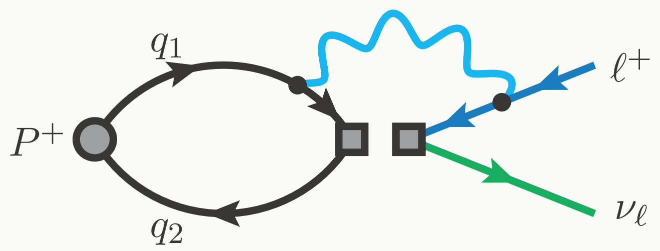
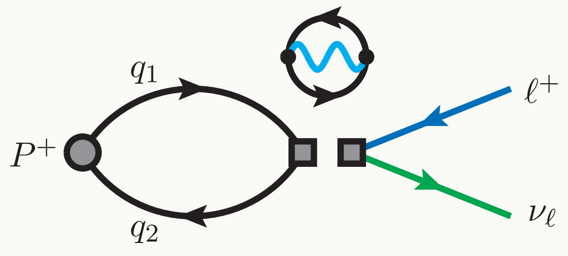
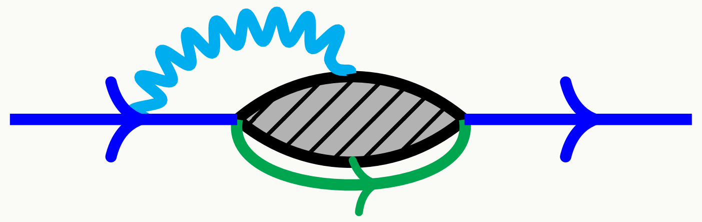
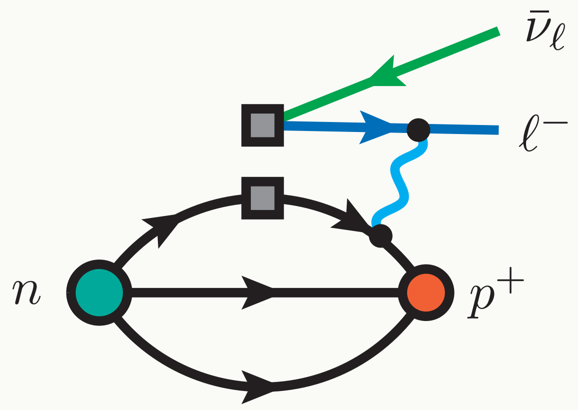
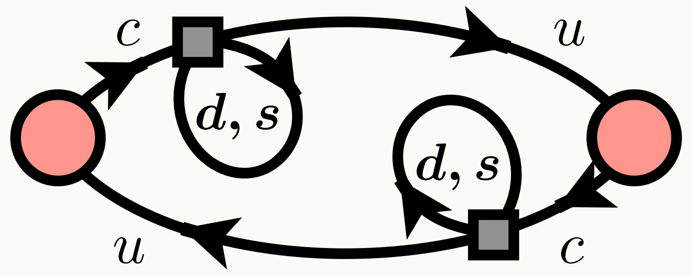










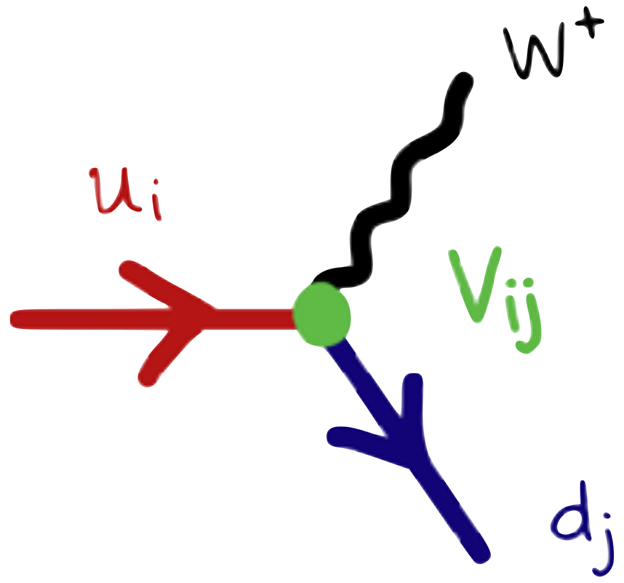
Long-distance hadronic dynamics for precision flavour phenomenology

Matteo Di Carlo

8th June 2026

Flavour physics

Flavour physics offers opportunities to test the Standard Model and probe new physics effects



$$V_{\mathrm{CKM}} = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix}$$

in the Standard Model:

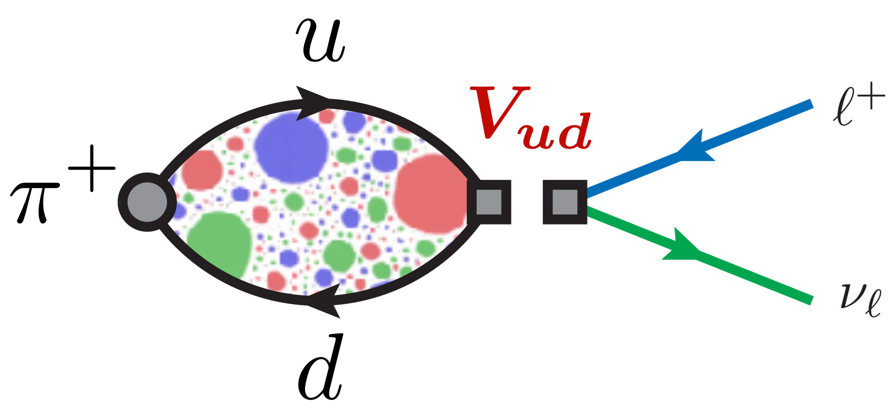
3 mixing angles + **1** CPV phase

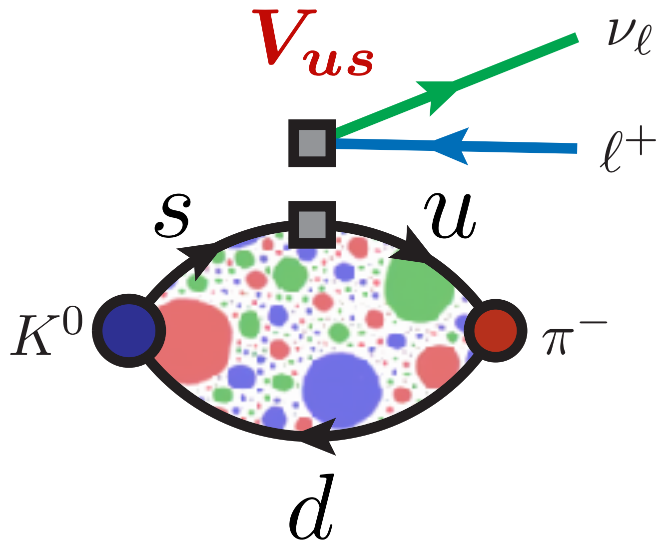
$$|V_{ud}|^2 + |V_{us}|^2 + |V_{ub}|^2 = 1$$

Matrix elements can be extracted e.g. from leptonic and semileptonic decays of hadrons

$$\underbrace{\frac{\Gamma[K \to \ell\nu_\ell(\gamma)]}{\Gamma[\pi \to \ell\nu_\ell(\gamma)]}}_{\text{experiments}} \propto \left|\frac{V_{us}}{V_{ud}}\right|^2 \underbrace{\left(\frac{f_K}{f_\pi}\right)^2}_{\text{QCD}}$$

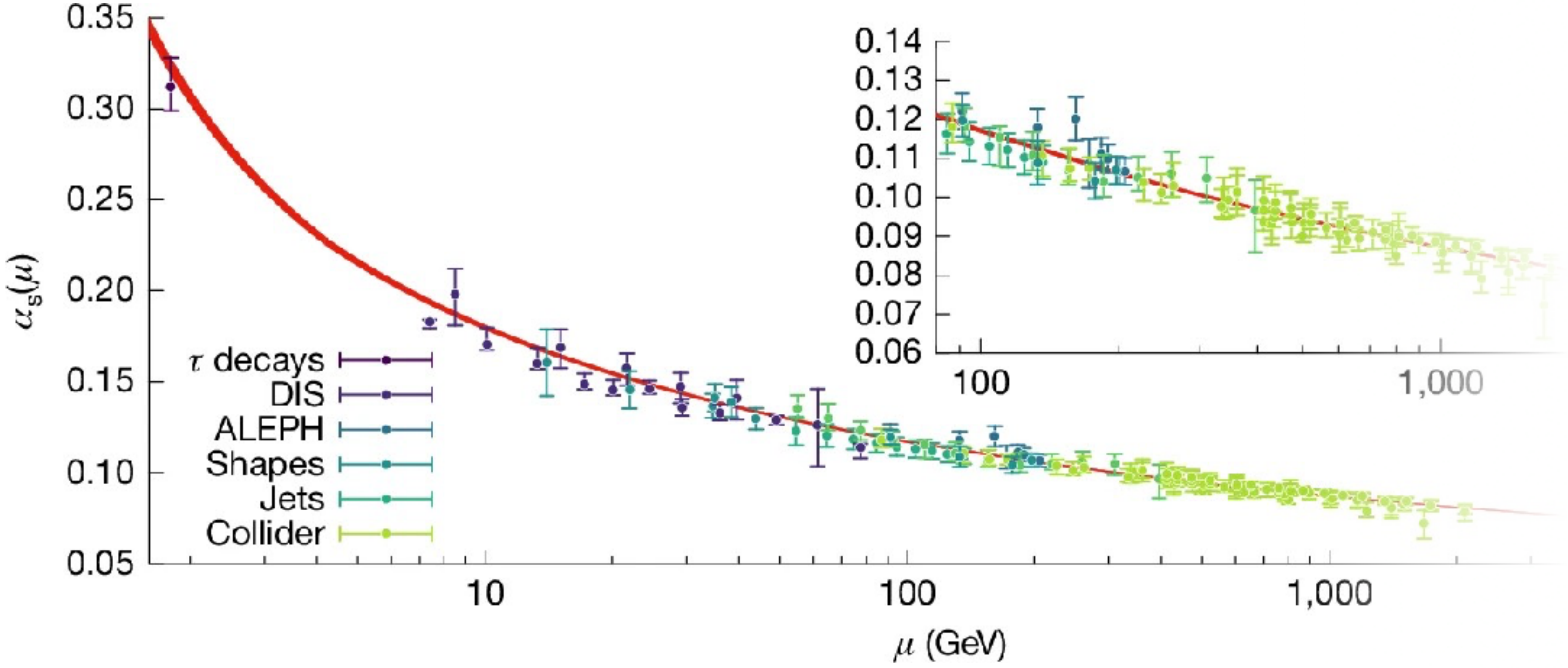
$$\underbrace{\Gamma[K \to \pi\ell\nu_\ell(\gamma)]}_{\text{experiments}} \propto |V_{us}|^2 \underbrace{|f_+^{K\pi}(0)|^2}_{\text{QCD}}$$





The strong coupling constant

The strong coupling constant $\alpha_s(Q^2)$ runs with the energy Q



M.Dalla Brida et al., Nature, 652 328-334 (2026)

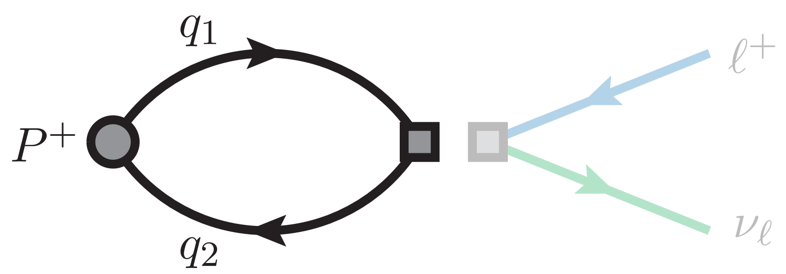
- At high energies $Q \sim m_Z$ the coupling is small:
  - perturbative expansion
  - **asymptotically free** quarks
- At small energies $Q \sim \Lambda_{\rm QCD}$ the coupling is strong:
  - non-perturbative
  - **confined** quarks

➜ **Lattice QCD**

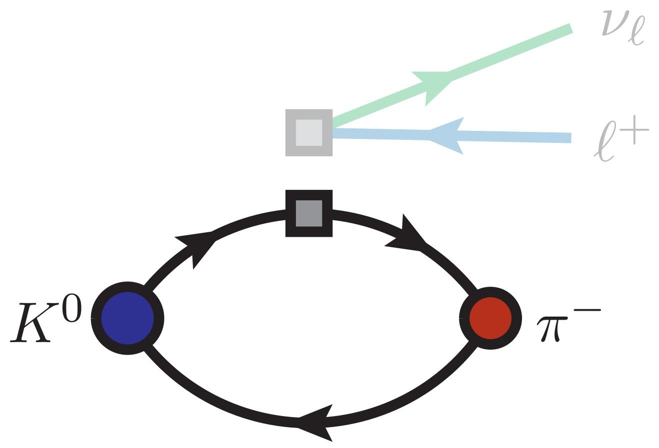
Lattice QCD

Novel frontiers

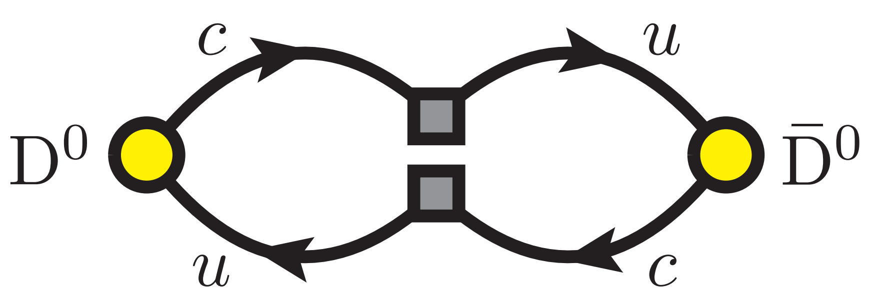
- Lattice QCD calculations are mature for standard **local hadronic matrix elements**:



$\langle 0|\bar{q}_2\gamma_\mu\gamma_5 q_1|P^+(p)\rangle$



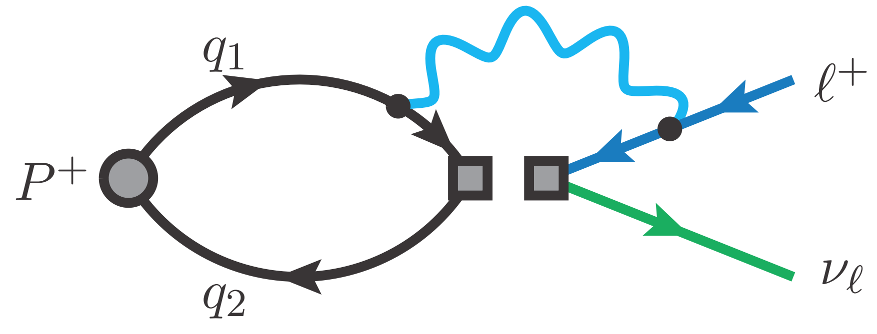
$\langle \pi(p_\pi)|\bar{s}\gamma_\mu u|K(p_K)\rangle$



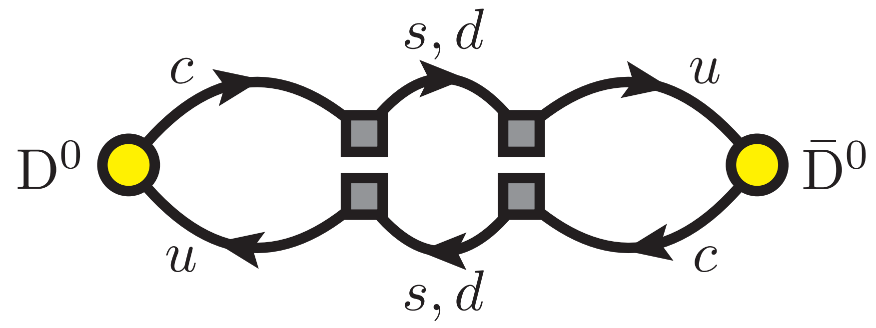
$\langle \bar{D}^0(p)|\mathcal{H}_{\mathrm{w}}^{\Delta C=2}|D^0(p)\rangle$

- Beyond local matrix elements: **long-distance processes**

QED corrections



Multi-hadron interactions



Cabibbo-Kobayashi-Maskawa matrix

$$V_{\mathrm{CKM}} = \begin{pmatrix} 0^+ \to 0^+ \ \beta \\ \pi_{\ell 2},\, \pi_{e3} & K_{\ell 2},\, K_{\mu 3}, \\ \tau \text{ decays} & B \to \tau\nu_\tau, \\ B \text{ semilept.} \\ D \to \ell\nu_\ell, \\ D \to \pi\ell\nu_\ell & D_s \to \ell\nu_\ell, \\ D \to K\ell\nu_\ell & B \text{ semilept.} \\ \text{meson mixing} & / \text{ rare decays} & / \ \ldots \end{pmatrix}$$

Cabibbo-Kobayashi-Maskawa matrix

In this talk...

1. Cabibbo anomaly

$$V_{\mathrm{CKM}} = \begin{pmatrix} 1-\frac{1}{2}\lambda^2 & \lambda & A\lambda^3(\rho-i\eta) \\ -\lambda & 1-\frac{1}{2}\lambda^2 & A\lambda^2 \\ A\lambda^3(1-\rho-i\eta) & -A\lambda^2 & 1 \end{pmatrix}$$

Cabibbo-Kobayashi-Maskawa matrix

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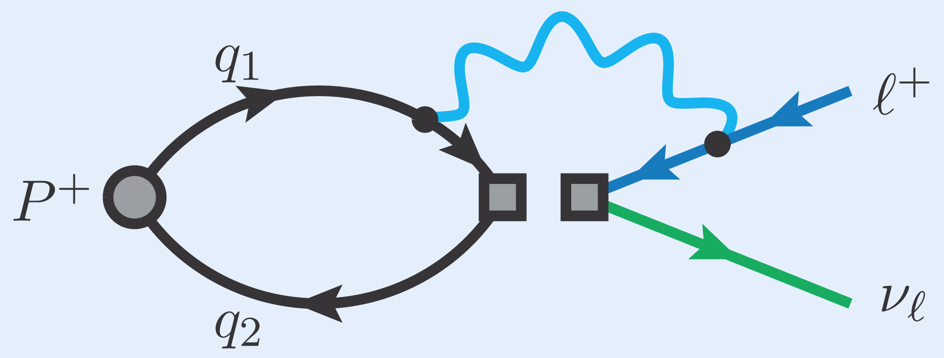
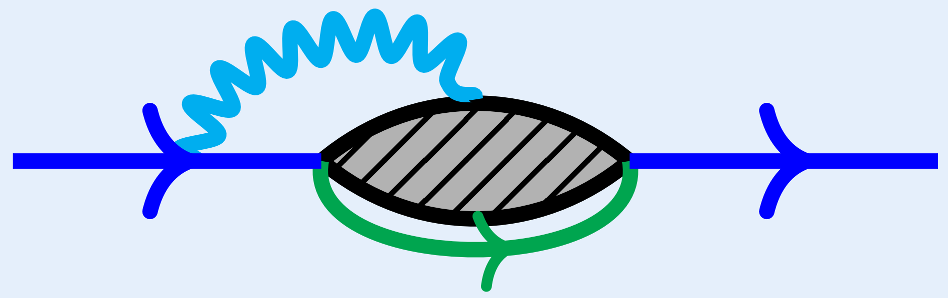
2. Neutral D-meson mixing

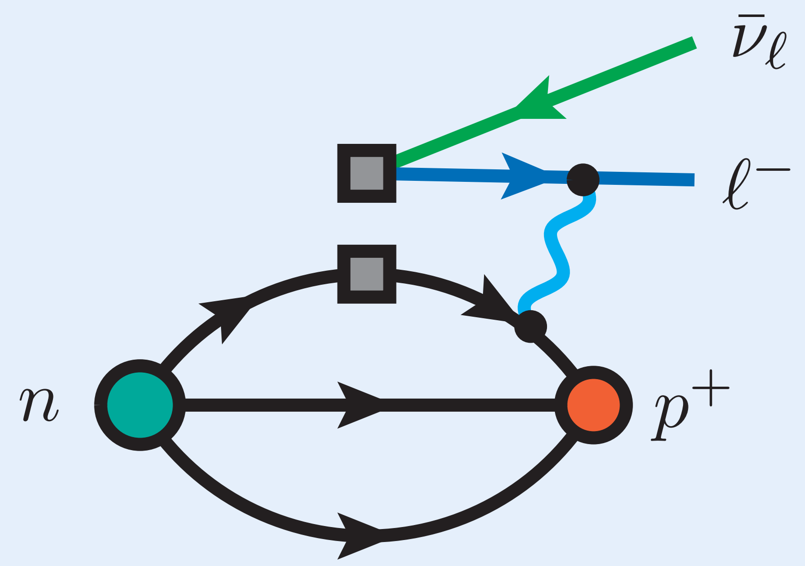
1. Cabibbo anomaly



Leptonic decays of hadrons



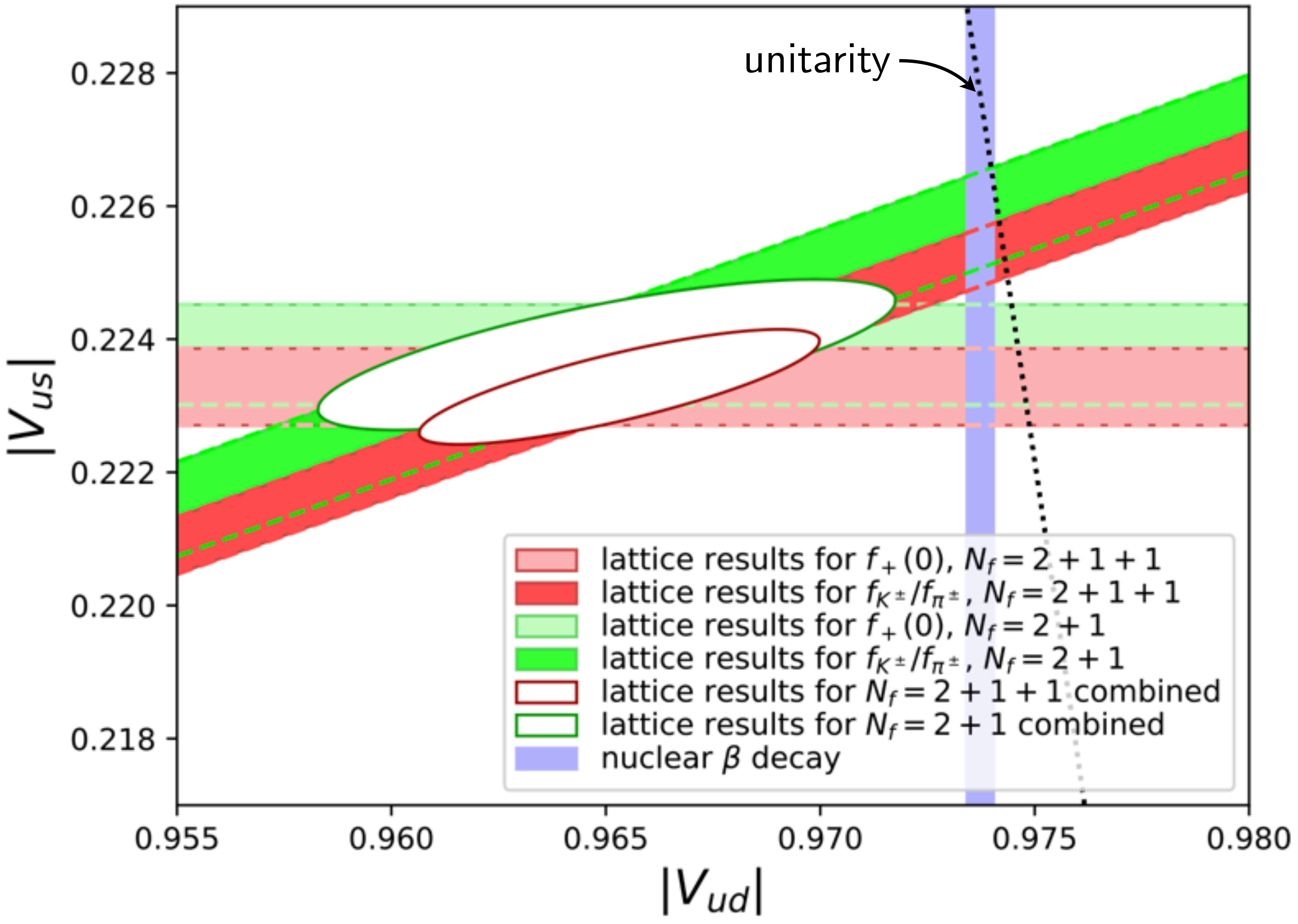
Hadronic decays of leptons



Semi-leptonic decays of hadrons

The Cabibbo "anomaly"

FLAG2024 FLAG Review 2024, [arXiv:2411.04268]



$$\frac{|V_{us}|}{|V_{ud}|}\frac{f_{K^\pm}}{f_{\pi^\pm}} = 0.27599(41)$$

M.Moulson, PoS CKM2016 (2017)
PDG, PTET 2022 (2022)

$$|V_{us}|\,|f_+^{K^0\pi^-}(0)| = 0.21654(41)$$

Different **tensions** in the V_{us}-V_{ud} plane:

$$|V_u|^2 - 1 = 2.8\sigma$$

$$|V_u|^2 - 1 = 3.1\sigma \qquad |V_u|^2 - 1 = 1.7\sigma$$

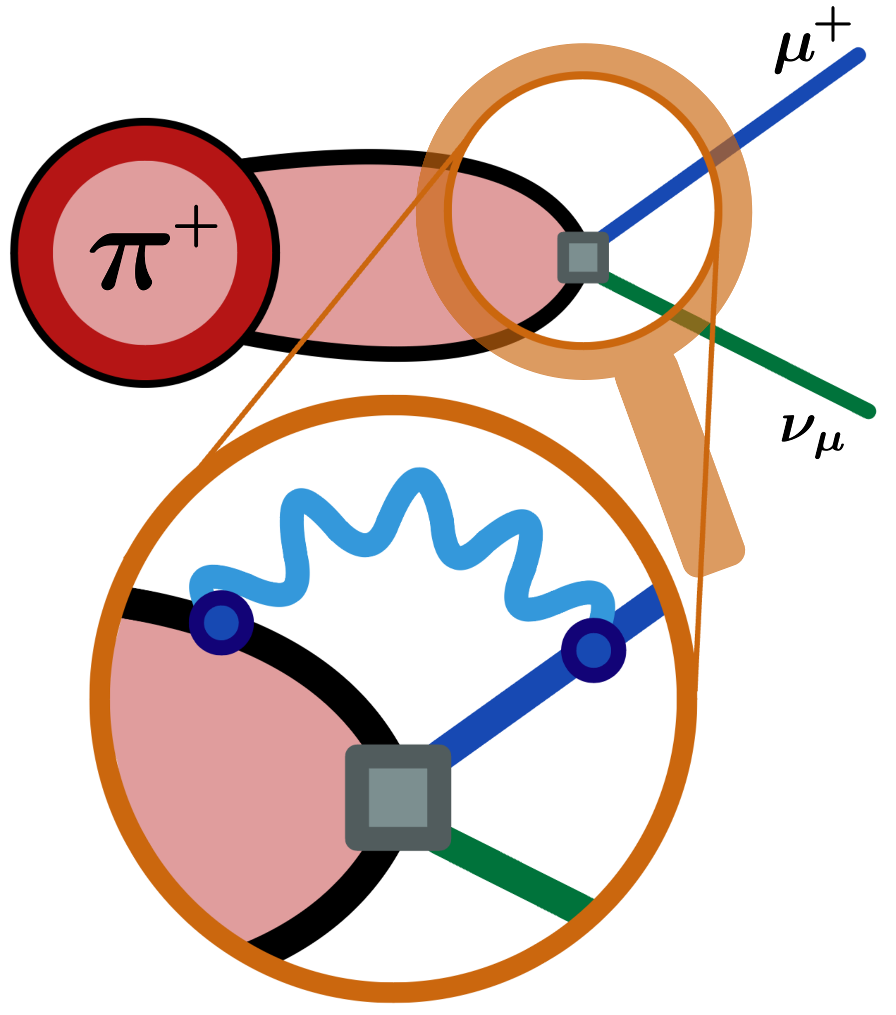
Lattice inputs ($Nf = 2+1+1$)

$$f_{K^\pm}/f_{\pi^\pm} = \mathbf{1.1934\,(19)}$$

$$f_+^{K\pi}(0) = \mathbf{0.9698\,(17)}$$

Such level of precision requires careful treatment of **QED and strong isospin-breaking** effects!

QED and isospin-breaking effects



Current level of precision requires the inclusion of isospin-breaking corrections due to

- **strong** effects $[m_u - m_d]_{\mathrm{QCD}} \neq 0$
- **electromagnetic** effects $\alpha \neq 0$

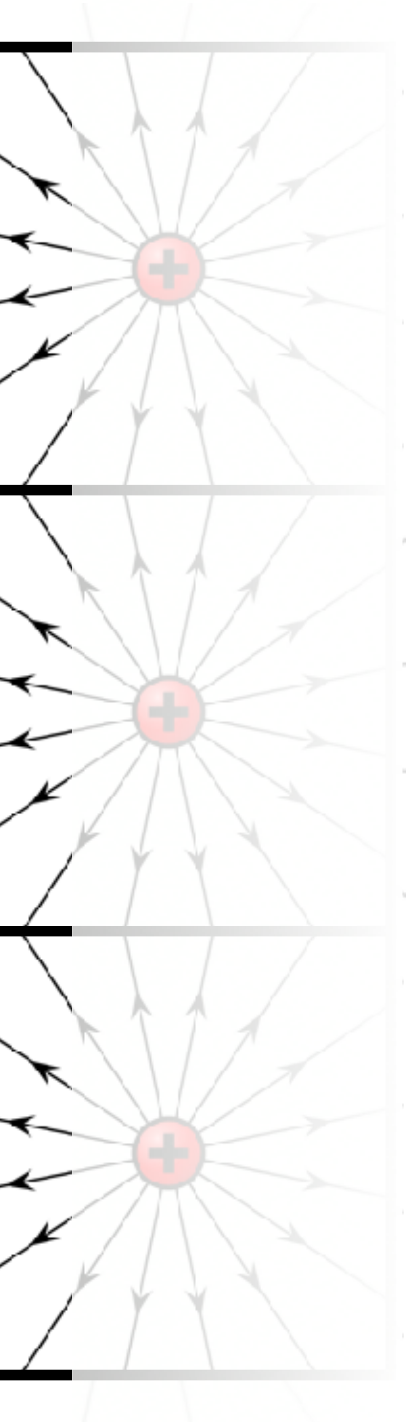
$\sim \mathcal{O}(1\%)$

$$\frac{\Gamma(K \to \ell\nu_\ell)}{\Gamma(\pi \to \ell\nu_\ell)} \propto \frac{|V_{us}|^2}{|V_{ud}|^2} \left(\frac{f_K}{f_\pi}\right)^2 (1 + \delta R_{K\pi}) \qquad \Gamma(K \to \pi\ell\nu_\ell) \propto |V_{us}|^2 \, |f_+^{K\pi}(0)|^2 \, \mathcal{I}_{K\pi}^\ell \, (1 + \delta R_{K\pi}^\ell)$$

- results currently quoted in the PDG come from χ**PT** V.Cirigliano & H.Neufeld, PLB 700 (2011)
- **non-perturbative** and **structure-dependent** quantities
- first-principles **lattice calculations** are possible

Lattice QCD + QED

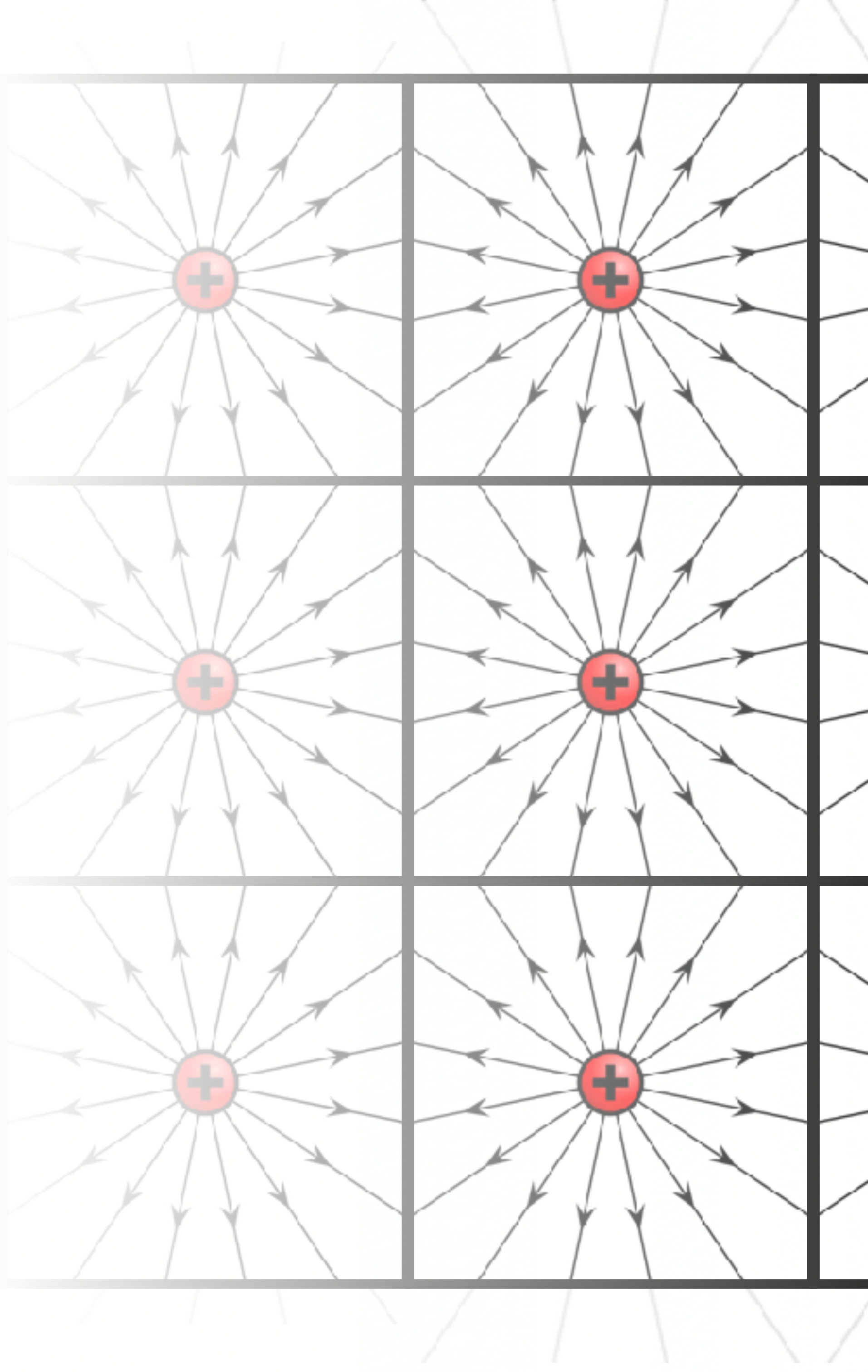
Computing **QED corrections on a finite-sized lattice** is challenging:



- **long-range interactions** don't like **finite volumes** with **periodic boundary conditions**
- **finite-volume effects** can be sizeable and power-like

  M.Hayakawa & S.Uno, PTP 120 (2008) / Z.Davoudi & M.Savage, PRD 90 (2014) / S.Borsanyi et al., Science 347 (2015)
- logarithmic **infrared divergences** arise in virtual/real decay rates

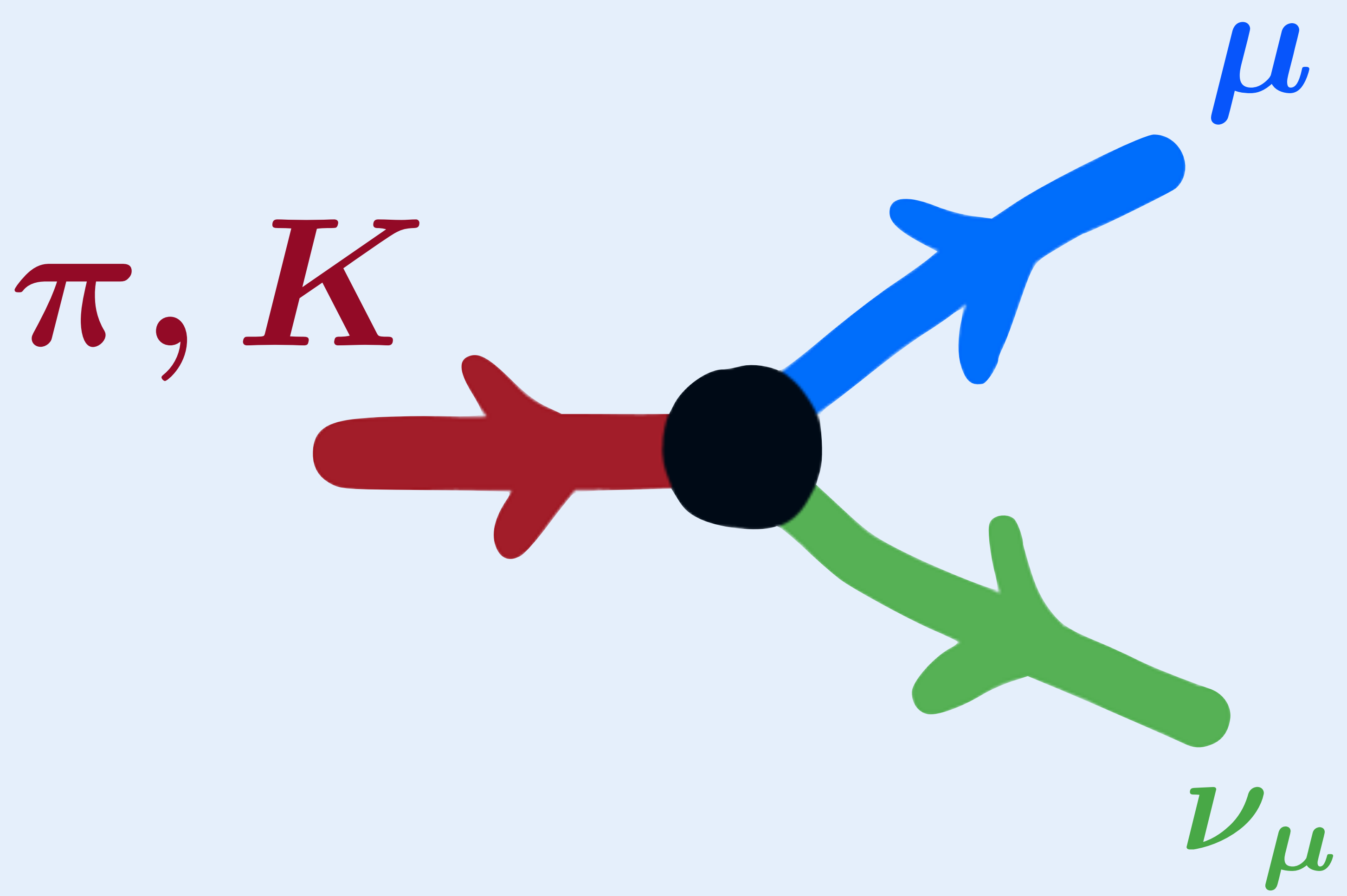
  V.Lubicz et al., PRD 95 (2017)



Problems well studied. **Different lattice QED formulations** proposed and used.

Analytical **calculations of finite-volume effects** possible in many cases.

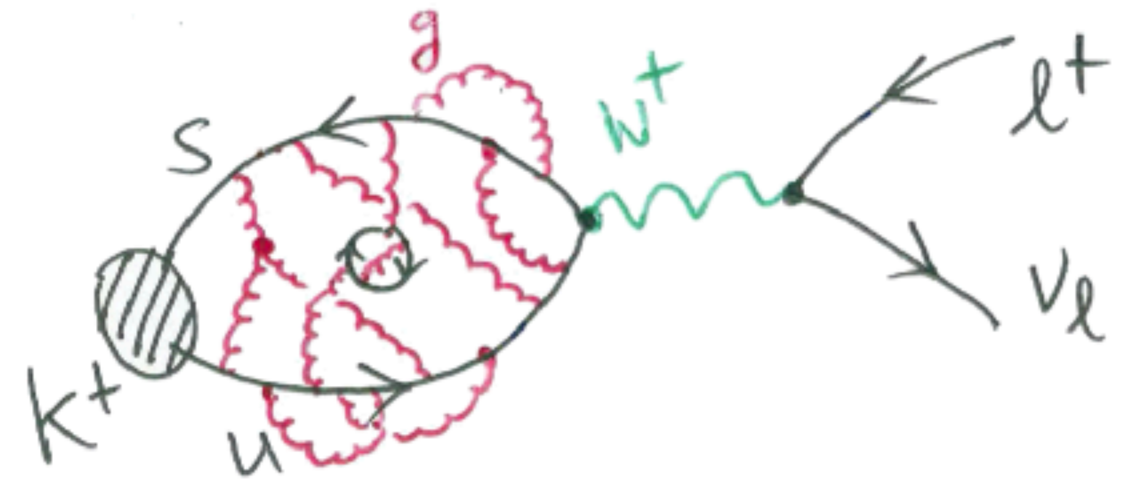
Leptonic decays of hadrons



MDC et al., PRD 100 (2019)
P.Boyle, MDC et al., JHEP 02 (2023)

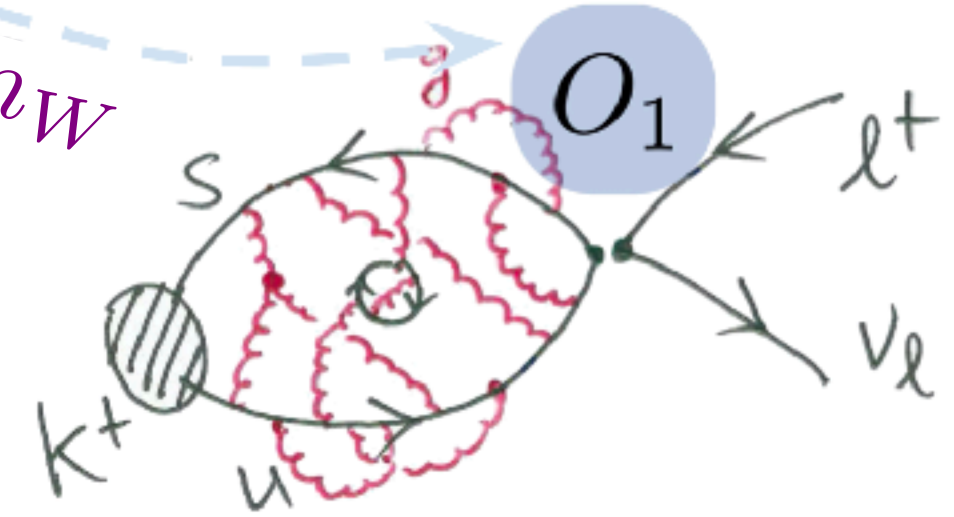
Leptonic decays of pseudoscalar mesons

Can be studied in an **effective Fermi theory** with the W-boson integrated out and the local interaction described by



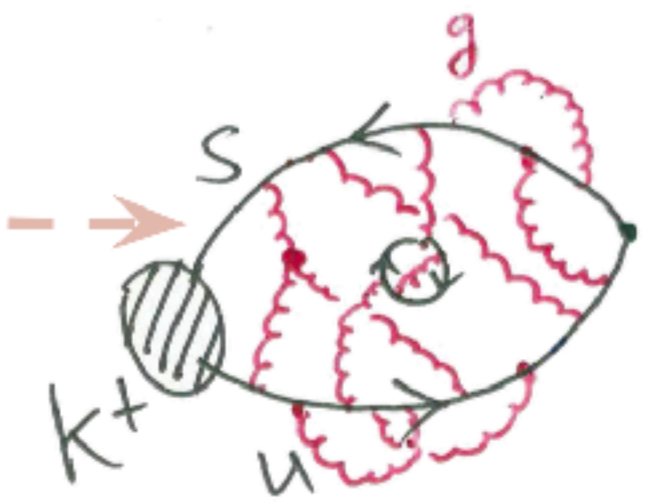
$$\mathcal{H}_{\text{eff}} = \frac{G_F}{\sqrt{2}} V^*_{q_1 q_2} \left[\bar{q}_2\, \gamma_\mu (1-\gamma_5)\, q_1\right] \left[\bar{\nu}_\ell\, \gamma^\mu (1-\gamma_5)\, \ell\right]$$

$1/a \ll m_W$



In the **PDG convention**, the tree-level decay rate takes the form

$$\Gamma_P^{\text{tree}} = \frac{G_F^2}{8\pi} m_\ell^2 \left(1 - \frac{m_\ell^2}{m_P^2}\right)^2 m_P \left[f_{P,0}\right]^2$$



with the non-perturbative dynamics encoded in the **decay constant**

$$\mathcal{Z}_0 \langle 0 | \bar{q}_2\, \gamma_0 \gamma_5\, q_1 | P, \mathbf{0} \rangle^{(0)} = i\, m_{P,0}\, f_{P,0}$$

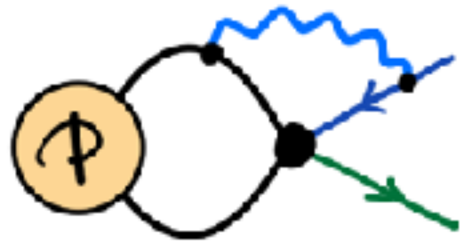
Leptonic decay rate at $\mathcal{O}(\alpha)$

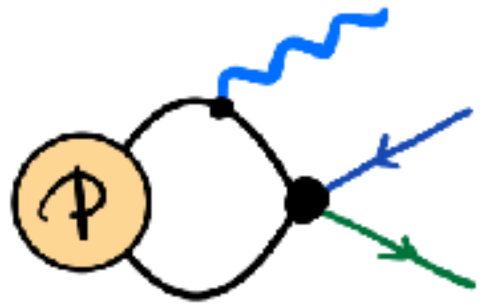
When including radiative corrections, a number of subtleties arise:

- The **decay constant** $f_{P,0}$ becomes an ambiguous and unphysical quantity: a **scheme** is required to give a meaning to "QCD"

- **IR divergences** appear in intermediate steps of the calculation

F. Bloch & A. Nordsieck, PR 52 (1937)
N. Carrasco et al., PRD 91 (2015)

$$\Gamma(P_{\ell 2}) = \lim_{\Lambda_{\mathrm{IR}}\to 0} \Big\{$$


$+$


$$\Big\}$$

IR finite — IR divergent — IR divergent

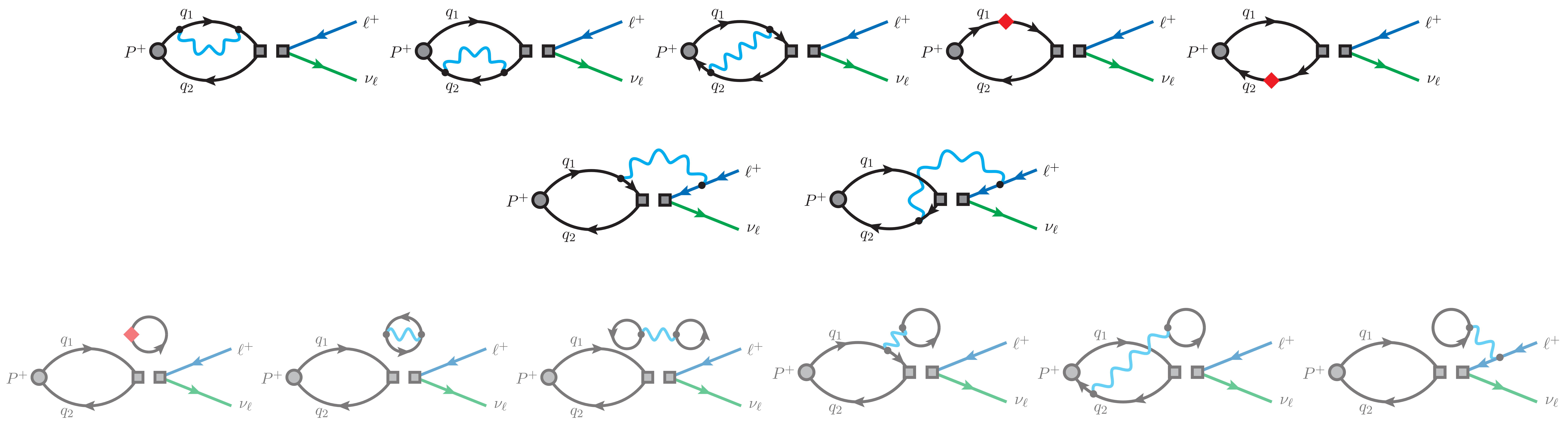
- **UV divergences**: need to include QED in the renormalization of the weak Hamiltonian

MDC et al., PRD 100 (2019)
M.Gorbahn et al., JHEP 01 (2023) / F.Moretti et al., [arXiv:2510.27648] / P.A.Boyle et al., [arXiv:2606.04729]

IB corrections to the decay amplitude

G.M.de Divitiis et al. [RM123], PRD 87 (2013)

Correlation functions in RM123 approach

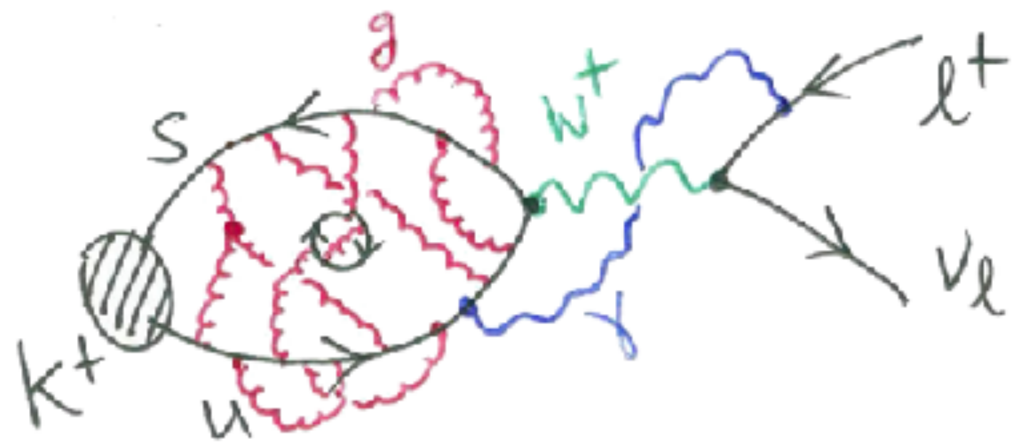


Current calculations have been performed in the **electro-quenched approximation (sea quarks electrically neutral).**

Work in progress to compute the remaining diagrams by different collaborations.

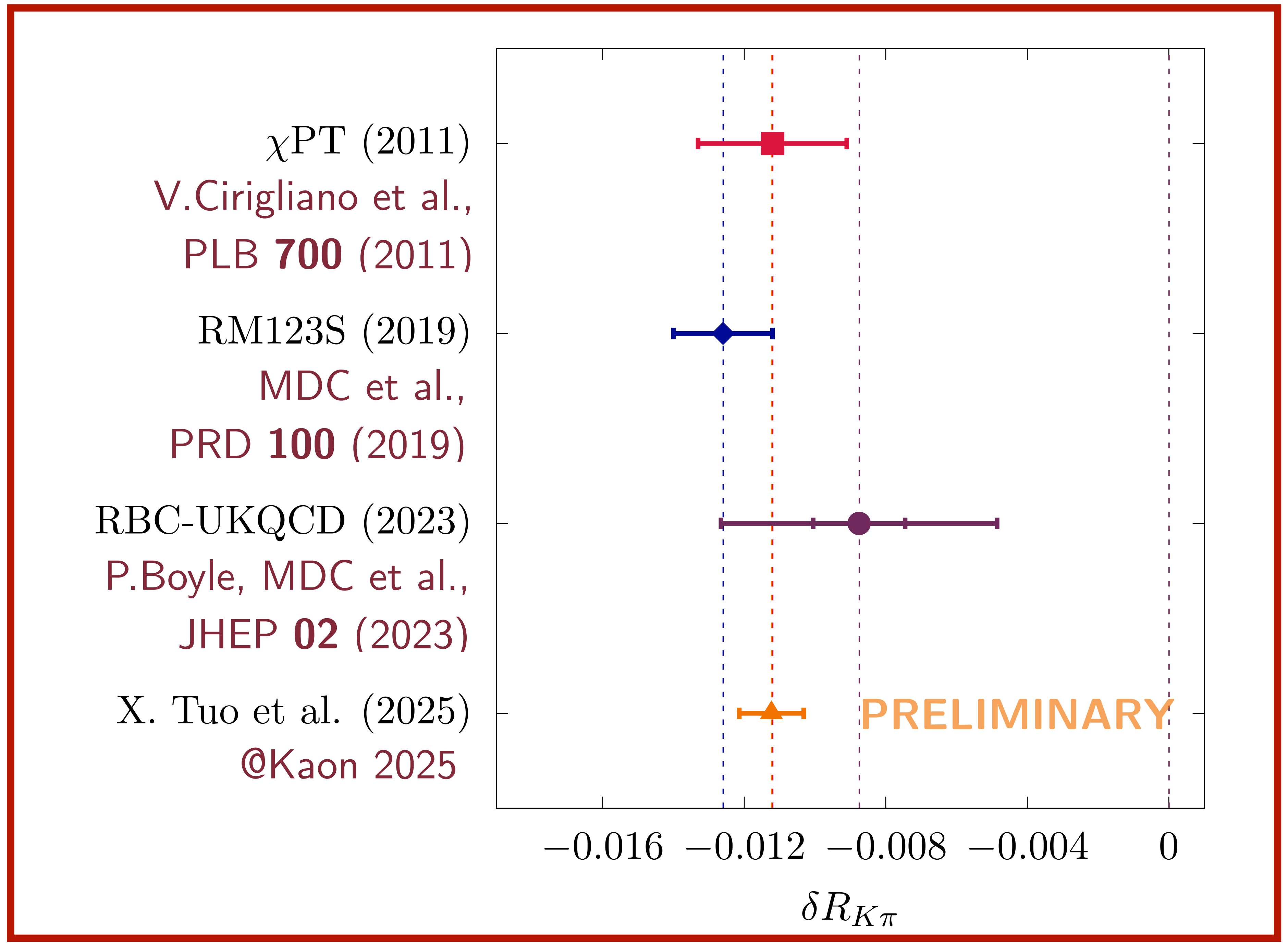
T.Harris et al., PoS LATTICE 2022 (2023) 013

Results for $\delta R_{K\pi}$



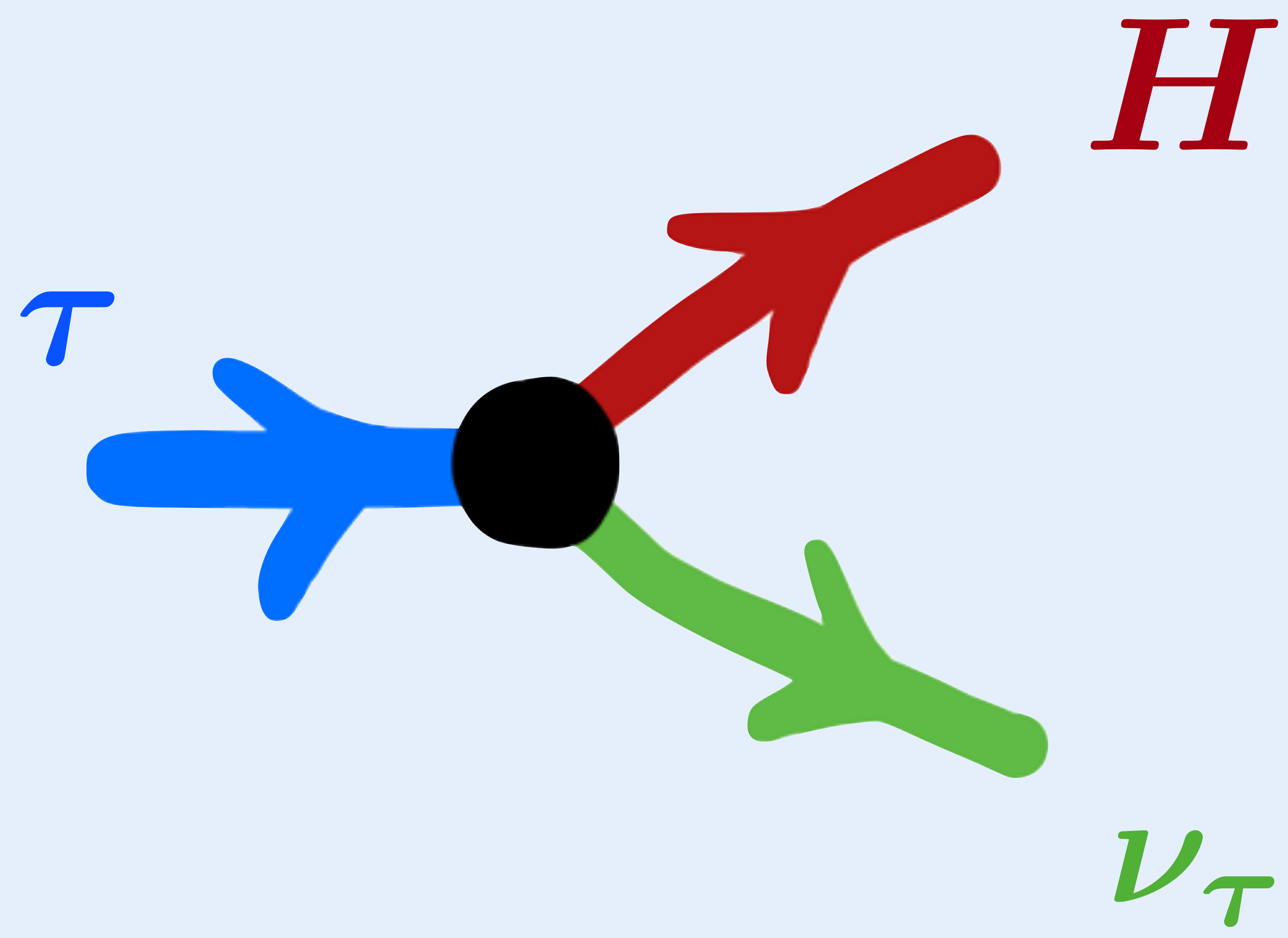
- ■ $\delta R_{K\pi} = -0.0112\,(21)$
- ◆ $\delta R_{K\pi} = -0.0126\,(14)$
- ● $\delta R_{K\pi} = -0.0086\,(13)(39)_{\text{vol.}}$
- ▲ $\delta R_{K\pi} = -0.01123\,(91)$

$$\frac{\Gamma(K \to \ell\nu_\ell)}{\Gamma(\pi \to \ell\nu_\ell)} \propto \frac{|V_{us}|^2}{|V_{ud}|^2} \left(\frac{f_K}{f_\pi}\right)^2 (1 + \delta R_{K\pi})$$



- $\delta R_{K\pi}$ **can be computed** from first principles non-perturbatively **on the lattice**!
- **RBC-UKQCD result** highlights the crucial importance of **finite-volume effects**
  Work in progress *to improve the result*
- **Errors on** $|V_{us}|/|V_{ud}|$ from **theoretical** inputs can become comparable with **experimental** ones

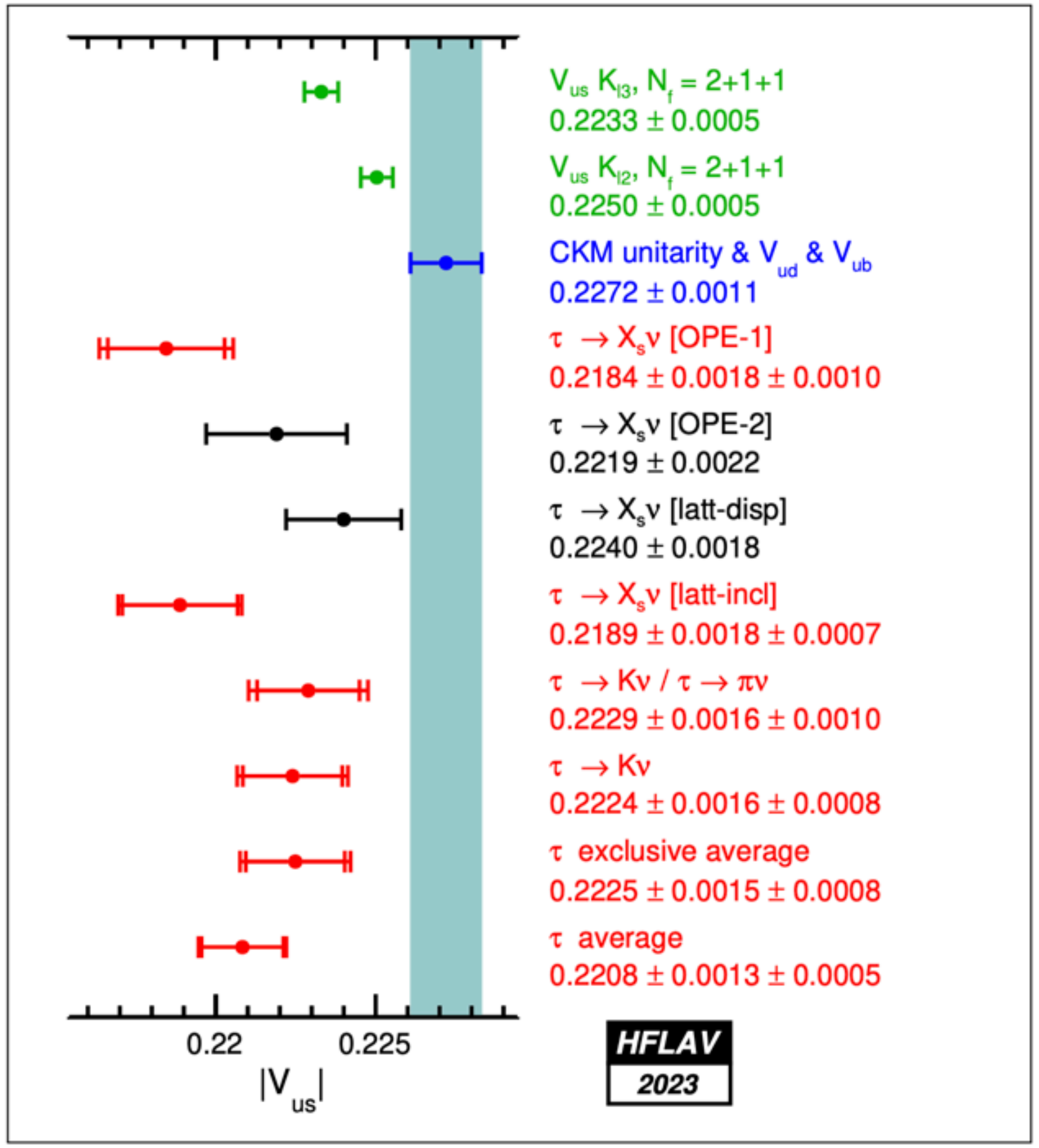
Hadronic decays of leptons



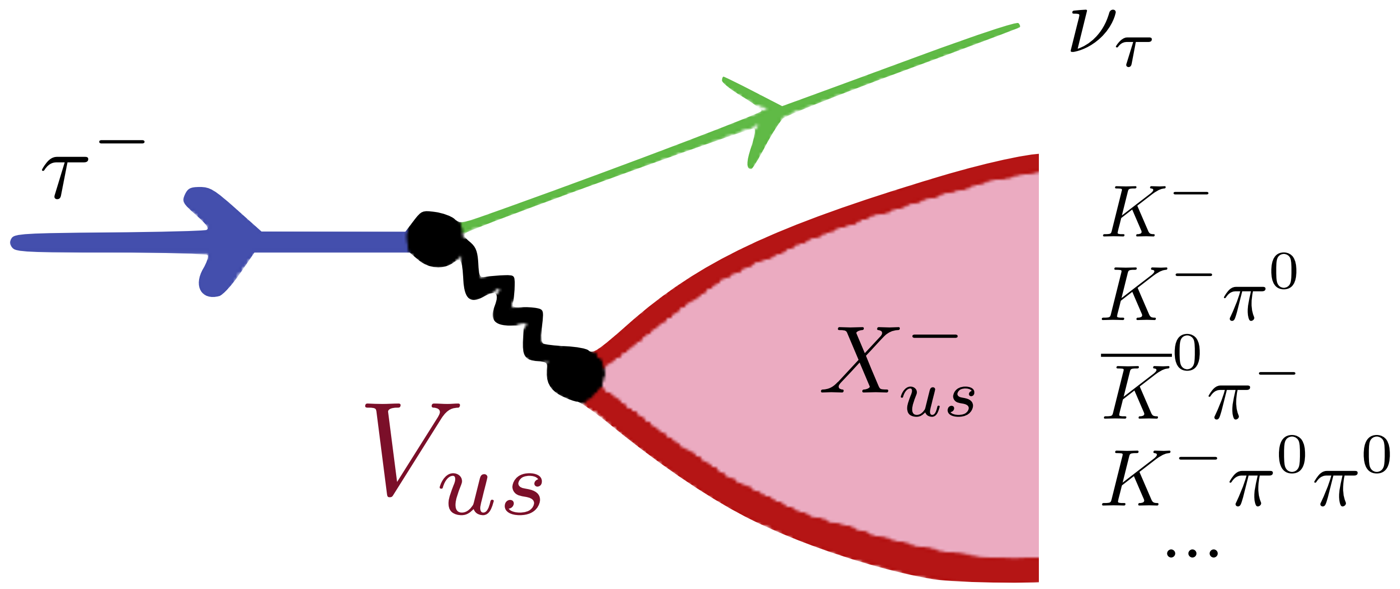
MDC et al., [arXiv:2603:29016]

Inclusive hadronic τ decays

Provide an alternative precise determination of $|V_{us}|$



HFLAV, [2411.18639]



Yet another **puzzle**:

- Inclusive decays give results **smaller** than that obtained imposing CKM unitarity
- Exclusive channels give results **larger** than inclusive, but **smaller** than CKM unitarity

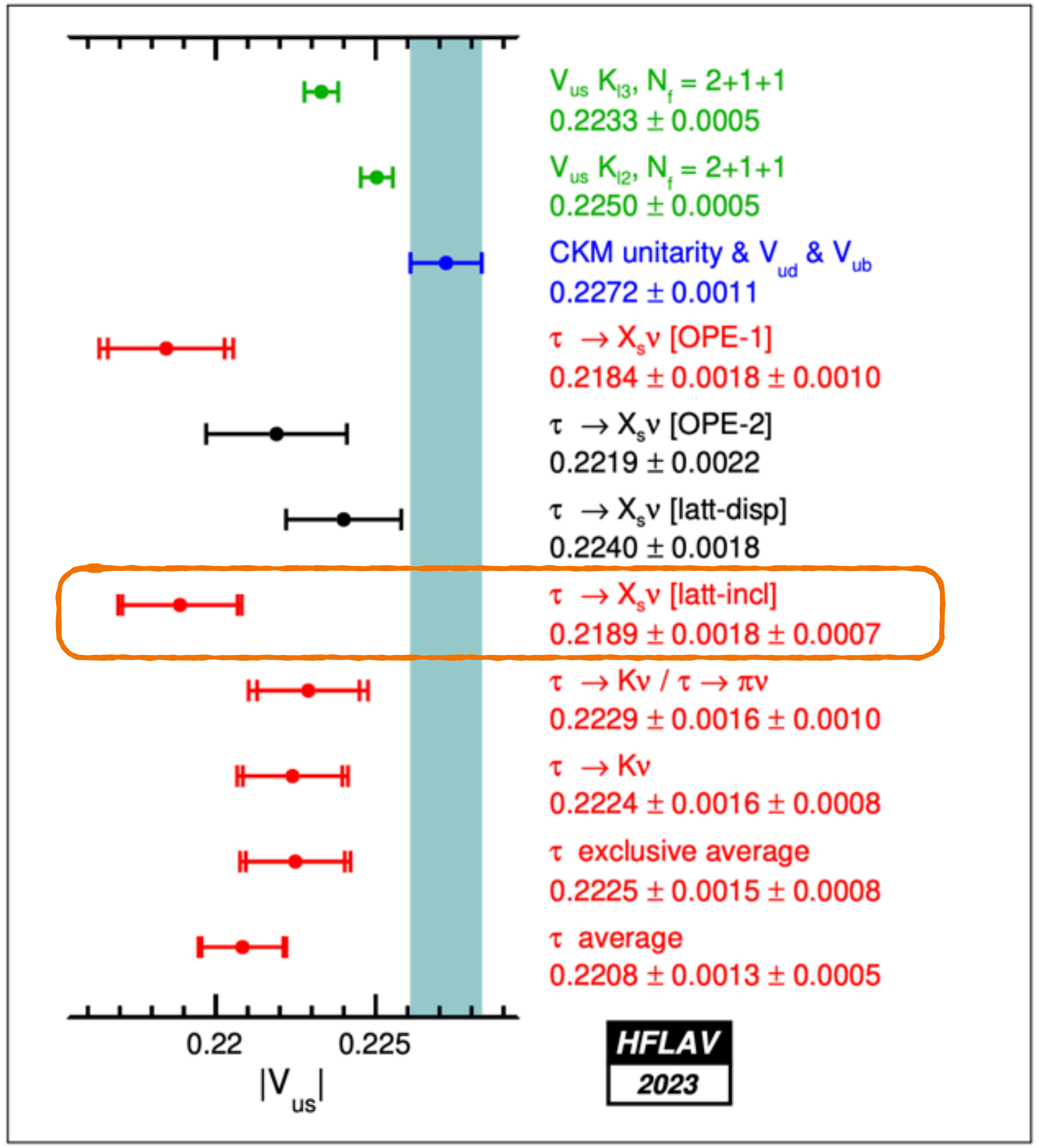
... underestimated exp. uncertainties?

... missing isospin-breaking effects?

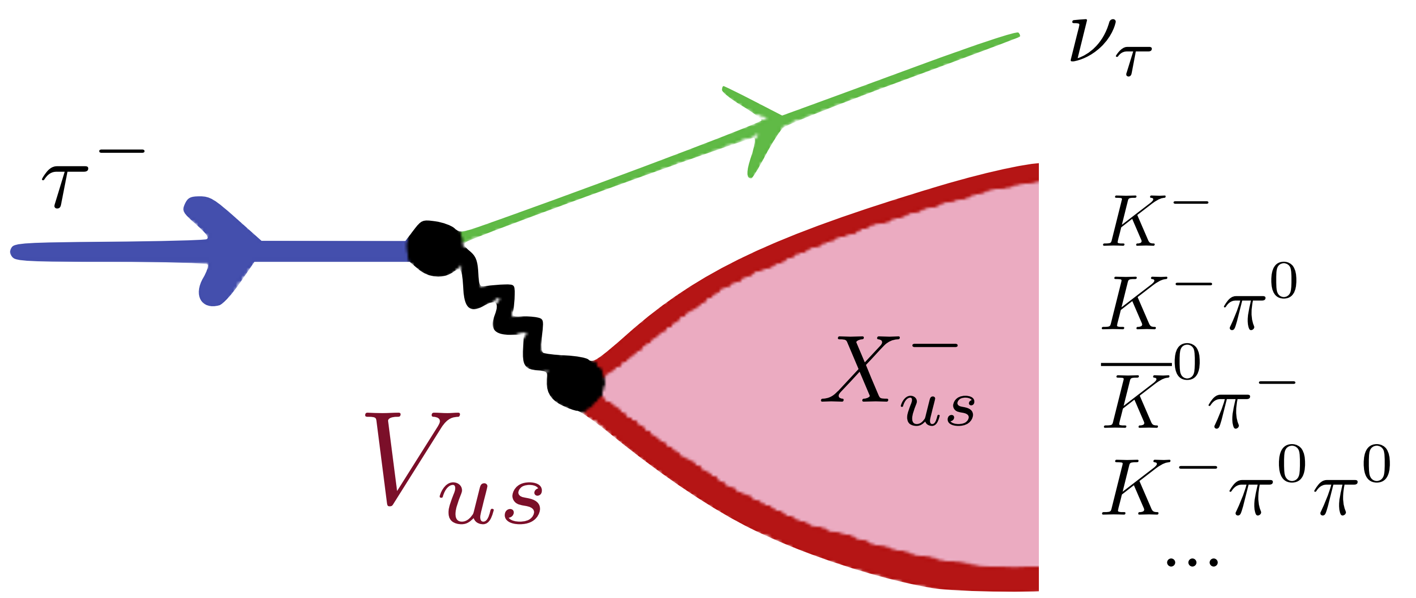
Inclusive hadronic τ decays

A.Evangelista et al. (ETMC), PRD 108 (2023)
C.Alexandrou et al. (ETMC), PRL 132 (2024)

Provide an alternative precise determination of $|V_{us}|$



HFLAV, [2411.18639]



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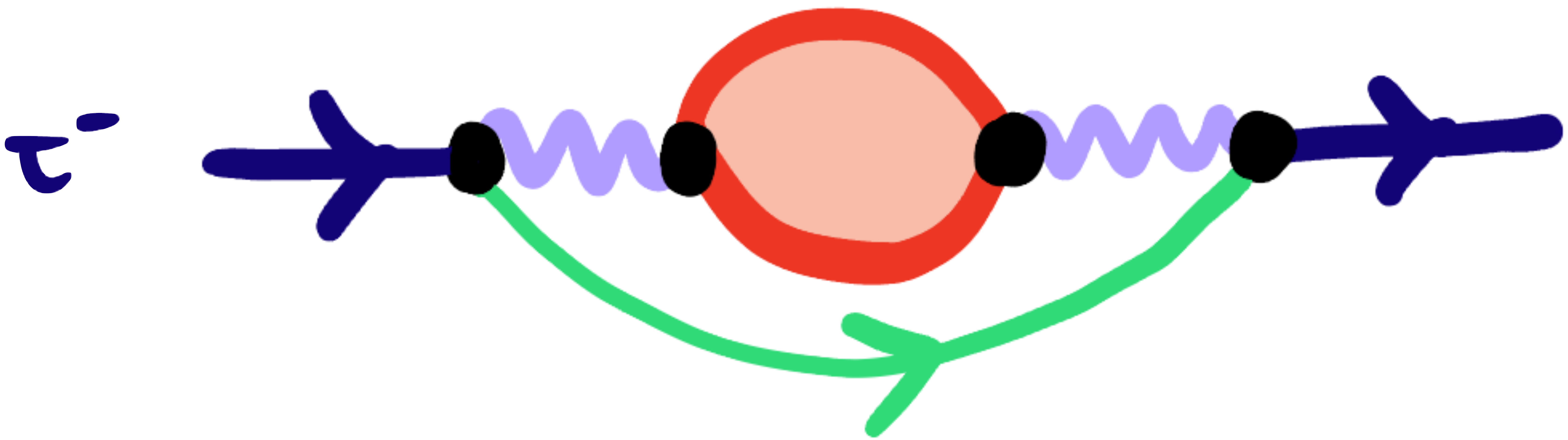
... missing isospin-breaking effects?

Inclusive decay rate in QCD

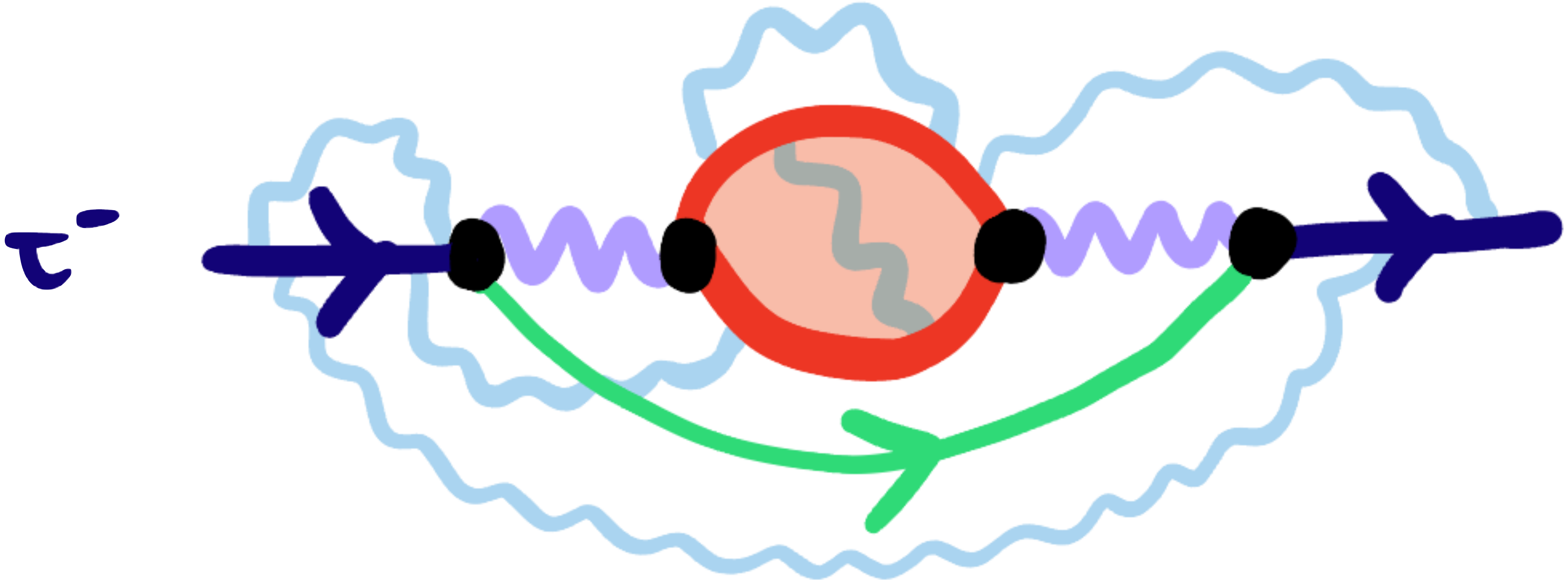
A.Evangelista et al. (ETMC), PRD 108 (2023)
C.Alexandrou et al. (ETMC), PRL 132 (2024)



$$\Gamma\big(\tau \to X_{us}\nu_\tau\big) = \frac{1}{m_\tau}\,\mathrm{Im}\big[\mathcal{M}(\tau \to \tau)\big]_{\mathrm{QCD}}$$

Inclusive decay rate in QCD+QED

A.Evangelista et al. (ETMC), PRD 108 (2023)
C.Alexandrou et al. (ETMC), PRL 132 (2024)

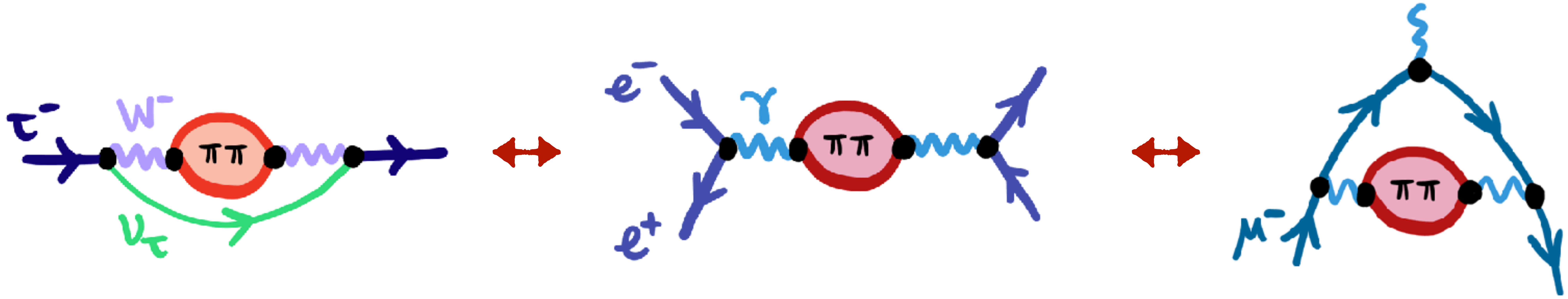


$$\Gamma\big(\tau \to X_{us}\nu_\tau[\gamma]\big) = \frac{1}{m_\tau}\,\mathrm{Im}\big[\mathcal{M}(\tau \to \tau)\big]_{\mathrm{QCD+QED}}$$

Hadronic τ decays and the muon g-2

R.Alemany, M.Davier, A.Hocker, EPJC 2 (1998)

In the **absence of isospin-breaking corrections**, τ decays (in *ud* channel) can be connected to the **hadronic contributions to the muon g-2** by an isospin rotation:



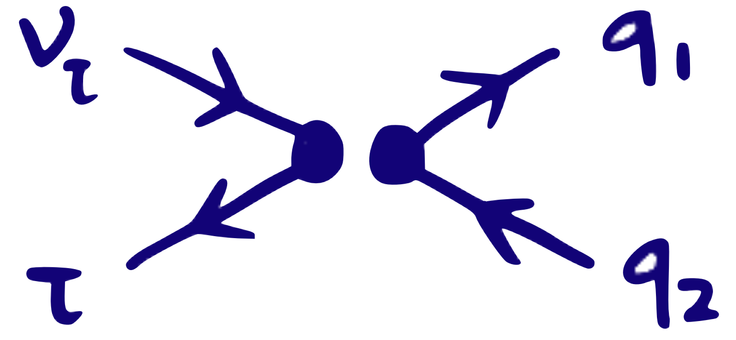
Inclusion of isospin breaking effects in the $\pi\pi$ **channel** is addressed in different ways:

- **chiral perturbation theory** — V.Cirigliano, G.Ecker & H.Neufeld, PLB 513 (2001) + JHEP 08 (2002)
- **dispersive methods** — see talks by S.Holz & M.Cottini at IJCLab Orsay (2025)
- **lattice QCD** — see talk by M.Bruno (RBC/UKQCD) at IJCLab Orsay (2025)

Inclusive decay rate in QCD+QED

Given the **weak effective Hamiltonian**

$$\mathcal{H}_W(x) = \frac{G_F S_{\rm EW}^{1/2}}{\sqrt{2}}\, \mathcal{O}_W(x) \equiv \frac{G_F S_{\rm EW}^{1/2}}{\sqrt{2}} \left[\bar{\tau}\gamma^\alpha(1-\gamma_5)\nu_\tau\right](x)\, j_\alpha(x) + \text{h.c.}$$



then the **inclusive decay rate in QCD+QED** is obtained as $\Gamma = \frac{G_F^2 S_{\rm EW}}{4m_\tau} |\mathcal{A}(m_\tau)|^2$

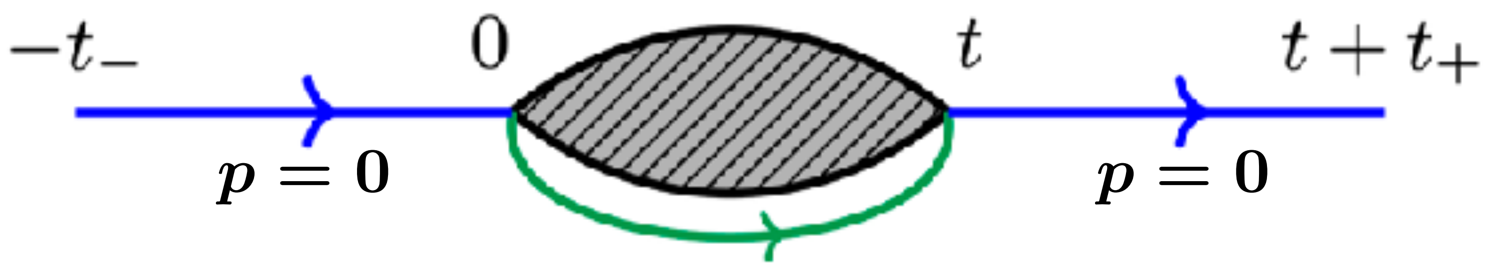
$$|\mathcal{A}(m_\tau)|^2 = \frac{1}{2}\sum_r \langle \tau(r,p)|\mathcal{O}_W(0)\,(2\pi)^4\,\delta^4(\hat{P}-p)\,\mathcal{O}_W(0)\,|\tau(r,p)\rangle_C$$
$$= \sum_X \left\{ \frac{1}{2}\sum_r |\langle \tau(r,p)|\mathcal{O}_W(0)|X(p)\rangle|^2 \right\}$$

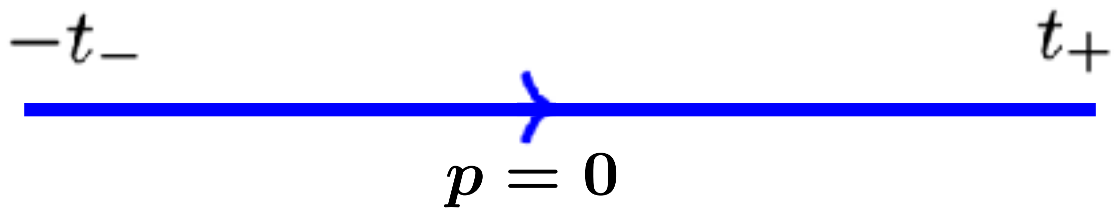
$|X(p)\rangle$ = *hadrons + photons*

Our goal: extract $|\mathcal{A}(m_\tau)|^2$ from Euclidean lattice correlators

Defining the Euclidean correlator

Let us consider **full QCD+QED theory** and define the following two correlation functions:

$\mathcal{C}(t_+, t, t_-) =$ 

$C_\tau(t_- + t_+) =$ 

With **any IR regulator**, the **ratio** of such traced correlators for large time separations gives

$$\mathcal{R}(t) \equiv 4m_\tau \frac{\mathrm{Tr}[\,\mathcal{C}(t_-, t, t_+)\,]}{\mathrm{Tr}[C_\tau(t_- + t_+)]} \xrightarrow{t_\pm \gg 0,\ t > 0} \int_0^\infty \frac{d\omega}{2\pi}\, e^{-\omega t}\, |\mathcal{A}(\omega)|^2$$

We plan to follow **two parallel approaches**:

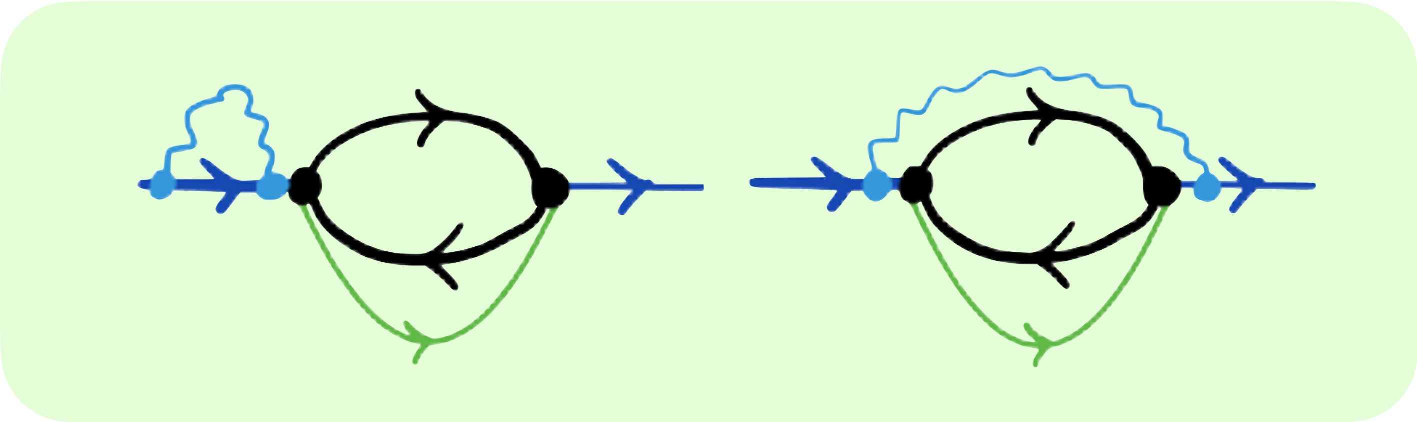
1. Full **QCD+QED** simulation (RC* ensembles)
2. **Perturbative "RM123" expansion** (ETMC ensembles)

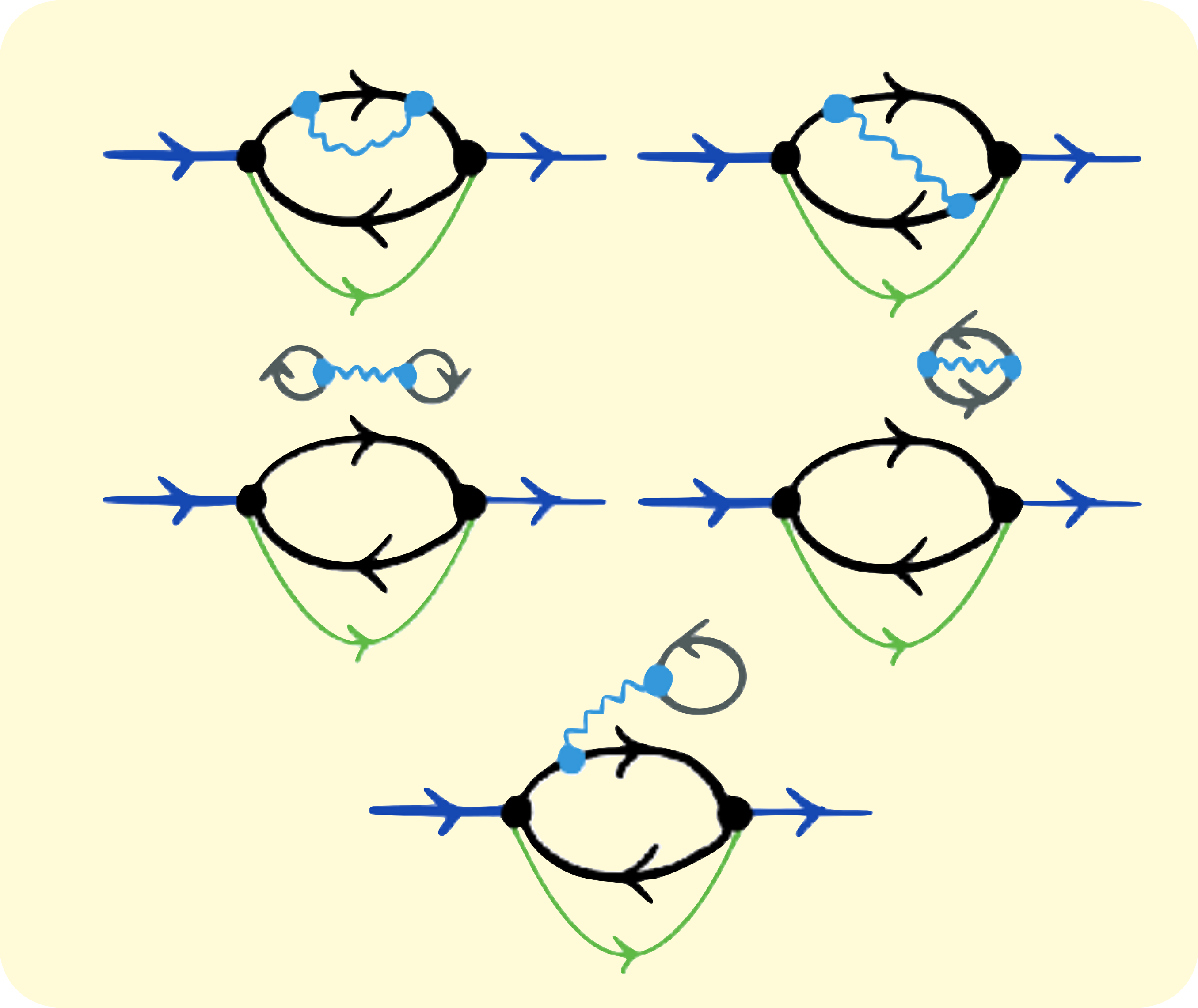
Expansion of the correlators at $\mathcal{O}(\alpha)$

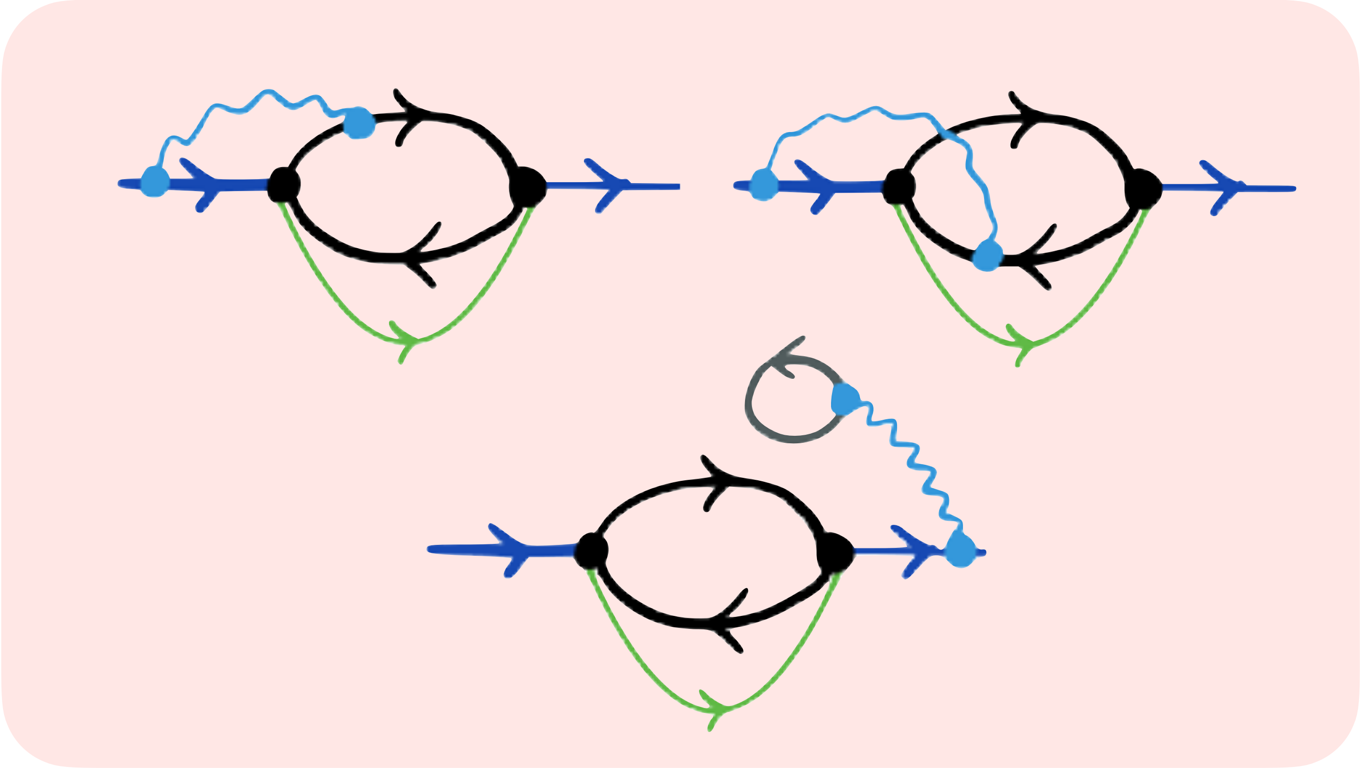
RM123 approach

Taking derivatives with respect to the (lepton & quarks) electric charges gives:

$$\mathcal{R}(t) = \big[\mathcal{R}(t)\big]_{\text{lep}} + \big[\mathcal{R}(t)\big]_{\text{fact}} + \big[\mathcal{R}(t)\big]_{\text{non-fact}}$$







Note that the 3 contributions are separately *infrared finite*

Spectral reconstruction

J.Barata, K.Fredenhagen, CMP 138 (1991)
. . .
M.T.Hansen, H.Meyer, D.Robaina, PRD 96 (2017)
S.Hashimoto, PTEP 2017 (2017)
M.Hansen, A.Lupo, N.Tantalo, PRD 99 (2019)
J.Bulava, M.T.Hansen, PRD 100 (2019)
P.Gambino, S.Hashimoto PRL 125 (2020)
. . .
J.Bulava et al., JHEP 07 (2022)
R.Frezzotti et al., PRD 108 (2023)
A.Patella, N.Tantalo, JHEP 01 (2025)
M.Bruno, L.Giusti, M.Saccardi, PRD 111 (2025)
. . .

$$\Gamma = \frac{G_F^2 S_{\rm EW}}{4m_\tau} |\mathcal{A}(m_\tau)|^2 \longleftrightarrow \mathcal{R}(t) = \int_0^\infty \frac{d\omega}{2\pi} e^{-\omega t} |\mathcal{A}(\omega)|^2$$

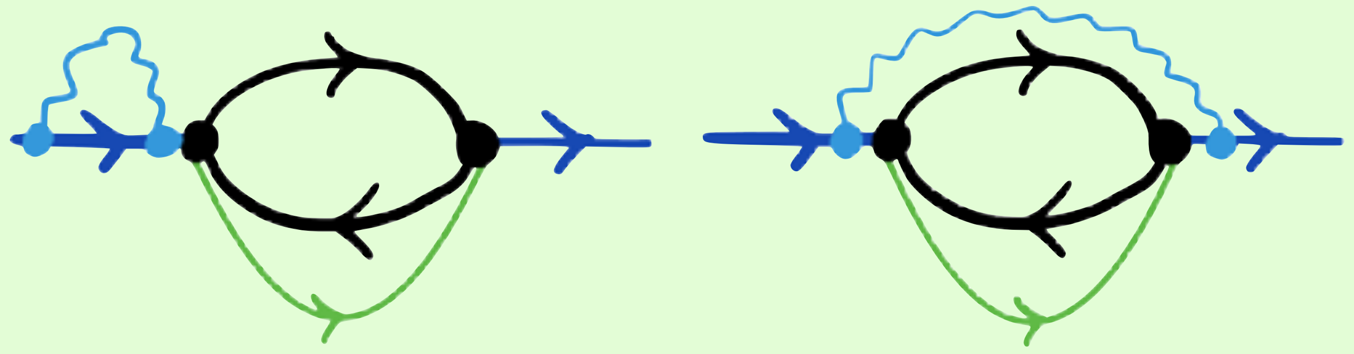
- Start from the relation $|\mathcal{A}(m_\tau)|^2 = \int_0^\infty \frac{d\omega}{2\pi} [2\pi\delta(\omega - m_\tau)] |\mathcal{A}(\omega)|^2$
- Define a **smeared kernel** such that $\lim_{\sigma\to 0} \Delta_\sigma(\omega, m_\tau) = 2\pi\,\delta(\omega - m_\tau)$
- **Approximate** kernel as sum of exponentials $\Delta_\sigma(\omega, m_\tau) \approx \sum_t g_t(\sigma, m_\tau)\, e^{-\omega t}$
- Use an optimisation strategy (e.g. **HLT method**) to obtain the coefficients $g_t(\sigma, m_\tau)$

$$\longrightarrow \quad |\mathcal{A}(m_\tau)|^2 \approx \lim_{\sigma\to 0} \sum_t g_t(\sigma, m_\tau)\, \mathcal{R}(t)$$

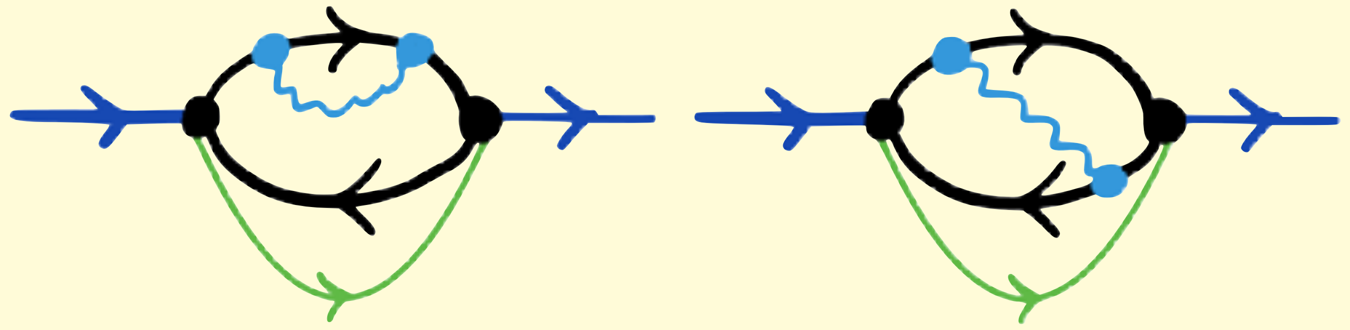
Expansion of the squared amplitude at $\mathcal{O}(\alpha)$

RM123 approach

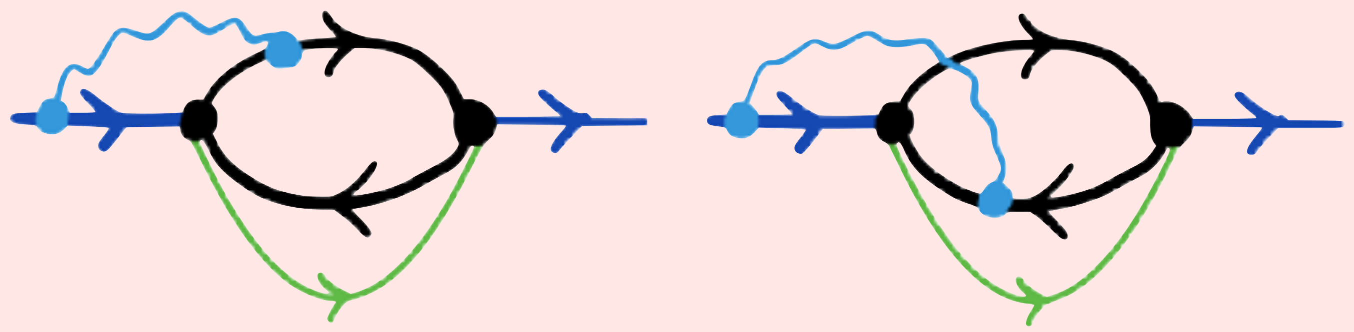
$$|\mathcal{A}(m_\tau)|^2 = |\mathcal{A}(m_\tau)|^2_{\text{lep}} + |\mathcal{A}(m_\tau)|^2_{\text{fact}} + |\mathcal{A}(m_\tau)|^2_{\text{non-fact}}$$



$$|\mathcal{A}(m_\tau)|^2_{\text{lep}} = \frac{m_\tau^6}{(8\pi^2)^2} \int_0^\infty ds \left[\delta\mathcal{K}_{\text{T}}(s)\rho_{\text{T}}(s) + \delta\mathcal{K}_{\text{L}}(s)\rho_{\text{L}}(s)\right]$$



$$|\mathcal{A}(m_\tau)|^2_{\text{fact}} = \frac{m_\tau^6}{8\pi^2} \int_0^\infty ds \left[\mathcal{K}_{\text{T}}(s)\rho_{\text{T}}^{\text{full}}(s) + \mathcal{K}_{\text{L}}(s)\rho_{\text{L}}^{\text{full}}(s)\right]$$



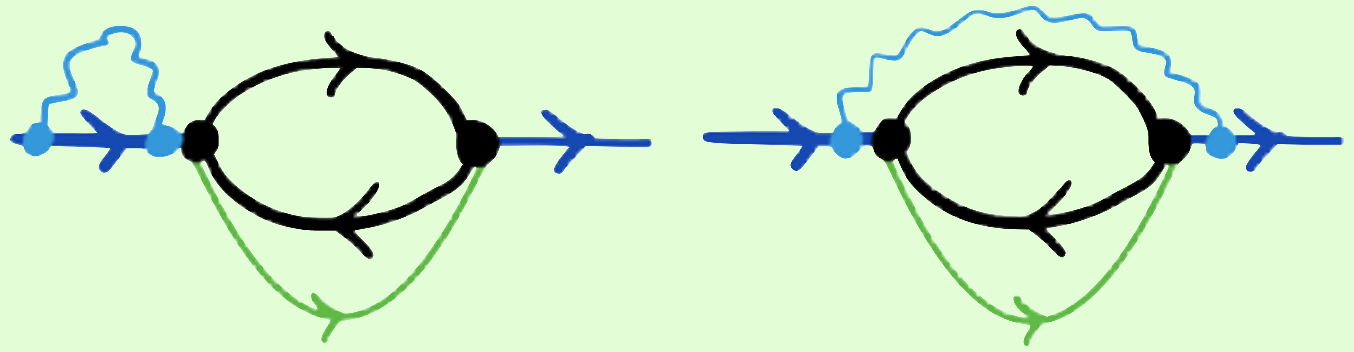
$$|\mathcal{A}(m_\tau)|^2_{\text{non-fact}} = \int_0^\infty d\omega\, \delta(\omega - m_\tau)\, |\mathcal{A}(\omega)|^2_{\text{non-fact}}$$

Expansion of the squared amplitude at $\mathcal{O}(\alpha)$

RM123 approach

$$|\mathcal{A}(m_\tau)|^2 = |\mathcal{A}(m_\tau)|^2_{\text{lep}} + |\mathcal{A}(m_\tau)|^2_{\text{fact}} + |\mathcal{A}(m_\tau)|^2_{\text{non-fact}}$$

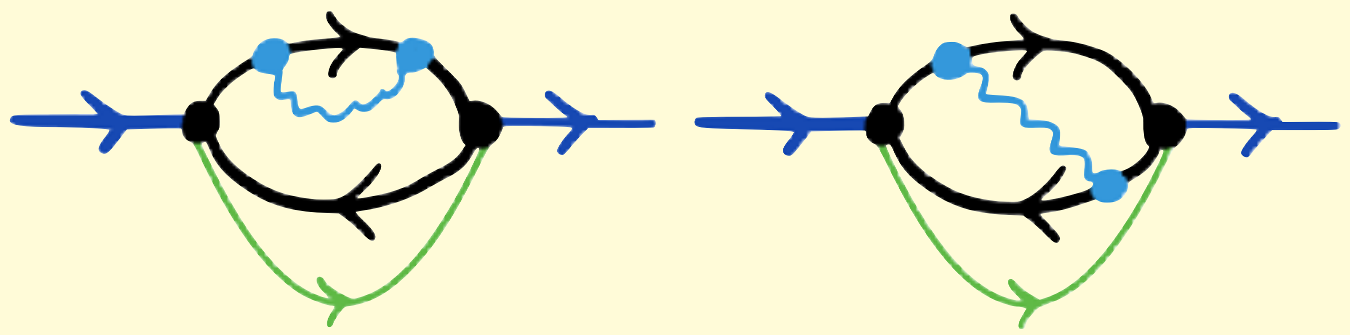
✔



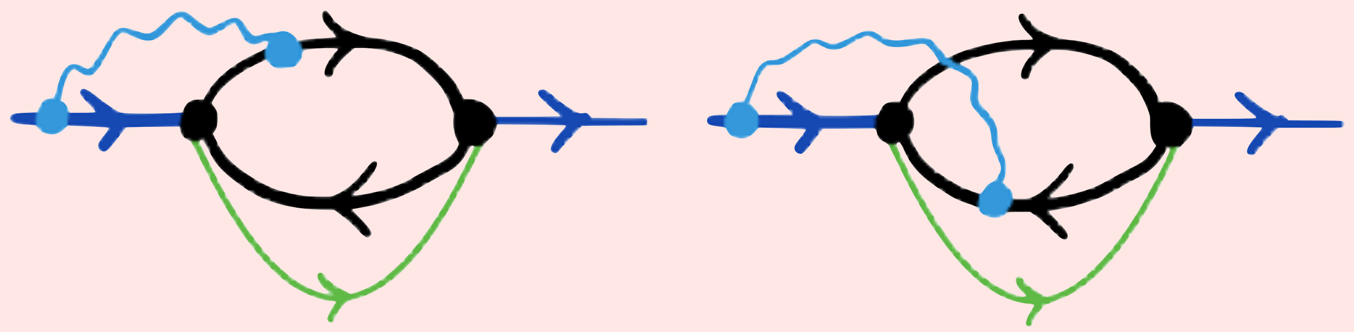
$$|\mathcal{A}(m_\tau)|^2_{\text{lep}} = \frac{m_\tau^6}{(8\pi^2)^2} \int_0^\infty ds \left[\delta\mathcal{K}_{\text{T}}(s)\rho_{\text{T}}(s) + \delta\mathcal{K}_{\text{L}}(s)\rho_{\text{L}}(s)\right]$$

MDC et al., [arXiv:2603:29016]

✔



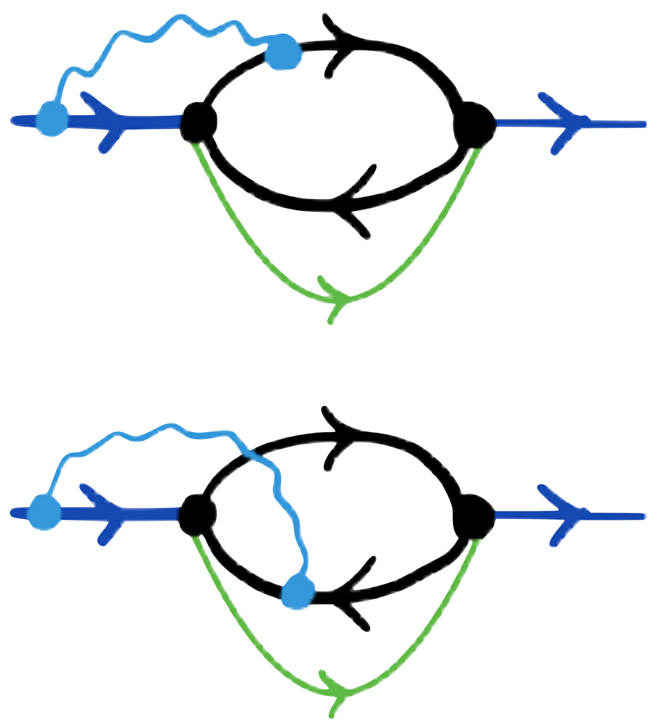
$$|\mathcal{A}(m_\tau)|^2_{\text{fact}} = \frac{m_\tau^6}{8\pi^2} \int_0^\infty ds \left[\mathcal{K}_{\text{T}}(s)\rho^{\text{full}}_{\text{T}}(s) + \mathcal{K}_{\text{L}}(s)\rho^{\text{full}}_{\text{L}}(s)\right]$$



$$|\mathcal{A}(m_\tau)|^2_{\text{non-fact}} = \int_0^\infty d\omega\, \delta(\omega - m_\tau)\, |\mathcal{A}(\omega)|^2_{\text{non-fact}}$$

Next steps

✦ **non-factorisable diagram**



$$\left[\mathcal{R}(t)\right]_{\text{non-fact}} = e_\tau \, \text{Re} \int d^3x \, \left[S_\nu(t, \boldsymbol{x}, 0) \right]_{ij} \, \left[\mathcal{K}_{ji}(t, \boldsymbol{x})\right]_{\text{non-fact}}$$

✦ **QCD+QED renormalisation of the correlator**

$\mathcal{H}_W = C^{\text{R}}(\mu)\, O_1^{\text{R}}(\mu)$ the weak Hamiltonian requires a QCD+QED renormalization.

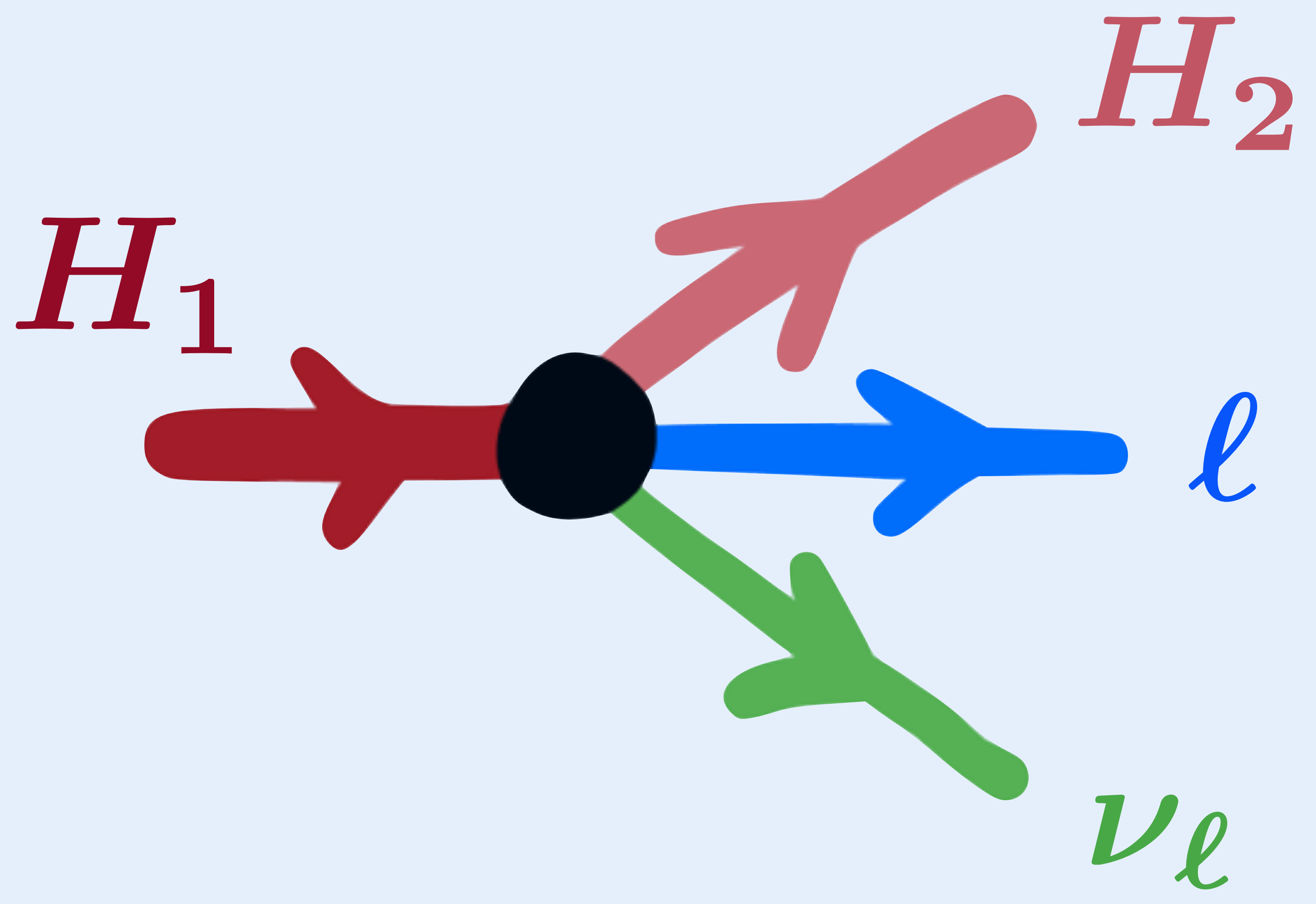
Electroweak matching with the SM studied in ***W-mass*** and ***$\overline{MS}$ scheme.***

A.Sirlin, NPB 196 (1982)
E.Braaten & C.S.Li, PRD 42 (1990)
M.Gorbahn et al., JHEP 01 (2023)
F.Moretti et al., [arXiv:2510.27648]

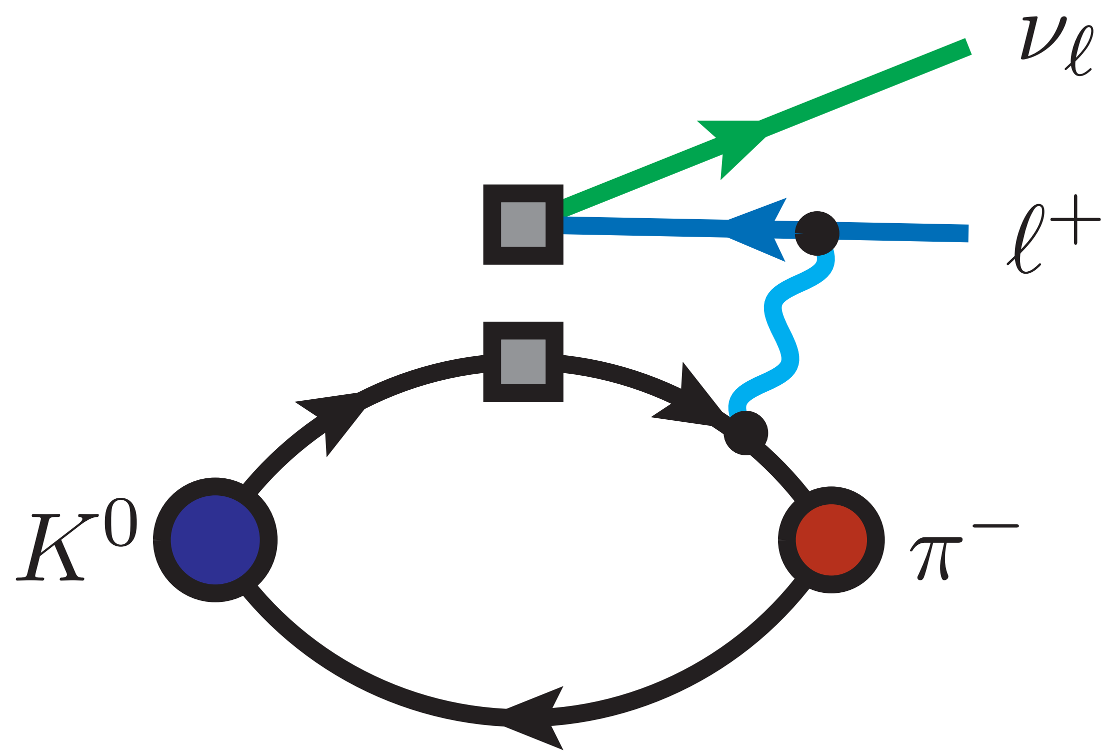
✦ **Physical extrapolations**

$$\lim_{\sigma\to 0} \lim_{L\to\infty} \lim_{a\to 0}$$

Semi-leptonic decays of hadrons



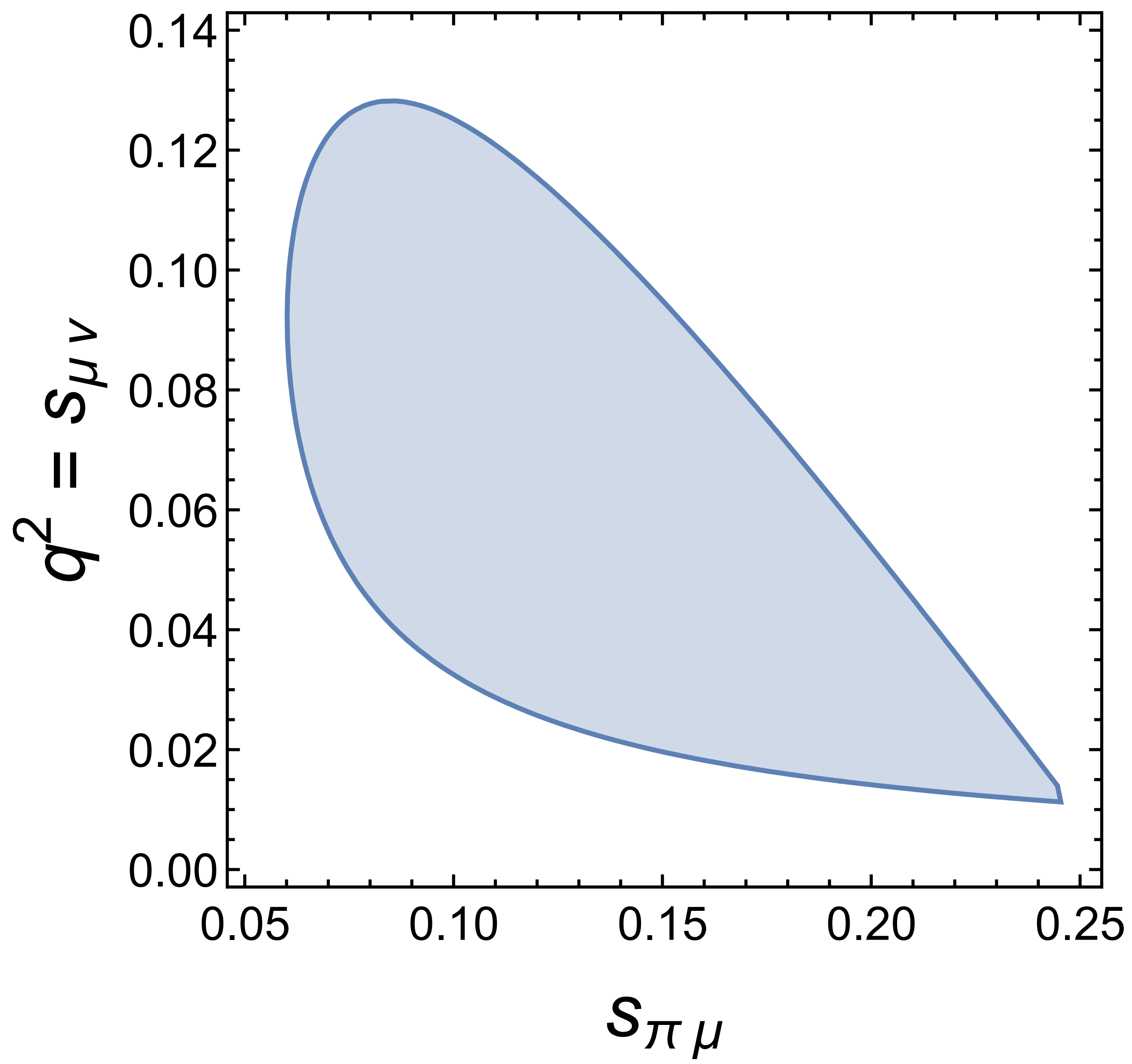
QED corrections to semileptonic decays



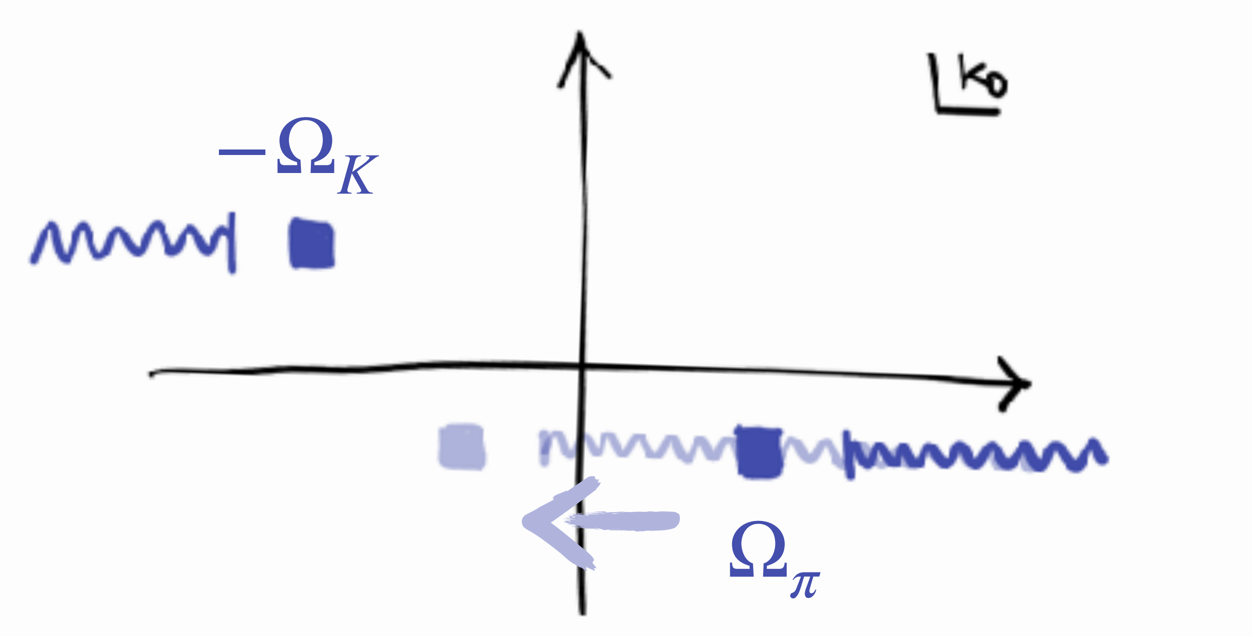
Additional **difficulties** arise compared to leptonic decays:

- integration over **three-body phase-space**
- problems of **analytical continuation** when intermediate on-shell states are lighter than external ones

$$\{\omega_\pi(\boldsymbol{p_\pi}+\boldsymbol{k})+\omega_\ell(\boldsymbol{p_\ell}-\boldsymbol{k})\}-\{\omega_\pi(\boldsymbol{p_\pi})+\omega_\ell(\boldsymbol{p_\ell})\}<0$$



$$H(k)=\int \mathrm{d}^4x\, \mathrm{e}^{ik\cdot x}\,\langle\pi(\boldsymbol{p}_\pi)|\,\mathrm{T}\{j_\mathrm{w}(0)j_\mathrm{em}(x)\}\,|K(\boldsymbol{p}_K)\rangle$$



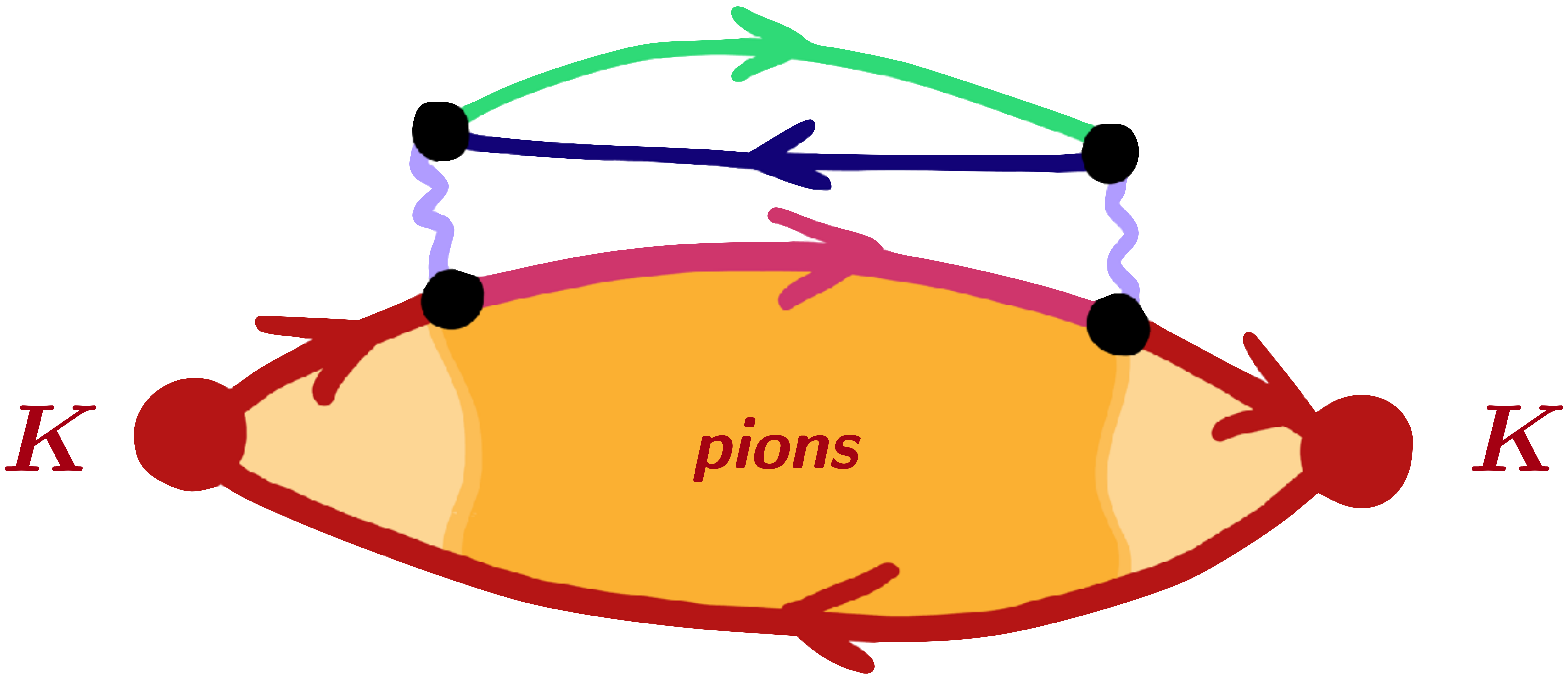
$$\Omega_K=\sqrt{m_K^2+\mathbf{k}^2}-m_K$$

$$\Omega_\pi=\sqrt{\omega_\pi^2+2\,\mathbf{p}_\pi\cdot\mathbf{k}+\mathbf{k}^2}-\omega_\pi$$

QED corrections to semileptonic decays

An inclusive approach?

Application to D_s mesons:
A.De Santis et al. (ETMC), PRD 112 (2025)

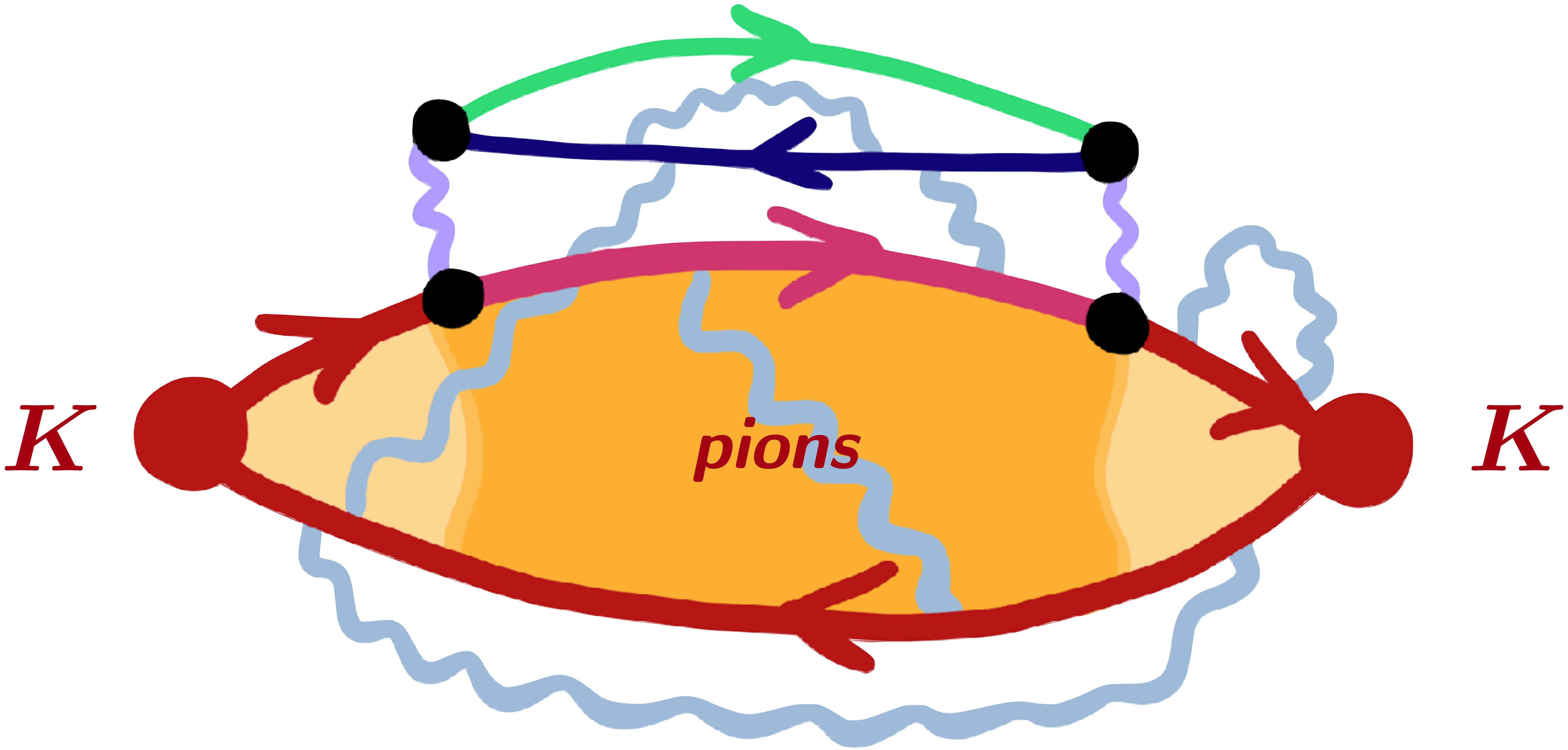


- possible approach for an **inclusive semileptonic K_{l3} decay rate**?

QED corrections to semileptonic decays

An inclusive approach?

Application to D_s mesons:
A.De Santis et al. (ETMC), PRD 112 (2025)



- possible approach for an **inclusive semileptonic K_{l3} decay rate**?

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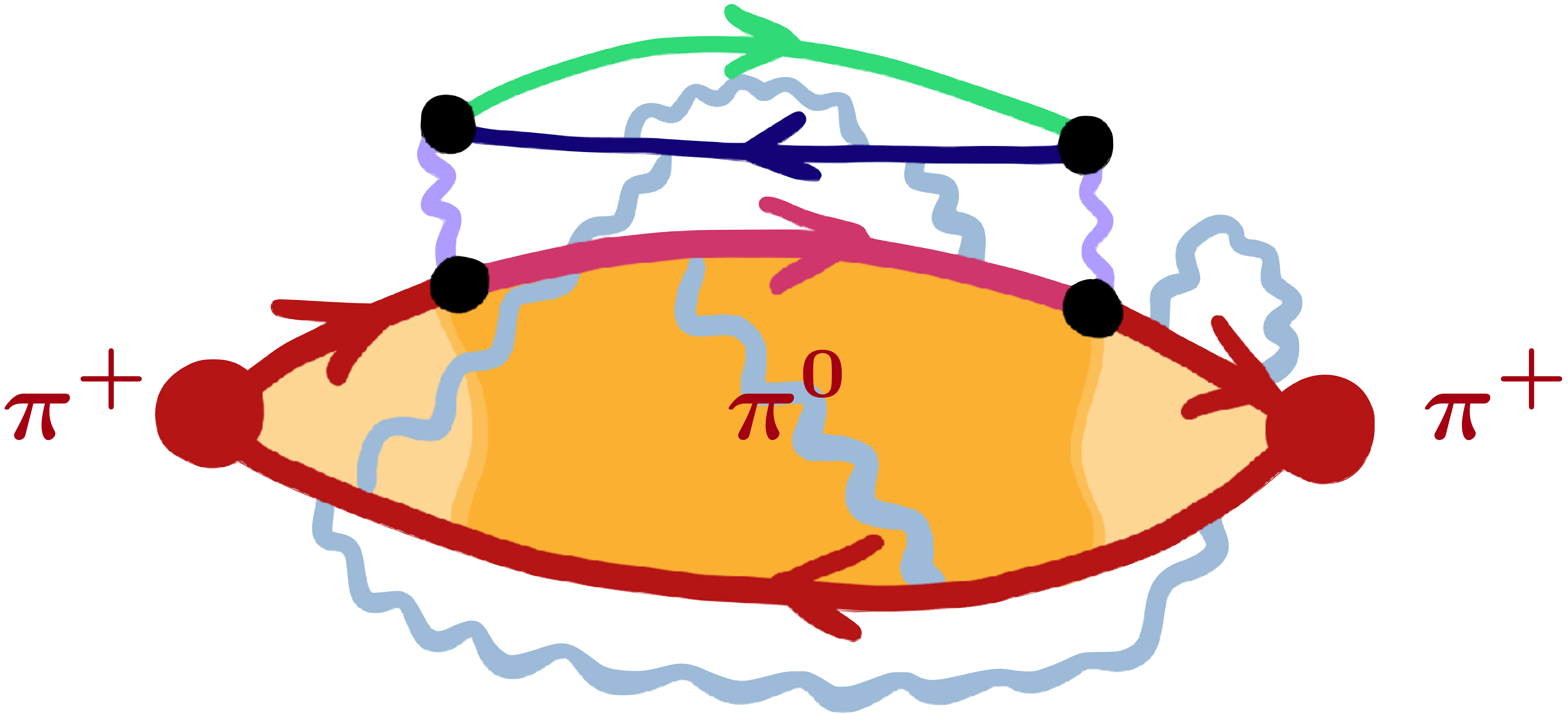
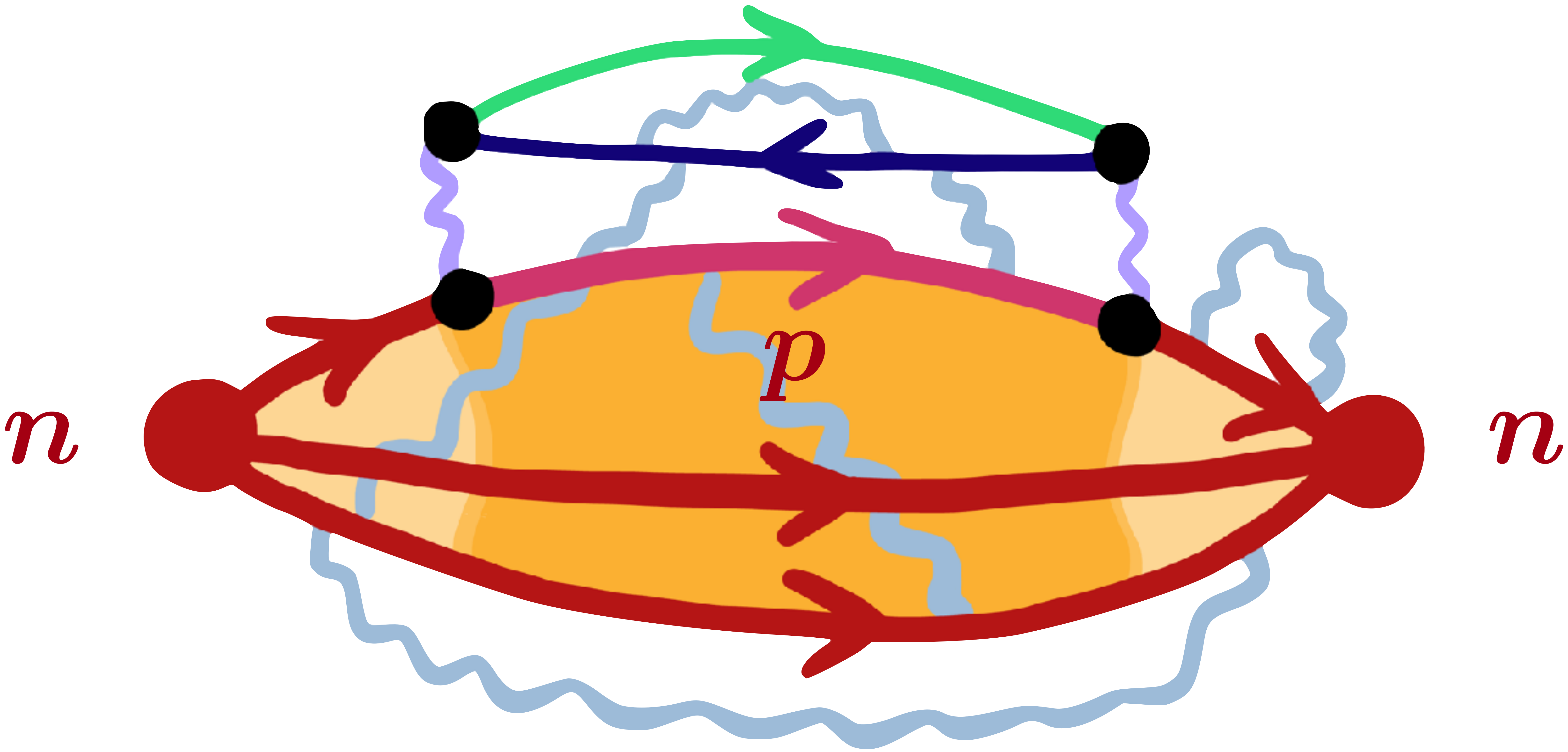


- possible approach for an **inclusive semileptonic *Kl3* decay rate**?
- channels where only a **single intermediate state** is allowed?

QED corrections to semileptonic decays

An inclusive approach?

Application to *Ds* mesons:
A.De Santis et al. (ETMC), PRD 112 (2025)



- possible approach for an **inclusive semileptonic *Kl3* decay rate**?
- channels where only a **single intermediate state** is allowed?

Cabibbo-Kobayashi-Maskawa matrix

In this talk...

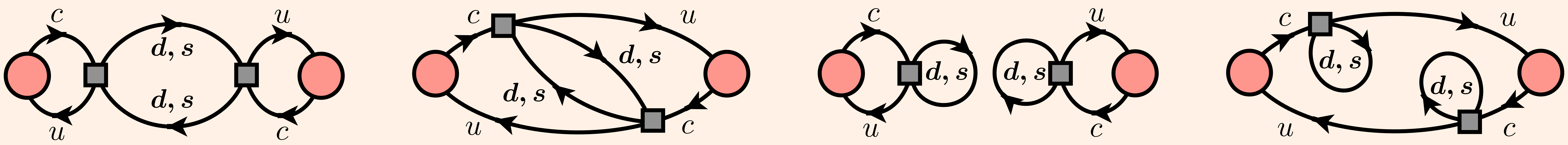
1. Cabibbo anomaly

$$V_{\mathrm{CKM}} = \begin{pmatrix} 1-\frac{1}{2}\lambda^2 & \lambda & A\lambda^3(\rho-i\eta) \\ -\lambda & 1-\frac{1}{2}\lambda^2 & A\lambda^2 \\ A\lambda^3(1-\rho-i\eta) & -A\lambda^2 & 1 \end{pmatrix}$$

$$\begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix}$$

2. Neutral D-meson mixing

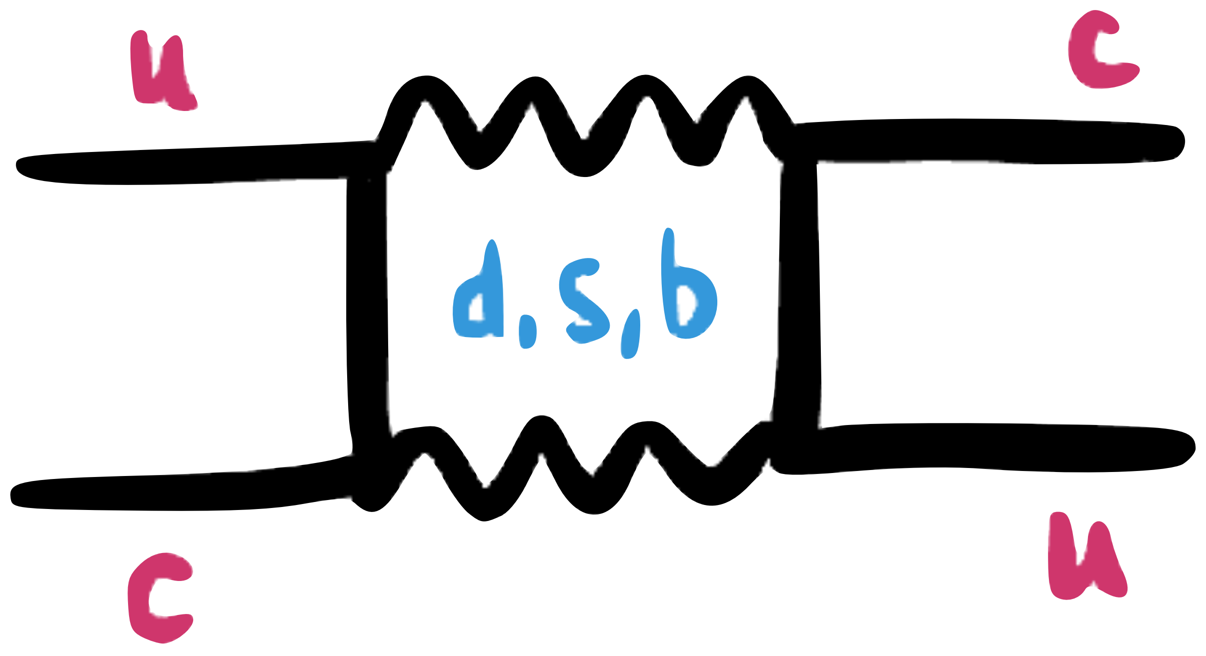
2. Neutral D-meson mixing

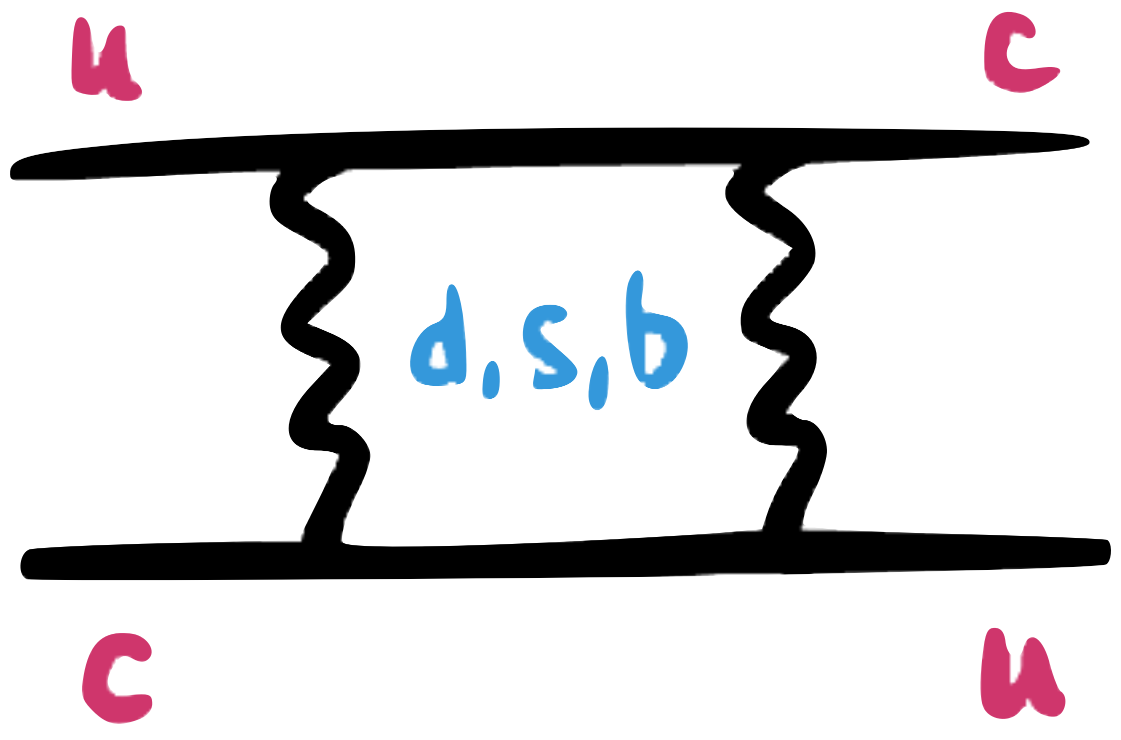


MDC, F.Erben & M.T.Hansen, JHEP 07 (2025)

Neutral *D*-meson mixing

- First evidence of ***D*-meson mixing** at Belle and Babar in 2007, later confirmed by CDF, **LHCb**, Belle II
- **Charm mixing is unique**: D^0 mesons are the only ones made of *up*-type quarks that can mix
- Mixing and CP violation are **suppressed** in the Standard Model (GIM & CKM)
- *D*-mixing amplitudes are dominated by **long-distance effects**, which are **poorly known**: theoretical estimates span *few orders of magnitude*!





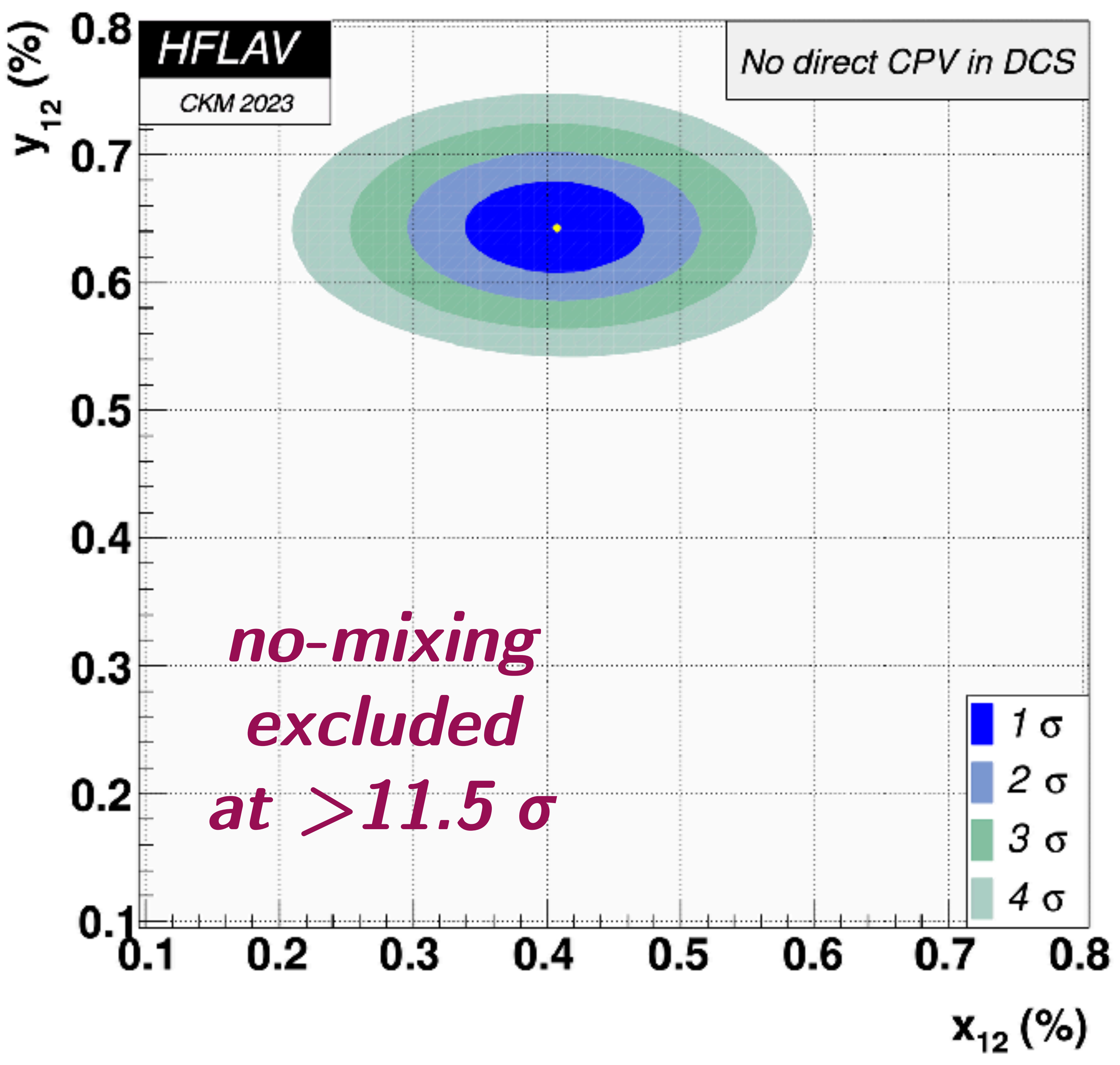
Mixing parameters

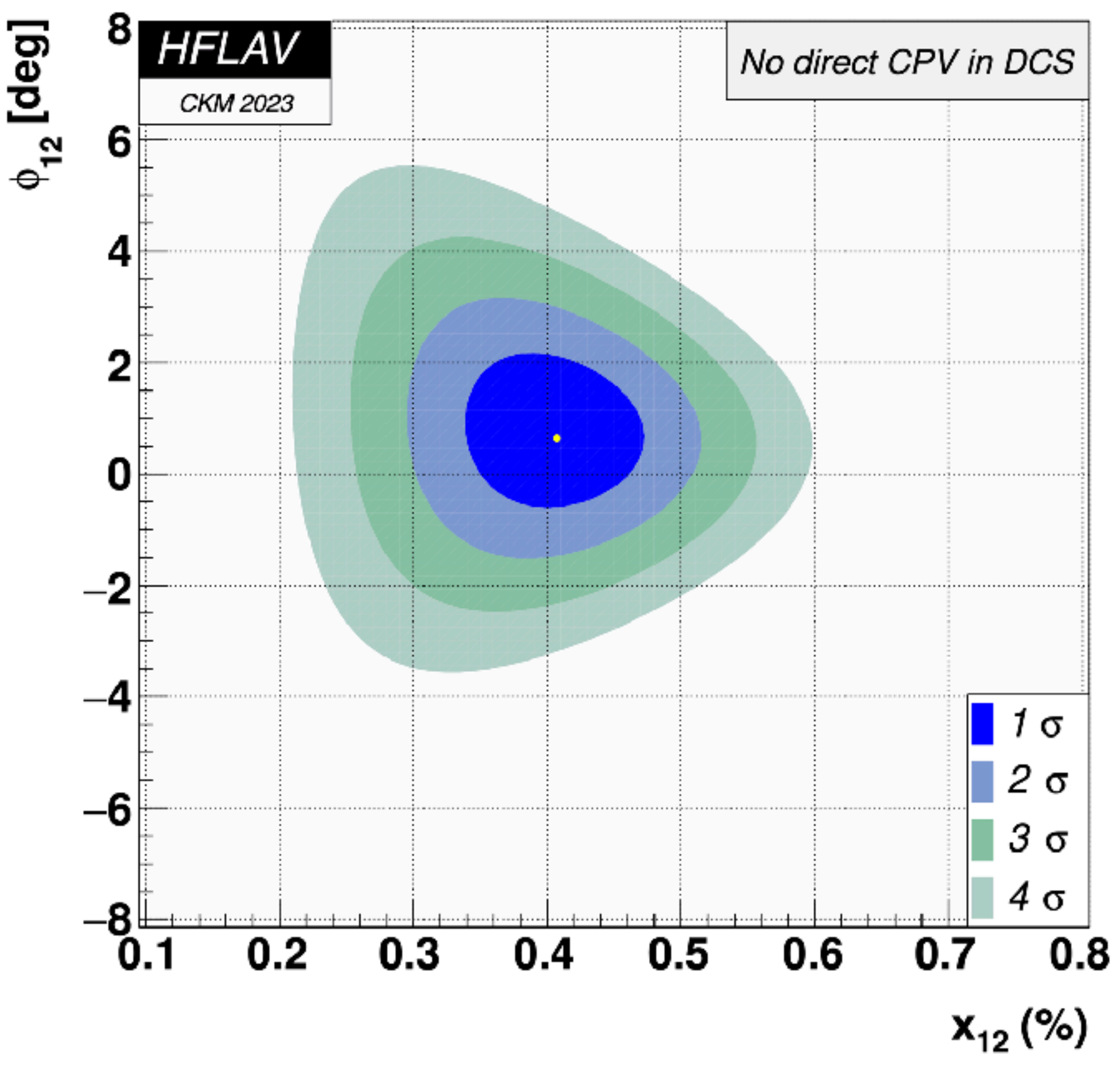
HFLAV Collaboration [2411.18639]

$$x_{12} = 2\frac{|M_{12}|}{\Gamma_D} = (0.407 \pm 0.044)\ \%$$

$$y_{12} = \frac{|\Gamma_{12}|}{\Gamma_D} = (0.643\,^{+0.024}_{-0.023})\ \%$$

$$\phi_{12} = \arg\left(\frac{M_{12}}{\Gamma_{12}}\right) = (0.65\,^{+0.92}_{-0.90})^\circ$$





Mixing amplitude

The **mixing amplitude** for the process $D^0 \to \bar{D}^0$ is given by the following ***S*-matrix** element:

$$\langle \bar{D}^0, \boldsymbol{p}'_D | \mathcal{S} - \mathbf{1} | D^0, \boldsymbol{p}_D \rangle = \langle \bar{D}^0, \boldsymbol{p}'_D | \, \mathrm{T} \, \exp\left\{ -\mathrm{i} \int \mathrm{d}^4x \, \mathcal{H}_{\mathrm{w}}(x) \right\} | D^0, \boldsymbol{p}_D \rangle$$

$$\equiv (2\pi)^4 \delta^4(P_{D^0} - P_{\bar{D}^0}) \, \mathrm{i}\mathcal{M}_{D^0 \to \bar{D}^0}$$

Expanding to **2nd order** in the weak Hamiltonian we get

$$\mathcal{M}_{D^0 \to \bar{D}^0} = - \langle \bar{D}^0, \boldsymbol{p}_D | \mathcal{H}_{\mathrm{w}}(0) | D^0, \boldsymbol{p}_D \rangle + \frac{\mathrm{i}}{2} \int \mathrm{d}^4x \, \langle \bar{D}^0, \boldsymbol{p}_D | \mathrm{T}\{ \mathcal{H}_{\mathrm{w}}(x) \mathcal{H}_{\mathrm{w}}(0) \} | D^0, \boldsymbol{p}_D \rangle$$

short-distance ($\Delta C = 2$)

long-distance ($\Delta C = 1$)

ETMC, PRD 92 (2015)
Fermilab/MILC, PRD 97 (2018)

Mixing amplitude

$$\left(\mathbf{M} - \frac{\mathrm{i}}{2}\mathbf{\Gamma}\right)_{12} = -\frac{1}{2m_D}\left[\mathcal{M}^{\mathsf{SD}}_{D^0\to\bar{D}^0} + \lim_{\epsilon\to 0}\int \frac{\mathrm{d}\omega}{2\pi}\,\frac{\rho(\omega)}{\omega - E_D - \mathrm{i}\epsilon}\right]$$

$$\rho(\omega) = \langle \bar{D}^0, \boldsymbol{p}_D |\, \mathcal{H}_{\mathrm{w}}(0)\,(2\pi)^4\delta(\hat{H}-\omega)\delta^3(\hat{\mathbf{P}}-\boldsymbol{p}_D)\,\mathcal{H}_{\mathrm{w}}(0)\,| D^0, \boldsymbol{p}_D\rangle$$

Theoretical roadblock

Traditional lattice methods: compute the contribution of **all states** to the spectral density

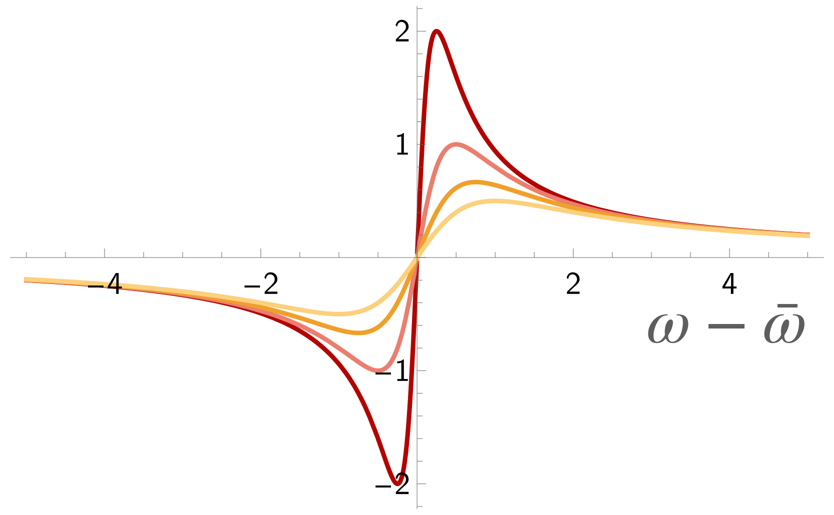
$$|n(\boldsymbol{p}_D)\rangle\langle n(\boldsymbol{p}_D)|$$

$$\boldsymbol{m_D \sim 13\, m_\pi \sim 3\, m_K}$$

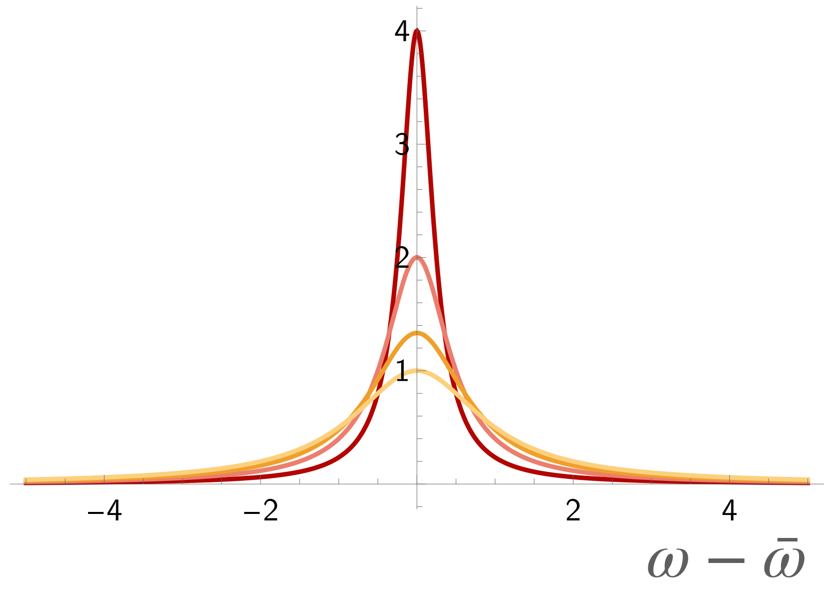
(7+ pages of hadronic modes listed in the PDG...)

Smeared spectral density

$$\mathcal{M}^{\mathsf{LD}}_{D^0\to\bar{D}^0} = \lim_{\epsilon\to 0}\int \frac{\mathrm{d}\omega}{2\pi}\,\frac{\rho(\omega)}{\omega - E_D - \mathrm{i}\epsilon}$$



$$\mathcal{K}^{\mathrm{R}}_{\epsilon}(\omega,\varpi) = \frac{\omega-\varpi}{(\omega-\varpi)^2+\epsilon^2}$$



$$\mathcal{K}^{\mathrm{I}}_{\epsilon}(\omega,\varpi) = \frac{\epsilon}{(\omega-\varpi)^2+\epsilon^2}$$

Consider now this quantity at finite **smearing parameter** ϵ :

$$\hat{\rho}(\varpi,\epsilon) = \int \frac{\mathrm{d}\omega}{2\pi}\,\frac{\rho(\omega)}{\omega-\varpi-\mathrm{i}\epsilon} = \int \frac{\mathrm{d}\omega}{2\pi}\,\rho(\omega)\,\mathcal{K}_{\epsilon}(\omega,\varpi) = \hat{\rho}^{\mathrm{R}}(\varpi,\epsilon) + \mathrm{i}\,\hat{\rho}^{\mathrm{I}}(\varpi,\epsilon)$$

→

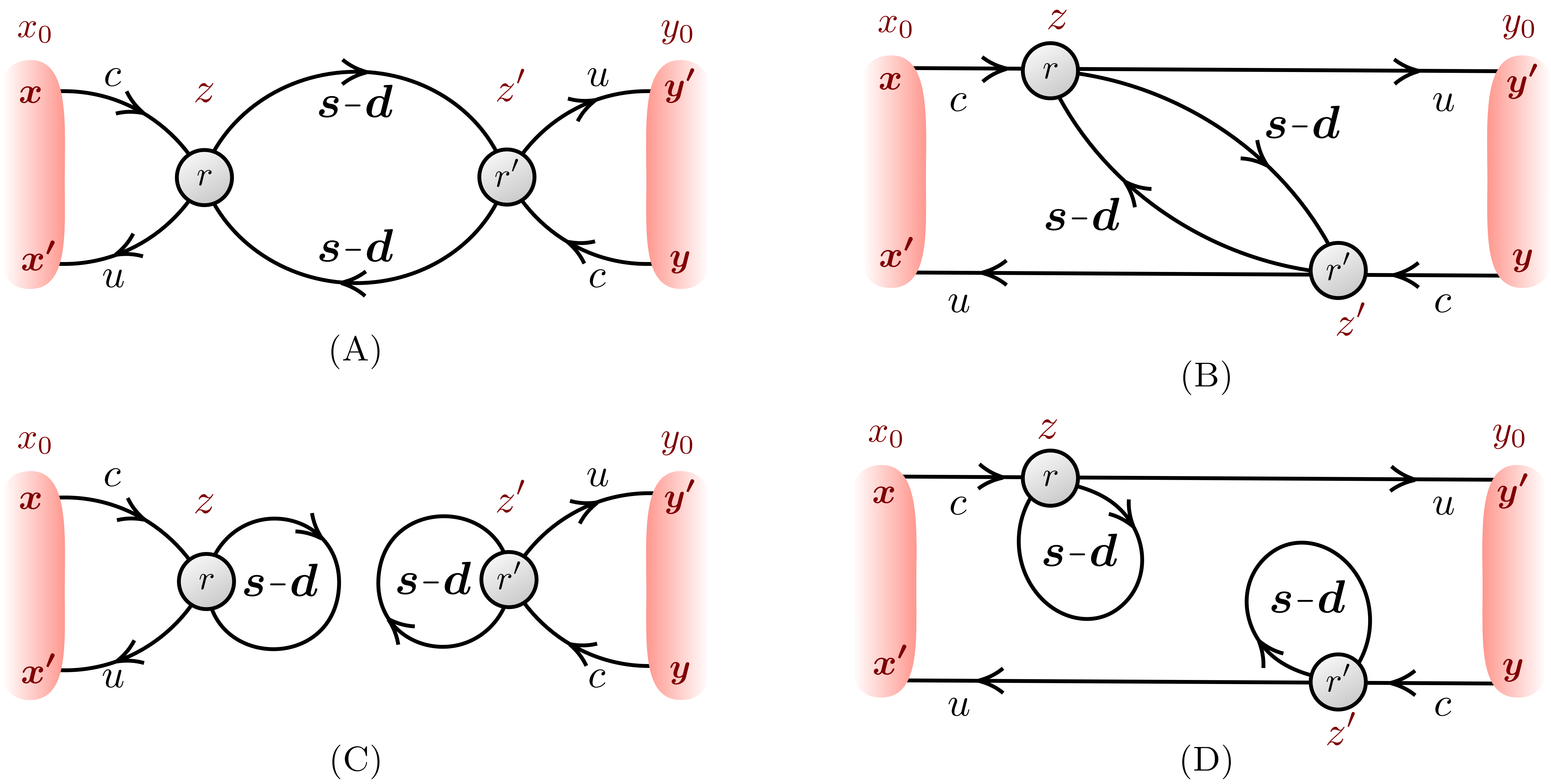
$$\mathrm{M}_{12} = -\frac{1}{2m_D}\mathcal{M}^{\mathsf{SD}}_{D^0\to\bar{D}^0} - \frac{1}{2m_D}\lim_{\epsilon\to 0}\hat{\rho}^{\mathrm{R}}(E_D,\epsilon)$$

$$\Gamma_{12} = \frac{1}{2m_D}\lim_{\epsilon\to 0} 2\,\hat{\rho}^{\mathrm{I}}(E_D,\epsilon)$$

... and apply the *same method as in inclusive hadronic tau decays.*

Lattice correlation functions

$$C_L(\tau) = 2E_D L^3 \int_L \mathrm{d}^3\boldsymbol{x}\, \mathrm{e}^{-E_D\tau} \langle \bar{D}^0, \boldsymbol{p}_D | \mathcal{H}_\mathrm{w}(\tau,\boldsymbol{x})\mathcal{H}_\mathrm{w}(0) | D^0, \boldsymbol{p}_D\rangle_L = \int \frac{\mathrm{d}\omega}{2\pi}\, \mathrm{e}^{-\omega\tau} \rho_L(\omega)$$



***s*-*d* structure** greatly simplifies the calculation and allows for variance reduction techniques

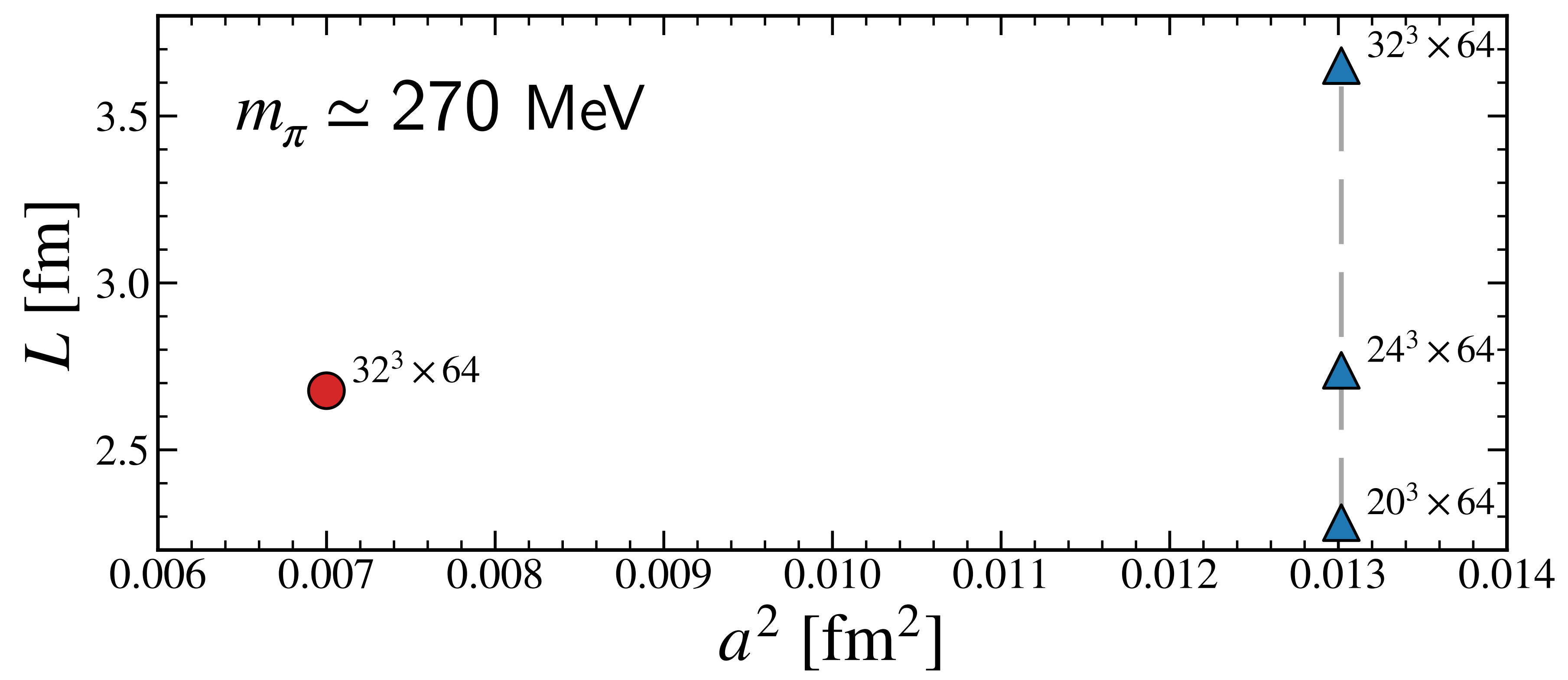
L.Giusti, T.Harris, A.Nada and S.Schaefer, EPJC 79 (2019)

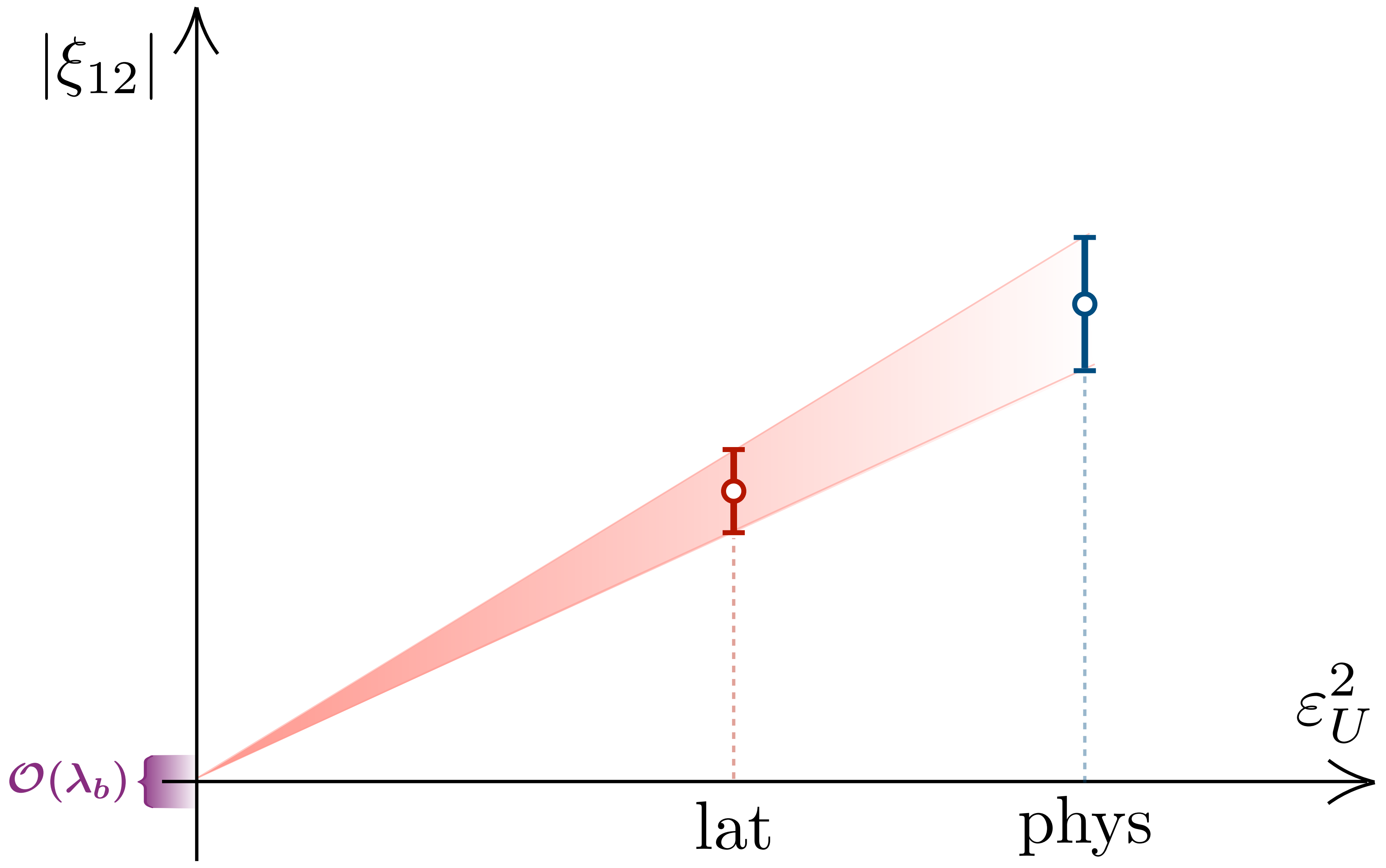
First lattice calculation

- Strategy proposed in recent publication
  MDC, F.Erben & M.T.Hansen, JHEP 07 (2025)
- Exploratory calculation ongoing:
  - 4 ensembles
  - 2 lattice spacings
  - 3 volumes
- Even *away from the physical point* we can obtain info on physical mixing parameters:

$$|\mathrm{M}_{12}|, |\Gamma_{12}| \propto \varepsilon_U^2 = (m_s - m_d)^2$$

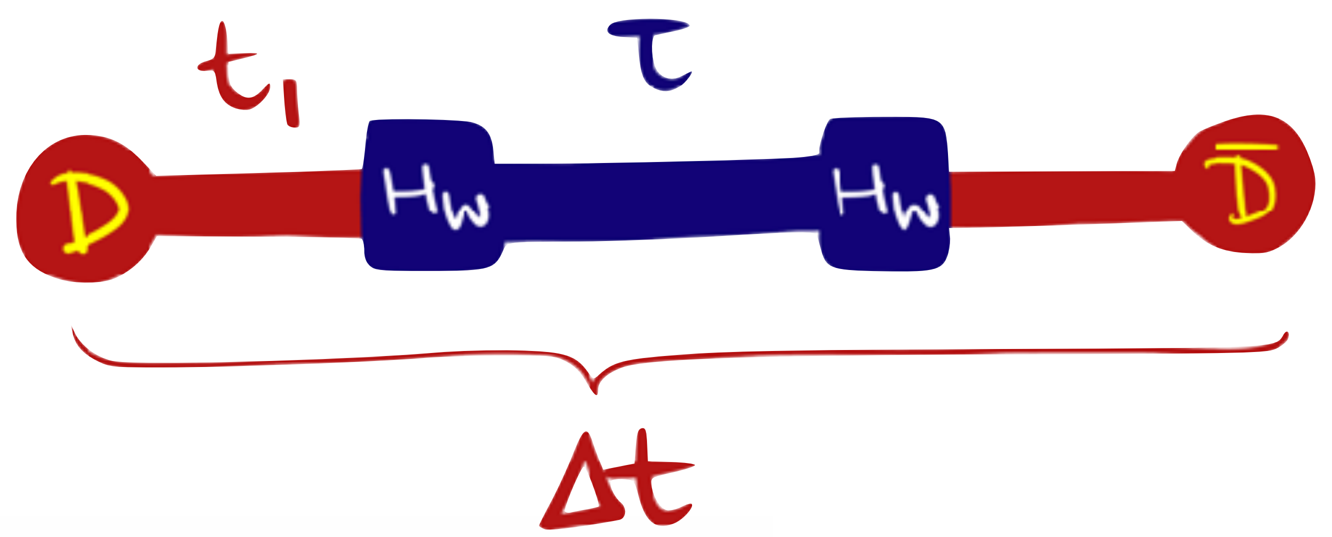
- Use of **chiral domain-wall fermions** essential for renormalisation of the weak operators

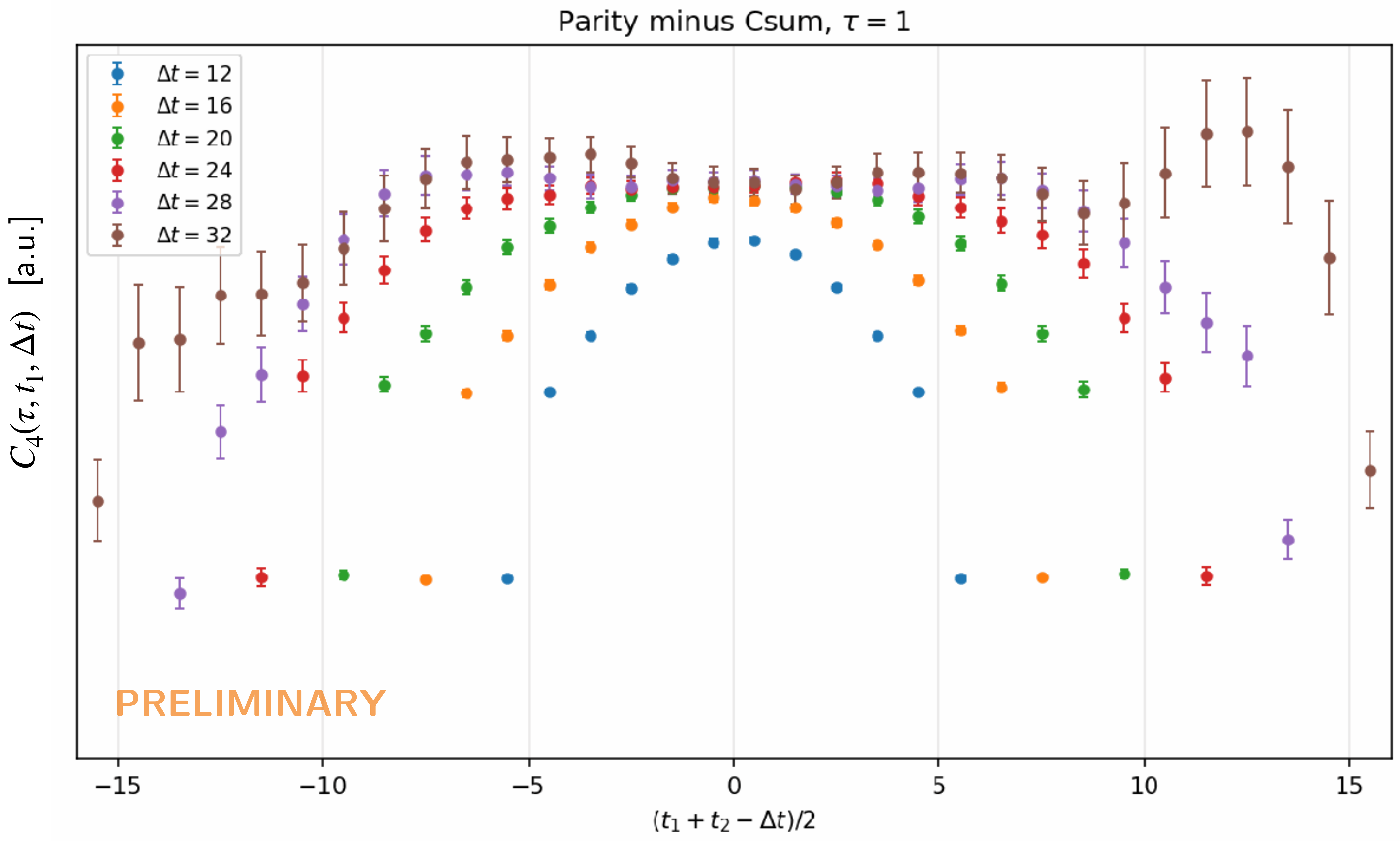




A first look at the data

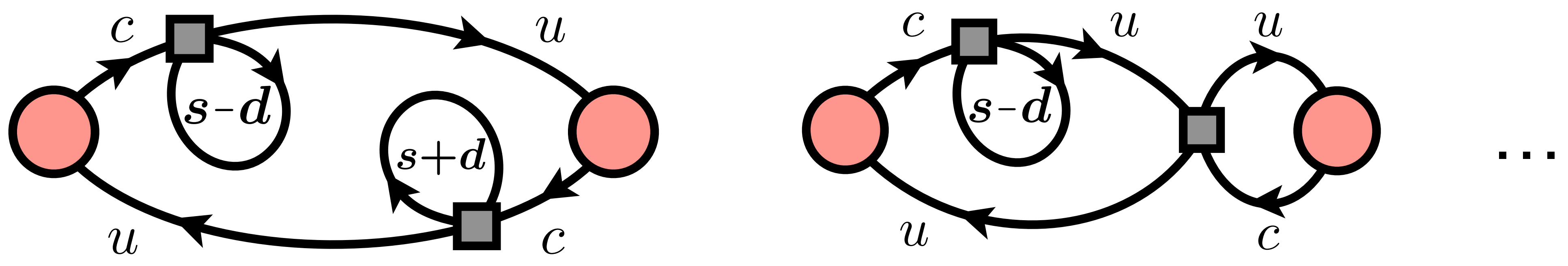
$$C_4(\tau, t_1, \Delta t) = \langle \phi_D(\Delta t)\, \mathcal{H}_{\rm w}(t_1+\tau)\, \mathcal{H}_{\rm w}(t_1)\, \phi_D(0) \rangle$$



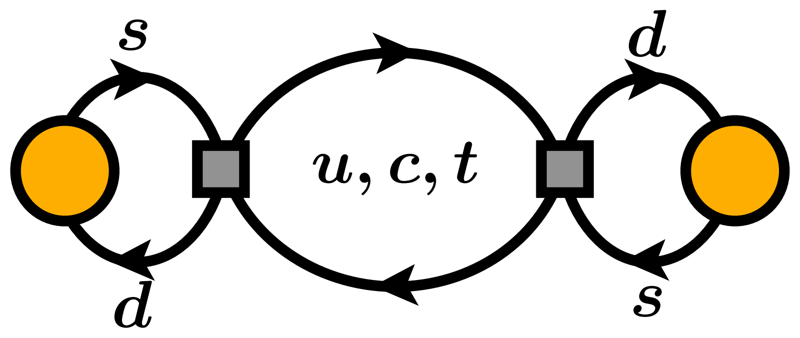


A long road ahead...

- Feasibility study ongoing at **unphysical masses**
- Planned extension to **physical point** ensembles
- Study of **CP violating phase** ϕ_{12} will require including penguin operators: *more correlation functions & more involved renormalisation mixing*



- The same method can be applied also to **kaon mixing**, offering an alternative to traditional approaches



N.H.Christ et al., PRD 88 (2013)
Z.Bai et al., PRL 113 (2014)
Z.Bai et al., PRD 109 (2024)

Conclusions

Long-distance hadronic dynamics becomes essential **for precision flavour phenomenology**

Lattice QCD is moving beyond local matrix elements:

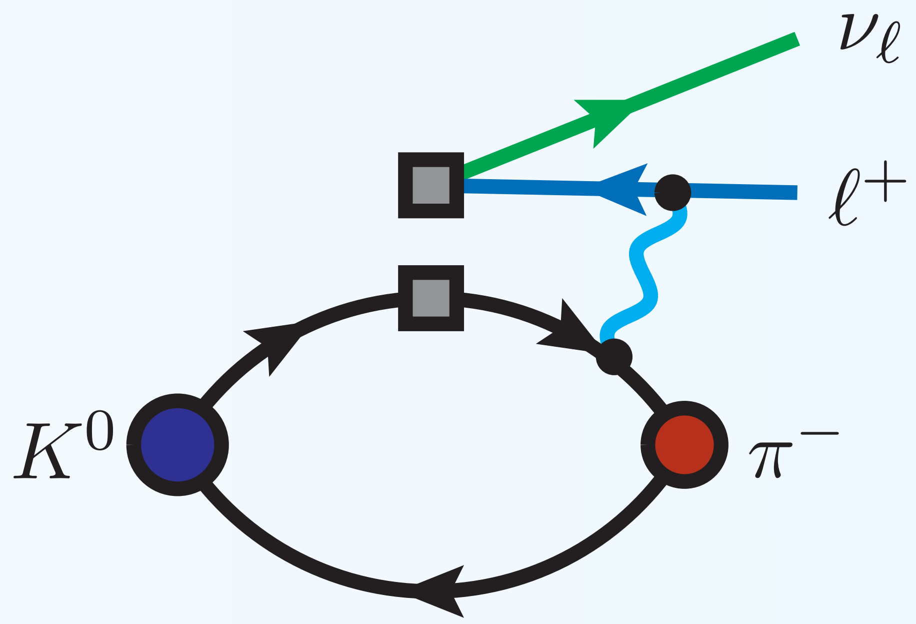
QED corrections | **inclusive rates** | **mixing amplitudes**

- The **Cabibbo "anomaly"** calls for controlled isospin-breaking effects and **lattice QCD+QED** calculations of weak decay amplitudes are possible
- Novel **spectral reconstruction** approaches allow calculations of **inclusive decay rates** with multi-particle final states in QCD+QED
- The same techniques pave the way for calculations previously considered unfeasible like **long-distance effects in neutral *D*-meson mixing amplitude**

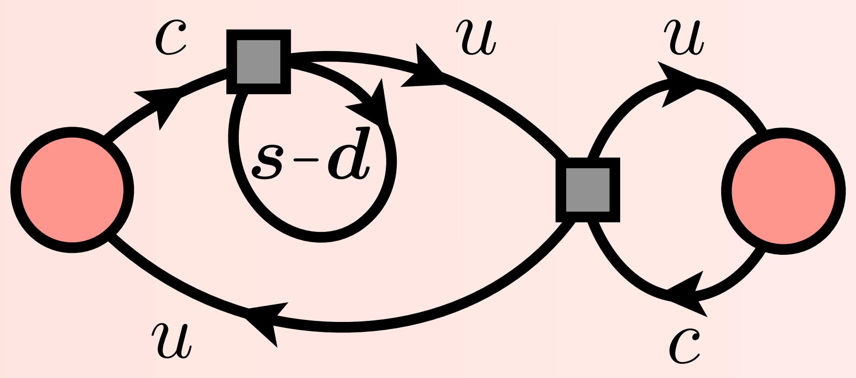
Future prospects

QED corrections to weak decays

- improve control of **finite-volume effects in leptonic decays**
- extend lattice QCD+QED calculations to **inclusive τ decays**
- investigate inclusive strategies for **semileptonic decay rates**



Neutral *D*-meson mixing



- complete the **first exploratory calculation** at unphysical masses
- move towards **physical-point** ensembles
- include ingredients needed to access **CP-violating phase**
- explore application to **kaon mixing**

Thank you