Analytic Continuation and Numerical Evaluation of Multivariate Hypergeometric Functions

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$${}_2F_1(a, b; c; x)$$

$$\sum_{m=0}^{\infty} \frac{(a)_m (b)_m}{(c)_m} \frac{x^m}{m!}$$

$$\frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-\ x\,t)^{-a}\,dt$$

$$[x(1-x)\partial_x^2 + (c-(a+b-1)x)\partial_x - a\,b] \bullet {}_2F_1 = 0$$

$$\frac{\Gamma(c)}{\Gamma(a)\Gamma(\mathrm{b})}\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\frac{\Gamma(a+s)\Gamma(b+s)\Gamma(-s)}{\Gamma(c+s)}(-x)^{-s}ds$$

Gauss Hypergeometric Function
(GHF)

Pochhammer symbol: $$(a)_m = \frac{\Gamma(a+m)}{\Gamma(a)}$$

When a is non-negative integer & m is positive integer:

$$(a)_m = a(a+1)\ldots(a+m-1) \qquad (1)_m = m!$$

Gauss hypergeometric function:

$$ {}_2F_1(a,b;c;x) = \sum_{m=0}^{\infty} \frac{(a)_m (b)_m}{(c)_m} \frac{x^m}{m!}$$

Valid for $|x| < 1$

- ❑ Integral form exists, Mellin-Barnes representation exists
- ❑ Solution of 2nd order ODE with three regular singular points
- ❑ Orthonormal polynomials can be expressed in terms of ${}_2F_1$

Multivariate Hypergeometric Functions

How about this?

$$F(x,y) = {}_2F_1(a,b;c;x) \ * \ {}_2F_1(a',b';c';y)$$

$$= \sum_{m=0}^{\infty} \frac{(a)_m (b)_m}{(c)_m} \frac{x^m}{m!} \quad * \sum_{n=0}^{\infty} \frac{(a')_n (b')_n}{(c')_n} \frac{y^n}{n!}$$

Nothing new here!

Multivariate Hypergeometric Functions

Appell functions:

$$F_1(a, b_1, b_2, c; x, y) = \sum_{m,n=0}^{\infty} \frac{(a)_{m+n}(b_1)_m(b_2)_n}{(c)_{m+n}} \frac{x^m y^n}{m!\,n!}$$

Valid for $|x| < 1 \wedge |y| < 1$

[P. Appell and J. Kampé de Fériet 1926]

$$F_2(a, b_1, b_2;\ c_1, c_2;\ x, y) = \sum_{m,n=0}^{\infty} \frac{(a)_{m+n}(b_1)_m(b_2)_n}{(c_1)_m(c_2)_n} \frac{x^m y^n}{m!\,n!}$$

Valid for $|x| + |y| < 1$

Appell F_3, F_4 Horn G_1, G_2, G_3, H_1, …, H_7

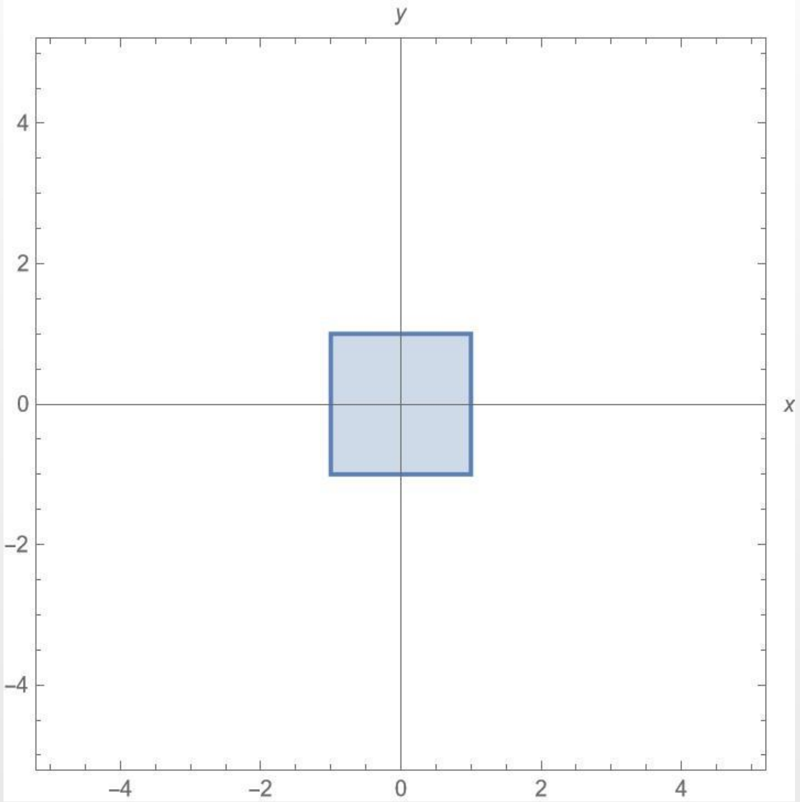
Kampé de Fériet function

} in 2 variables

Lauricella-Saran functions, Srivastava functions, ….

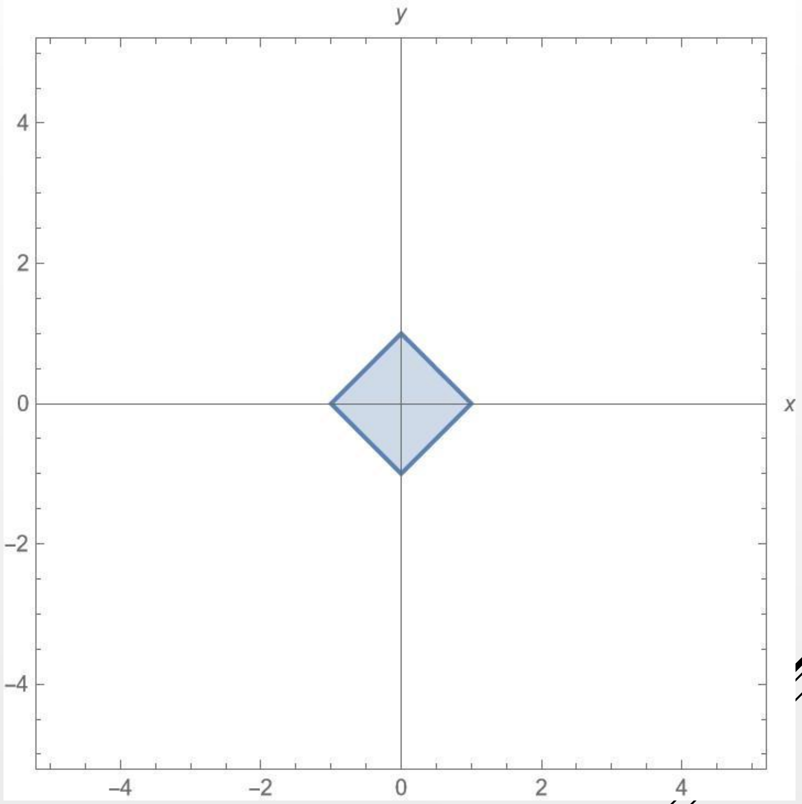
Multivariate Hypergeometric Functions

Region Of Convergence (ROC)



$|x| < 1 \wedge |y| < 1$

ROC of Appell F_1

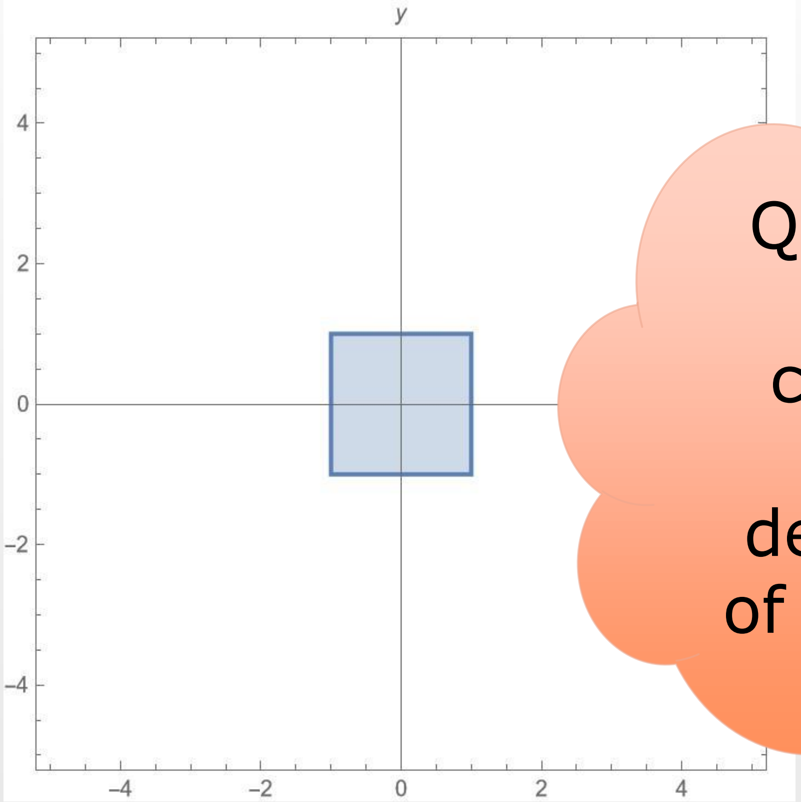


$|x| + |y| < 1$

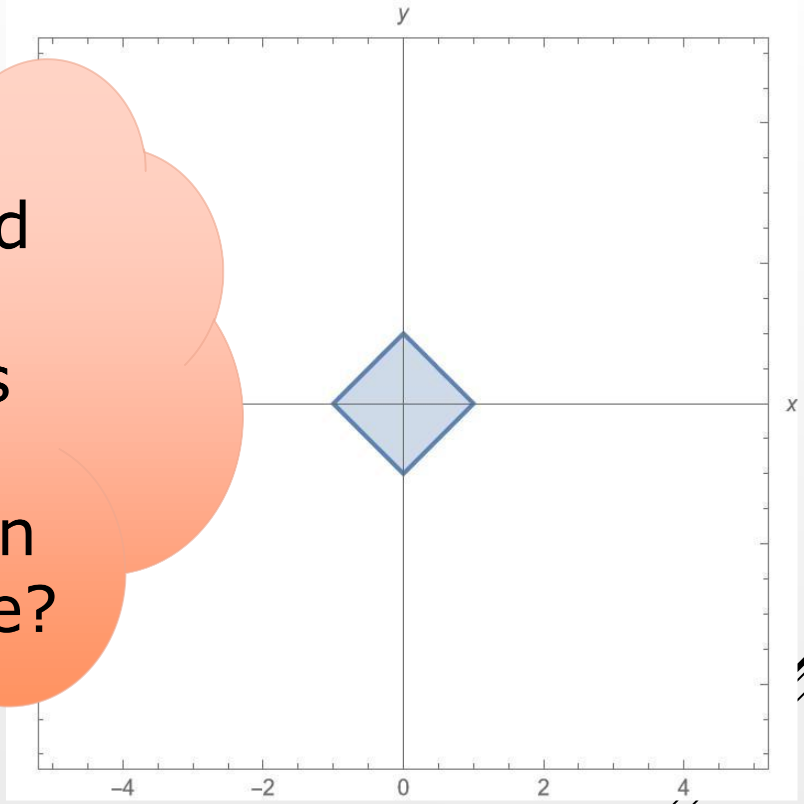
ROC of Appell F_2

Multivariate Hypergeometric Functions

Region Of Convergence (ROC)



Q: How to find analytic continuations beyond the defining region of convergence?



$|x| < 1 \wedge |y| < 1$

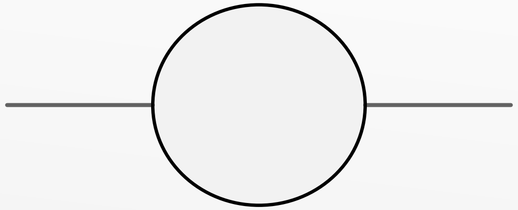
ROC of Appell F_1

$|x| + |y| < 1$

ROC of Appell F_2

Application 1

Feynman Integrals

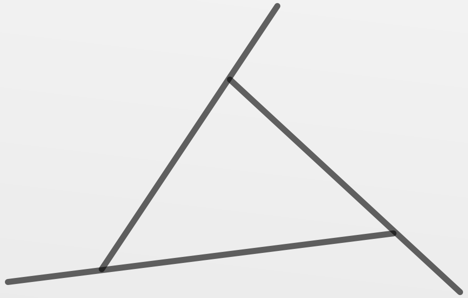


One loop two point function (Passarino Veltman B_0)

$$= \sum \text{prefactor} \times \ F_4(\dots\dots\dots\dots; x, y\ ) \qquad ; x = \frac{m_1^2}{p^2}, y = \frac{m_2^2}{p^2}$$

$$F_4(1, \epsilon, 2-\epsilon, \epsilon; x, y\ ), \qquad F_4(\epsilon, 2\epsilon - 1, \epsilon, \epsilon; x, y\ ), \dots$$

$$\boxed{d = 4\ - 2\,\epsilon}$$

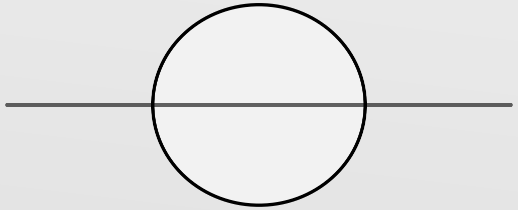


One loop three-point function

$$= \sum \text{prefactor} \times \ F_2(\dots\dots\dots\dots; x, y\ ) \qquad ; x = \frac{m_1^2}{m_2^2}, y = \frac{q_1^2}{m_2^2}$$

$$F_2(1+\epsilon, 1,1, \epsilon+1, 2-\epsilon, \epsilon; x, y\ ), \qquad F_2(1, 1-\epsilon, 1, 1-\epsilon, 2-\epsilon; x, y\ ), \dots$$

[Anastasiou, C. and Glover, E. W. Nigel and Oleari, C.; Nucl. Phys. B 572 (2000)]

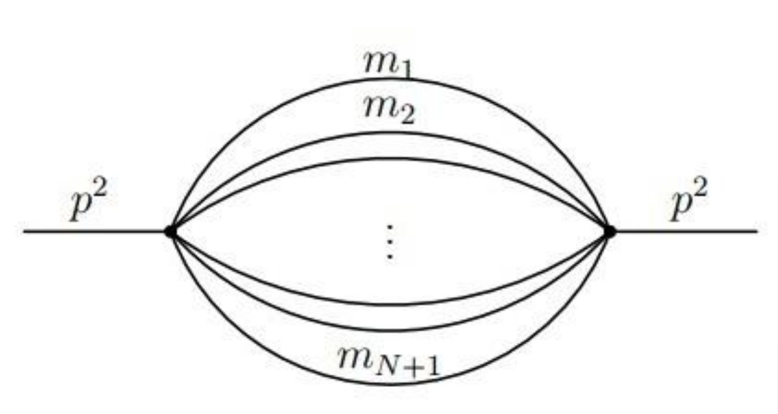


Two loop sunset $= \sum \text{prefactor} \times F_c^3(\dots\dots\dots\dots; x, y, z\ )$

[Berends, Frits A. and Buza, M. and Bohm, M. and Scharf, R., Z. Phys. C 63 (1994)]

Application 1

Feynman Integrals

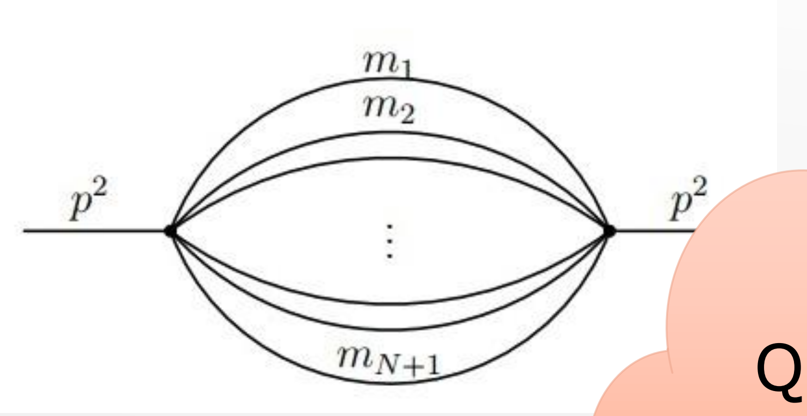


$$I_{1,1,\ldots,N+1}(p^2, m_1^2, \ldots, m_{N+1}^2) \sim \sum F_C^{(N)}\left(\frac{D-k(D-2)}{2}, \frac{2-k(D-2)}{2}; \{c_j^{(\mathcal{I})}\}_{j=1}^{N+1}; z_1, \ldots, z_{N+1}\right)$$

$$z_i = m_i^2/m_{N+1}^2 \text{ for } i = 1, \ldots, N,\ z_{N+1} = p^2/m_{N+1}^2.$$

Application 1

Feynman Integrals



Q: How to find ϵ expansion of such functions?

$$I_{1,1,\dots,N+1}(p^2, m_1^2, \dots,$$

$$\frac{2-k(D-2)}{2};\ \{c_j^{(\mathcal{I})}\}_{j=1}^{N+1};\ z_1, \dots, z_{N+1}\Big)$$

$z_i = m_i^2/m_{N+1}^2$ for $i = 1, \dots, N$, $z_{N+1} = p^2/m_{N+1}^2$.

Application 2

Angular Integrals

Lorentz invariant phase space integral (PSI)

$$\int \mathrm{dPS}_{2,P} = \int \frac{\mathrm{d}^{D-1}k_1}{(2\pi)^{D-1}2k_1^0} \int \frac{\mathrm{d}^{D-1}k_2}{(2\pi)^{D-1}2k_2^0} (2\pi)^D \delta^D (P - k_1 - k_2)$$

$$\int \mathrm{dPS}_{2,P} = \frac{\Gamma(1-\varepsilon)}{(4\pi)^{2-\varepsilon}\,\Gamma(1-2\varepsilon)} \left(P^2\right)^{-\varepsilon} \int \mathrm{d}\Omega_{k_1k_2} \,.$$

$$\mathrm{d}\Omega_{k_1k_2} \equiv \mathrm{d}\theta_1 \sin^{1-2\varepsilon}\theta_1 \mathrm{d}\theta_2 \sin^{-2\varepsilon}\theta_2$$

Neerven parametrization:

$$I_D^{(j,l)}(a,b,A,B,C) = \int \mathrm{d}\Omega_{k_1k_2} \frac{1}{(a+b\cos\theta_1)^j (A+B\cos\theta_1+C\sin\theta_1\cos\theta_2)^l} \,.$$

Somogyi parametrization

$$\int \mathrm{d}\Omega_{k_1k_2} \frac{1}{(v_1\cdot k)^j (v_2\cdot k)^l} \,,$$

Application 2
Angular Integrals

G Somogyi, J.Math.Phys. 52 (2011) 083501

$$\Omega_{j_1,\dots,j_n} = \int \mathrm{d}\Omega_{d-1}(q) \frac{1}{(p_1\cdot q)^{j_1}\dots(p_n\cdot q)^{j_n}},$$

$$\mathrm{d}\Omega_{d-1}(q) = \prod_{k=1}^{n} \mathrm{d}(\cos\vartheta_k)\,(\sin\vartheta_k)^{-k+1-2\epsilon}\mathrm{d}\Omega_{d-1-n}(q),$$

Result exists in terms of multi-fold Mellin-Barnes integrals

$$\Omega_{j_1,\dots,j_n}(\{v_{kl}\};\epsilon) = 2^{2-j-2\epsilon}\pi^{1-\epsilon}\frac{1}{\prod_{k=1}^{n}\Gamma(j_k)\Gamma(2-j-2\epsilon)} \times \int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty}\left[\prod_{k=1}^{n}\prod_{l=k}^{n}\frac{\mathrm{d}z_{kl}}{2\pi\mathrm{i}}\Gamma(-z_{kl})\,(v_{kl})^{z_{kl}}\right]\left[\prod_{k=1}^{n}\Gamma(j_k+z_k)\right]\Gamma(1-j-\epsilon-z).$$

2 denominator massless case:

$$\Omega_{j,k}(v_{12},v_{11},0;\epsilon) = 2^{2-j-k-2\epsilon}\pi^{1-\epsilon}\frac{\Gamma(1-k-\epsilon)}{\Gamma(2-k-2\epsilon)}v_{12}^{-j} \times F_1\left(j,1-k-\epsilon,1-k-\epsilon,2-k-2\epsilon,\frac{2v_{12}-1-\sqrt{1-4v_{11}}}{2v_{12}},\frac{2v_{12}-1+\sqrt{1-4v_{11}}}{2v_{12}}\right)$$

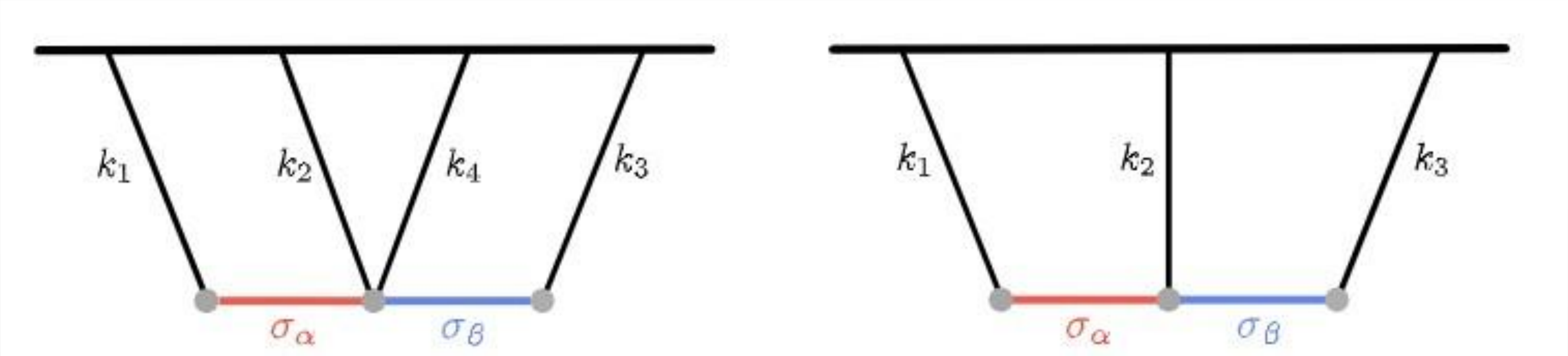
2 denominator massive case:

$$\Omega_{j,\ell}(v_1,v_2) = \frac{2\pi\, v_{12}^{1-j-\ell-\varepsilon}}{1-2\varepsilon} F_B^{(3)}\left(\tfrac{j}{2},\tfrac{\ell}{2},\tfrac{3-j-\ell}{2}-\varepsilon;\ \tfrac{j+1}{2},\tfrac{\ell+1}{2},\tfrac{1-j-\ell}{2}-\varepsilon;\ \tfrac{3}{2}-\varepsilon;\ x_1,x_2,x_3\right)$$

V. E. Lyubovitskij, F. Wunder, and A. S. Zhevlakov., JHEP 06 (2021)

Application 3

Cosmological & Holographic Correlators



Double massive exchange diagrams

[S. Aoki, L. Pinol, F. Sano, M. Yamaguchi, and Y. Zhu, JHEP 09 (2024), p.176]

$$\langle \varphi_{\mathbf{k_1}} \varphi_{\mathbf{k_2}} \varphi_{\mathbf{k_3}} \varphi_{\mathbf{k_4}} \rangle' \qquad \langle \varphi_{\mathbf{k_1}} \varphi_{\mathbf{k_2}} \varphi_{\mathbf{k_3}} \rangle'$$

φ : Inflation fluctuation field

3 pt / 4 pt functions $\sim \sum$ Multivariate Hypergeometric Functions

Problem Statements

1. How to find analytic continuations of multivariate hypergeometric functions?

2. How to find expansion of hypergeometric functions about their parameters?

Approach 1:

a. Derive analytic continuations first
b. Perform expansion around parameters

Approach 2:

a. Perform expansion around parameters first
b. Find analytic continuations of the resulting functions in the expansion

Problem 1: Analytic Continuations

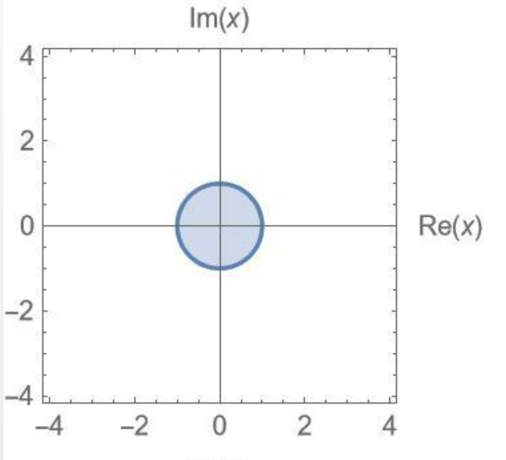
Analytic Continuations of GHF

$$ {}_2F_1(a,b;c;x) = \frac{\Gamma(c)\Gamma(\mathrm{c-a-b})}{\Gamma(c-a)\Gamma(\mathrm{c-b})}\, {}_2F_1(a,b;a+b+1-c;1-x) $$

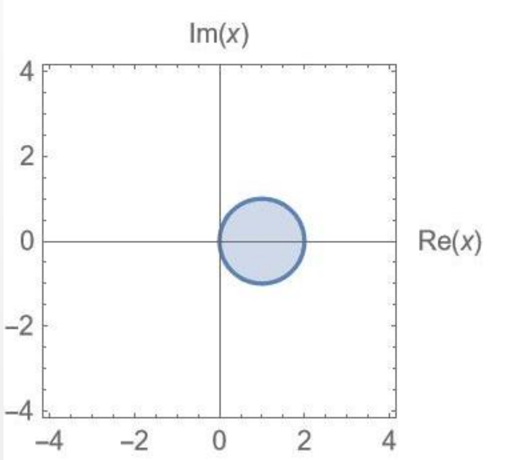
$$ +(1-x)^{c-a-b}\frac{\Gamma(c)\Gamma(\mathrm{a+b}-c)}{\Gamma(a)\Gamma(\mathrm{b})}\, {}_2F_1(c-a,c-b;\ 1+c-a-b;1-x) $$

Analytic Continuations of GHF

$$ {}_2F_1(a,b;c;x) = \sum \text{prefactor} \times \ {}_2F_1(\dots\dots\dots\dots;1-x) $$

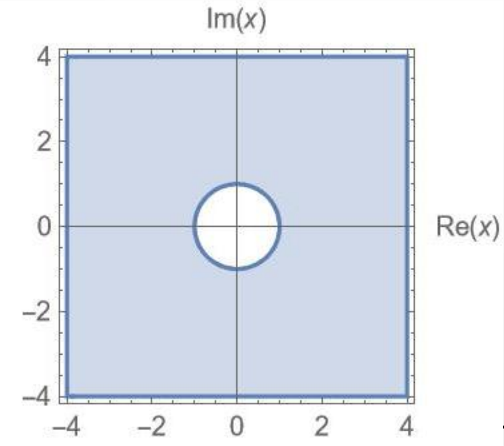


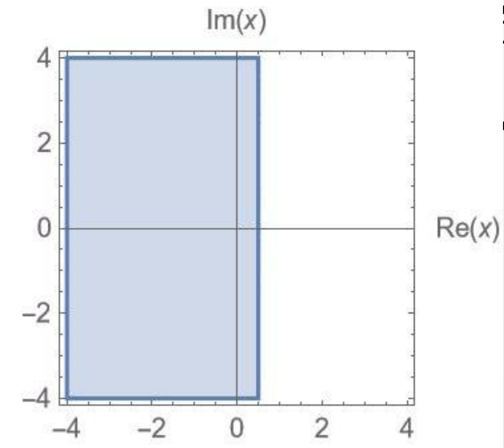
$|x| < 1$



$|1-x| < 1$

$\left|\frac{1}{x}\right| < 1$



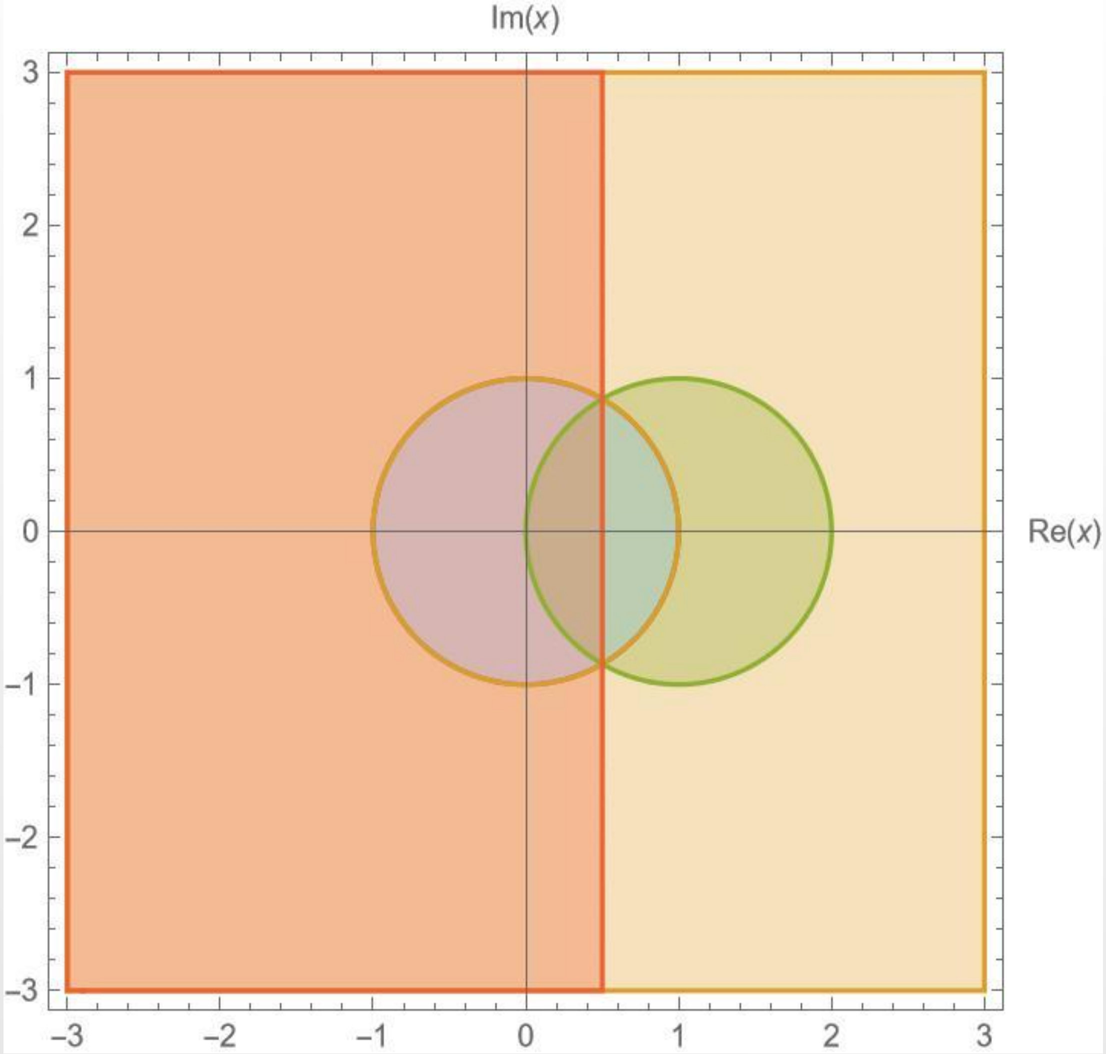


$\left|\frac{x}{x-1}\right| < 1$

$$ {}_2F_1(a,b;c;x) = \sum \text{prefactor} \times \ {}_2F_1\left(\dots\dots\dots\dots;\frac{1}{x}\right) $$

$$ {}_2F_1(a,b;c;x) = \text{prefactor} \times \ {}_2F_1\left(\dots\dots\dots\dots;\frac{x}{x-1}\right) $$

Analytic Continuations of GHF



1999 Appell F1 in Mathematica 4.0

2004 f1: a code to compute Appell's F1 hypergeometric function

F. D Colavecchia and G Gasaneo; Comput. Phys. Commun. 157 (2004)

2017 Appell F1, F2, F3, F4 included in Maple 2017

2021 On the evaluation of the Appell F2 double hypergeometric function

B. Ananthanarayan, S. Bera, S. Friot, O. Marichev, and T. Pathak; Comput. Phys. Commun. 284 (2023), p. 108589

2023 Appell F2, F3, F4 included in Mathematica 13.3

2024 Analytic continuations and numerical evaluation of the Appell F1, F3, Lauricella FD3 and Lauricella-Saran FS3 and their application to Feynman integrals

S. Bera and T. Pathak; Comput. Phys. Commun. 306 (2025) 109386

2025 PrecisionLauricella: Package for numerical computation of Lauricella functions depending on a parameter (F1,F2,F3 and FA3, FB3, FD3)

M.A. Bezuglov, B.A. Kniehl, A.I. Onishchenko and O.L. Veretin; Comput.Phys.Commun. 316 (2025) 109812

⋮

Analytic Continuations

The Method Of Olsson [P. O. M. Olsson, J. Math. Phys. 5 (1964), Pp. 420–430]

$$F_2(a, b_1, b_2;\ c_1, c_2;\ x, y) = \sum_{m,n=0}^{\infty} \frac{(a)_{m+n}(b_1)_m(b_2)_n}{(c_1)_m(c_2)_n} \frac{x^m y^n}{m!\,n!}$$

$$= \sum_{n=0}^{\infty} \frac{(a)_n(b_2)_n}{(c_2)_n} \frac{y^n}{n!}\, {}_2F_1(a+n, b_1; c_1; x\ )$$

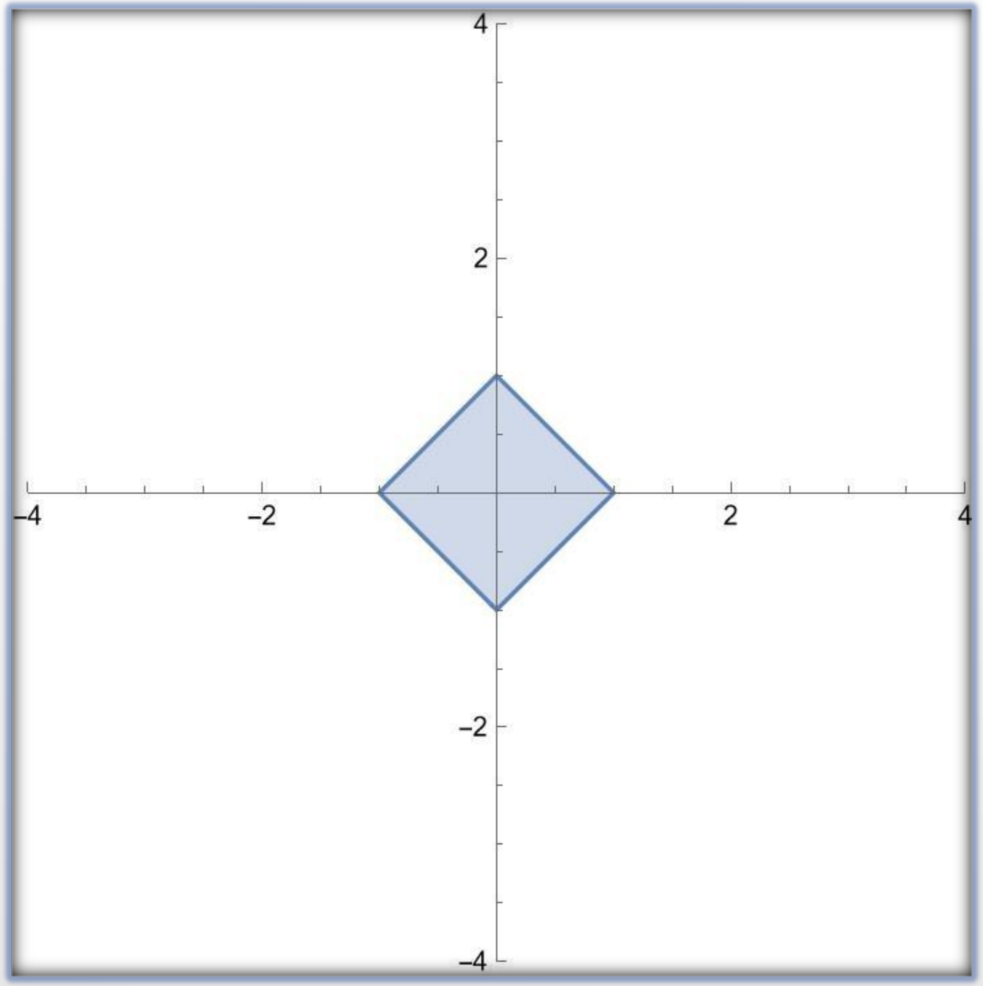
Using the analytic continuations of ${}_2F_1$, one can find analytic continuations of F_2

The method of Olsson implemented in **Olsson.wl** [B. Ananthanarayan, **SB**, S. Friot, and T. Pathak, Comput. Phys. Commun. 300 (2024), p. 109162]

companion package **ROC2.wl** to find region of convergence of bivariate series

Applied to find analytic continuations of $F_1, F_2, F_3, H_1, H_5, FD(3), FS(3)$

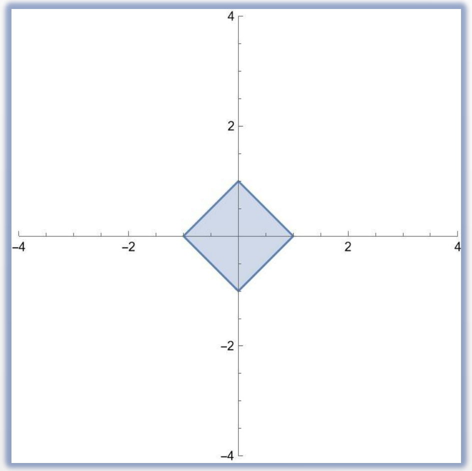
Analytic Continuations of F_2



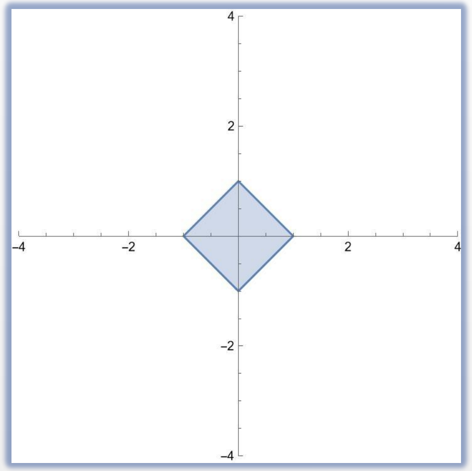
Defining domain of convergence of Appell F2

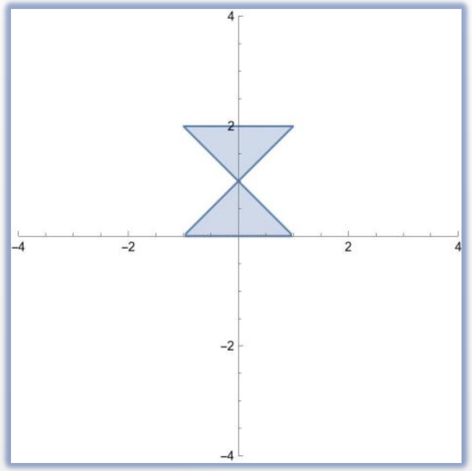
$|x| + |y| < 1$

Analytic Continuations of F_2

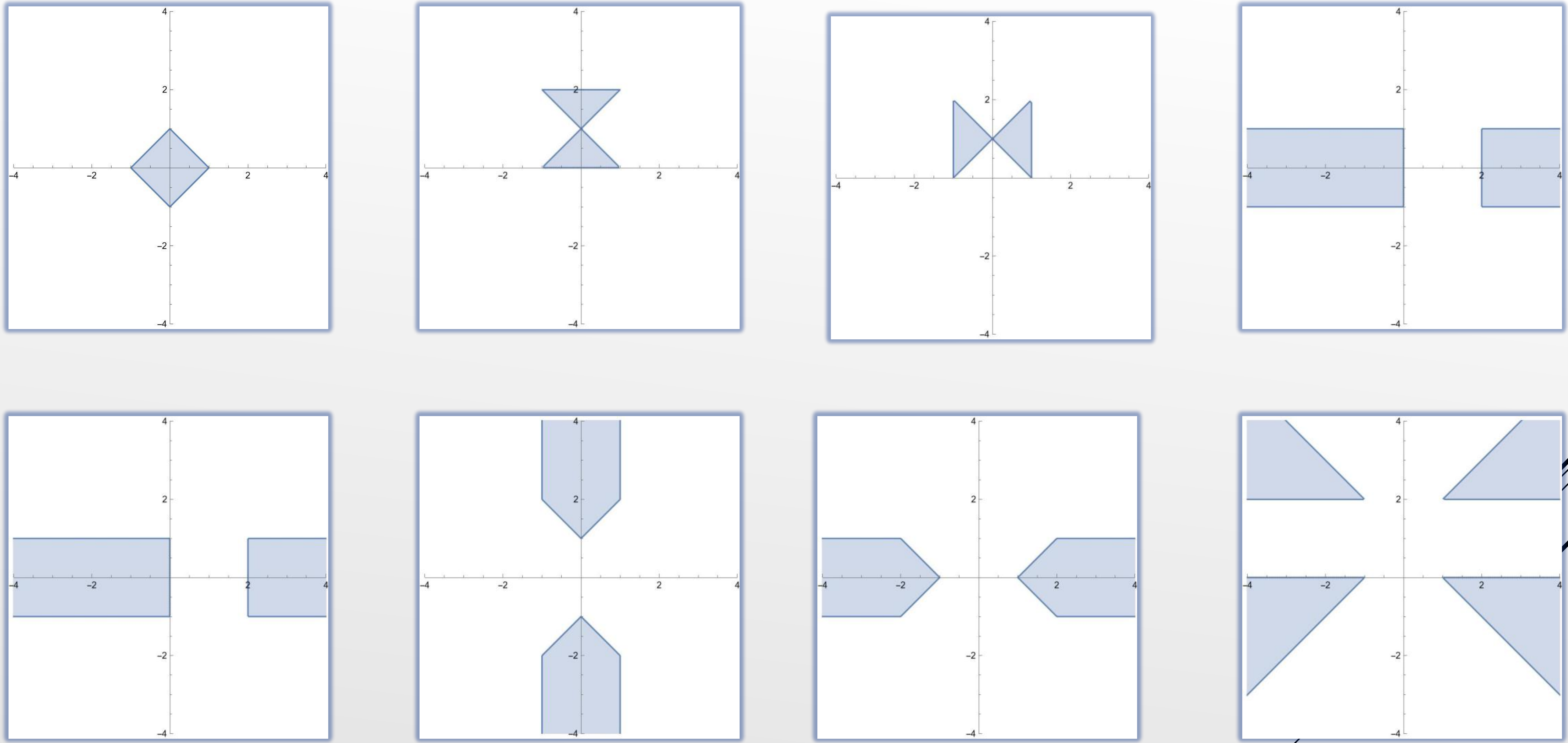


Analytic Continuations of F_2



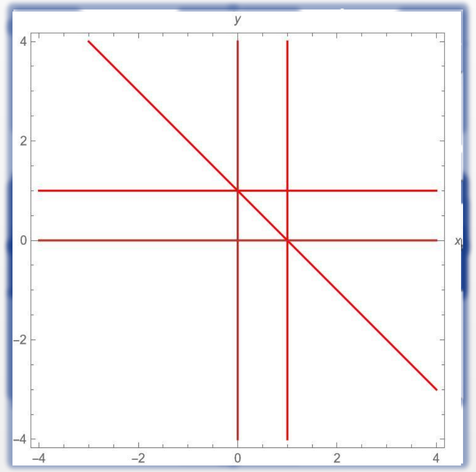


Analytic Continuations of F_2

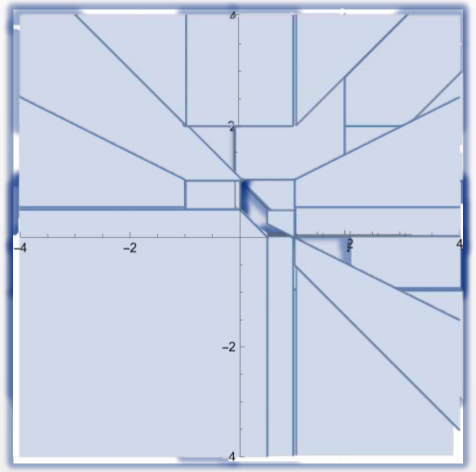


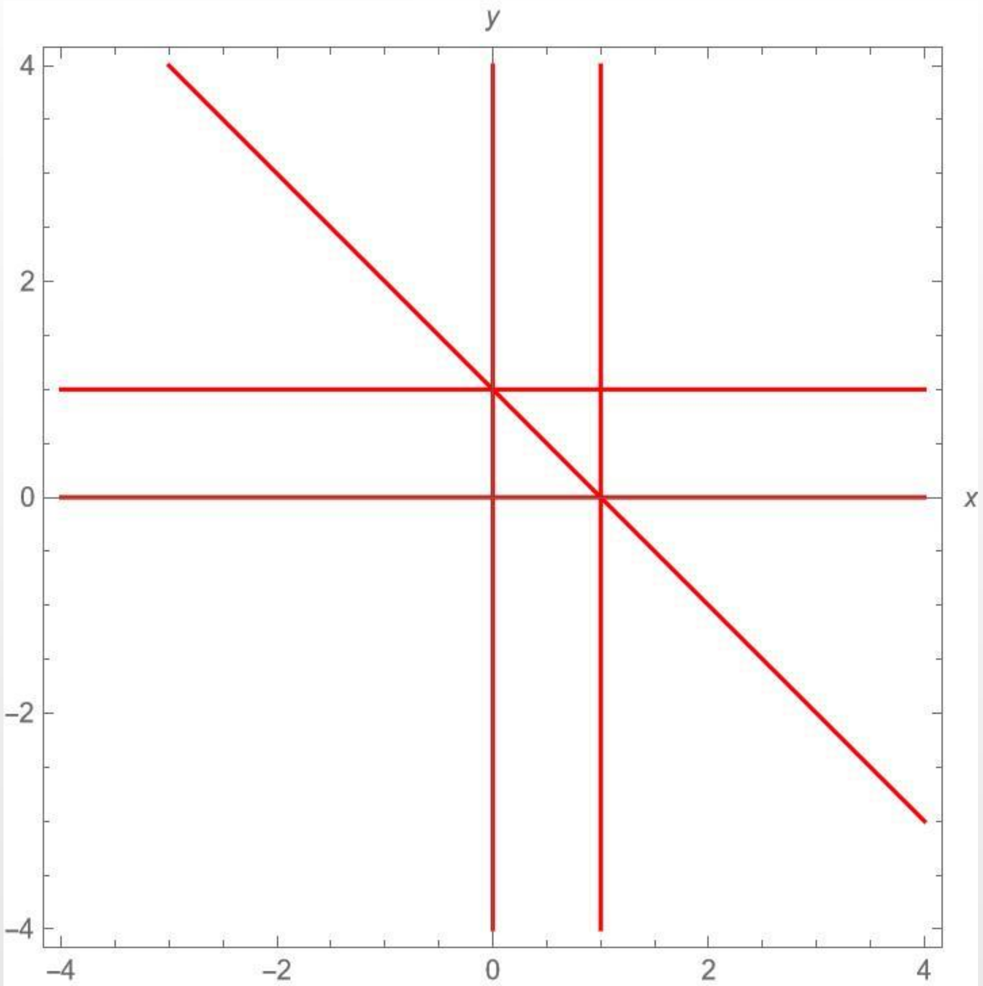
+…= total 44

Analytic Continuations of F_2



Analytic Continuations of F_2





$x = 0$
$y = 0$
$x = 1$
$y = 1$
$x + y = 1$

Singularities of Appell F2

AppellF2 Package

```
In[1]:= << AppellF2.wl

AppellF2.wl v1.0
Authors : Souvik Bera & Tanay Pathak
```

[B. Ananthanarayan, **SB**, S. Friot, O. Marichev, and T. Pathak
Comput. Phys. Commun. 284 (2023), p. 108589]

parameters | (x, y) | Precision, upper limit of summation

```
In[2]:= AppellF2[1.2, 1.23, 4.13, 5.7, 2.773, 1.3, 3.1, 20, 100]

valid series : {{7}, {10}, {21}, {26}, {37}, {38}}

convergence rates :{{0.44185902695642353820542803519, 37}, {0.44956345927019833753209725152, 7}, {0.47448422406017966133586719222, 21},
  {0.81484221048546334095945717296, 10}, {0.92764579225393021824102261507, 26}, {1.05658450414215910171093065978, 38}}

selected series : 37

Out[2]= -0.10991301435506057035 4 + 0.075673136129987512686 i
```

Applicable for generic complex values of parameters, real values of (x, y)

Gelfand Kapranov Zelevinsky System

(GKZ System)

Multivariate Hypergeometric functions satisfy PDEs

GKZ systems = A set of PDEs $H_A(\nu)$

$$H_A(\nu) \bullet I(\nu) = 0$$

Solutions for $I(\nu)$ are series of hypergeometric type

Feynman integrals in QFT, Witten diagrams in CFT, cosmological correlators

~ solutions of GKZ systems

Problem 2: Expansion around parameter

2002 Symbolic expansion of transcendental functions

S. Weinzierl; Comput. Phys. Commun. 145 (2002) 357

2006 HypExp: A Mathematica package for expanding hypergeometric functions around integer-valued parameters

T. Huber and D. Maitre; Comput. Phys. Commun. 175 (2006) 122

XSummer: Transcendental functions and symbolic summation in form

S. Moch and P. Uwer; Comput. Phys. Commun. 174 (2006) 759

2008 HypExp 2, Expanding Hypergeometric Functions about Half-Integer Parameters

T. Huber and D. Maitre; Comput. Phys. Commun. 178 (2008) 755

2013 Analytic and Algorithmic Aspects of Generalized Harmonic Sums and Polylogarithms

J. Ablinger, J. Blümlein and C. Schneider; J. Math. Phys. 54 (2013) 082301

NumExp: Numerical epsilon expansion of hypergeometric functions

Z.-W. Huang and J. Liu; Comput. Phys. Commun. 184 (2013) 1973

2024 MultiHypExp: A Mathematica package for expanding multivariate hypergeometricfunctions in terms of multiple polylogarithms

S. Bera; Comput. Phys. Commun. 297 (2024) 109060

Expansion of hypergeometric functions in terms of polylogarithms with a nontrivial change of variables

M. A. Bezuglov and A. I. Onishchenko; Theor.Math.Phys. 219 (2024) 3, 871-896

ϵ-Expansion Of Hypergeometric Functions

The differential equation associated with hypergeometric functions can be brought to a Pfaff system.

a, b, c depends of ϵ

$$[x(1-x)\partial_x^2 + (\ldots)\partial_x - a\, b] \bullet {}_2F_1 = 0$$

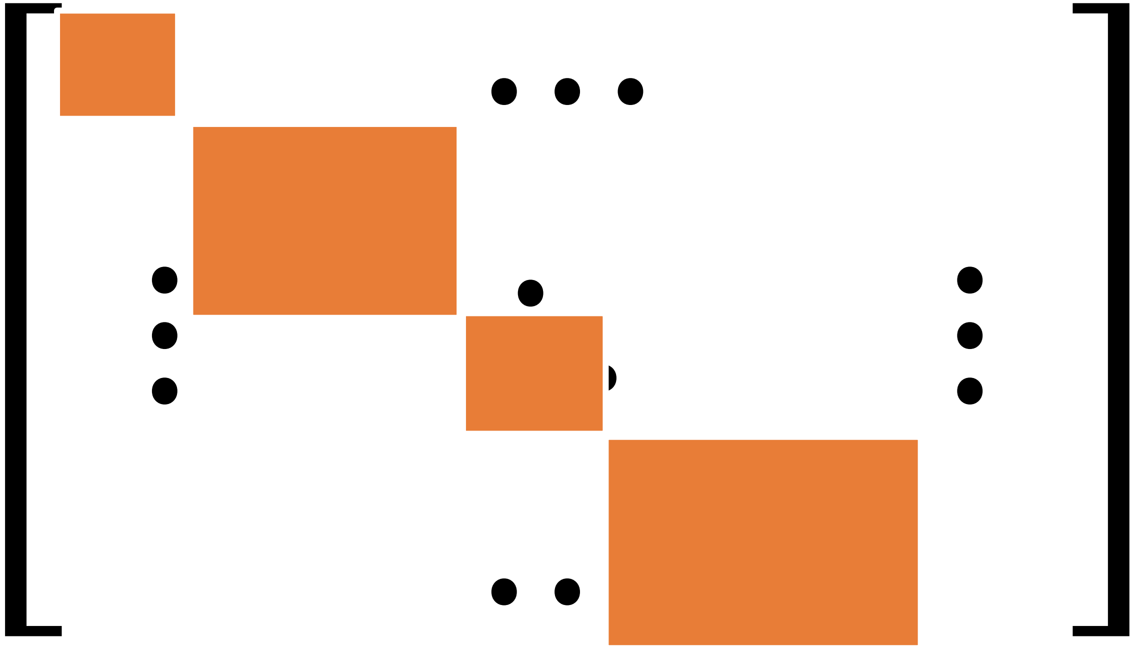
~ Integration-by-parts identities for Feynman integrals

Consider $g = (2F_1, x\, \partial_x\, {}_2F_1)$ ~ Master integrals

$$dg = \left[\Omega_{2\times 2}(\epsilon, x) \right] g$$

ϵ-B

Th
bro

$dI =$  I

ls

Consider $g = (2F_1, x\,\partial_x\,{}_2F_1)$ ~ Master integrals

$$dg = \left[\ \Omega_{2\times2}(\epsilon, x)\ \right] g$$

ϵ-Expansion Of Hypergeometric Functions

Change of basis: $g = \ T\, g'$ brings the DE to the *canonical form*

[J. Henn, *Phys.Rev.Lett.* 110 (2013) 251601]

$$dg' = \epsilon\, \Omega'(x)\, g' \qquad \Omega' = T^{-1}\Omega T \ - T^{-1} dT$$

With the boundary condition $g = (1,0)$,
the system can be solved order by order in ϵ

$$\boxed{{}_2F_1(\epsilon, -\epsilon; 1+\epsilon; x) = 1 - \epsilon^2\, Li_2(x) + \epsilon^3 \left(Li_3(x) + Li_{1,2}(1,x)\right) + O(\epsilon^4)}$$

MultiHypExp

```
In[17]:= AppendTo[$Path, sourcepath];
         << MultiHypExp.wl

         MultiHypExp v1.0
         Author : Souvik Bera (souvikbera@iisc.ac.in)
```

[**SB**, Comput. Phys. Commun. 297 (2024), p. 109060]

Expansion of

$F_1(\epsilon, \epsilon, \epsilon, 1+\epsilon; x, y)$

```
In[20]:= SeriesExpand[F1, {e, e, e, 1 + e} , {x, y} , e , 4]
```

```
Out[20]= 1 + (-G[0, 1, x] - G[0, 1, y]) e^2 +
   (G[1, y] × G[0, 1, x] - G[1, y] × G[0, y, x] + G[0, 0, 1, y] +
     G[0, 1, 1, x] + G[0, 1, 1, y] + G[0, y, 1, x]) e^3 + O[e]^4
```

Applicable for certain two and three variable hypergeometric functions around integer valued parameters.

Bottlenecks

In some cases, a transformation of variables is needed to bring the Pfaff system to the canonical form

Elliptic integrals arise

$$ {}_2F_1\left(\frac{1}{2},\frac{1}{2};1;x^2\right) = \frac{2}{\pi}\,K(x) $$

$K(x)$ is the complete elliptic functions of first find

$$ K(x) = \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-x^2t^2)}} $$

$$ {}_2F_1\left(\frac{1}{2}+\epsilon,\frac{1}{2}+\epsilon;1+\epsilon;x^2\right) = \frac{2}{\pi}\,K(x) + (??)\epsilon + (???)\epsilon^2 + \ \dots $$

Numerical Approach

Step 1: Obtain the Pfaffian system

Step 2: Find the boundary condition

Step 3: Numerically solve the system

Step 1: Obtain The Pfaffian System

Appell F_2

$$F_2(a, b_1, b_2; c_1, c_2; x, y) = \sum_{m,n=0}^{\infty} \frac{(a)_{m+n}(b_1)_m(b_2)_n}{(c_1)_m(c_2)_n} \frac{x^m y^n}{m!\, n!} = \sum_{m,n=0}^{\infty} A(m, n)\, x^m y^n,$$

ratios of neighbouring coefficients

$$P_i(\mathbf{m}) := \frac{A(\mathbf{m} + \mathbf{e}_i)}{A(\mathbf{m})} = \frac{g_i(\mathbf{m})}{h_i(\mathbf{m})}, \qquad i = 1, \ldots, n,$$

For F_2

$$P_1(m, n) = \frac{(b_1 + m)(a + m + n)}{(m + 1)(c_1 + m)}, \qquad P_2(m, n) = \frac{(b_2 + n)(a + m + n)}{(n + 1)(c_2 + n)}.$$

Step 1: Obtain The Pfaffian System

The annihilators are:

$$L_i \;=\; h_i(\boldsymbol{\theta})\,\frac{1}{x_i} \;-\; g_i(\boldsymbol{\theta})\,, \qquad\qquad i=1,\ldots,n\,,$$

For F_2

$$L_1 = -ab_1 + \Big(c_1 - x(a+b_1+1)\Big)\partial_x - b_1 y\,\partial_y - xy\,\partial_x\partial_y - (x-1)x\,\partial_{xx},$$
$$L_2 = -ab_2 + \Big(c_2 - y(a+b_2+1)\Big)\partial_y - b_2 x\,\partial_x - xy\,\partial_x\partial_y - (y-1)y\,\partial_{yy}.$$

The basis: $(\,\theta_x = x\,\partial_x, \theta_y = y\,\partial_y)$

$$\mathbf{g} = \Big(F_2,\, \theta_x F_2,\, \theta_y F_2,\, \theta_x\theta_y F_2\Big)^T,$$

$$\mathrm{d}\mathbf{g} = (\Omega_1\,\mathrm{d}x + \Omega_2\,\mathrm{d}y)\,\mathbf{g}.$$

Step 1: Obtain The Pfaffian System

$$\Omega_1 = \begin{pmatrix} 0 & \frac{1}{x} & 0 & 0 \\ -\frac{ab_1}{x-1} & \frac{ax+b_1x-c_1+1}{x(1-x)} & -\frac{b_1}{x-1} & \frac{1}{1-x} \\ 0 & 0 & 0 & \frac{1}{x} \\ \frac{ab_1b_2y}{(x-1)(x+y-1)} & \frac{b_2y(a+b_1-c_1+1)}{(x-1)(x+y-1)} & \frac{b_1\left(b_2y-(x-1)(a-c_2+1)\right)}{(x-1)(x+y-1)} & \frac{b_2xy-(x-1)\left(ax+b_1x-c_2x+c_1y-c_1+x-y+1\right)}{(x-1)\,x\,(x+y-1)} \end{pmatrix},$$

$$\Omega_2 = \begin{pmatrix} 0 & 0 & \frac{1}{y} & 0 \\ 0 & 0 & 0 & \frac{1}{y} \\ -\frac{ab_2}{y-1} & -\frac{b_2}{y-1} & \frac{ay+b_2y-c_2+1}{y(1-y)} & \frac{1}{1-y} \\ \frac{ab_1b_2x}{(y-1)(x+y-1)} & \frac{b_2\left(b_1x-(y-1)(a-c_1+1)\right)}{(y-1)(x+y-1)} & \frac{b_1x(a+b_2-c_2+1)}{(y-1)(x+y-1)} & \frac{b_1xy-(y-1)\left(ay+b_2y+c_2x-c_1y-c_2-x+y+1\right)}{(y-1)\,y\,(x+y-1)} \end{pmatrix}.$$

The annihilators are differentiated sufficient times and solved

To bypass ‘intermediate expression swell’, FiniteFlow is used

[T. Peraro. JHEP 12 (2016), p. 030] [JHEP 07 (2019), p. 031]

Step 2: Obtain The Boundary Condition

$$F_2(a, b_1, b_2; c_1, c_2; x, y) = \sum_{m,n=0}^{\infty} \frac{(a)_{m+n}(b_1)_m(b_2)_n}{(c_1)_m(c_2)_n} \frac{x^m y^n}{m!\,n!} = \quad 1 + (\dots)x + (\dots)y + (\dots)x\,y + \ \dots$$

So, $F_2(\dots, 0{,}0) = 1$

$$\mathbf{g} = \left(F_2,\ \theta_x F_2,\ \theta_y F_2,\ \theta_x\theta_y F_2\right)^T, \qquad (\ \theta_x = x\,\partial_x, \theta_y = y\,\partial_y)$$

So, $\theta_x\, F_2(\dots, 0{,}0) = 0$

$$g(0{,}0) = (1{,}0{,}0{,}0)$$

Step 3: Numerical Solution

Pfaff system can be solved using the Frobenius method

- DiffExp [M. Hidding, Comput. Phys. Commun. 269 (2021), p. 108125]
- LINE [R. M. Prisco, J. Ronca, and F. Tramontano, JHEP 07 (2025)]
- SeaSyde [T. Armadillo, R. Bonciani, S. Devoto, N. Rana, and A. Vicini, Comput. Phys. Commun. 282 (2023)]
- AMFlow DE solver [X. Liu and Y.-Q. Ma, Comput. Phys. Commun. 283 (2023),p. 108565]

Or using Bulirsch-Stoer (BS) algorithm

- Implementation in C++, Julia, Python,…

Or using Runge-Kutta (RK) algorithm

- Implementation in C++, Julia, Python,…

Step 3: Numerical Solution

Parameterize the path:

$$x_i(\eta) = \frac{x_{0,i}}{1+\eta}, \qquad i = 1, \ldots, n.$$

PDE → ODE

$$\frac{\mathrm{d}\mathbf{g}}{\mathrm{d}\eta} = M(\eta)\,\mathbf{g},$$

$$M(\eta) = \sum_{i=1}^{n} \frac{\partial x_i}{\partial \eta} \Omega_i\big(\mathbf{x}(\eta)\big) = -\sum_{i=1}^{n} \frac{x_{0,i}}{(1+\eta)^2} \Omega_i\big(\mathbf{x}(\eta)\big).$$

In the η plane, the boundary point lies at infinity

↓ Transport

The target point lies at the origin.

Using AMFlow DE solver

Crosschecked against results from BS algorithm

HyperPrecision

arXiv:2605.XYZ

HyperPrecision

A Mathematica package for High-Precision Numerical Evaluation of Multivariate Hypergeometric Functions

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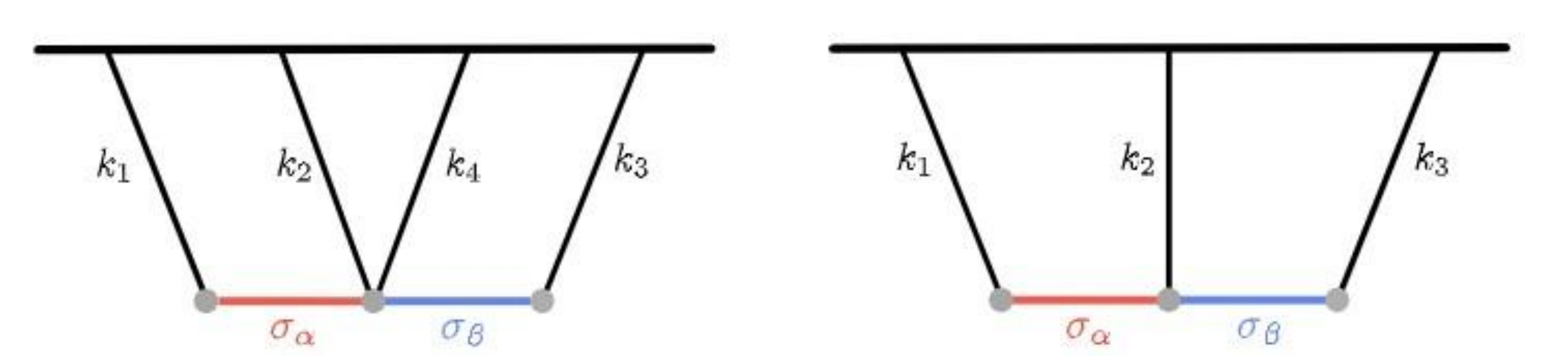
github.com/HyperPrecision/HyperPrecision

```
In[]:=  HypFunctionExpand[AppellF2[2, 3/2, 1 + ϵ, 4, -1 - ϵ, 3, 11/3], {ϵ, 1},
    10]
Out[]=  (0.528662817 - 4.194390019 I) + 0.5149686376/ϵ
        - (10.978138236 + 4.834942296 I) ϵ
```

Cosmological Correlators

Closed-form expressions for correlators involving massive-field exchange



Double massive exchange diagrams

[S. Aoki, L. Pinol, F. Sano, M. Yamaguchi, and Y. Zhu, JHEP 09 (2024), p.176]

$$\Sigma(u,v) = 2\,\mathrm{Re}\Big[I^{+-+}(u,v) + I^{++-}(u,v) + I^{-++}(u,v) + I^{+++}(u,v)\Big]\,,$$

The seed integrals $I^{+-+}, I^{++-}, \ldots$ contain Appell F_4, Kampé de Fériet functions

Cosmological Correlators

Normalized shape function

$$\frac{S(k, x_2 k, x_3 k)}{S(k,k,k)} = \frac{1}{\Sigma(1,1)}\left[\frac{x_3}{x_2^2}\,\Sigma\!\left(\frac{1}{x_2},\frac{x_3}{x_2}\right) + x_2 x_3\,\Sigma(x_2,x_3) + \frac{x_2}{x_3^2}\,\Sigma\!\left(\frac{1}{x_3},\frac{x_2}{x_3}\right)\right],$$

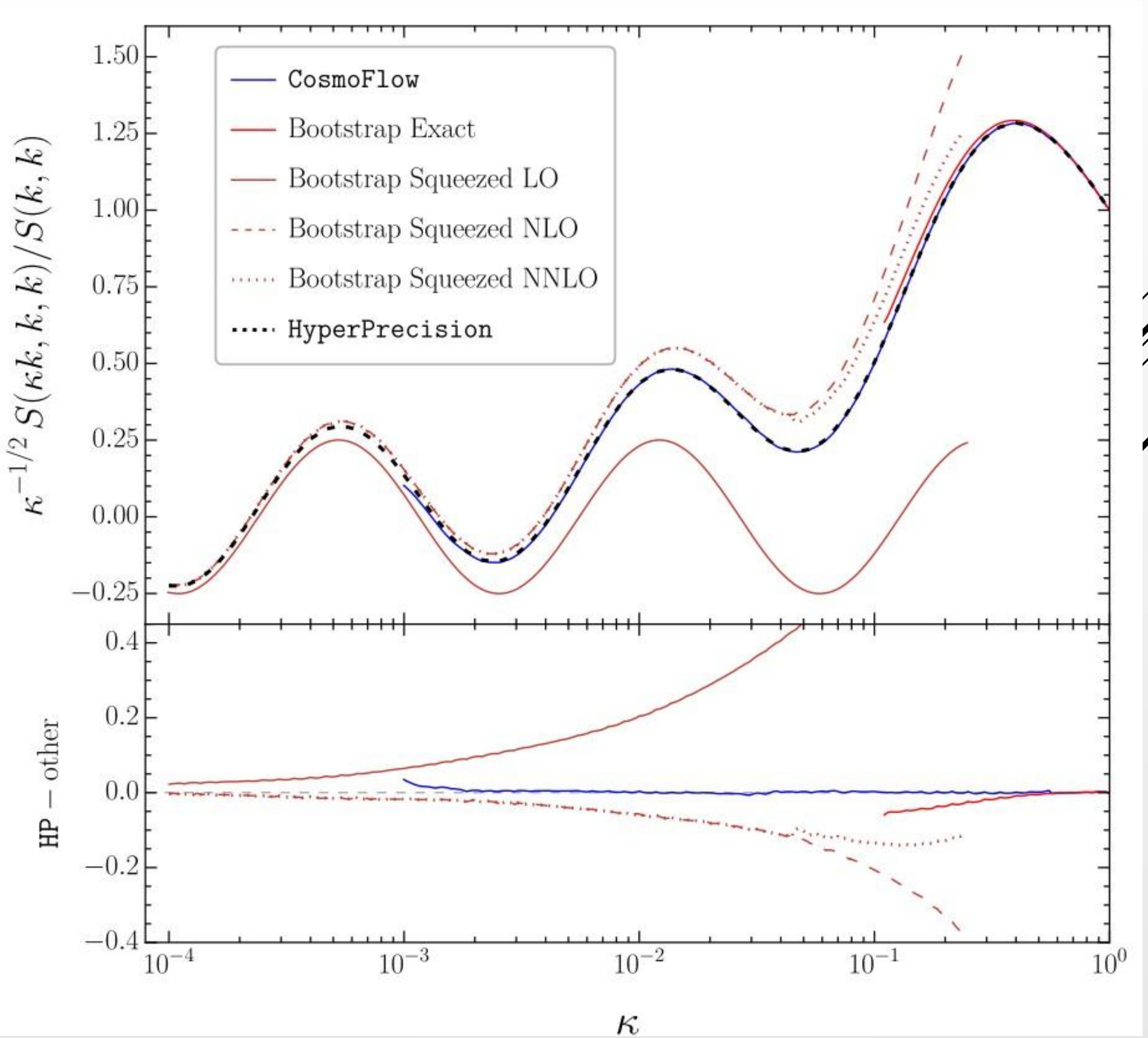
Contains 27 double variable series

$$\sim (\dots.)\, F_4\left(\dots;\frac{1}{x_3^2},\frac{x_2^2}{x_3^2}\right)$$

$$+ (\dots.)\, F_4(\dots; x_2^2, x_3^2)$$

$$+ \dots$$

$$+ (\dots.)\, KdF\left(\dots;\frac{2}{1+x_2+x_3},\frac{2\,x_2}{1+x_2+x_3}\right)$$



Summary

- HyperPrecision can find expansion of multivariate hypergeometric functions for real values of arguments with high precision
- Complex values of the arguments can be done
- For faster numerical evaluation, we plan to include a C++ interface of BS algorithm

Multivariate hypergeometric functions appear in many branches of physics and mathematics.
We are hopeful that HyperPrecision will be a useful tool for the broader scientific community.

Thank You!