturbation theory

- Leading order chiral Lagrangian for $\Sigma = \exp(i\Phi/f)$ [Lee, Sharpe, PRD 60, 1999]:

$$\mathcal{L}(\Sigma) = \frac{a^4 f^2}{8}\mathrm{Tr}\left(\partial_\mu\Sigma\partial_\mu\Sigma^\dagger\right) - \frac{a^4}{4}B\, f^2\mathrm{Tr}\left(\mathcal{M}\Sigma + \mathcal{M}\Sigma^\dagger\right) + \frac{a^4 m_0^2}{24}\left(\mathrm{Tr}\Phi\right)^2 + \mathcal{V},$$

- f, B, m_0 are low energy constants that exist in the continuum limit.

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- f, B, m_0 are low energy constants that exist in the continuum limit.
- Potential $\mathcal{V} = \mathcal{U} + \mathcal{U}'$ due to **four-fermion operators (dimension six)**.
- $\mathcal{U}'$ consists of **two-trace** terms affecting **only flavor-diagonal** pion masses.
- $\mathcal{U}$ consists of **single-trace** terms affecting **flavor offdiagonal** pion masses.

$$\begin{aligned}
-\mathcal{U} &= a^6 \sum_k C_k \mathcal{O}_k \\
&= a^6 C_1 \mathrm{Tr}\left(\xi_5 \Sigma \xi_5 \Sigma^\dagger\right) \\
&+ a^6 C_3 \frac{1}{2} \sum_\nu \left[\mathrm{Tr}\left(\xi_\nu \Sigma \xi_\nu \Sigma\right) + \mathrm{Tr}\left(\xi_\nu \Sigma^\dagger \xi_\nu \Sigma^\dagger\right)\right] \\
&+ a^6 C_4 \frac{1}{2} \sum_\nu \left[\mathrm{Tr}\left(\xi_{\nu 5} \Sigma \xi_{5\nu} \Sigma\right) + \mathrm{Tr}\left(\xi_{\nu 5} \Sigma^\dagger \xi_{5\nu} \Sigma^\dagger\right)\right] \\
&+ a^6 C_6 \sum_{\mu<\nu} \mathrm{Tr}\left(\xi_{\mu\nu} \Sigma \xi_{\nu\mu} \Sigma^\dagger\right).
\end{aligned}$$

- **For HISQ, only C_4 is substantial**; $C_1, C_3, C_6 \approx 0$. [Lee, Sharpe, PRD 60, 1999].

SChPT-inspired model for the pion spectrum for aHISQ (fail)

- Isotropic SChPT prediction (with factor $\frac{16a_\sigma^2}{f^2}$ absorbed in C_i)

$$\delta M_{45}(\xi) = C_1 + 3C_3 + \frac{1}{\xi^2} C_4 + 3\frac{1}{\xi^2} C_6$$

$$\delta M_{i5}(\xi) = C_1 + \left(2 + \frac{1}{\xi^2}\right) C_3 + C_4 + \left(2 + \frac{1}{\xi^2}\right) C_6$$

$$\delta M_{ij}(\xi) = \left(1 + \frac{1}{\xi^2}\right) C_3 + C_4 + 2C_6$$

$$\delta M_{i4}(\xi) = 2\left(C_3 + C_4 + \left(1 + \frac{1}{\xi^2}\right) C_6\right)$$

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$$\delta M_{4}(\xi) = C_1 + \frac{1}{\xi^2} C_3 + 3C_4 + 3\frac{1}{\xi^2} C_6$$

$$\delta M_{s}(\xi) = \left(3 + \frac{1}{\xi^2}\right) C_3 + \left(3 + \frac{1}{\xi^2}\right) C_4$$

where factors $\frac{1}{\xi^2}$ are introduced for each pair of ξ_4 ($a^6 \to a_\sigma^4 a_\tau^2 = \frac{a_\sigma^6}{\xi^2}$).

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  where factors $\frac{1}{\xi^2}$ are introduced for each pair of ξ_4 ($a^6 \to a_\sigma^4 a_\tau^2 = \frac{a_\sigma^6}{\xi^2}$).
- This is incompatible with the observed aHISQ spectrum, since
  1. even tastes $\xi_i\xi_5$,$\xi_i\xi_j$,ξ_4 would have to stay constant,
  2. odd tastes $\xi_4\xi_5$,$\xi_i\xi_j$,ξ_i,1 would have to approach new even partners as $\frac{1}{\xi^2}$.
- **Seven** splittings vs **four** LECs: this model is not flexible enough!

SChPT-inspired model for the pion spectrum for aHISQ (success)

- **Remedy**: ξ dependence and breakup into $C_i(\xi)$ for ξ_i and $D_i(\xi)$ for ξ_4.

$$
\begin{aligned}
-\frac{\mathcal{U}_{ani}}{a_\sigma^6} &= C_1(\xi)\mathrm{Tr}\left(\xi_5\Sigma\xi_5\Sigma^\dagger\right) \\
&+ C_3(\xi)\frac{1}{2}\sum_j\left[\mathrm{Tr}\left(\xi_j\Sigma\xi_j\Sigma\right) + \mathrm{Tr}\left(\xi_j\Sigma^\dagger\xi_j\Sigma^\dagger\right)\right] \\
&+ D_3(\xi)\frac{1}{2}\left[\mathrm{Tr}\left(\xi_4\Sigma\xi_4\Sigma\right) + \mathrm{Tr}\left(\xi_4\Sigma^\dagger\xi_4\Sigma^\dagger\right)\right] \\
&+ C_4(\xi)\frac{1}{2}\sum_j\left[\mathrm{Tr}\left(\xi_{j5}\Sigma\xi_{5j}\Sigma\right) + \mathrm{Tr}\left(\xi_{j5}\Sigma^\dagger\xi_{5j}\Sigma^\dagger\right)\right] \\
&+ D_4(\xi)\frac{1}{2}\left[\mathrm{Tr}\left(\xi_{45}\Sigma\xi_{54}\Sigma\right) + \mathrm{Tr}\left(\xi_{45}\Sigma^\dagger\xi_{54}\Sigma^\dagger\right)\right] \\
&+ C_6(\xi)\sum_{j<k}\mathrm{Tr}\left(\xi_{jk}\Sigma\xi_{kj}\Sigma^\dagger\right) \\
&+ D_6(\xi)\sum_j\mathrm{Tr}\left(\xi_{j4}\Sigma\xi_{4j}\Sigma^\dagger\right).
\end{aligned}
$$

- **Seven** anisotropic splittings vs **seven** anisotropic LECs, can be inverted!
- Obvious limiting behavior $C_i(\xi \to 1) = D_i(\xi \to 1)$ must be satisfied.
- Expected the same for a genuine derivation [Luo, PRD 57, 1998]; [Lee, Sharpe, PRD 60, 1999].

Anisotropic taste pion splittings

$$
\begin{aligned}
\delta M_{45}(\xi) &= C_1(\xi) + 3C_3(\xi) + D_4(\xi) + 3D_6(\xi),\\
\delta M_{i5}(\xi) &= C_1(\xi) + 2C_3(\xi) + D_3(\xi) + C_4(\xi) + 2C_6(\xi) + D_6(\xi),\\
\delta M_{ij}(\xi) &= C_3(\xi) + D_3(\xi) + C_4(\xi) + D_4(\xi) + 2C_6(\xi) + 2D_6(\xi),\\
\delta M_{i4}(\xi) &= 2C_3(\xi) + 2C_4(\xi) + 2C_6(\xi) + 2D_6(\xi),\\
\delta M_{i}(\xi) &= C_1(\xi) + C_3(\xi) + 2C_4(\xi) + D_4(\xi) + 2C_6(\xi) + D_6(\xi),\\
\delta M_{4}(\xi) &= C_1(\xi) + D_3(\xi) + 3C_4(\xi) + 3D_6(\xi),\\
\delta M_{s}(\xi) &= 3C_3(\xi) + D_3(\xi) + 3C_4(\xi) + D_4(\xi).
\end{aligned}
$$

- **Even tastes**: $I_\sigma = 1, 2, 3$ **times** $C_4(\xi)$ **and unbalanced spatial links.**
- **Odd tastes**: $I_\sigma = 0, 1, 2, 3$ **times** $C_4(\xi)$ **and unbalanced spatial links and** $I_\tau = 1$ **times** $D_4(\xi)$ **and one unbalanced temporal link.**
- **Other LECs contribute**, too, and, in particular, $C_1(\xi), C_3(\xi), D_6(\xi)$ contribute to $\delta M_{45}(\xi)$, which must vanish in the $\xi \to \infty$ limit!

Anisotropic LECs

$$C_1(\xi) = \frac{1}{8}\left(\delta M_{45}(\xi) + 3\delta M_{i5}(\xi) - 3\delta M_{ij}(\xi) - 3\delta M_{i4}(\xi) + 3\delta M_{i}(\xi) + \delta M_{4}(\xi) - \delta M_{s}(\xi)\right),$$

$$C_3(\xi) = \frac{1}{8}\left(\delta M_{45}(\xi) + \delta M_{i5}(\xi) - \delta M_{ij}(\xi) + \delta M_{i4}(\xi) - \delta M_{i}(\xi) - \delta M_{4}(\xi) + \delta M_{s}(\xi)\right),$$

$$D_3(\xi) = \frac{1}{8}\left(-\delta M_{45}(\xi) + 3\delta M_{i5}(\xi) + 3\delta M_{ij}(\xi) - 3\delta M_{i4}(\xi) - 3\delta M_{i}(\xi) + \delta M_{4}(\xi) + \delta M_{s}(\xi)\right),$$

$$C_4(\xi) = \frac{1}{8}\left(-\delta M_{45}(\xi) - \delta M_{i5}(\xi) - \delta M_{ij}(\xi) + \delta M_{i4}(\xi) + \delta M_{i}(\xi) + \delta M_{4}(\xi) + \delta M_{s}(\xi)\right),$$

$$D_4(\xi) = \frac{1}{8}\left(\delta M_{45}(\xi) - 3\delta M_{i5}(\xi) + 3\delta M_{ij}(\xi) - 3\delta M_{i4}(\xi) + 3\delta M_{i}(\xi) - \delta M_{4}(\xi) + \delta M_{s}(\xi)\right),$$

$$C_6(\xi) = \frac{1}{8}\left(-\delta M_{45}(\xi) + \delta M_{i5}(\xi) + \delta M_{ij}(\xi) + \delta M_{i4}(\xi) + \delta M_{i}(\xi) - \delta M_{4}(\xi) - \delta M_{s}(\xi)\right),$$

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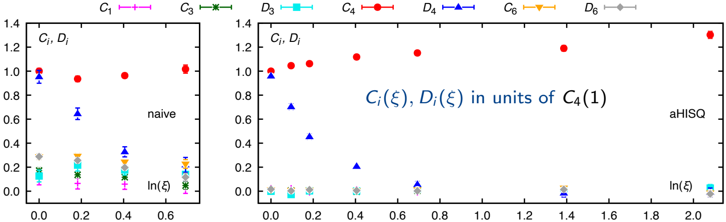
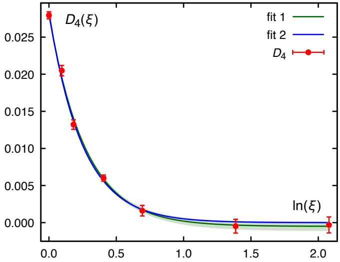
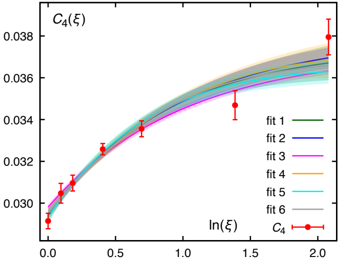


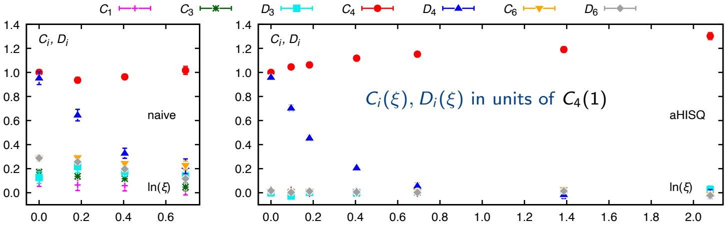
Anisotropic LECs





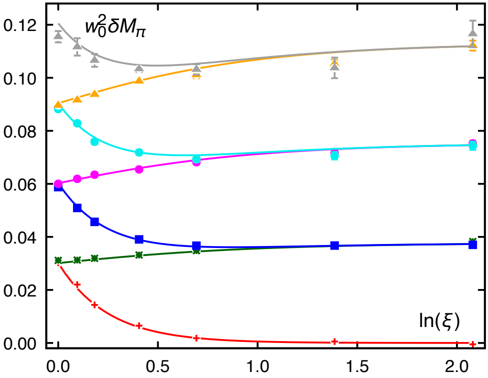
fit2: $D_4(\xi) = \frac{d}{\xi^4}, \ \frac{\chi^2}{dof} = 0.9$

fit6: $C_4(\xi) = c\,\frac{5\xi^2 - 1 - \ln(\xi)}{4\xi^2}, \ \frac{\chi^2}{dof} = 1.4$



Full model of anisotropy dependence

$$\delta M_\pi(\xi) = c \left(l_\sigma \cdot \frac{5\xi^2 - 1 - \ln(\xi)}{4\xi^2} + l_\tau \cdot \frac{1}{\xi^4} \right)$$



- Factor $\frac{5}{4}$ enforces the **isotropic multiplets**.
- Isotropic limit is not perfectly equidistant: ratios 1:1.94:2.92:3.82 instead of 1:2:3:4.
- Deviation: C_1, C_3, C_6!
- Fit: $c = 0.030129(31)$ @ $\frac{\chi^2}{dof} = 3.2$ vs isotropic result $C_4 = 0.02914(37)$

- RMS pion splitting reduces only by $\frac{\delta M_\pi(\xi \to \infty)}{\delta M_\pi(\xi \to 1)} = \frac{15}{16}$.
- Match $C_4(\xi), D_4(\xi)$ and $\delta E(\xi), f(\xi)$ for aHISQ (assume all other $C_i, D_i \approx 0$):

$$\delta E(\xi) = \frac{1}{4}(3C_4(\xi) + D_4(\xi)) \quad \text{and} \quad f(\xi) = 3\,\frac{C_4(\xi) - D_4(\xi)}{3C_4(\xi) + D_4(\xi)}.$$

Summary (1)

- **Anisotropic LQCD** is a powerful tool, well-established for spectroscopy (HadSpec) or thermodynamics (FASTSUM) with Wilson clover fermions.
- **Anisotropic LQCD** with doubled fermions: in early stage, target $\xi = 4 - 8$.
- **Ongoing work:** quenched or $N_\mathrm{f} = 3$ naive 4-stout staggered backgrounds:
  - gauge anisotropy tuning,
  - fermion anisotropy tuning,
  - effects of Symanzik improvement and link smoothness (i.e. smearing and anisotropy) on taste patterns.
- **Anisotropy tuning** with gradient flow is full of suprises: **SWC scheme**.
- **Anisotropic staggered** naive or aHISQ:
  - **Tuning** is simple for improved staggered: $\frac{\xi_0^f}{\xi^f} \simeq 1$ is mass independent!
  - **Taste multiplets** rearrange dramatically to new degeneracy pattern.
  - Identified a physically/SχPT motivated, **empirical one-parameter model** for taste rearrangement.
- Next steps:
  - More anistropies
  - Finer lattices
  - The dynamical setup is confirmed for naive staggered [Levkova et al., PRD 73, 2006].

Summary (2)

- **Karsten-Wilczek fermions** (unimproved or improved):
  - Anisotropic counterterms and anisotropy from gauge fields: for $\xi > 1$ **anisotropy tuning** turned out more challenging than expected
  - Two different types of **taste-symmetry breaking**, change differently under **Naik improvement** (type 1) vs **anisotropy or smearing** (type 2).
  - Much smaller **taste-symmetry breaking** achievable than staggered.
  - Prospects: finite density, order of the chiral transition, $U(1)_A$ restoration.

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Thank you for your attention!

Lattice gauge theory at finite density

- Fermion fields are **Grassmann variables** that cannot be sampled from.
- Formally **integrate out fermion bilinears**, then sample stochastically

$$\langle X \rangle_{\mathrm{LQCD}} = \frac{1}{\mathcal{Z}} \frac{1}{N} \sum_{\{U\}}^{N} X[U] \prod_{f} \det\left(M_f[U]\right) \mathrm{e}^{-S_g[U]} + \mathcal{O}\left(\frac{1}{\sqrt{N}}\right) .$$

- $\det\left(M_f[U]\right) \geq 0$, since $M_f[U]$ is γ_5-hermitian: $M_f[U]\gamma_5 = \gamma_5 M_f[U]^\dagger$.

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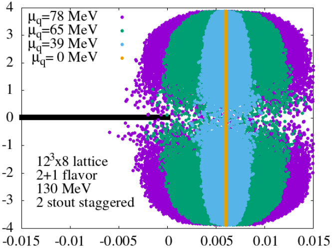
$$M_f[A,\mu_f](x) = \not{D}[A](x) + m_f + \mu_f\gamma_4 \quad \text{(continuum)}$$

is not γ_5-hermitian unless $\mathrm{Re}\,\mu_f = 0$: $M_f[A,\mu_f]\gamma_5 = \gamma_5 M_f[A,-\mu_f]^\dagger$.



[Borsanyi et al, PRD 109,2024]

- $\mu_f = 0$ eigenvalues real or in **complex conjugate pairs.**

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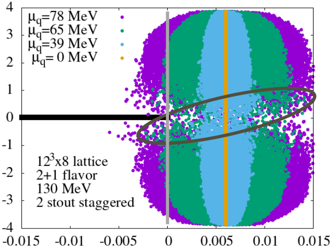
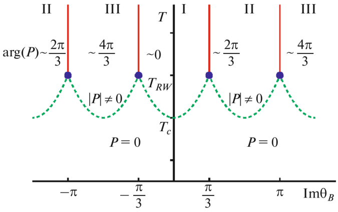


[Borsanyi et al, PRD 109,2024]

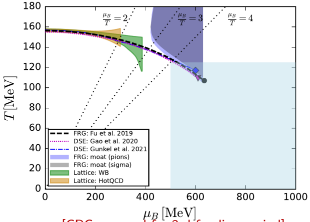
- $\mu_f = 0$ eigenvalues real or in **complex conjugate pairs.**
- $\mu_f \neq 0$ **opposite tilt** for complex conjugate pairs.
- At $\mu_f \simeq m_\pi/2$, some **real parts cross zero** making the sign problem manifest.

Working around the sign problem

- **Analytic continuation** from $\mu_f = \mathrm{i}|\mu_f|$, which can be simulated directly, since $\det(M_f[U, \pm\mathrm{i}|\mu_f|]) \geq 0$. But the range in $\mathrm{i}|\mu_f|$ is limited by the **Roberge-Weiss transitions** at $|\frac{\mu_f}{T}| = \frac{\pi}{N_c}$ between Polyakov loop sectors.
- **Taylor expansion** of $\det(M_f[U, \mu_f])$ in $\frac{\mu_f^2}{T^2}$, which can be done at $\mu_f = 0$. But higher orders are costly and noisy, breakdown far from the CEP.
- Mix of approaches: **Taylor expansion** of $\det(M_f[U, \pm\mathrm{i}|\mu_f|])$ in $\frac{\mu_f^2}{T^2}$ at $\mu_f = \mathrm{i}|\mu_f|$, fix noisy higher orders by fitting lattice data at $\mu_f = \mathrm{i}|\mu_f|$.



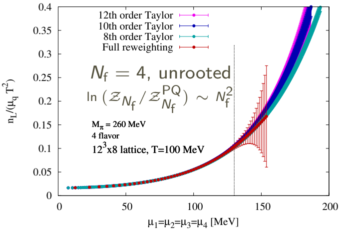
[Rogalev et al, PPNL 20,2023]



[CRC proposal for 3rd funding period]

Phase-quenched QCD simulations

- Phase- or sign-quenched QCD $\det(M_f[U,\mu_f]) \to |\det(M_f[U,\mu_f])| \geq 0$: [A09]
  strict bound $\mathcal{Z}^{PQ} \geq \mathcal{Z}$, **tight at large** T **or large chemical potential** μ_f,
  [Moore, Gorda, JHEP 12,2023];[Navarrete et al, PRD 110,2024]
  but it is far off if T **and chemical potential** μ_f are both at most moderate.
- Reweighted phase- or sign-quenched QCD (phase as part of observable):
  Okay for **unrooted** fermions,



[Borsanyi et al, PRD 109,2024]

- Quark density-to-chemical potential ratio

$$\frac{\hat{n}_L}{\hat{\mu}_q} \equiv \frac{1}{\hat{\mu}_q}\frac{\partial \hat{p}}{\partial \hat{\mu}_q} = \frac{1}{\mu_q TV}\frac{\partial \log \mathcal{Z}}{\partial \mu_q}\ , \quad \hat{n}_L = n_L/T^3\ , \quad \hat{\mu}_q = \mu_q/T\ .$$

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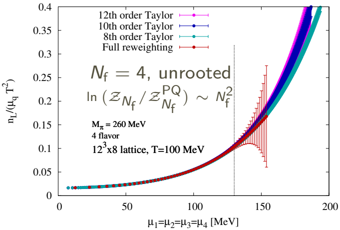
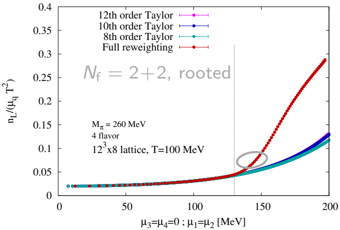
  Fails for **rooted** fermions, **convergence to wrong result** at $\mu_f \simeq m_\pi/2$!



[Borsanyi et al, PRD 109,2024]



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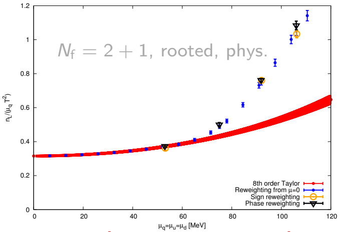
- Reweighted phase- or sign-quenched QCD (phase as part of observable):
  Okay for **unrooted** fermions, **no overlap problem** until moderate μ_f.
  Fails for **rooted** fermions, **convergence to wrong result** at $\mu_f \simeq m_\pi/2$!



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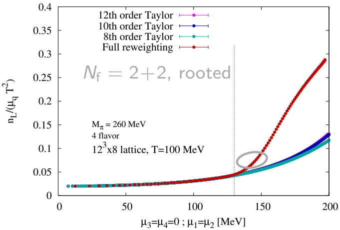

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Minimal Doubling and Karsten-Wilczek fermions

- (Standard, i.e. no Naik) **KW fermions** [Karsten, PLB 104, 1981]; [Wilczek, PRL 59, 1987]:

$$2D^{KW}_{xy}[U] = \sum_{\mu=1}^{3} \gamma_\mu \left[U_{x,\mu}\delta_{x+\hat{\mu},y} - U^\dagger_{x-\hat{\mu},\mu}\delta_{x-\hat{\mu},y}\right]$$
$$+ \xi_0^f \gamma_4 \left[U_{x,4}\delta_{x+\hat{4},y} - U^\dagger_{x-\hat{4},4}\delta_{x-\hat{4},y}\right]$$
$$- r\mathrm{i}\gamma_4 \sum_{\mu=1}^{3} \left[U_{x,\mu}\delta_{x+\hat{\mu},y} + U^\dagger_{x-\hat{\mu},\mu}\delta_{x-\hat{\mu},y} - 2\delta_{xy}\right]$$

- **Anisotropic counterterms** needed [Capitani et al., JHEP 09, 2010]; [Weber, PhD thesis, 2015]!
- Temporal staples in **smearing** w. **anisotropy factors** $\rho_{i4} = \xi^2$. [Borsanyi, 2012]

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- Momentum space representation

$$
D^{KW}(k) = \mathrm{i} \left[\sum_{\mu=1}^{3} \gamma_\mu \sin(a_\sigma k_\mu) + \xi_0^f \gamma_4 \sin(a_\tau k_4) + r\gamma_4 \sum_{\mu=1}^{3} \left[1 - \cos(a_\sigma k_\mu) \right] \right] .
$$

- Propagator poles at $a_\sigma k_j = 0$, $a_\tau k_4 = 0, \pi$ $\Rightarrow$ minimal doubling @ $r^2 > \frac{1}{4}$!

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- Propagator poles at $a_\sigma k_j = 0$, $a_\tau k_4 = 0, \pi \Rightarrow$ minimal doubling @ $r^2 > \frac{1}{4}$!
- Charge conjugation $\mathcal{C}$, **time reflection $\mathcal{T}$ and time reversal Θ:** $r \to -r$.
- *Obvious* **discrete** symmetries: cubic symmetry, parity, $\mathcal{CT}$, $\mathcal{C}\Theta$ (and $\mathcal{T}\Theta$).
- **Continuous** symmetries: $U(1)_V$ and $U(1)_A$. $U(1)_A$ with chiral SSB.

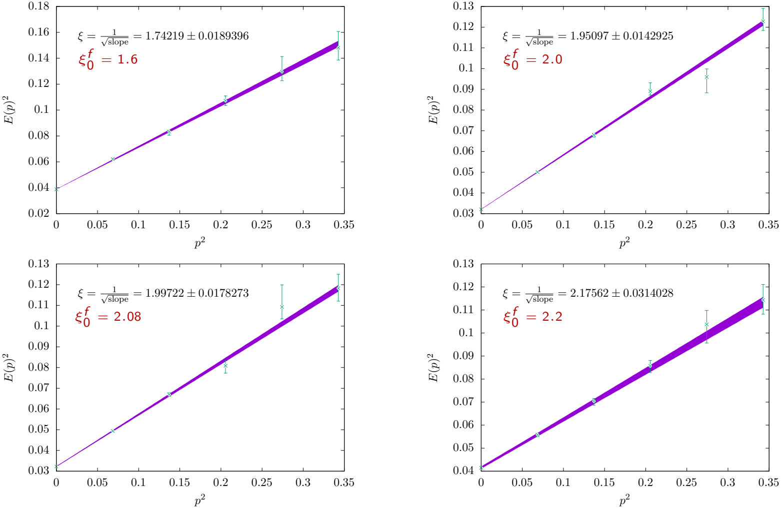
Spatially Naik-improved Karsten-Wilczek fermions

- **Naik KW fermions** [Naik, NPB 316, 1989];[Borsanyi et al., PRD 111, 2025];[Bakenecker, BSc thesis, 2024]:

$$2D^{NKW}_{xy}[U] = \sum_{\mu=1}^{3}\gamma_\mu\left[s_1^\sigma\left[U_{x,\mu}\delta_{x+\hat{\mu},y} - U^\dagger_{x-\hat{\mu},\mu}\delta_{x-\hat{\mu},y}\right] + s_3^\sigma\left[U^L_{x,\mu}\delta_{x+3\hat{\mu},y} - U^{L\dagger}_{x-3\hat{\mu},\mu}\delta_{x-3\hat{\mu},y}\right]\right]$$
$$+\,\xi_0^f\gamma_4\left[s_1^\tau\left[U_{x,4}\delta_{x+\hat{4},y} - U^\dagger_{x-\hat{4},4}\delta_{x-\hat{4},y}\right] + s_3^\tau\left[U^L_{x,4}\delta_{x+3\hat{4},y} - U^{L\dagger}_{x-3\hat{4},\mu}\delta_{x-3\hat{4},y}\right]\right]$$
$$-\,r\mathrm{i}\gamma_4\sum_{\mu=1}^{3}\left[c_1\left[U_{x,\mu}\delta_{x+\hat{\mu},y} + U^\dagger_{x-\hat{\mu},\mu}\delta_{x-\hat{\mu},y} - 2\delta_{xy}\right]\right.$$
$$\left.+c_3\left[U^L_{x,\mu}\delta_{x+\hat{\mu},y} + U^{L\dagger}_{x-\hat{\mu},\mu}\delta_{x-\hat{\mu},y} - 2\delta_{xy}\right]\right]\;.$$

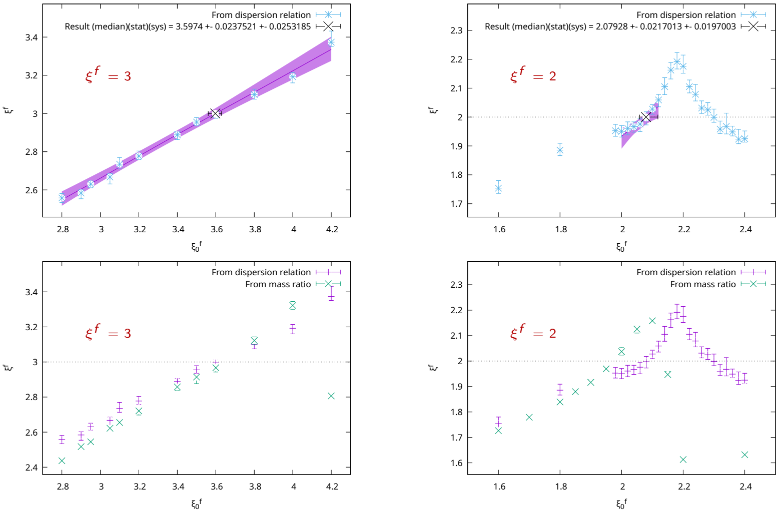
- **Ameliorate the symmetry breaking** $\mathcal{O}(a_\sigma) \to \mathcal{O}(a_\sigma^3)$ via $c_1 = \frac{9}{8}$, $c_3 = -\frac{1}{8}$!
- Choose coefficients $s_1^\sigma = \frac{9}{8}$, $s_3^\sigma = -\frac{1}{24}$ for **spatial** $\mathcal{O}(a_\sigma^2)$ **improvement**.
- Preserve **two-step transfer matrix: unimproved** in time ($s_1^\tau = 1$, $s_3^\tau = 0$).
  Yet we ameliorate $\mathcal{O}(a_\tau^2)$ lattice artifacts by using **anisotropy** $\xi^f \gg 1$.
- Same smeared links for longlinks $U^L_{x,\mu}$, but different coefficients $c_3 \neq s_3^\sigma$.
- Symmetry characteristics are inherited from (standard) KW fermions.

Dispersion relation with spatial Naik KW fermions



- Temporal dispersion relation for spatial Naik KW up to $\left(\frac{L \cdot p}{2\pi}\right)^2 \leq 5$.
- Errors are factor 2-3 larger (vs naive staggered on quenched).

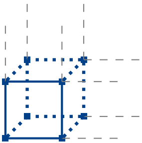
Anisotropy tuning with spatial Naik KW fermions



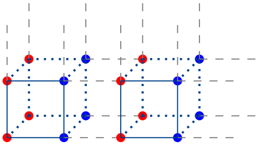
- $\frac{\xi_0^f}{\xi^f} - 1 \ll 0.1$ at $\xi = 1$, $\frac{\xi_0^f}{\xi^f} - 1 \lesssim 0.1$ at $\xi = 2 : w$, $|\frac{\xi_0^f}{\xi^f} - 1| \simeq 0.2$ at $\xi = 3, 4$
- Multiple solutions at $\xi = 2, 4$, but not at $\xi = 1, 3$. Why?

Spin-taste structure of Karsten-Wilczek fermions

Lattice for bosons



KW fermion lattice



- KW fermions: 4-**component field** represents 4×2 spin-taste d.o.f., spread out over a **KW fermion hypersite** of 2 boson sites **separated in time**.

Spin-taste structure of Karsten-Wilczek fermions

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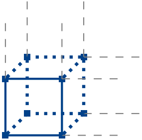

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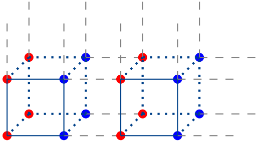

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$$S^{\gamma}_{j,xy} \equiv \gamma_j \delta_{xy}, \quad S^{\gamma}_{4,xy} \equiv \gamma_4 X_{x,4}[U] \delta_{x \pm \hat{4},y}$$

- KW representation of $\mathfrak{su}(2)$ **taste matrices**: $S^{\xi}_{1,xy} = -\mathrm{i} S^{\xi}_{2,xx} S^{\xi}_{3,xy}$,

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- $S^{\gamma}_{\mu,xy}$ and $S^{\xi}_{j,xy}$ satisfy exact **algebras even in the interacting theory**.

Spin-taste structure of Karsten-Wilczek fermions

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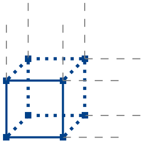

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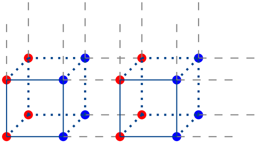

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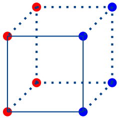
- $S^{\gamma}_{\mu,xy}$ and $S^{\xi}_{j,xy}$ satisfy exact **algebras even in the interacting theory**.
- S^{ξ}_2 yields $r \to -r$. The **extra term is explicitly isovector** at tree-level!
- **Mirror fermion symmetry** $\mathcal{T} S^{\xi}_2$ [Pernici, PLB 346, 1995]; $\mathcal{C} S^{\xi}_2$ and ΘS^{ξ}_2 related.
- **KW action** is invariant under S^{ξ}_3, which is just a **discrete translation**.

KW spin-taste operators

- **Spin-taste operators** via chaining multiple $S^{\gamma}_{\mu,xy}$ for $\mu = 1,2,3$, $S^{\gamma}_{4,xy}$ and $S^{\xi}_{j,xy}$.
- **Spatial vector** $S^{\gamma}_{\mu,xy}$ commute w. all $S^{\xi}_{j,xy}$.

KW hypersite



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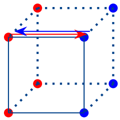

- $U(1)_A$ chiral SSB is linked to the **site-local spin-γ_5 taste-ξ_3**

$$\gamma_5 = S^{\gamma}_{5,xy} S^{\xi}_{3,yx} \ .$$

- There is yet another oscillating **site-local spin-γ_5 taste-ξ_1**

$$\gamma_4(-1)^{x_4} = \mathrm{i} S^{\gamma}_{5,xy} S^{\xi}_{1,yx} = -S^{\gamma}_{5,xy} S^{\xi}_{3,yx} S^{\xi}_{2,xx} \ .$$

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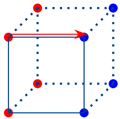

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- S^{ξ}_{2} yields $r \to -r$: 1st type of **anisotropic taste-symmetry breaking** exists already **at tree level**, but is reduced by the **spatial Naik improvement**.
- **Uncancelled** $U_{x,4}$: 2nd type of **anisotropic taste-symmetry breaking.**
- $S^{\gamma}_{5,xy} = \gamma_5 U_{x,4} \delta_{x\pm\hat{4},y}$ or $S^{\gamma}_{5,xy} S^{\xi}_{2,xx} = -\mathrm{i}\gamma_4(-1)^{x_4} U_{x,4} \delta_{x\pm\hat{4},y}$ are **site-split spin-γ_5** (**taste-singlet** or **taste-ξ_2**, respectively). 1st and 2nd for **taste-ξ_2!**

KW spin-taste structure

- **Taste hierarchy** $\xi_3 < \{\xi_1 \gtrless 1\} < \xi_2$ with unclear ordering $\xi_1 \gtrless 1$, depends on balance between **spatial Naik improvement**, **smearing** and **anisotropy.**
- **Anisotropy or smearing** are expected to cleanly reorder **taste hierarchy to** $\xi_3 \lesssim 1 < \xi_1 \lesssim \xi_2$**.** Eventually $\xi \to \infty$: $2, 2$-fold **degeneracies**.

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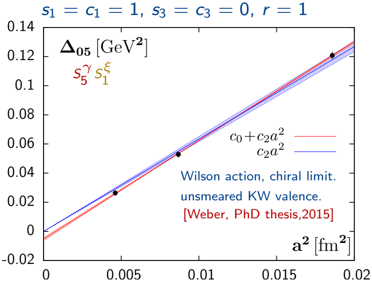
- **Anisotropy or smearing** are expected to cleanly reorder **taste hierarchy to $\xi_3 \lesssim 1 < \xi_1 \lesssim \xi_2$.** Eventually $\xi \to \infty$: $2, 2$-fold **degeneracies**.
- S_2^ξ: $r \to -r$ and S_3^ξ: $S_f^{KW} \to S_f^{KW}$: while the continuum/naive part of the KW operator is **taste singlet**, but the **extra term is taste-S_3^ξ**.
- **KW Noether currents** inevitably **mix different taste structures**.
- Conserved **vector current** $V_{\mu,xy}$ [Capitani et al., JHEP 09, 2010]:
  - $V_{\mu,xy}$, $\mu < 4$ has **taste-singlet** $S_{\mu,xy}^\gamma$ and $\frac{\mathrm{i}r}{a_\sigma}$-suppressed **taste-$S_3^\xi$**: $\gamma_4 = S_{4,xy}^\gamma S_{3,yx}^\xi$ parts (ignoring spatial shifts).
  - $V_{4,xy}$ is **taste-singlet** $S_{4,xy}^\gamma$ $\Rightarrow$ **unmixed conserved baryon charge**.
- Conserved **axial current** $A_{\mu,xy}$ [Capitani et al., JHEP 09, 2010]
  - $A_{\mu,xy}$, $\mu < 4$ has **taste-S_3^ξ**: $S_{\mu,xx}^\gamma \gamma_5 = S_{\mu,xx}^\gamma S_{5,xz}^\gamma S_{3,zx}^\xi$ and $\frac{\mathrm{i}r}{a_\sigma}$ suppressed **taste-singlet** $\gamma_4\gamma_5 = S_{4,xz}^\gamma S_{5,zx}^\gamma$ parts (ignoring spatial shifts).
  - $A_{4,xy}$ is **taste-S_3^ξ**: $S_{4,xy}^\gamma \gamma_5 = S_{4,xu}^\gamma S_{5,uv}^\gamma S_{3,vy}^\xi$ $\Rightarrow$ **unmixed Goldstone pion**.

Taste-symmetry breaking in pseudoscalar masses



- **Type 1 TSB** is mass-independent, but does not extrapolate just as a^2.

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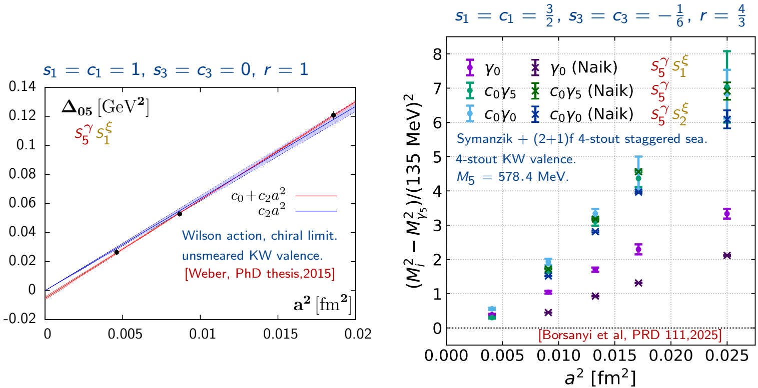

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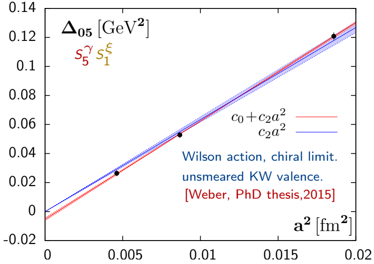
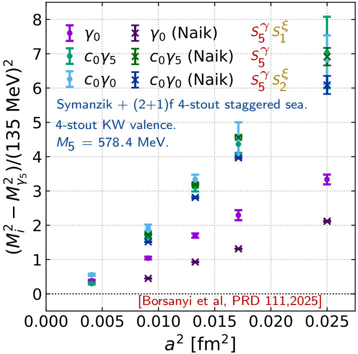
Taste-symmetry breaking in pseudoscalar masses

$s_1 = c_1 = 1,\ s_3 = c_3 = 0,\ r = 1$



$s_1 = c_1 = \frac{3}{2},\ s_3 = c_3 = -\frac{1}{6},\ r = \frac{4}{3}$



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Any	Naik:	$\xi_3 < \xi_1 < \xi_2 < 1$
Coarse	no Naik:	$\xi_3 < \xi_1 < \xi_2 \simeq 1$
Fine	no Naik:	$\xi_3 < \xi_1 < 1 < \xi_2$
Finest	no Naik:	$\xi_3 < 1 < \xi_1 < \xi_2$?