Lattice QCD for precision physics

Nikolai Husung

Introduction to lattice QCD

Continuum limit / extrapolation

Symanzik Effective Field Theory (SymEFT)

Summary and outlook

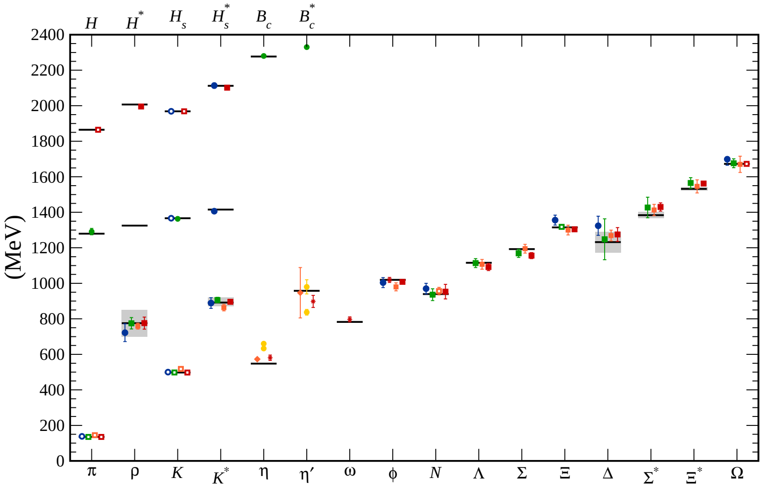
Introduction to lattice QCD

Goal: Make *non-perturbative* predictions for physical observables in (Euclidean) QCD

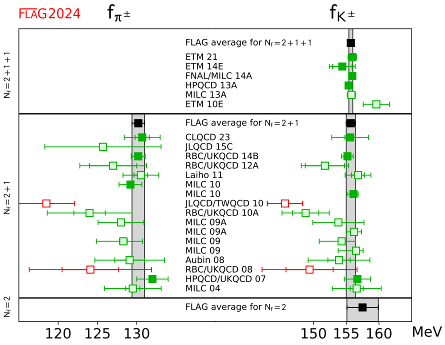
$$S_{\mathrm{QCD}} = \int \mathrm{d}^4 x \left\{ -\frac{1}{2g_0^2}\,\mathrm{tr}(F_{\mu\nu}F_{\mu\nu}) + \bar{\Psi}(\gamma_\mu D_\mu + M)\Psi \right\},$$

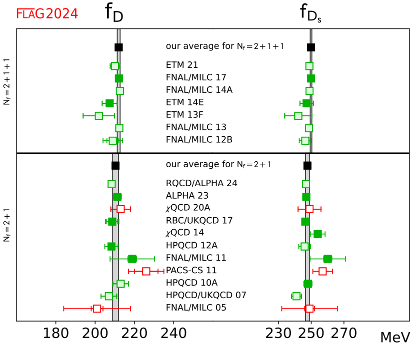
where $F_{\mu\nu} = [D_\mu, D_\nu]$ and $D_\mu = \partial_\mu + A_\mu^a T^a,\ T^a \in \mathfrak{su}(N)$.

Observables: Hadron spectrum [Kronfeld, 2012]



Observables: Pseudoscalar decay constant [Aoki et al., 2026]





$$ip_\mu f_X = \langle 0|A_\mu|X(p)\rangle$$

	Continuum	$\leftrightarrow$	Lattice
Free action	$\int \mathrm{d}^4x\, \bar{\Psi}\gamma_\mu\partial_\mu\Psi$		$a^4 \sum_x \bar{\Psi}(x)\gamma_\mu \frac{\Psi(x+a\hat{\mu}) - \Psi(x-a\hat{\mu})}{2a}$

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Gauge invariance	$\Downarrow$	$\Psi(x) \to \Omega(x)\Psi(x)$	$\Downarrow$

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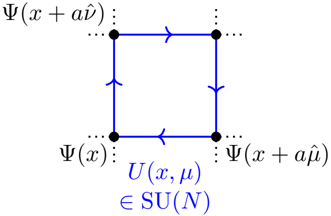
$U(x,\mu)$ transforms like a Wilson line ("gauge transporter").

The lattice spacing acts as a momentum cutoff $|p_\mu| \leq \pi/a$.



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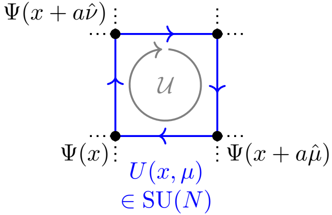
$$S_{\text{lattice}} = \frac{2}{g_0^2}\sum_x \mathrm{Re}\,\mathrm{tr}(1 - \mathcal{U}(x)) + a^4 \sum_x [\bar{\Psi}\hat{D}\Psi](x)$$

where $\hat{D}$ is a discretised lattice Dirac operator.



Simulation

Evaluate high-dimensional path integral via Markov-Chain Monte-Carlo at fixed g_0

$$Z = \int \mathcal{D}U \underbrace{\int \mathcal{D}\bar{\Psi}\mathcal{D}\Psi \, \exp(-S_{\text{lattice}}[U, \bar{\Psi}, \Psi])}_{\text{probability distribution}}$$

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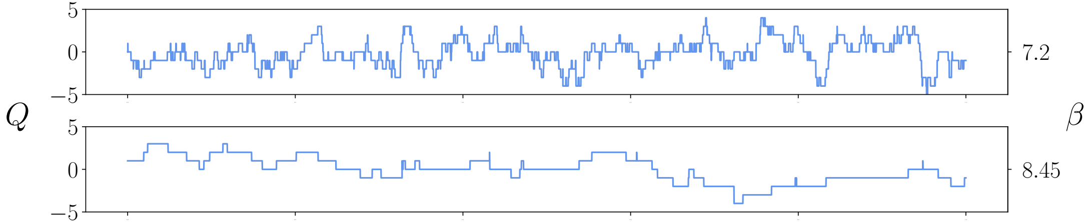
Various sources of errors, e.g.,

- **Statistical** $\propto 1/\sqrt{\#\text{MC samples}}$, but Markov-Chain introduces *autocorrelation*.

Example: Topological charge $\langle Q \rangle = 0$ U(1) model [Albandea et al., 2021]



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  $Lm_\pi \gg 4$ ensures (exponential) suppression of finite volume interactions.

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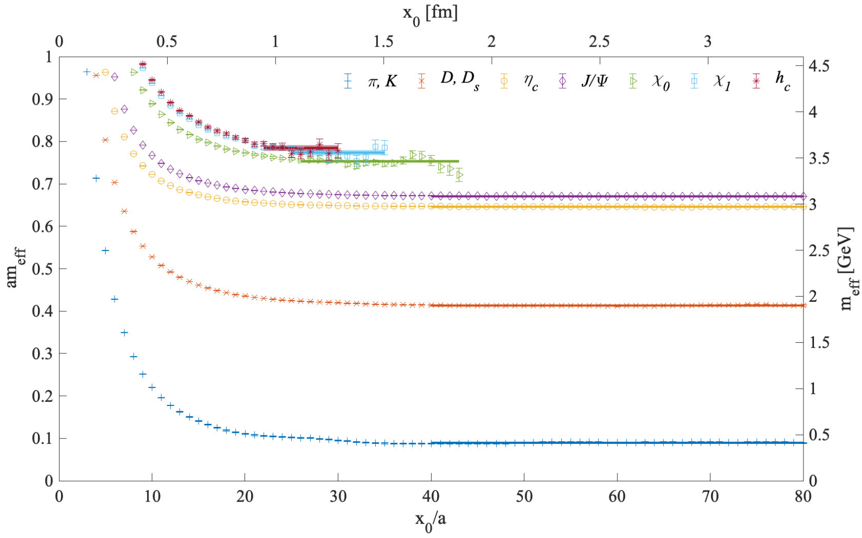

[Höllwieser et al., 2020]

Ensemble B

$N_f = 3 + 1$ QCD

Wilson quarks

144×48^3 lattice

$\beta = 3.43$

$a \approx 0.0428\,\mathrm{fm}$

open boundaries in time

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- Various **extrapolations** and **interpolations**, e.g., $m_\pi \to m_\pi^{\text{phys}}$, fixed physical distances, ... $\boldsymbol{a \searrow 0}$ All simulations are performed with regulator in place, $a > 0$.

Continuum limit / extrapolation

Lattice discretisations break symmetries

Spacetime: By construction the lattice regularisation breaks O(4) symmetry down to discrete rotations. $\rightsquigarrow$ Only 90° rotations persist on a hypercubic lattice.

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Chiral: Naively one ends up with 2^4 flavours (doublers)

$$a\tilde{\hat{D}}^{\mathrm{free}}_{\mathrm{naive}}(p) = \sum_\mu \gamma_\mu \sin(ap_\mu) \overset{!}{=} 0, \quad p \in]-\pi/a, \pi/a]^4$$

No-go theorem [Nielsen, Ninomiya, 1981]: For any local lattice Dirac operator without doublers

$$\left\{\hat{D}, \gamma_5\right\} \neq 0.$$

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$$\{\hat{D}, \gamma_5\} \neq 0.$$

$\Rightarrow$ More involved renormalisation of composite fields: energy-momentum tensor, 4-quark operators, ...

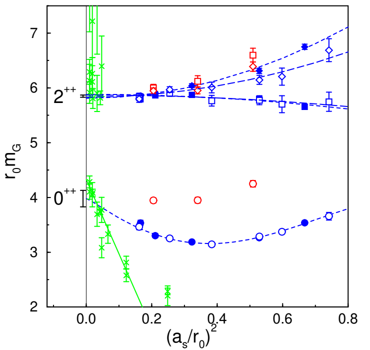
Various formulations for the discretised lattice Dirac operator, e.g.,

	Quark action	Flavour symmetries	Comment
✗	Naive [Wilson, 1974]	$\mathrm{SU}(N_\mathrm{f})_\mathrm{L} \times \mathrm{SU}(N_\mathrm{f})_\mathrm{R} \times U(1)_\mathrm{B}$	Doublers $\Rightarrow N_\mathrm{f} = 16n$
✓	Wilson [Wilson, 1975]	$\mathrm{SU}(N_\mathrm{f})_\mathrm{V} \times U(1)_\mathrm{B}$	additive mass renormalisation
✓	Ginsparg-Wilson [Ginsparg, Wilson, 1982]	$\mathrm{SU}(N_\mathrm{f})_\mathrm{L} \times \mathrm{SU}(N_\mathrm{f})_\mathrm{R} \times U(1)_\mathrm{B}$	lattice chiral symmetry [Lüscher, 1998]
✗	Staggered [Kogut, Susskind, 1975]	$\mathrm{Shift} \times U(1)_\mathrm{B} \times U(1)_\mathrm{A}$	$a > 0$: Flavour-changing. Doublers $\Rightarrow N_\mathrm{f} = 4n$
✗	Rooted Staggered [Gottlieb et al., 1987]	$\mathrm{SU}(N_\mathrm{f})_\mathrm{V} \times U(1)_\mathrm{B} \times U(1)_\mathrm{A}$	$a > 0$: Locality? Universality class?

✓: proven to be perturbatively renormalisable [Reisz, 1989; Reisz, Rothe, 2000]

Simulations at different lattice spacings

Universality: Gaussian fixed-point of lattice QCD corresponds to the continuum limit.
$\Rightarrow$ All lattice formulations yield continuum QCD if Ward identities of broken symmetries are restored by renormalisation. (conjecture!)



Plot taken from [Weisz, 2010] with data from [Morningstar, Peardon, 1997].

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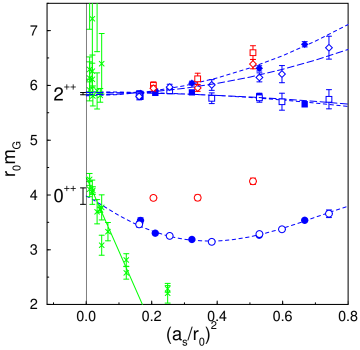
Problem: Numerical results are only known at discrete values of the lattice spacing $a > 0$. Typically $a \in [0.04, 0.11]\,\mathrm{fm}$.

Eventually we are interested in the continuum value!



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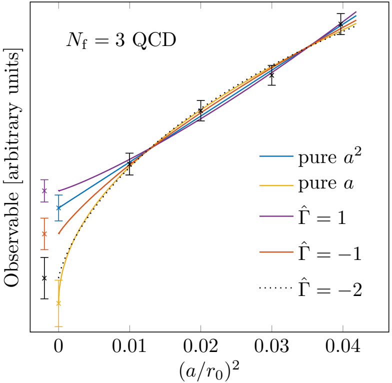
Lattice artifacts in lattice QCD



In an asymptotically free theory, like QCD, leading lattice artifacts are of the form (up to factors of $\log \bar{g}(1/a)$)

$$\frac{\mathcal{P}(a)}{\mathcal{P}(0)} = 1 + a^{n_{\min}} \sum_i [\bar{g}^2(1/a)]^{\hat{\Gamma}_i} c_i$$

$$+ \mathrm{O}(a^{n_{\min}+1}, a^{n_{\min}} \bar{g}^{2\hat{\Gamma}_i+2}(1/a), \ldots)$$

$\hat{\Gamma}_i$ can be negative and distinctly nonzero
$\Rightarrow$ impact on convergence.

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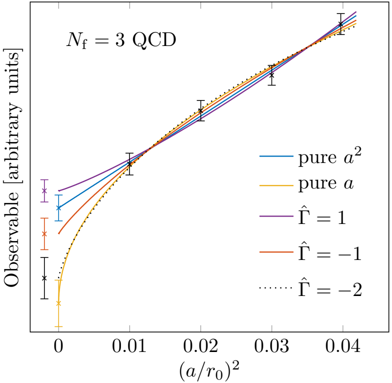

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$$\frac{\mathcal{P}(a)}{\mathcal{P}(0)} = 1 + a^{n_\mathrm{min}} \sum_i \left[\bar g^2(1/a)\right]^{\hat\Gamma_i} c_i + \mathrm{O}\left(a^{n_\mathrm{min}+1}, a^{n_\mathrm{min}} \bar g^{2\hat\Gamma_i+2}(1/a), \ldots\right)$$

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$\Rightarrow$ impact on convergence.

Warning example: 2d O(3) non-linear sigma model **$\min \hat\Gamma_i = -3$** [Balog et al., 2009, 2010]

$\Rightarrow$ **Compute $\hat\Gamma_i$ in QCD to gain better control over continuum extrapolation.**

Symanzik Effective Field Theory (SymEFT)

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Lattice action: $$\Delta S = a^{n_{\min}} \int \mathrm{d}^4 x \sum_i \bar{\omega}_i(g_0) Q_i(x) + \ldots, \quad Q = \mathcal{O} \cup \mathcal{E}$$

Discretised local field: $$\Delta J(x) = a^{n_{\min}} \sum_i \bar{\nu}_i(g_0) J_i(x) + \ldots$$

$\mathcal{O}_i$: on-shell basis (sufficient for spectral quantities) [Lüscher et al., 1996]
$\mathcal{E}_i$: EOM-vanishing basis (only needs to be tracked for local fields)
J_i: Lattice artifacts of the specific discretisation of the local field.

All operator bases are constrained by the symmetries and transformation properties realised on the lattice.

Operator basis: action

Occurring operators $\mathcal{O}_i$ must comply with symmetries of the lattice formulation, e.g. DWF $\rightsquigarrow$ Ginsparg-Wilson fermions

- Local SU(N) gauge symmetry,
- C-, P- and T-symmetry,
- broken O(4) symmetry due to reduced rotation symmetry,
- approximate $\mathrm{SU}(N_\mathrm{f})_\mathrm{L} \times \mathrm{SU}(N_\mathrm{f})_\mathrm{R} \times \mathrm{U}(1)_\mathrm{V}$ flavour symmetry for massless QCD.

Remark: Require minimal basis for physical matrix elements ("on-shell")
$\Rightarrow$ use continuum EOMs to reduce set of operators [Lüscher et al., 1996], e.g.

$$-\frac{2}{g_0^2}\,\mathrm{tr}(D_\mu F_{\mu\nu} D_\rho F_{\rho\nu}) \overset{\mathrm{EOM}}{=} \bar{\Psi}\gamma_\mu D_\nu F_{\nu\mu}\Psi \overset{\mathrm{EOM}}{=} g_0^2(\bar{\Psi}\gamma_\mu T^a \Psi)^2$$

But need to keep minimal EOM vanishing basis $\mathcal{E}$ for local fields!

(Massless) minimal operator basis

Lattice action with flavours $\Psi = (\mathrm{u}, \mathrm{d}, \mathrm{s}, \ldots)$.

pure gauge [Lüscher, Weisz, 1985] $\mathrm{O}(a)$ improved Wilson-like [Sheikholeslami, Wohlert, 1985]
Staggered [Follana et al., 2007; Husung, 2026a]

$\mathrm{O}(a)$ $\frac{i}{4}\bar{\Psi}\gamma_{\mu\nu}F_{\mu\nu}\Psi$

$\mathrm{O}(a^2)$ $\frac{1}{g_0^2}\sum_\mu \mathrm{tr}(D_\mu F_{\mu\nu} D_\mu F_{\mu\nu})$ $\sum_\mu \bar{\Psi}\gamma_\mu D_\mu^3 \Psi$ $g_0^2(\bar{\Psi}\gamma\otimes\tau\Psi)^2$ $g_0^2(\bar{\Psi}\gamma\otimes\tau T^a\Psi)^2$

$\frac{1}{g_0^2}\mathrm{tr}(D_\mu F_{\nu\rho} D_\mu F_{\nu\rho})$

$\gamma\otimes\tau \in \{\mathbb{1}, \gamma_5, \gamma_\mu, \gamma_5\gamma_\mu, i\sigma_{\mu\nu}\}\otimes\mathbb{1} \cup \{\mathbb{1}, \gamma_5, \gamma_{\rho\sigma}\}\otimes\{\tau_\mu, \tau_5\tau_\mu\}\cup\ldots$

$\Rightarrow$ **7, 12 or 35 operators at $\mathrm{O}(a^2)$** **No on-shell $\mathrm{O}(a)$ terms for Staggered action.**

(Massless) minimal operator basis

Axial vector $A_\mu^{kl} = \bar{q}_k \gamma_\mu \gamma_5 q_l$ for Wilson quarks [Husung, 2025]

Basis again constrained by transformation properties.

$\mathrm{O}(a)$	$\partial_\mu \bar{q}_k \gamma_5 q_l$	[Lüscher et al., 1996; Bhattacharya et al., 2004, 2006]	
$\mathrm{O}(a^2)$	$\bar{q}_k \gamma_\rho \tilde{F}_{\rho\mu} q_l$	$\delta_{\mu\rho\lambda} \partial_\rho^2 \mathrm{A}_\lambda^{kl}$	$\partial^2 \mathrm{A}_\mu^{kl}$
	$\frac{\delta_{kl}}{g_0^2} \mathrm{tr}(D_\rho F_{\rho\lambda} \tilde{F}_{\mu\lambda})$	$\frac{\delta_{kl}}{g_0^2} \partial_\mu \mathrm{tr}(F_{\nu\rho} \tilde{F}_{\nu\rho})$	$\frac{\delta_{kl} \delta_{\mu\nu\rho\sigma}}{g_0^2} \mathrm{tr}(D_\nu F_{\rho\lambda} \tilde{F}_{\sigma\lambda})$

Advertisement: Finding minimal bases according to canonical mass-dimension and discrete transformation properties has been automated in a `Python` package.
`https://github.com/nikolai-husung/opbasis` [Husung, 2026b]

How to proceed with minimal operator bases from SymEFT?

1. **Scaling analysis**
   - Compute 1-loop anomalous dimensions to rule out severely negative $\hat{\Gamma}_i$ for the various operators.
   - Use $\hat{\Gamma}_i$ for ansätze in continuum extrapolations.

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1. **Scaling analysis**
   - Compute 1-loop anomalous dimensions to rule out severely negative $\hat{\Gamma}_i$ for the various operators.
   - Use $\hat{\Gamma}_i$ for ansätze in continuum extrapolations.
2. **Symanzik improvement**
   - Restore symmetries at fixed order in the lattice spacing by properly tuning coefficients of "irrelevant" operators.
   - Beyond $\mathrm{O}(a)$ continuum-symmetry-preserving artifacts remain. **Ongoing work:** Restoration of O(4) symmetry at $\mathrm{O}(a^2)$ in pure gauge.
   - Special case: tree-level improvement.

Example: Hadron masses

$$\Delta S = a^{n_{\min}} \int \mathrm{d}^4 x \sum_i \bar{\omega}_i(g_0)\mathcal{O}_i(x) + \ldots$$

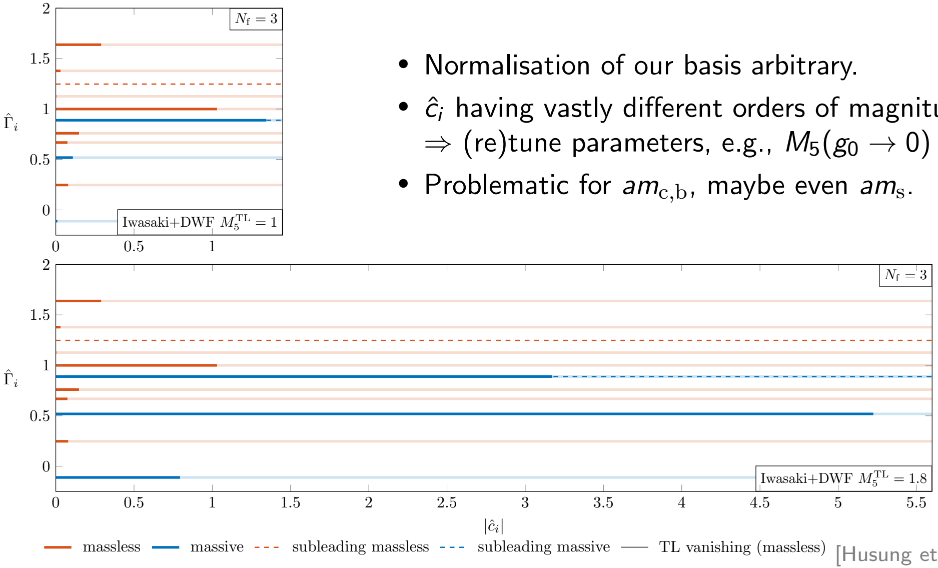
$$am_\pi(a) = \lim_{t\to\infty} \log \frac{C^\pi_{2\mathrm{pt}}(t)}{C^\pi_{2\mathrm{pt}}(t+a)} \qquad C^\pi_{2\mathrm{pt}}(t) = \langle 0|\hat{\pi}(t)\hat{\pi}^\dagger(0)|0\rangle$$

$$\frac{m_\pi(a)}{m_K(a)} = \frac{m_\pi}{m_K}\left\{1 - a^{n_{\min}} \sum_i \hat{c}_i [2b_0\bar{g}^2(1/a)]^{\hat{\Gamma}^{\mathcal{B}}_i} \left(\frac{m^\pi_{i;\mathrm{RGI}}}{m_\pi} - \frac{m^K_{i;\mathrm{RGI}}}{m_K}\right) + \ldots\right\}$$

where $\hat{\Gamma}^{\mathcal{B}}_i = (\gamma^{\mathcal{B}}_0)_i/(2b_0) + n^{\mathcal{B}}_i$ can be obtained from 1-loop running of the operators $\mathcal{O}_i$

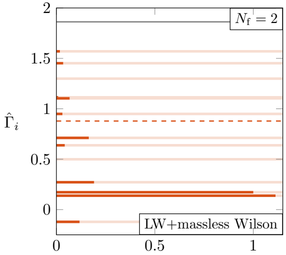
$$\mu \frac{\mathrm{d}\mathcal{O}_{i;\overline{\mathrm{MS}}}(\mu)}{\mathrm{d}\mu} = -(\gamma^{\mathcal{O}}_0)_{ik}\bar{g}^2(\mu)\mathcal{O}_{k;\overline{\mathrm{MS}}}(\mu)$$

and a change of basis $\mathcal{O} \to \mathcal{B}$ s.t. $\gamma^{\mathcal{B}}_0$ is diagonal.

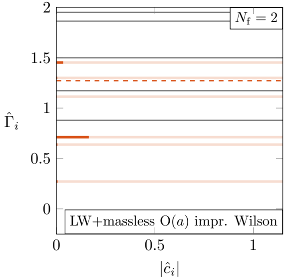


- Normalisation of our basis arbitrary.
- $\hat{c}_i$ having vastly different orders of magnitude.
  $\Rightarrow$ (re)tune parameters, e.g., $M_5(g_0 \to 0) \to 1$.
- Problematic for $am_{\mathrm{c,b}}$, maybe even am_{s}.

[Husung et al., 2022]



$\Downarrow$ Symanzik $\mathrm{O}(a)$ improvement $\Downarrow$



Continuum symmetry of massless QCD in a finite volume

$\bar{\Psi} \to i\bar{\Psi}\gamma_5\tau^j, \quad \Psi \to i\gamma_5\tau^j\Psi$ with Pauli matrix τ^j.

$S_1 = a \int d^4x \, i[\bar{\Psi}\sigma_{\mu\nu}F_{\mu\nu}\Psi](x)$

$\langle \ldots S_1 \rangle \to -\langle \ldots S_1 \rangle = 0$

$\langle \ldots S_1^2 \rangle \to +\langle \ldots S_1^2 \rangle \neq 0$

$\Rightarrow$ Double insertion of S_1 impacts $\mathrm{O}(a^2)$ lattice artifacts.

Example: Pion decay constant

$$\Delta S = a^{n_{\min}} \int \mathrm{d}^4x \sum_i \bar{\omega}_i(g_0) Q_i(x) + \ldots, \quad Q = \mathcal{O} \cup \mathcal{E}$$

$$\Delta J(x) = a^{n_{\min}} \sum_i \bar{\nu}_i(g_0) J_i(x) + \ldots$$

$$\frac{Z_{\mathrm{A}}(a\mu)\langle 0|\mathrm{A}_0(x;a)|\pi(\mathbf{0})\rangle}{\lim_{a'\searrow 0}[m_\pi f_\pi](a')} = 1 + a^{n_{\min}} \sum_i d_i \frac{\langle 0|(\mathrm{A}_0)_{i;\overline{\mathrm{MS}}}(x)|\pi(\mathbf{0})\rangle}{\lim_{a'\searrow 0}[m_\pi f_\pi](a')} + \left(\begin{smallmatrix}\text{corrections}\\ \text{from } Z_{\mathrm{A}}\end{smallmatrix}\right)$$

$$- a^{n_{\min}} \sum_i c_i \int \mathrm{d}^4y \frac{\langle 0|\mathrm{A}_0(x) Q_{i;\overline{\mathrm{MS}}}(y)|\pi(\mathbf{0})\rangle_c}{\lim_{a'\searrow 0}[m_\pi f_\pi](a')} + \ldots$$

(Contact-terms will affect matching coefficients d_i)

Example: Pion decay constant

$$\Delta S = a^{n_{\min}} \int \mathrm{d}^4x \sum_i \bar{\omega}_i(g_0) Q_i(x) + \ldots, \quad Q = \mathcal{O} \cup \mathcal{E}$$

$$\Delta J(x) = a^{n_{\min}} \sum_i \bar{\nu}_i(g_0) J_i(x) + \ldots$$

$$\frac{Z_{\mathrm{A}}(a\mu)\langle 0|\mathrm{A}_0(x;a)|\pi(\mathbf{0})\rangle}{\lim_{a'\searrow 0}[m_\pi f_\pi](a')} = 1 + a^{n_{\min}} \sum_i \hat{d}_i [2b_0\bar{g}^2(1/a)]^{\hat{\Gamma}_i^{\mathrm{A}}} \frac{\langle 0|(\mathcal{A}_0)_{i;\mathrm{RGI}}(x)|\pi(\mathbf{0})\rangle}{\lim_{a'\searrow 0}[m_\pi f_\pi](a')} + \binom{\text{corrections}}{\text{from } Z_{\mathrm{A}}}$$

$$- a^{n_{\min}} \sum_i \hat{c}_i [2b_0\bar{g}^2(1/a)]^{\hat{\Gamma}_i^{\mathcal{B}}} \int \mathrm{d}^4y \frac{\langle 0|\mathrm{A}_0(x)\mathcal{B}_{i;\mathrm{RGI}}(y)|\pi(\mathbf{0})\rangle_{c\,\text{contact div. subtracted}}}{\lim_{a'\searrow 0}[m_\pi f_\pi](a')} + \ldots$$

$(\mathrm{A}_0)_i \rightarrow (\mathcal{A}_0)_i$ s.t. $\hat{\Gamma}_i^{\mathrm{A}} = \frac{(\gamma_0^{\mathcal{A}})_i}{2b_0} + n_i^{\mathrm{A}}$

Fermion bilinears: Leading powers

massive only if $k = l$

$\mathcal{J}^{kl}$ at $\mathrm{O}(a)$	N_f	$\hat{\gamma}^{\mathcal{J}}$
scalar ($k \neq l$ only)	2	0.414
	3	0.444
pseudo-scalar	2	−0.586, 0.414
	3	−0.556, 0.444
vector	2	0.138, 0.414
	3	0.148, 0.444
axial-vector	2	−0.414, 0.414
	3	−0.444, 0.444
tensor	2	−0.138, 0.414
	3	−0.148, 0.444

$+$ **Powers from SymEFT action!** [Husung et al., 2022; Husung, 2023]

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$\mathcal{J}^{kl} \sim \frac{\delta_{kl}}{g_0^2} \mathrm{tr}(F_{\mu\nu}\tilde{F}_{\mu\nu})$

Contact term with $i\bar{\Psi}\sigma_{\mu\nu}F_{\mu\nu}\Psi$ gives rise to TL contributions.

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$\mathcal{J}^{kl}_\mu \sim \partial_\mu \mathrm{P}^{kl}$, $\mathcal{J}^{kl}_{\mu\nu} \sim \partial_\mu \mathrm{V}^{kl}_\nu - \partial_\nu \mathrm{V}^{kl}_\mu$
Likely present at TL for discretisations spanning multiple lattice sites.

Opposite chirality.
$\Rightarrow$ Suppressed for light quarks **in finite volume**. May still impact $\mathrm{O}(a^2)$.

$\mathcal{J}^{kl}_{\mu\nu}$ can arise at TL via contact terms with $i\bar{\Psi}\sigma_{\mu\nu}F_{\mu\nu}\Psi$.

+ Powers from SymEFT action! [Husung et al., 2022; Husung, 2023]

Fermion bilinears: Leading powers

massive only if $k = l$ massive, only if $k \neq l$ and non-degenerate

$\mathcal{J}^{kl}$ at $\mathrm{O}(a^2)$	N_f	$\hat{\gamma}^{\mathcal{J}}$
scalar ($k \neq l$ only)	2	0, 0.483, 0.828
	3	0, 0.519, 0.889
pseudo-scalar	2	−0.172, 0, 0.483, 0.828
	3	−0.111, 0, 0.519, 0.889
vector	2	0, 0.368, 0.552, 0.575, 0.828
	3	0, 0.395, 0.593, 0.617, 0.889
axial-vector	2	−1, 0, 0.368, 0.506, 0.552, 0.559, 0.575, 0.828, 1.085
	3	−1, 0, 0.395, 0.593, 0.595, 0.617, 0.889, 1.244
tensor	2	0, 0.276, 0.46, 0.563, 0.69, 0.828
	3	0, 0.296, 0.494, 0.605, 0.741, 0.889

+ Powers from SymEFT action! [Husung et al., 2022; Husung, 2023]

Summary and outlook

- Are available lattice spacings sufficiently small? Typically $\alpha_{\overline{\mathsf{MS}}}(1/a) \in [0.21, 0.32]$
BUT: $am_{\mathrm{c;RGI}} \gtrsim 0.3$, $am_{\mathrm{b;RGI}} \gtrsim 1.4$.

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  BUT: $am_{\mathrm{c;RGI}} \gtrsim 0.3$, $am_{\mathrm{b;RGI}} \gtrsim 1.4$.
- Leading powers for Wilson and GW quarks (massless, $N_\mathrm{f} = 0, 2, 3$, J bilinear):

$$\mathrm{O}(a)\begin{cases} \hat{\Gamma}_i^{\mathcal{B}} \gtrsim 0.07 \\ \hat{\Gamma}_i^{J^{k\neq l}} \gtrsim -0.5 \\ \hat{\Gamma}_i^{J^{k=l}} \gtrsim -0.6 \end{cases} \quad \mathrm{O}(a^2)\begin{cases} \hat{\Gamma}_i^{\mathcal{B}} \gtrsim 0.2\,(-0.2\text{ w/o SW-term}) \\ \hat{\Gamma}_i^{J^{k\neq l}} \geq 0 \\ \hat{\Gamma}_i^{J^{k=l}} \geq -1 \end{cases} \Rightarrow \min \hat{\Gamma}_i \gg -3$$

- **Hierarchy in matching coefficients** can hint to better parameter tuning.
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- **Hierarchy in matching coefficients** can hint to better parameter tuning.
- Explicit Symanzik improvement is favourable over automatic improvement.
- Proposed minimal strategy assuming $\mathrm{O}(a^2)$:
  - Add ansätze $a^2[2b_0\bar{g}^2(1/a)]^{\hat{\Gamma}}$ with $\hat{\Gamma} \in \{\hat{\Gamma}_i\}$.
  - Try **also** higher order corrections a^3 or a^4.

Caveat: Results only applicable to on-shell matrix elements. Insufficient for, e.g., RI-(S)MOM type renormalisation schemes or integrated correlation functions.

	Quark action	Flavour symmetries	Comment / Scaling analysis
✓	Naive [Wilson, 1974]	$\mathrm{SU}(N_\mathrm{f})_\mathrm{L} \times \mathrm{SU}(N_\mathrm{f})_\mathrm{R} \times \mathrm{U}(1)_\mathrm{B}$	GW w/ 16 degenerate flavours
✓	Wilson [Wilson, 1975]	$\mathrm{SU}(N_\mathrm{f})_\mathrm{V} \times \mathrm{U}(1)_\mathrm{B}$	[Husung et al., 2020; Husung, 2023]
✓	Ginsparg-Wilson [Ginsparg, Wilson, 1982]	$\mathrm{SU}(N_\mathrm{f})_\mathrm{L} \times \mathrm{SU}(N_\mathrm{f})_\mathrm{R} \times \mathrm{U}(1)_\mathrm{B}$	[Husung, 2023]
✓	Staggered [Kogut, Susskind, 1975]	$\mathrm{Shift} \times \mathrm{U}(1)_\mathrm{B} \times \mathrm{U}(1)_\mathrm{A}$	[Husung, 2026a]
✗	Rooted Staggered [Gottlieb et al., 1987]	$\mathrm{SU}(N_\mathrm{f})_\mathrm{V} \times \mathrm{U}(1)_\mathrm{B} \times \mathrm{U}(1)_\mathrm{A}$	$a > 0$: Locality? Universality class?

✓: SymEFT asymptotic-scaling analysis available

Outlook (ongoing)

- Calculation of 1-loop anomalous dimensions[1] and finding minimal bases[2] mostly automated.
- Possible extensions of the SymEFT analysis:
  - Gradient flow for full QCD (also flowed fermions?).
  - QCD+QED: enlarges the minimal operator basis further.
  - Other local fields, e.g., energy-momentum tensor.
  - Static quarks?
- **Non-perturbative** restoration of O(4) symmetry at $\mathrm{O}(a^2)$.
- SymEFT not directly applicable to contact terms in integrated correlation functions.
  $\rightsquigarrow$ Establish how to incorporate **finite** contact terms.

[1] https://github.com/nikolai-husung/Symanzik-QCD-workflow
[2] https://github.com/nikolai-husung/opbasis

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